A class of graphs with large rankwidth

Chính Hoàng, Nicolas Trotignon

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Abstract

We describe several graphs with arbitrarily large rankwidth (or equivalently with arbitrarily large cliquewidth). Korpelainen, Lozin, and Mayhill [Split permutation graphs, *Graphs and Combinatorics*, 30(3):633–646, 2014] proved that there exist split graphs with Dilworth number 2 with arbitrarily large rankwidth, but without explicitly constructing them. Our construction provides an explicit construction. Maffray, Penev, and Vušković [Coloring rings, *CoRR*, abs/1907.11905, 2019] proved that graphs that they call rings on n sets can be colored in polynomial time. Our construction shows that for some fixed integer $n \geq 3$, there exist rings on n sets with arbitrarily large rankwidth. When $n \geq 5$ and n is odd, this provides a new construction of even-hole-free graphs with arbitrarily large rankwidth.

1 Introduction

The *cliquewidth* of a graph is an integer intended to measure how complex is the graph. It was defined by Courcelle, Engelfriet and Rozenberg in [6] and is successful in the sense that many hard problems on graphs become tractable on graph classes of bounded cliquewidth [7]. This includes for instance finding the largest clique or independent set, and deciding if a colouring with at most k colors exists (for fixed $k \in \mathbb{N}$). This makes cliquewidth particularly interesting in the study of algorithmic properties of hereditary graph classes.

The notion of *rankwidth* was defined by Oum and Seymour in [16], where they use it for an approximation algorithm for cliquewidth. They also show that rankwidth and cliquewidth are equivalent, in the sense that a graph class has bounded rankwidth if, and only if, it has bounded cliquewidth. In the rest of this article, we only use rankwidth, and we therefore refer to results in the literature with this notion, even if in the original papers, the notion of cliquewidth is used (recall the two notions are equivalent as long as we care only for a class being bounded or not by the parameter).

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Determining whether a given class of graphs has bounded rankwidth has been well studied lately, and let us survey the main results in this direction. For every classe of graphs defined by forbidding one or two graphs as induced subgraphs, it is known whether the class has bounded or unbounded rankwidth, apart for a very small number of open cases, see [8] for the most recent results.

Similar classifications were obtained for chordal graphs [3] and split graphs [2]. Recall that a *chordal graph* is a graph such that every cycle of length at least 4 has a chord, and a *split graph* is a graph whose vertex set can be partitioned into a clique and a stable set.

A graph is *even-hole-free* if every cycle of even length has a chord. Determining whether several subclasses of even-hole-free graphs have bounded rankwidth also attracted some attention. See [17, 13, 5, 4] for subclasses of bounded rankwidth and [1, 18] for subclasses with unbounded rankwidth.

When G is a graph and x a vertex of G, we denote by $N(x)$ the set of all neighbors of x. We set $N[x] = N(x) \cup \{x\}$. The *Dilworth number* of a graph G is the maximum number of vertices in a set D such that for all distinct $x, y \in D$, the two sets $N(x) \setminus N[y]$ and $N(y) \setminus N[x]$ are non-empty. In [11], it is proved that graphs with Dilworth number 2 and arbitrarily large rankwidth exist. It should be pointed out that these graphs are split, and that only their existence is proved, no explicit construction is given.

For an integer $n \geq 3$, a *ring on n sets* is a graph G whose vertex set can be partitioned into n cliques $X_1, \dots, X_n$, with three additional properties:

- For all $i \in \{1, \dots, n\}$ and all $x, x' \in X_i$, either $N(x) \subseteq N(x')$ or $N(x') \subseteq N(x)$.
- For all $i \in \{1, \dots, n\}$ and all $x \in X_i$, $N(x) \subseteq X_{i-1} \cup X_i \cup X_{i+1}$ (where the addition of subscripts is modulo n).
- For all $i \in \{1, \dots, n\}$, there exists a vertex $x \in X_i$ that is adjacent to all vertices of $X_{i-1} \cup X_{i+1}$.

Rings were studied in [15], where a polynomial time algorithm to color them is given. The notion of ring in [15] is slightly more restricted than ours (at least 4 sets are required), but it makes no essential difference here. Observe that the Dilworth number of a ring on n sets is at most n. As explained in [13], a construction from [10] shows that there exist rings with arbitrarily large rankwidth. Also in [12], it is proved that the so-called *twisted chain graphs*, that are similar in some respect to rings on 3 sets, have unbounded rankwidth. However, for any fixed integer n, it is not known whether there exist rings on n sets with arbitrarily large rankwidth.

Our main result is a new way to build graphs with arbitrarily large rankwidth. The construction has some flexibility, so it allows us to reach several goals. First, we give split graphs with Dilworth number 2 with arbitrarily large rankwidth, and we describe them explicitly. By "tuning" the construction differently, we will show that for each integer $n \geq 3$, there exist rings on n sets with arbitrarily large rankwidth. For odd integers n, this provides new even-hole-free graphs with arbitrarily large rankwidth. It should be pointed out that our construction does not rely on modifying a grid (a classical method to obtain graphs with arbitrarily large rankwidth).

In Section 2, we recall the definition of rankwidth together with lemmas related to it. In Section 3, we give the main ingredients needed to construct our graphs, called *carousels*, to be defined in Section 4. In Section 5, we give an overview of the proof that carousels have unbounded rankwidth. In Section 6, we give several technical lemmas about the rank of matrices that arise form partitions of the vertices in carousels. In Sections 7 and 8, we prove that carousels have unbounded rankwidth (we need two sections because there are two kinds of carousels, the even ones and the odd ones). In Section 9, we show how to tune carousels in order to obtain graphs split graphs with Dilworth number 2 or rings. We conclude the paper by open questions in Section 10.

2 Rankwidth

When G is a graph and (Y, Z) a partition of some subset of $V(G)$, we denote by $M_{G,Y,Z}$ the matrix M whose rows are indexed by Y, whose columns are indexed by Z and such $M_{y,z} = 1$ when $yz \in E(G)$ and $M_{y,z} = 0$ when $yz \notin E(G)$. We define $\text{rank}_G(Y, Z) = \text{rank}(M_{G,Y,Z})$, where the rank is computed on the binary field. When the context is clear, we may refer to $\text{rank}(G_{Y,Z})$ as the *rank* of (Y, Z).

A *tree* is a connected acyclic graph. A *leaf* of a tree is a node incident to exactly one edge. For a tree T, we let $L(T)$ denote the set of all leaves of T. A tree node that is not a leaf is called *internal*. A tree is *cubic*, if it has at least two nodes and every internal node has degree 3.

