On the semi-proper orientations of graphs

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Abstract

A *weighted orientation* of a graph G is a pair (D, w) where D is an orientation of G and w is an *arc-weighting* of D, that is an application $A(D) \to \mathbb{N} \setminus \{0\}$. The *in-weight* of a vertex v in a weighted orientation (D, w), denoted by $S_{(D,w)}(v)$, is the sum of the weights of arcs with head v in D. A *semi-proper orientation* is a weighted orientation such that two adjacent vertices have different in-weights. The *semi-proper orientation number* of a graph G, denoted by $\overrightarrow{\chi_s}(G)$, is $\min_{(D,w)\in\Gamma} \max_{v\in V(G)} S_{(D,w)}(v)$, where Γ is the set of all semi-proper orientations of G. A semi-proper orientation (D, w) of a graph G is *optimal* if $\max_{v\in V(G)} S_{(D,w)}(v) = \overrightarrow{\chi_s}(G)$. In this work, we show that every graph G has an optimal semi-proper orientation (D, w) such that the weight of each arc is 1 or 2. We then give some bounds on the semi-proper orientation number: we show $\left\lceil \frac{\mathrm{Mad}(G)}{2} \right\rceil \leq \overrightarrow{\chi_s}(G) \leq \left\lceil \frac{\mathrm{Mad}(G)}{2} \right\rceil + \chi(G) - 1$ and $\left\lceil \frac{\delta^*(G)+1}{2} \right\rceil \leq \overrightarrow{\chi_s}(G) \leq 2\delta^*(G)$ for all graph G, where $\mathrm{Mad}(G)$ and $\delta^*(G)$ are the maximum average degree and the degeneracy of G, respectively. We then deduce that the maximum semi-proper orientation number of a tree is 2, of a cactus is 3, of an outerplanar graph is 4, and of a planar graph is 6. Finally, we consider the computational complexity of associated problems: we show that determining whether $\overrightarrow{\chi_s}(G) = \overrightarrow{\chi}(G)$ is NP-complete for planar graphs G with $\overrightarrow{\chi_s}(G) = 2$; we also show that deciding whether $\overrightarrow{\chi_s}(G) \leq 2$ is NP-complete for planar bipartite graphs G.

1 Introduction

Terminology and notation generally follow [15]. However some standard definitions that are relevant for this paper as well as non-usual definitions are given in Section 2. The graphs have no parallel edges and no loops and the digraphs have no parallel arcs and no loops. We denote by $[n]$ the set of integers $\{1, \ldots, n\}$.

1.1 Proper orientation

An orientation of a graph G is *proper* if any two adjacent vertices have different in-degrees. The *proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}(G)$, is the minimum of the maximum in-degree taken over all proper orientations of the graph G. In other words, the values of the in-degrees define a proper vertex colouring of G. Thus,

$$\chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G) \tag{1}$$

The existence of proper orientations was implicitly demonstrated by Borowiecki, Grytczuk and Pilśniak in [8], who established $\overrightarrow{\chi}(G) \leq \Delta(G)$. Afterwards, the proper orientation number was introduced in [1]. Since then, the proper orientation number has been studied by several authors, for instance see [1, 3, 4, 5, 6, 12]. In [4], it is shown that the proper orientation number of a tree is at most 4 and that there exists a tree whose proper orientation number is at most 4. A natural question is to ask how it can be generalized.

Problem 1.1. Which graph classes containing the trees have bounded proper-orientation number?

Araujo *et al.* [5] proved that this is the case for cacti (their proper orientation number is bounded by 7), but left open the question for the class of outerplanar graphs. The question for the more general class of planar graphs was already asked in [4].

Problem 1.2.
- Does there exists a constant c_1 such that $\vec{\chi}(G) \leq c_1$ for every outerplanar graph G?
- Does there exists a constant c_2 such that $\vec{\chi}(G) \leq c_2$ for every planar graph G?

In [4], the authors also asked the following problem.

Problem 1.3. Is the proper-orientation number upper bounded by a function of the treewidth or the maximum average degree?

Those questions seem highly non-trivial. One of the reasons is that, contrary to many other parameters like the chromatic number, the proper-orientation number is not monotonic. Recall that a graph parameter γ is *monotonic* if $\gamma(H) \leq \gamma(G)$ for every (induced) subgraph H of G.

