Optimal Control Techniques for Sampled-Data Control Systems with Medical Applications

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1. Thèse CIFRE : **The Mathematical Frame and the Application to functional Muscular Control**, 2020–

Control system of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad x(T) = x_T$$

and an optimal control of the Mayer type

$$\min_{u(\cdot)} \quad \varphi(x(T))$$

with $\varphi : \mathbb{R}^n \to \mathbb{R}$.

- *Permanent Control :* $u : [0, T] \mapsto U$, $u \in \mathcal{U}$ where $\mathcal{U}$ are the absolutely continuous maps valued in U.
- *Sampled Data Case* $u \in \mathcal{U}_{sampled}$: fix $n \in \mathbb{N}$,
  - n sampling times
$$0 < t_1 < \cdots < t_n < T.$$
  - $(n+1)$-amplitudes
$$\eta = (\eta_0, \ldots, \eta_n) \in [0,1]^{n+1}$$

  The control is constant over $[t_i, t_{i+1}]$.

Numerical schemes

In the **permanent case**, the optimal control can be computed using

- *Direct scheme :* the problem is transformed into a finite dimensional optimization problem using
  - discretization scheme for the dynamics
  - discretization scheme for the control
- *Indirect scheme :* the problem can be analyzed using Pontryagin Maximum Principle which leads to Hamiltonian equations

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

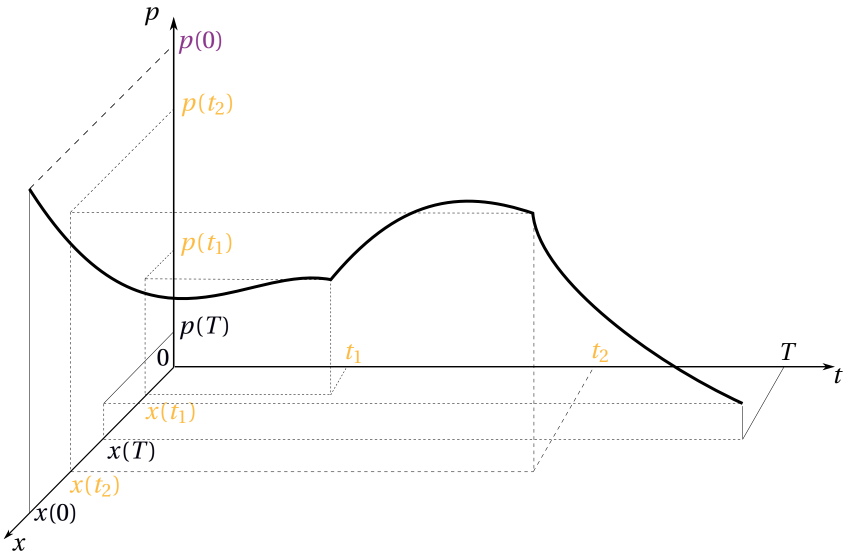
$$H(x,p,u) = \max_{v \in U} H(x,p,v)$$

with $H(x,p,v) = p \cdot f(x,v)$, p being the adjoint vector satisfying

$$p(T) = p_0 \frac{\partial \varphi}{\partial x}(x(T)) \qquad \text{(transversality condition).}$$

This necessary optimality condition can be handled using a *shooting method*

Shooting method



Force-Fatigue muscular model [2]

FES input i. Dirac impulses $\boldsymbol{\delta}$ at times $t = 0, t_1, t_2, \ldots, t_N$.

$$i(t) = \sum_{i=0}^{N} R_i \eta_i \boldsymbol{\delta}(t - t_i), \quad \eta_i \in [0,1]$$

where

$$R_i := \begin{cases} 1, & \text{for } i = 0, \\ 1 + (\bar{R} - 1)\exp\left(-\dfrac{t_i - t_{i-1}}{\tau_c}\right), & \text{for } i = 1, \ldots, N, \end{cases}$$

takes into account the *tetanic* contraction.

FES signal E_s.

$$E_s(t) = \frac{1}{\tau_c} \sum_{i=0}^{N} R_i \eta_i \boldsymbol{H}(t - t_i) \exp\left(-\frac{t - t_i}{\tau_c}\right)$$

$\boldsymbol{H}$: Heaviside

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2. based on the Ding et al. / Hill-Huxley model

The FES signal drives the evolution of the dynamics :

$$\dot{C}_N(t) = -\frac{C_N(t)}{\tau_c} + E_s(t; t_i, \eta_i),$$
$$\dot{F}(t) = -F(t)\, \boldsymbol{\gamma(t)} + A\, \boldsymbol{\beta(t)}.$$

where the Hill functions are given by

$$\boldsymbol{\beta(t)} := \frac{C_N(t)}{K_m + C_N(t)}, \quad \text{and} \quad \boldsymbol{\gamma(t)} := \frac{1}{\tau_1 + \tau_2 \beta(t)}.$$

(A, K_m, τ_1, τ_2) are the fatigue parameters.