A *tree decomposition* of a graph G is a cubic tree T, such that $L(T) = V(G)$. Note that if $|V(G)| \leq 1$, then G has no tree decomposition. For every edge $e \in E(T)$, $T \setminus e$ has two connected components, Y_e and Z_e (that we view as trees). The *width* of an edge $e \in E(T)$ is defined as $\text{rank}(M_{G,L(Y_e),L(Z_e)})$. The *width* of T is the maximum width over all edges of T. The *rankwidth* of G, denoted by $\text{rw}(G)$, is the minimum integer k, such that there is a tree decomposition of G of width k. If $|V(G)| \leq 1$, we let $\text{rw}(G) = 0$.

Let G be a graph. A partition (Y, Z) of $V(G)$ is *balanced* if

$$|V(G)|/3 \leq |Y|, |Z| \leq 2|V(G)|/3$$

and *unbalanced* otherwise. An edge of a tree decomposition T is (un)-balanced if the partition $(L(Y_e), L(Z_e))$ of G as defined above is (un)-balanced.

Lemma 2.1 *Every tree decomposition T of a graph G has a balanced edge.*

Proof. For every edge $e \in E(T)$, removing e from T yields two components Y_e and Z_e. We orient e from Y_e to Z_e if $2\,|L(Y_e)| < |L(Z_e)|$. If there is a non-oriented edge e, then e is balanced. So, assume that all edges are oriented. Since T is a tree, some node $s \in V(T)$

must be a sink. Note that s cannot be a leaf. But T is cubic, so each of the three subtrees obtained from T by deleting s contains less than $\frac{1}{3}$ of the vertices of $L(T)$, a contradiction.□

Interestingly, we do not need the full definition of the rankwidth of a graph, the following property is enough for our purpose.

Lemma 2.2 *Let G be a graph and $r \geq 1$ an integer. If every partition (Y, Z) of $V(G)$ with rank less than r is unbalanced, then* $\mathrm{rw}(G) \geq r$.

Proof. Suppose for a contradiction that $\mathrm{rw}(G) < r$. Then there exists a tree decomposition T of G with width less that r. Consider a balanced edge e of T, and let Y_e and Z_e be the two connected components of $T \setminus e$. Then, $(L(Y_e), L(Z_e))$ is a partition of $V(G)$ that is balanced and has rank less than r, a contradiction to our assumptions. □

We do not need many definitions from linear algebra. In fact the following basic lemma and the fact that the rank does not increase when taking submatrices are enough for our purpose. A (0-1) $n \times n$ matrix M is *diagonal* if $M_{i,j} = 1$ whenever $i = j$ and $M_{i,j} = 0$ whenever $i \neq j$. It is *antidiagonal* if $M_{i,j} = 0$ whenever $i = j$ and $M_{i,j} = 1$ whenever $i \neq j$. It is *triangular* if $M_{i,j} = 1$ whenever $i \geq j$ and $M_{i,j} = 0$ whenever $i < j$.

Lemma 2.3 *For every integer $r \geq 1$, the rank of a diagonal, antidiagonal or triangular $r \times r$ matrix is at least $r - 1$ (in fact it is $r - 1$ when r is odd and the matrix is antidiagonal, and it is r otherwise).*

Proof. Clear. □

A (0-1) $n \times n$ matrix M is *near-triangular* if $M_{i,j} = 1$ whenever $i > j$ and $M_{i,j} = 0$ whenever $i < j$. The values on the diagonal are not restricted.

Lemma 2.4 *For every integer $r \geq 1$, the rank of a near-triangular $2r \times 2r$ matrix M is at least r.*

Proof. The submatrix N of M formed by the rows of even indexes and the columns of odd indexes is a triangular $r \times r$ matrix (formally for every $i, j \in \{1, \dots r\}$, $N_{i,j} = M_{2i,2j-1}$). By Lemma 2.3, $\mathrm{rank}(M) \geq \mathrm{rank}(N) = r$. □

3 Matchings, antimatchings and crossings

In this section, we describe the particular adjacencies that we need to build our graphs. Suppose that a graph G contains two disjoint ordered sets of vertices of same cardinality k, say $X = \{u^1, \dots, u^k\}$ and $X' = \{v^1, \dots, v^k\}$.

- We say that (G, X, X') is a *regular matching* when:

  for all $j, j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j = j'$.

- We say that (G, X, X') is a *regular antimatching* when:

  for all $j, j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j \neq j'$.

- We say that (G, X, X') is a *regular crossing* when:

  for all $j, j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j + j' \geq k + 1$.

- We say that (G, X, X') is a *expanding matching* when:

  for all $j, j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' = 2j$ or $j' = 2j + 1$.

- We say that (G, X, X') is a *expanding antimatching* when:

  for all $j, j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' \neq 2j$ and $j' \neq 2j + 1$.

- We say that (G, X, X') is a *expanding crossing* when:

  for all $j, j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $2j + j' \geq 2k + 2$.

- We say that (G, X, X') is a *skew expanding matching* when k equal 2 modulo 4 and:

  For all $j \in \{1, \ldots, (k-2)/4\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' = 2j$ or $j' = 2j + 1$; and

  For all $j \in \{(k-2)/4 + 1, \ldots, (3k+2)/4\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \notin E(G)$;

  For all $j \in \{(3k+2)/4 + 1, \ldots, k\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' = 2j - k - 2$ or $j' = 2j - k - 1$.

- We say that (G, X, X') is a *skew expanding antimatching* when k equal 2 modulo 4 and:

  for all $j \in \{1, \ldots, (k-2)/4\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' = 2j$ or $j' = 2j + 1$;

  for all $j \in \{(k-2)/4 + 1, \ldots, (3k+2)/4\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$; and

  for all $j \in \{(3k+2)/4 + 1, \ldots, k\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' = 2j - k - 2$ or $j' = 2j - k - 1$.