We should mention that interest in proper orientations stems from their connection to the 1-2-3 Conjecture [11] that says: "If G is a graph with no connected component having exactly two vertices, then its edges can be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights". For more information about 1-2-3-Conjecture and its variants see [2, 7, 9, 14].

1.2 Semi-proper Orientation

Motivated by the proper orientations and the 1-2-3 Conjecture, we investigate the semi-proper orientations. A *weighted orientation* of graph G is a pair (D, w) where D is an orientation of G and w is an arc-weighting $A(D) \to \mathbb{N}\backslash\{0\}$. A *semi-proper orientation* is a weighted orientation such that for every two adjacent vertices u and v, $S_{(D,w)}(v) \neq S_{(D,w)}(u)$, where $S_{(D,w)}(x)$ is the sum of the weights of arcs with head x in D. The *semi-proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}_s(G)$, is $\min_{(D,w)\in\Gamma} \max_{v\in V(G)} S_{(D,w)}(v)$, where Γ is the set of all semi-proper orientations of G. Every proper orientation of a graph G is a semi-proper orientation where the weight of each arc is 1. Consequently, by (1), we have

$$\chi(G) - 1 \leq \overrightarrow{\chi}_s(G) \leq \overrightarrow{\chi}(G) \leq \Delta(G) \tag{2}$$

A semi-proper orientation (D, w) is *optimal* if $\max_{v\in V(G)} S_{(D,w)}(v) = \overrightarrow{\chi}_s(G)$.

In this work, we first show that every graph G has an optimal semi-proper orientation (D, w) such that for each arc $a \in A(D)$, we have $w(a) \in \{1, 2\}$.

Theorem 1.4. *Every graph has an optimal semi-proper orientation such that the weight of each arc is* 1 *or* 2.

In Section 4, we prove that the semi-proper orientation number is monotonic. Next, in Section 5, we give some bounds on $\overrightarrow{\chi_s}$. We first establish the following bounds in terms of the maximum average degree Mad and the chromatic number χ of the graph.

Theorem 1.5. $\left\lceil \frac{\mathrm{Mad}(G)}{2} \right\rceil \leq \overrightarrow{\chi_s}(G) \leq \left\lceil \frac{\mathrm{Mad}(G)}{2} \right\rceil + \chi(G) - 1$ *for any graph* G.

A *planar graph* is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. An *outerplanar graph* is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing. A planar graph has maximum average degree less than 6 and chromatic number at most 4; an outerplanar graph has maximum average degree less than 4, and chromatic number at most 3; a tree has maximum average degree less than 2, and chromatic number at most 2. Hence Theorem 1.5 immediately yields the following, which answers in the affirmative to the analogue of Problem 1.2 for $\overrightarrow{\chi_s}$.

Corollary 1.6. *(i)* $\overrightarrow{\chi_s}(G) \leq 6$ *for every planar graph* G.

(ii) $\overrightarrow{\chi_s}(G) \leq 4$ *for every outerplanar graph* G.

(iv) $\overrightarrow{\chi_s}(T) \leq 2$ *for every tree* T.

A graph G is a *cactus* if every 2-connected component of G is either an edge or a cycle. Clearly, every cactus is an outerplanar graph, so its semi-proper orientation number is at most 4. However, one shows in Subsection 5.2 that it is indeed at most 3.

In Subsection 5.3, we derive the following bounds in terms of the degeneracy δ^* and treewidth tw.

Theorem 1.7. $\left\lceil \frac{\delta^*(G)+1}{2} \right\rceil \leq \overrightarrow{\chi_s}(G) \leq 2\delta^*(G) \leq 2\,\mathrm{tw}(G)$ *for every graph* G.

In Section 6, we show that the above upper bounds are tight. More specifically, we prove that the following types of graphs exist.

- planar graphs G with $\overrightarrow{\chi_s}(G) = 6$ (Proposition 6.1).
- outerplanar graphs G with $\overrightarrow{\chi_s}(G) = 4$ (Proposition 6.3).
- cacti C with $\overrightarrow{\chi_s}(C) = 3$ (Proposition 6.5).
- k-degenerate graphs G with $\overrightarrow{\chi_s}(G) \leq 2k$ for every positive integer k (Proposition 6.6).