The problem fits in the sampled data control frame with :

- $u_0 = \eta_0 e^{-t/\tau_c}/\tau_c$ on $[0, T]$
- $u_1 = u_0(t_1) + \eta_1 R_1 e^{-(t-t_1)/\tau_c}/\tau_c$ on $[t_1, T]$

  $\vdots$

Note that in this form each control splits into

- *Head* : restricting to $[t_i, t_{i+1}]$
- *Tail* : restricting to $[t_i, T]$

Optimal control problems considered.

- **(OCP1)** $\max_{t_i,\eta_i} F(T)$
- **(OCP2)** $\min_{t_i,\eta_i} \int_0^T |F(t) - F_{ref}|^2 \mathrm{d}t$ (F_{ref} : reference force).

Main theoretical results

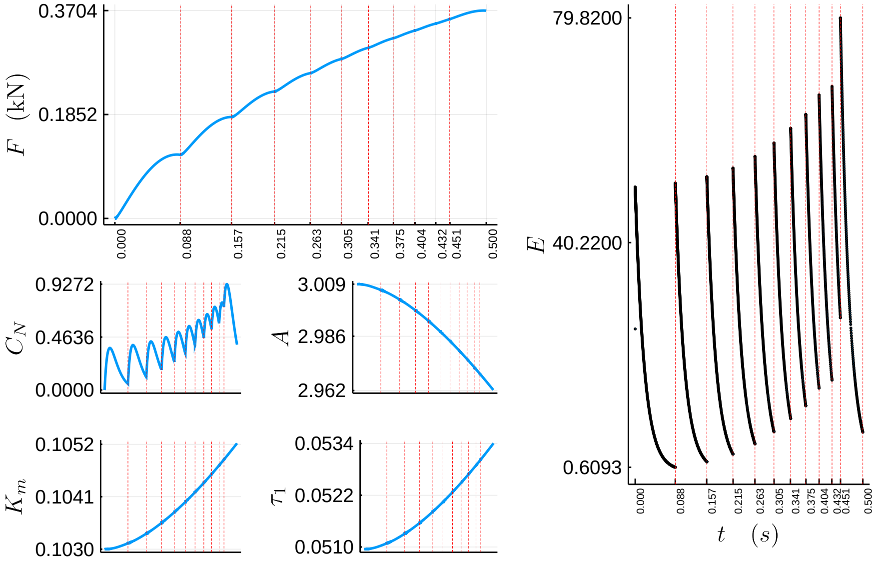
- $$c_N(T) = \frac{1}{\tau_c} \sum_{i=0}^{n} e^{-(t-t_i)/\tau_c} (T - t_i) R_i$$

  $F(T)$ is a piecewise C^∞ mapping with respect to η_i, t_i and the problem fits in a finite dimensional optimization problem.
- *The non-fatigue model* for a sequence of train, fatigue parameters P satisfy a dynamics of the form

  $$\dot{P}(t) = \frac{P(t) - P_{rest}}{\tau_p} + \alpha_p F(t).$$

- First order necessary optimality conditions can be obtained adapting [5] and are described in [2]. They were numerically implemented and compared with a direct optimization scheme in [3].

Numerical results using direct method



Theorem (see [2])

If $(\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)$ **is optimal**, *then there exists* p *satisfying the co-state equation and the transversality condition.*
Moreover, the necessary conditions are :

(i) *the inequality*

$$\left(\int_{t_i^*}^{T} p_1(s) b(s)\,\mathrm{d}s\right) \tilde{\eta}_i \leq 0,$$

for all $i = 0, \ldots, n$ *and all admissible perturbation* $\tilde{\eta}_i$ *of* η_i^* *;*

(ii) *and the inequality*

$$\boldsymbol{NC_i} := \Big(-p_1(t_i^*) b(t_i^*) G(t_{i-1}^*, t_i^*) \eta_i^* + b(-t_i^*) \eta_i^* \int_{t_i^*}^{T} p_1(s) b(s)\,\mathrm{d}s$$
$$+ b(-t_i^*)(\bar{R} - 1) \eta_{i+1}^* \int_{t_{i+1}^*}^{T} p_1(s) b(s)\,\mathrm{d}s \Big) \tilde{t}_i \leq 0,$$

for all $i = 1, \ldots, n$ *and all admissible perturbation* $\tilde{t}_i$ *of* t_i^*.

Work in progress

- We need efficient algorithms (real-time application) :
  - Explicit expression of
    $$(t_1,\ldots,t_n,\eta_0,\ldots,\eta_n) \to F(T)$$
    to apply a direct optimization scheme
  - LQ methods
  - MPC methods
- Online parameter estimation coupled with optimization methods [1]
- Extension to the non-isometric case with joint angle variable to produce a motion [4]

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[3] T. Bakir, B. Bonnard B., L. Bourdin and J. Rouot, Direct and Indirect Methods to Optimize the Muscular Force Response to a Pulse Train of Electrical Stimulation, preprint, hal-02053566.

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