- We say that (G, X, X') is a *skew expanding crossing* when k equal 2 modulo 4 and:

  for all $j \in \{1, \ldots, (k-2)/4\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $2j + j' \geq k$;

  for all $j \in \{(k-2)/4 + 1, \ldots, k/2\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' \geq k/2 + 1$;

  for all $j \in \{k/2 + 1, \ldots, (3k+2)/4\}$ and $j' \in \{1, \ldots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $j' \geq k/2 - 1$; and

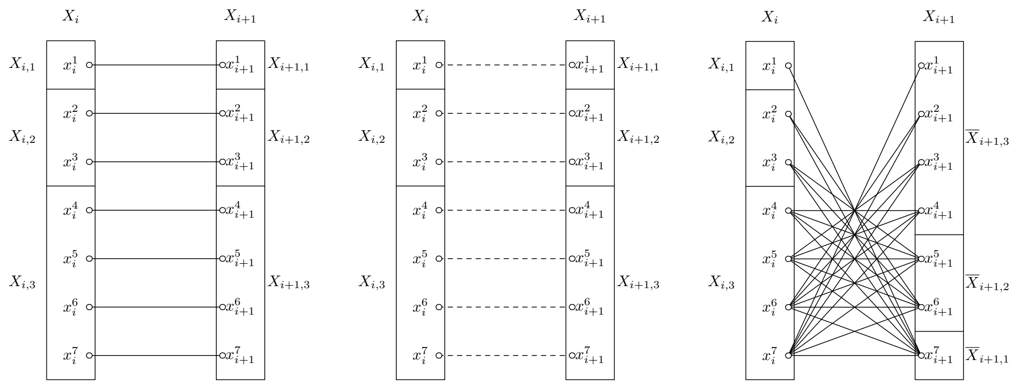

Figure 1: A regular matching, a regular antimatching and a regular crossing (on the regular antimatching, only non-edges are represented)

for all $j \in \{(3k+2)/4+1, \dots, k\}$ and $j' \in \{1, \dots, k\}$, $u^j v^{j'} \in E(G)$ if and only if $2j + j' - 2 \geq 2k$.

A triple (G, X, X') is *regular* if it is a regular matching, a regular antimatching or a regular crossing, see Figure 1. On the figures, vertices are partitioned into boxes and several sets receive names. These will be explained in the next section.

A triple (G, X, X') is *expanding* if it is an expanding matching, an expanding antimatching or an expanding crossing, see Figure 2.

A triple (G, X, X') is *skew expanding* if it is a skew expanding matching, a skew expanding antimatching or a skew expanding crossing, see Figure 3.

A triple (G, X, X') is a *parallel triple* if it is a matching or an antimatching (regular, expanding or skew expanding). A triple (G, X, X') is a *cross triple* if it is a crossing (regular, expanding or skew expanding).

4 Carousels

The graphs that we construct are called *carousels* and are built from $n \geq 3$ sets of vertices of equal cardinality $k \geq 1$: $X_1, \dots, X_n$. So, let G be a graph such that $V(G) = X_1 \cup \dots \cup X_n$. Throughout the rest of the paper, the subscripts for sets X_i's are considered modulo n. The graph G is a *carousel on n sets of cardinality k* if:

(i) (G, X_1, X_2) is a regular crossing;

(ii) for every $i \in \{2, \dots, n-1\}$, (G, X_i, X_{i+1}) is a regular triple and

(iii) (G, X_n, X_1) is an expanding triple or a skew expanding triple.

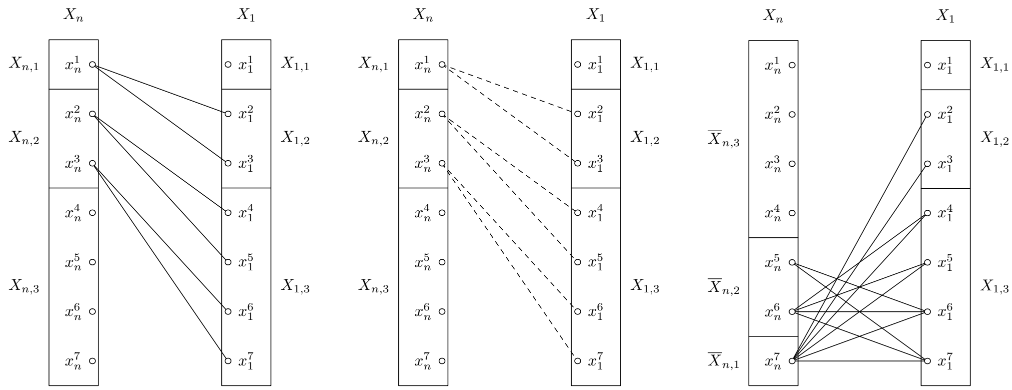

Figure 2: An expanding matching, an expanding antimatching and an expanding crossing

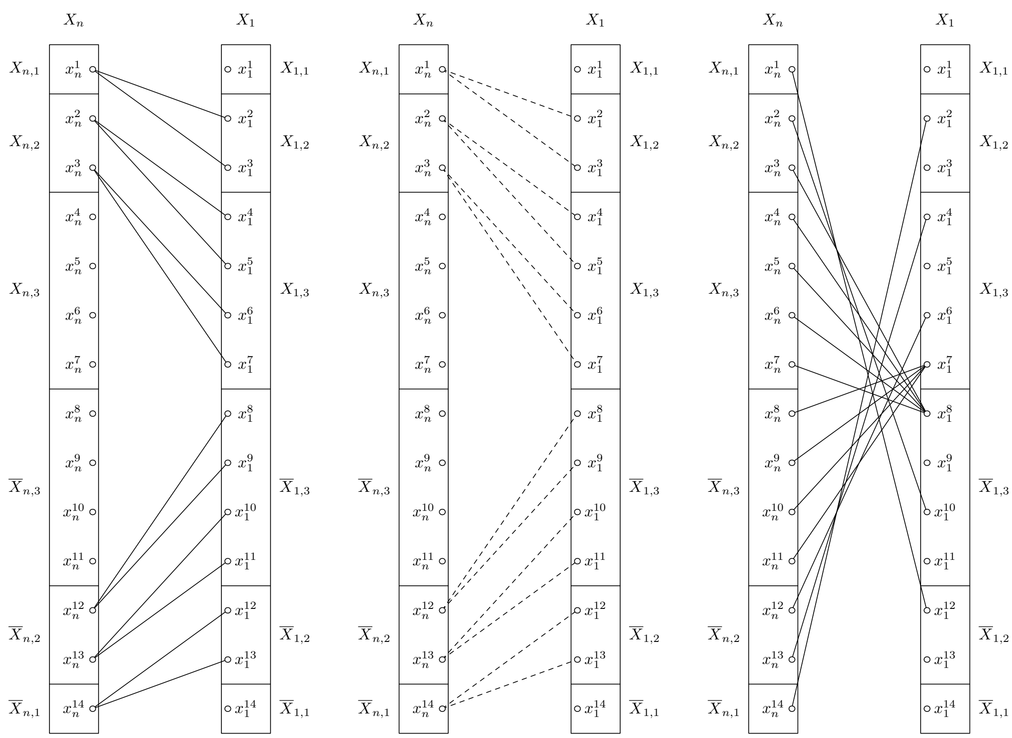

Figure 3: A skew expanding matching, a skew expanding antimatching and a skew expanding crossing (on the crossing, for each $i \in \{1, \ldots, k\}$, only the edge $x_n^i x_1^j$ such that j is minimum is represented)

Whether (G, X_n, X_1) should be an expanding triple or a skew expanding triple depends on how many crossing triples there are among the triples (G, X_i, X_{i+1}) for $i \in \{1, \dots, n\}$. Let us explain this.