Next, we consider the complexity aspects of the semi-proper orientation number. By definition $\overrightarrow{\chi_s} \leq \overrightarrow{\chi}$ and there are many graphs whose proper orientation number is greater than their semi-proper orientation number (for example the trees with proper orientation number 4). We first show that it is NP-complete to decide whether a planar graph G satisfies $\overrightarrow{\chi_s}(G) = \overrightarrow{\chi}(G)$. More precisely we show the following.

Theorem 1.8. *It is NP-complete to decide whether a given planar graph* G *with* $\overrightarrow{\chi}_s(G) = 2$ *has a proper orientation number* 2.

It was shown [4] that it is NP-complete to decide whether the proper orientation number of a given planar bipartite graph is less than or equal to 3. We improve this hardness result for semi-proper orientation number and proper orientation number of bipartite graphs.

Theorem 1.9. *(1) It is NP-complete to decide whether a given planar bipartite graph* G *satisfies* $\overrightarrow{\chi}_s(G) \leq 2$.
(2) It is NP-complete to decide whether a given planar bipartite graph G *satisfies* $\overrightarrow{\chi}(G) \leq 2$.

This shows that determining the semi-proper orientation number of a graph is NP-hard. However, since the degeneracy can be computed in polynomial-time, Theorem 1.7 implies that the semi-proper orientation number can be 4-approximated. Moreover, we describe in the proof of this theorem, a simple greedy procedure that returns a semi-proper orientation (D, w) of a graph G with maximum in-weight at most $2\delta^*(G) \leq 2\overrightarrow{\chi_s}(G)$.

2 Definitions and preliminaries

Let D be a digraph. If (u, v) is an arc, we say that u *dominates* v and write $u \to v$. The *tail* of (u, v) is u and its *head* is v. Let v be a vertex of D. The *out-neighbourhood* of v, denoted by $N_D^+(v)$, is the set of vertices u such that $v \to u$. The *in-neighbourhood* of v, denoted by $N_D^-(v)$, is the set of vertices u such that $u \to v$. The *out-degree* $d_D^+(x)$ (resp. the *in-degree* $d_D^-(x)$) is $|N_D^+(v)|$ (resp. $|N_D^-(v)|$). The maximum in-degree of D is $\Delta^-(D) = \max\{d_D^-(x) \mid x \in V(D)\}$.

An *arc-weighting* of a digraph D is an application $w : A(D) \to \mathbb{N}$. A *weighted digraph* is a pair (D, w), where D is a digraph and w an arc-weighting of D. Let (D, w) be a weighted digraph (D, w). The *in-weight* of a vertex v in (D, w), denoted by $S_{(D,w)}(v)$, is the sum of the weights of arcs with head v in D: $S_{(D,w)}(v) = \sum_{u \in N^-(v)} w(uv)$. The *maximum in-weight* of (D, w) is $\Sigma(D, w) = \max_{v \in V(D)} S_{(D,w)}(v)$. A weighted digraph is *semi-proper* if two adjacent vertices have different in-weights. The *semi-proper number* of a digraph D, denoted by $\mu(D)$, is the minimum maximum in-weight over all semi-proper weighted digraphs on D: $\mu(D) = \min\{\Sigma(D,w) \mid (D,w) \text{ is a semi-proper orientation}\}$.

Let G be a graph. A *(proper) k-colouring* of G is a mapping $c : V(G) \to \{1,\dots,k\}$ such that $c(u) \neq c(v)$ for every edge $uv \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G admits a k-colouring.

Consider a graph $G = (V,E)$, and let $S \subset V$ be any subset of vertices of G. Then, the induced subgraph on the set of vertices S, denoted by $G\langle V(S)\rangle$, is the graph whose vertex set is S and whose edge set consists of all the edges in E that have both endpoints in S.

The *average degree* of G is $\mathrm{Ad}(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} d(v) = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G $\mathrm{Mad}(G) = \max\{Ad(H) \mid H \text{ is a subgraph of } G\}$.