Let $s \geq 1$ be an integer. An *even carousel of order s on n sets* is a carousel on n sets of cardinality k such that:

(i) $k = 2^s - 1$;

(ii) the total number of crossing triples among the triples (G, X_i, X_{i+1}) for $i \in \{1, \dots, n\}$ is even and

(iii) (G, X_n, X_1) is an expanding triple.

Let $s \geq 1$ be an integer. An *odd carousel of order s on n sets* is a carousel on n sets of cardinality k such that:

(i) $k = 2 \times (2^s - 1)$ (so k is equal to 2 modulo 4);

(ii) the total number of crossing triples among the triples (G, X_i, X_{i+1}) for $i \in \{1, \dots, n\}$ is odd and

(iii) (G, X_n, X_1) is a skew expanding triple.

In carousels, the edges inside the sets X_i or between sets X_i and X_j such that $j \notin \{i-1, i, i+1\}$ are not specified, they can be anything. Our main result is the following.

Theorem 4.1 *For every integers $n \geq 3$ and $r \geq 1$, there exists an integer s such that every even carousel and every odd carousel of order s on n sets has rankwidth at least r.*

5 Outline of the proof

In this section, we provide an outline of the proof of the main theorem. The detail will be given in later sections.

For all $i \in \{1, ..., n\}$, we let $X_i = \{x_i^1, ..., x_i^k\}$. There is a symmetry in every set X_i and we need some notation for it. Let f be the function defined for each integer j by $f(j) = k - j + 1$. We will use an horizontal bar to denote it as follows. When x_i^j is a vertex in some set X_i, we denote by $\overline{x}_i^j$ the vertex $x_i^{f(j)}$ and by $\overline{x}_{i+1}^j$ the vertex $x_{i+1}^{f(j)}$. We use a similar notation for sets of vertices: if $S_i \subseteq X_i$ then

$$\overline{S}_i = \{\overline{x}_i^j | x_i^j \in S_i\}$$

and

$$\overline{S}_{i+1} = \{\overline{x}_{i+1}^j | x_i^j \in S_i\}.$$

Note that $\overline{\overline{a}} = a$ for any object a such that $\overline{a}$ is defined.

Each set X_i is partitioned into parts and this differs for the even and the odd case.

When G is an even carousel, we designate by induction each set X_i as a *top set* or a *botom set*. The set X_1 is by definition a top set, and the status of the next ones change at every cross triple. More formally, for every $i \in \{1, \dots, n-1\}$:

- If X_i is top set and (G, X_i, X_{i+1}) is a parallel triple, then X_{i+1} is a top set.
- If X_i is top set and (G, X_i, X_{i+1}) is a cross triple, then X_{i+1} is a botom set.
- If X_i is botom set and (G, X_i, X_{i+1}) is a parallel triple, then X_{i+1} is a botom set.
- If X_i is botom set and (G, X_i, X_{i+1}) is a cross triple, then X_{i+1} is a top set.

Then, for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, s\}$, we set $X_{i,j} = \{x_i^{2^{j-1}}, \dots, x_i^{2^j-1}\}$. Observe that for some fixed i, the $X_{i,j}$'s (resp. the $\overline{X}_{i,j}$'s) form a partition of X_i. We view every top set X_i as partitioned by the $X_{i,j}$'s and every botom set X_i as partitioned by the $\overline{X}_{i,j}$'s.

In an odd carousel of order s, there are also top sets and botom sets (defined as above), but they are all partitioned in the same way. For all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, s\}$, we set $X_{i,j} = \{x_i^{2^{j-1}}, \dots, x_i^{2^j-1}\}$. Observe that for some fixed i, the $X_{i,j}$'s and $\overline{X}_{i,j}$'s form a partition of X_i.

To prove Theorem 4.1, we fix the integers n and r. We then compute a large integer s (depending on n and r) and we consider a carousel G of order s on n sets. We then study the behavior of a partition (Y, Z) of $V(G)$ of rank less than r, our goal being to prove that it is unbalanced (this proves the rankwidth of G is at least r by Lemma 2.2).

To check whether a partition is balanced, we need a notation to measure how many elements of Y some set contains. For an integer $m \geq 0$, a set $S \subseteq V(G)$ receive *label* Y_m if

$$(m-1)r < |S \cap Y| \leq mr.$$

Note that S has label Y_0 if and only if $S \subseteq Z$. Having label Y_m means that $\lceil |S \cap Y|/r \rceil = m$. Observe that for every subset S of $V(G)$, there exists a unique integer $m \geq 0$ such that S has label Y_m.

The first step of the proof is to note that when going from x_1^1 to x_1^k (so in X_1), there are not too many changes from Y to Z or from Z to Y. Because a big number of changes would imply that the rank between X_1 and X_2 is high, this is formally stated and proved in Lemma 6.1. For this to be true, we need that (G, X_1, X_2) is a crossing, this is why there is no flexibility in the definition for the adjacency between X_1 and X_2 in the definition of carousels.

Since k (the common cardinality of the sets X_i's) is large enough by our choice of s, we then know that in X_1 there must be large intervals of consecutive vertices in Y or in Z, say in Z up to symmetry. So, one of the sets that partition X_1 is fully in Z, say $X_{1,t}$ for some large integer t, and has therefore label Y_0. The rest of the proof shows that this label propagates in the rest of the graph. By a propagation lemma (stated in the next section), we prove that $X_{2,t}$ (if (G, X_1, X_2) is a parallel triple) or $\overline{X}_{2,t}$ (if (G, X_1, X_2) is a cross triple) has a label very close to Y_0, namely Y_0 or Y_1. This is because a larger label, say, Y_m with $m > 1$, would give rise to a matrix of rank at least r between Y and Z. And we continue to apply the propagation mechanism until we reach $X_{n,1}$ (or $\overline{X}_{n,1}$). The label may be Y_0, Y_1, ..., Y_{n-1}, but not larger.

Now, we apply other propagation lemmas to handle the triple (G, X_n, X_1) (that is expanding or skew expanding by definition of carousels). Here the adjacency is designed so that some part that is twice larger than $X_{i,t}$, namely $X_{1,t+1}$ if G is an even carousel, or $\overline{X}_{1,t+1}$ if G is an odd carousel, has a label being not too large.

For even carousels, by repeating the procedure above, we prove that for each $i \in \{1, \dots, n\}$, $X_{i,s-1}$ and $X_{i,s}$ have a label Y_m with m not too large. And since the size of the sets $X_{i,j}$ is exponential in j, these two sets $X_{i,s-1}$ and $X_{i,s}$ represent a proportion more than $\frac{3}{4}$ of all the set X_i. And since $X_{i,s-1}$ and $X_{i,s}$ have label Y_m with m small, they contains mostly vertices from Z, so that the partition (Y, Z) is unbalanced.

For odd carousels, the proof is similar, except that we prove that $X_{i,s-1}$, $X_{i,s}$, $\overline{X}_{i,s-1}$ and $\overline{X}_{i,s}$ have label Y_m with m being not too large. These $4n$ sets represent a proportion more than $\frac{3}{4}$ of all the set X_i.

In the next section, we supply the detail of this proof.

6 Blocks and propagation

Throughout the rest of this section, $n \geq 3$, $s \geq 1$ and G is a an even or an odd carousel of order s on n sets. Also, we assume $r \geq 2$ is an integer and (Y, Z) is a partition of $V(G)$ of rank less than r.

Suppose $X = \{x^1, \dots, x^k\}$ is an ordered set and (Y, Z) is a partition of X. We call *interval* of X any subset of X of the form $\{x^i, x^{i+1}, \dots, x^j\}$. A *block of* X w.r.t. (Y, Z) is any non-empty interval of X that is fully contained in Y or in Z and that is maximal w.r.t. this property. Clearly, X is partitioned into its blocks w.r.t. (Y, Z).

Lemma 6.1 *X_1 has less than $8r$ blocks w.r.t. (Y, Z).*

Proof. Suppose that X_1 has at least $8r$ blocks, and let B_1, ..., B_{8r} be the $8r$ first ones. Up to symmetry, we may assume that for every $i \in \{1, \dots, 4r\}$, $B_{2i-1} \subseteq Y$ and $B_{2i} \subseteq Z$. For each $i \in \{1, \dots, 4r\}$, we choose some element $y_1^i \in B_{2i-1}$ and some element $z_1^i \in B_{2i}$. We denote by $\overline{y}_2^i$ and $\overline{z}_2^i$ the corresponding vertices in X_2 of y_1^i and z_1^i respectively. Formally, $y_1^i = x_1^j$ for some integer $j \in \{1, \dots, k\}$, $\overline{y}_2^i = \overline{x}_2^j = x_2^{k-j+1}$ and the definiton of $\overline{z}_2^i$ is similar. We set $X_2' = \{\overline{y}_2^1, \overline{z}_2^1, \dots, \overline{y}_2^{4r}, \overline{z}_2^{4r}\}$.

Consider $i, j \in \{1, \dots, 4r\}$, $u \in \{y_1^i, z_1^i\}$ and $v \in \{\overline{y}_2^j, \overline{z}_2^j\}$. From the definition of regular crossings, we have the following key observation:

- If $i+j < 4r+1$, then $uv \notin E(G)$.
- If $i+j > 4r+1$, then $uv \in E(G)$.
- Note that if $i+j = 4r+1$, the adjacency between u and v is not specified, it depends on whether u is y_1^i or z_1^i and on whether v is $\overline{y}_2^j$ or $\overline{z}_2^j$.

Suppose first that $|Y \cap X_2'| \geq |Z \cap X_2'|$. Then, there exist at least $2r$ sets among $\{\overline{y}_2^1, \overline{z}_2^1\}, \ldots \{\overline{y}_2^{4r}, \overline{z}_2^{4r}\}$ that have a non-empty intersection with Y. This means that there exist $2r$ distinct integers $i_1, \ldots, i_{2r}$ such that for every $j \in \{1, \ldots, 2r\}$, some vertex from $\{\overline{y}_2^{i_j}, \overline{z}_2^{i_j}\}$ is in Y. We denote by v_j such a vertex, and let $Y' = \{v_1, \ldots, v_{2r}\}$. We then set $Z' = \{z_1^{i_1}, \ldots, z_1^{i_{2r}}\}$.

By definition, $Z' \subseteq Z$ and $Y' \subseteq Y$. Also, by the key observation above, the matrix $M_{G[Y' \cup Z'], Y', Z'}$ is a near-triangular $2r \times 2r$ matrix. By Lemma 2.4, it has rank at least r. This proves that $\mathrm{rank}_G(Y, Z) \geq r$, a contradiction.

When $|Z \cap X_2| \geq |Y \cap X_2|$, the proof is similar. □

The following lemma explains how labels propagate in regular triples (represented in Figure 1).

Lemma 6.2 *Let $m \geq 0$, $1 \leq i < n$ and $1 \leq j \leq s$ be integers.*

(i) Suppose that (G, X_i, X_{i+1}) is a regular parallel triple. If $X_{i,j}$ has label Y_m, then $X_{i+1,j}$ has label $Y_{\max(m-1,0)}$, Y_m or Y_{m+1}.

If $\overline{X}_{i,j}$ has label Y_m, then $\overline{X}_{i+1,j}$ has label $Y_{\max(m-1,0)}$, Y_m or Y_{m+1}.

(ii) Suppose that (G, X_i, X_{i+1}) is a regular cross triple. If $X_{i,j}$ has label Y_m, then $\overline{X}_{i+1,j}$ has label $Y_{\max(m-1,0)}$, Y_m or Y_{m+1}.

If $\overline{X}_{i,j}$ has label Y_m, then $X_{i+1,j}$ has label $Y_{\max(m-1,0)}$, Y_m or Y_{m+1}.

Proof. We first deal with the case when (G, X_i, X_{i+1}) is a cross triple and $X_{i,j}$ has label Y_m. Suppose that the conclusion does not hold. This means that $\overline{X}_{i+1,j}$ has label Y_j with $j > m+1$ or $0 \leq j < m-1$.