We say that a graph G is *k-degenerate* if each of its subgraphs has a vertex of degree at most k. The *degeneracy* of G, denoted by $\delta^*(G)$, is the minimum integer k such that G is k-degenerate. In symbols, $\delta^*(G) = \max\{\delta(H) \mid H \text{ is a subgraph of } G\}$, where $\delta(H)$ is the minimum degree of H. It is well-known that

$$\delta^*(G) \leq \mathrm{Mad}(G) \leq 2\delta^*(G).$$

Moreover, it is folklore that every graph has an ordering $(v_1,\dots,v_n)$ of its vertices such that v_i has at most $\delta^*(G)$ neighbours in $\{v_1,\dots,v_{i-1}\}$ for all $2 \leq i \leq n$. In particular, it implies that

$$\chi(G) \leq \delta^*(G) + 1.$$

A set A of vertices is *complete* to another set B if ab is an edge for all $a \in A$ and $b \in B$.

A vertex v in a graph G is *k-simpilicial* if it has degree k and its neighbours form a clique. A graph G is a *k-tree* if either $G = K_{k+1}$ (the complete graph on $k+1$ vertices), or G contains a k-simplicial vertex v and $G - v$ is also a k-tree. The *treewidth* $\mathrm{tw}(G)$ of a graph G can be defined as follows:

$$\mathrm{tw}(G) = \min\{k \mid G \text{ is a spanning subgraph of a } k\text{-tree}\}.$$

By definition, every k-tree is k-degenerate, and so are all its subgraphs. Hence for every graph G we have

$$\delta^*(G) \leq \mathrm{tw}(G).$$

An *orientation* D of a graph G is a digraph obtained from the graph G by replacing each edge by exactly one of the two possible arcs with the same endvertices. A *weighted orientation* of G is a weighted digraph (D,w) where D is an orientation of G. A *semi-proper orientation* of G is a weighted orientation of G that is semi-proper. The *semi-proper orientation number*, denoted by $\overrightarrow{\chi_s}(G)$, is the minimum of the maximum in-weight over all semi-proper orientations of G:

$$\begin{aligned}\overrightarrow{\chi_s}(G) &= \min\ \{\Sigma(D,w) \mid (D,w) \text{ is a semi-proper orientation of } G\}\\ &= \min\ \{\mu(D) \mid D \text{ is an orientation of } G\}\end{aligned}$$

3 Optimal semi-proper orientation with weights in $\{1,2\}$

In this section, we prove Theorem 1.4 which states that every graph has an optimal semi-proper orientation such that the weight of each arc is 1 or 2.

Proof of Theorem 1.4. We prove the theorem by contradiction.

In an optimal semi-proper orientation of G, if the weight of an arc a is greater than 2, then we say that a is *bad*. Let t be the minimum number of bad arcs in an optimal semi-proper orientation of G, and let $\mathcal{F}$ be the set of optimal semi-proper orientations with exactly t bad arcs.

Suppose for a contradiction that $t > 0$. Let (D,w) be an optimal semi-proper orientation in $\mathcal{F}$ such that the sum of the weights of the bad arcs is minimum. Let b be the sum of the weights of bad arcs in (D,w).

Let uv be a bad arc in (D, w). Let $\mathcal{R}_v$ be the set of vertices z such that there is a directed path from v to z in D. Among all vertices in $\mathcal{R}_v$ let p be a vertex with minimum in-weight. (i.e. $S_{(D,w)}(p) = \min\{S_{(D,w)}(z) | z \in \mathcal{R}_v\}$). We distinguish two cases.

<u>**Case 1.**</u> $S_{(D,w)}(p) = S_{(D,w)}(v)$. In this case, the in-weight of v is minimum over all vertices in $\mathcal{R}_v$. Let $u_1, \dots, u_k$ be the in-neighbours of v. Without loss of generality assume that $u = u_1$. Also, let $o_1, \dots, o_r$ be the out-neighbours of v. So, we have $o_1, \dots, o_r \in \mathcal{R}_v$.

Claim 3.1. *$S_{(D,w)}(v) < S_{(D,w)}(o_i)$ for all $i \in [r]$.*

Proof. Let $i \in [r]$. By our assumption $S_{(D,w)}(v) \leq S_{(D,w)}(o_i)$, and since $S_{(D,w)}$ is a proper colouring, $S_{(D,w)}(v) \neq S_{(D,w)}(o_i)$. Hence $S_{(D,w)}(v) < S_{(D,w)}(o_i)$. ■

Claim 3.2. *$w(u_i v) = 2$ for all $2 \leq i \leq k$.*

Proof. To the contrary assume that there is $2 \leq i \leq k$ such that $w(u_i v) \neq 2$. We distinguish two cases.