If $j > m+1$, then $|\overline{X}_{i+1,j} \cap Y| > (m+1)r$. Since $|X_{i,j} \cap Y| \leq mr$, we have $|\overline{X}_{i+1,j} \cap Y| - |X_{i,j} \cap Y| \geq r+1$. So there exists a subset S_i of $X_{i,j}$ such that $|S_i| = r+1$, $S_i \subseteq Z$ and $\overline{S}_{i+1} \subseteq Y$. The matrix $M_{G[S_i \cup \overline{S}_{i+1}], S_i, \overline{S}_{i+1}}$ is triangular and has rank at least r by Lemma 2.3, a contradiction to (Y, Z) being a partition of $V(G)$ of rank less than r.

If $0 \leq j < m-1$, then $|\overline{X}_{i+1,j} \cap Y| \leq (m-2)r$. Since $|X_{i,j} \cap Y| > (m-1)r$, we have $|X_{i,j} \cap Y| - |\overline{X}_{i+1,j} \cap Y| \geq r+1$. So, there exists a subset S_i of $X_{i,j}$ such that $|S_i| = r+1$, $S_i \subseteq Y$ and $\overline{S}_{i+1} \subseteq Z$. The matrix $M_{G[S_i \cup \overline{S}_{i+1}], S_i, \overline{S}_{i+1}}$ is triangular and has rank at least r by Lemma 2.3, a contradiction to (Y, Z) being a partition of $V(G)$ of rank less than r.

All the other cases are similar. When $\overline{X}_{i,j}$ has label Y_m, the proof is symmetric. When (G, X_i, X_{i+1}) is a regular matching, we obtain a diagonal matrix, and when (G, X_i, X_{i+1}) is a regular antimatching, we obtain a antidiagonal matrix. □

The following lemma describes how labels propagate in expanding triples (see Figure 2).

Lemma 6.3 *Let $m \geq 0$ and j be integers such that $1 \leq j \leq s-1$.*

(i) Suppose that (G, X_n, X_1) is a parallel expanding triple. If $X_{n,j}$ has label Y_m, then $X_{1,j+1}$ has label $Y_{\max(m-2,0)}$, $Y_{\max(m-1,0)}$, Y_m, Y_{m+1} or Y_{m+2}.

(ii) Suppose that (G, X_n, X_1) is a cross expanding triple. If $\overline{X}_{n,j}$ has label Y_m, then $X_{1,j+1}$ has label $Y_{\max(m-2,0)}$, $Y_{\max(m-1,0)}$, Y_m, Y_{m+1} or Y_{m+2}.

Proof. We first deal with the case when (G, X_n, X_1) is a cross expanding triple and $\overline{X}_{n,j}$ has label Y_m. Suppose that the conclusion does not hold. This means that $X_{1,j+1}$ has label Y_j with $j > m+2$ or $0 \leq j < m-2$.

If $j > m+2$, then $|X_{1,j+1} \cap Y| > (m+2)r$. Since $|\overline{X}_{n,j} \cap Y| \leq mr$, we have $|X_{1,j+1} \cap Y| - |\overline{X}_{n,j+1} \cap Y| \geq 2r+1$. So there exists a subset $\overline{S}_n$ of $\overline{X}_{n,j}$ such that $|\overline{S}_n| = 2r+1$, $\overline{S}_n \subseteq Z$ and $S_1 \subseteq Y$. So, $\overline{S}_n$ contains a subset $\overline{S'}_n$ such that $|\overline{S'}_n| = r+1$, $\overline{S'}_n \subseteq Z$, $S'_1 \subseteq Y$ and the matrix $M_{G[\overline{S'}_n \cup S'_{1,j+1}], \overline{S'}_n, S'_{1,j+1}}$ is triangular and has rank at least r by Lemma 2.3, a contradiction to (Y, Z) being a partition of $V(G)$ of rank less than r.

If $0 \leq j < m-2$, then $|X_{1,j+1} \cap Y| \leq (m-3)r$. Since $|\overline{X_{n,j}} \cap Y| > (m-1)r$, we have $|\overline{X_{n,j}} \cap Y| - |X_{1,j+1} \cap Y| \geq 2r+1$. So, there exists a subset $\overline{S}_n$ of $\overline{X}_{n,j}$ such that $|\overline{S}_n| = 2r+1$, $\overline{S}_i \subseteq Y$ and $S_1 \subseteq Z$. So, $\overline{S}_n$ contains a subset $\overline{S'}_n$ such that $|\overline{S'}_n| = r+1$, $\overline{S'}_n \subseteq Y$, $S'_1 \subseteq Z$ and the matrix $M_{G[\overline{S'}_n \cup S'_{1,j+1}], \overline{S'}_n, S'_{1,j+1}}$ is triangular and has rank at least r by Lemma 2.3, a contradiction to (Y, Z) being a partition of $V(G)$ of rank less than r.

When (G, X_n, X_1) is a parallel expanding triple, the proof is similar. We obtain a diagonal, or anti-diagonal matrix of rank at least r, a contradiction. □

The following lemma describes how labels propagate in skew expanding triples (see Figure 3). We omit the proof since it is similar to the previous one.

Lemma 6.4 *Let $m \geq 0$ and j be integers such that $1 \leq j \leq s-1$.*

(i) Suppose that (G, X_n, X_1) is a parallel skew expanding triple.

If $X_{n,j}$ has label Y_m, then $X_{1,j+1}$ has label $Y_{\max(m-2,0)}$, $Y_{\max(m-1,0)}$, Y_m, Y_{m+1} or Y_{m+2}.

If $\overline{X}_{n,j}$ has label Y_m, then $\overline{X}_{1,j+1}$ has label $Y_{\max(m-2,0)}$, $Y_{\max(m-1,0)}$, Y_m, Y_{m+1} or Y_{m+2}.

(ii) Suppose that (G, X_i, X_{i+1}) is a cross triple.

If $X_{n,j}$ has label Y_m, then $\overline{X}_{1,j+1}$ has label $Y_{\max(m-2,0)}$, $Y_{\max(m-1,0)}$, Y_m, Y_{m+1} or Y_{m+2}.

If $\overline{X}_{n,j}$ has label Y_m, then $X_{1,j+1}$ has label $Y_{\max(m-2,0)}$, $Y_{\max(m-1,0)}$, Y_m, Y_{m+1} or Y_{m+2}.

7 Even carousels

In this section, $n \geq 3$ and $r \geq 2$ are fixed integers. We prove Theorem 4.1 for even carousels. We therefore look for an integer s such that every even carousel of order s on n sets has rankwidth at least r. We define s as follows. Let q be an integer such that:

$$2^{q+8r-1} \geq 10(n+1)(q+8r+1)r \tag{1}$$

Clearly q exists. We set $s = q + 8r + 1$. We now consider an even carousel G of order s on n sets. To prove that it has rankwidth at least r, it is enough by Lemma 2.2 to prove that every partition (Y, Z) of $V(G)$ with rank less than r is unbalanced, so let us consider (Y, Z) a partition of $V(G)$ or rank less than r.

Lemma 7.1 *There exists $t \in \{q, \dots, q + 8r - 1\}$ such that $X_{1,t}$ has label Y_0 or Z_0.*

Proof. Otherwise, for every $t \in \{q, \dots, q+8r-1\}$ the set $X_{1,i}$ is an interval of X_1 that must contain elements of Y and elements of Z. Hence, the $8r$ sets $X_{1,q}$, …, $X_{1,q+8r-1}$ show that the interval $S = \cup_{j=q}^{q+8r-1} X_{1,j}$ has at least $8r$ blocks, a contradiction to Lemma 6.1. □

Up to symmetry, we may assume that there exists $t \in \{q, \dots, q+8r-1\}$ such that $X_{1,t}$ has label Y_0. We denote by $\widetilde{X}_{i,j}$ the set that is equal to $X_{i,j}$ if X_i is a top set, and that is equal to $\overline{X}_{i,j}$ if X_i is a botom set.

The idea is now to apply Lemmas 6.2 and 6.3 repeatedly to show that for all $i \in \{1, \dots, n\}$ and for all $j \in \{t, \dots, q+8r+1\}$, the set $\widetilde{X}_{i,j}$ has a label Y_m with m not too big. We obtain that $m \leq (n+1)(8r+1)$, because there are at most $(n-1)(8r+1)$ applications of Lemmas 6.2 that each augments the index of the label by at most 1, and $8r+1$ applications of Lemma 6.3 that each augment the index of the label by at most 2.

Let us now prove that (Y, Z) is unbalanced. To do so, we focus on the sets $\widetilde{X}_{i,j}$ for $i \in \{1, \dots, n\}$ and $j \in \{s-1, s\}$. We denote by X their union. By elementary properties of powers of 2, we have $|X| > \frac{3}{4}|V(G)|$. Also, by the discussion above, each of the set $X_{i,j}$ has label Y_m with $m \leq (n+1)(8r+1)$. Hence, for $i \in \{1, \dots, n\}$ and $j \in \{8r, 8r+1\}$, $|X_{i,j} \cap Y| \leq (n+1)(q+8r+1)r$. So, by inequality (1), the proportion of vertices from Z in X is at least $\frac{9}{10}$. Hence,

$$|Z| \geq \frac{3}{4} \times \frac{9}{10}|V(G)| = \frac{27}{40}|V(G)| > \frac{2}{3}|V(G)|.$$

The partition (Y, Z) is therefore unbalanced as claimed.

8 Odd carousels

In this section, $n \geq 3$ and $r \geq 2$ are fixed integers. We prove Theorem 4.1 for odd carousels. We therefore look for an integer s such that every odd carousel of order s on n sets has rankwidth at least r. We define s as follows. Let q be an integer such that:

$$2^{q+16r-1} \geq 10(n+1)(q+16r+1)r \tag{2}$$

Clearly q exists. We set $s = q + 16r + 1$. We now consider a skew carousel G of order s on n sets. To prove that it has rankwidth at least r, it is enough by Lemma 2.2 to prove that every partition (Y, Z) of $V(G)$ with rank less than r is unbalanced, so let us consider (Y, Z) a partition of $V(G)$ of rank less than r.

Lemma 8.1 *There exists $t \in \{q, \dots, q + 16r - 1\}$ such that $X_{1,t} \cup X_{1,t+1}$ has label Y_0 or Z_0.*

Proof. Otherwise, for every $i \in \{q, \dots, q + 16r - 2\}$ the set $X_{1,i} \cup X_{1,i+1}$ is an interval of X_1 that must contain elements of Y and elements of Z. Hence, the $8r$ sets $X_{1,q} \cup X_{1,q+1}$, $X_{1,q+2} \cup X_{1,q+3}$, $\dots$, $X_{1,q+16r-2} \cup X_{1,q+16r-1}$ show that the interval of X_1 $S = \cup_{j=q}^{q+16r-1} X_{1,j}$ has at least $8r$ blocks a contradiction to Lemma 6.1. □

Up to symmetry, we may assume that there exists $t \in \{q, \dots, q + 16r - 2\}$ such that $X_{1,t} \cup X_{1,t+1}$ has label Y_0. We denote $\widetilde{X}_{i,j}$ the set that is equal to $X_{i,j}$ if X_i is a top set, and that is equal to $\overline{X}_{i,j}$ if X_i is a botom set.

The idea is now to apply Lemmas 6.2 and 6.4 repeatedly to show that all $i \in \{1, \dots, n\}$ and all $j \in \{t, \dots, q + 16r + 1\}$, the sets $X_{i,j}$ and $\overline{X}_{i,j}$ both have a label Y_m with m not too big. We obtain that $m \leq (n + 1)(16r + 1)$, because there are at most $(n - 1)(16r + 1)$ applications of Lemmas 6.2 that each augment the index of the label by at most 1, and $16r + 1$ applications of Lemma 6.3 that each augment the index of the label by at most 2.

Let us now prove that (Y, Z) is unbalanced. To do so, we focus on the $4n$ sets $X_{i,j}$, $\overline{X}_{i,j}$ for $i \in \{1, \dots, n\}$ and $j \in \{s - 1, s\}$. We denote by X their union. By elementary properties of powers of 2, we have $|X| > \frac{3}{4}|V(G)|$. Also, each of the set $X_{i,j}$ or $\overline{X}_{i,j}$ has label Y_m with $m \leq (n+1)(16r+1)$. Hence, for $i \in \{1, \dots, n\}$ and $j \in \{s-1, s\}$, $|X_{i,j} \cap Y| \leq (n + 1)(q + 16r + 1)r$ (and a similar inequality holds for $\overline{X}_{i,j}$). So, by inequality (2), the proportion of vertices from Z in X is at least $\frac{9}{10}$. Hence,

$$|Z| \geq \frac{3}{4} \times \frac{9}{10}|V(G)| = \frac{27}{40}|V(G)| > \frac{2}{3}|V(G)|.$$

The partition (Y, Z) is therefore unbalanced as claimed.

9 Applications

In this section, we explain how to tune our carousels to obtain the results announced in the introduction.

Split permutation graphs with Dilworth number 2

In [11], it is proved that there exist split permutation graphs with Dilworth number 2 and arbitrarily large rankwidth. Theorem 3 in [11] states that *the class of split permutation graphs is precisely the class of split graphs of Dilworth number at most 2*. Therefore, it is enough for us to study split graphs with Dilworth number 2.

We now provide an explicit construction. Consider an even carousel on 4 sets X_1, X_2, X_3, X_4 and the following additional properties:

- $X_1 \cup X_3$ is a clique.
- $X_2 \cup X_4$ is a stable set.
- The triples (G, X_1, X_2), (G, X_2, X_3), (G, X_3, X_4); are regular crossings.
- The triple (G, X_4, X_1) is an expanding crossing.

It is clear that graphs constructed in that way are split graphs because $X_1 \cup X_3$ is a clique and $X_2 \cup X_4$ is a stable set. To check that they have Dilworth number 2, it is enough to notice that for all vertices $x, y \in X_1 \cup X_2$, either $N(x) \setminus N[y] = \emptyset$ or $N(y) \setminus N[x] = \emptyset$, and for all vertices $x, y \in X_3 \cup X_4$, either $N(x) \setminus N[y] = \emptyset$ or $N(y) \setminus N[x] = \emptyset$.

Rings

We consider a carousel on $n \geq 3$ sets, all are cliques cliques, with all triples being crossings. If n even, this is an even carousel; and if n is odd, this is an odd carousel. For a fixed n, this gives rings with arbitrarily large rankwidth.

Flexibility of carousels

Observe that in this section, we never use matchings and antimatchings, that were included in the definition for possible later use of carousels. More generality could be allowed (for instance by not forcing (G, X_1, X_2) to be crossing, and instead forcing some (G, X_i, X_{i+1}) to be crossing). This might be useful, but we felt that it would make the explanation too complicated.

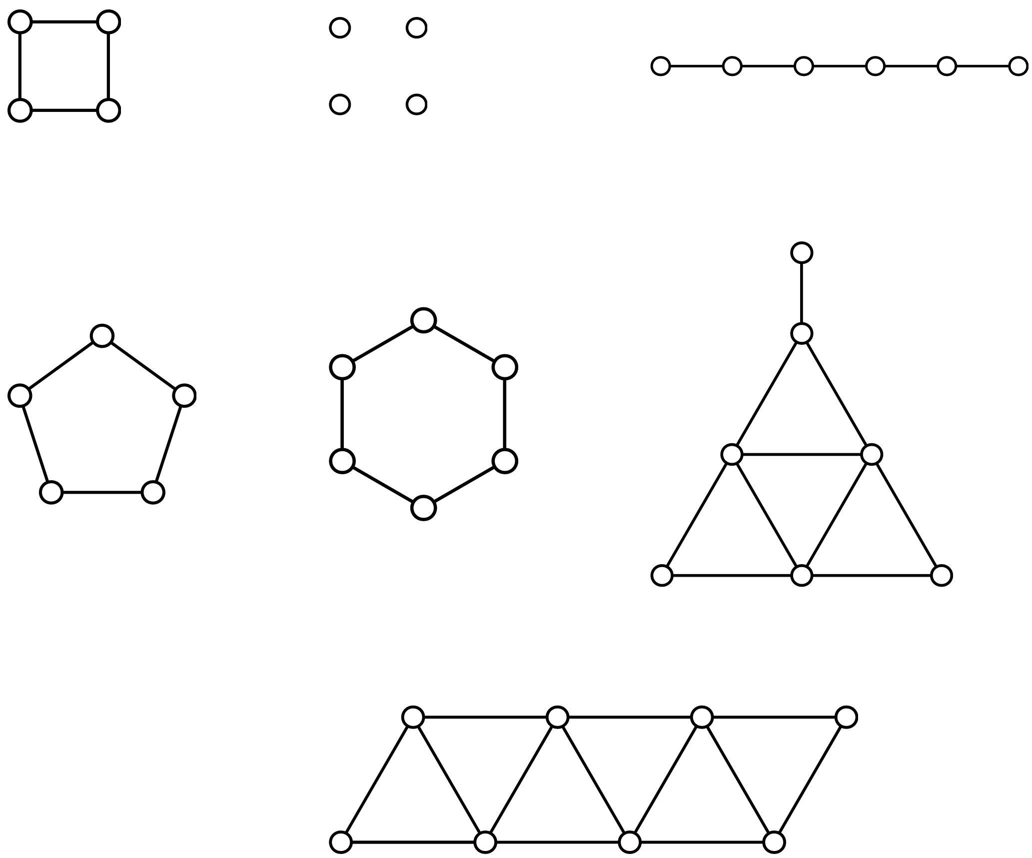

Figure 4: Graphs that are not induced subgraphs of rings on 3 sets, and that are minimal with respect to this property

10 Open questions

It would be nice to characterize the class of induced subgraphs of rings on n sets by forbidden induced subgraphs. For $n = 3$ this seems to be non-trivial. It might be easier for larger n. On Figure 4, a list of obstructions for $n = 3$ is given but we do not know whether it is complete.

Let $S_{1,2,3}$ be the graph represented on Figure 5, and let C the class of graphs that contain no triangle and no $S_{1,2,3}$ as an induced subgraph. A famous open question [8] is to determine whether graphs in C have bounded rankwidth. We tried many ways to tune our construction, but each time, there was either a triangle or an $S_{1,2,3}$ in it. So, to the best of our knowledge, our construction does not help to prove that the rankwidth is unbounded for this class. Similarly, whatever tuning we try, it seems that our construction does not help to solve the open problems from [2, 3, 8]. This might be because we did not try hard

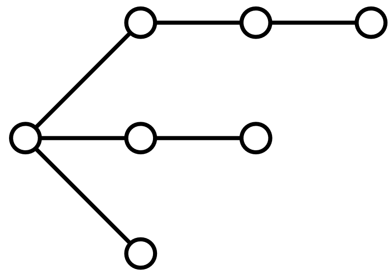

Figure 5: $S_{1,2,3}$

enough to tune our construction, or because it is not the right approach. And of course, the rankwidth might be bounded for all these classes.

In [9], a conjecture is given, stating that in each class of graphs that is closed under taking induced subgraphs and where the rankwidth is unbounded, there should be an infinite sequence of graphs $G_1, G_2, \dots$ such that for each pair of distinct integers i, j, G_i is not an induced subgraph of G_j. This was later disproved, see [14], but we wonder whether carousels may provide new counter-examples. This is difficult to check, because it not clear whether "big" carousels contain smaller ones as induced subgraphs.

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