- If $w(u_i v) = 1$, then consider the arc-weighting w' of D defined by

$$w'(a) = \begin{cases} w(uv) - 1, & \text{if } a = uv, \\ w(u_i v) + 1, & \text{if } a = u_i v, \\ w(a), & \text{otherwise.} \end{cases}$$

Clearly, (D, w') is an optimal semi-proper orientation that has at most t bad arcs and also the sum of the weights of its bad arcs is less than b. This is a contradiction.

- If $w(u_i v) > 2$, consider the arc-weighting w' of D defined by

$$w'(a) = \begin{cases} 1, & \text{if } a = uv, \\ w(u_i v) + w(uv) - 1, & \text{if } a = u_i v, \\ w(a), & \text{otherwise.} \end{cases}$$

Clearly, (D, w') is an optimal semi-proper orientation with $t - 1$ bad arcs, a contradiction. This completes the proof of Claim 3.2. ■

Claim 3.3. *$S_{(D,w)}(o_i) > 2k + 1$ for all $i \in [r]$.*

Proof. We have $w(u_1 v) = \alpha > 2$, so by Claim 3.2, we have $S_{(D,w)}(v) > 2k$. Thus, by Claim 3.1, we have $S_{(D,w)}(o_i) > 2k + 1$. ■

Now, we are ready to obtain a contradiction for Case 1. Let q be such that $k \leq q \leq 2k$ and $q \notin \{S_{(D,w)}(u_i) : 1 \leq i \leq k\}$. For each arc $u_i v$, $1 \leq i \leq k$, we choose $Var(u_i v)$ in $\{1, 2\}$ such that $\sum_{i=1}^{k} Var(u_i v) = q$. This is clearly possible. Now let w' be the arc-weighting of D defined by

$$w'(a) = \begin{cases} Var(u_i v), & \text{if } a = u_i v, \ 1 \leq i \leq k, \\ w(a), & \text{otherwise.} \end{cases}$$

By Claim 3.3, and the way that we choose q, it is clear that (D, w') is an optimal semi-proper orientation with $t - 1$ bad arcs, a contradiction.

<u>**Case 2.**</u> $S_{(D,w)}(p) \neq S_{(D,w)}(v)$. By the definition of $\mathcal{R}_v$ there is a directed path $\mathcal{P} = (v, z_1, \dots, z_l, p)$ from the vertex v to the vertex p. Let $u_1, \dots, u_k$ be the in-neighbours of p. Without loss of generality assume that $z_l = u_1$. (Note that we can have the case where the only directed path from v to p is the arc vp. In that case we assume that $u_1 = v$.) Next, let $o_1, \dots, o_r$ be the out-neighbours of p.

Claim 3.4. $S_{(D,w)}(p) \geq w(u_1 p) + 2k - 3$.

Proof. To the contrary suppose that $S_{(D,w)}(p) \leq w(u_1p) + 2k - 4$. So there are distinct indices j, j' greater than 1 such that $w(u_jp) = w(u_{j'}p) = 1$. We are going to reverse the arcs in $\mathcal{P}$, changing the weights of some arcs to ensure that the in-weights of vertices remain unchanged. More specifically, let D' be the orientation of G obtained from D by reversing the arcs of $\mathcal{P}$ and let w' be the arc-weighting of D' defined as follows:

$$w'(a) = \begin{cases} w(uv) - 1, & \text{if } a = uv, \\ 1, & \text{if } a = z_1v, \\ w(vz_1), & \text{if } a = z_2z_1, \\ w(z_{i-1}z_i), & \text{if } a = z_{i+1}z_i,\ 2 \leq i < l, \\ w(z_{l-1}z_l), & \text{if } a = pz_l, \\ 2, & \text{if } a = u_jp, \\ w(u_1p), & \text{if } a = u_{j'}p \\ w(a), & \text{otherwise.} \end{cases}$$

Note that for every vertex f we have $S_{(D,w)}(f) = S_{(D',w')}(f)$. Thus (D', w') is an optimal semi-proper orientation of G. Moreover it has at most t bad arcs and the sum of the weights of the bad arcs is $b - 1$. This is a contradiction. ■

Claim 3.5. *$S_{(D,w)}(o_i) > 2k - 2$ for all $i \in [r]$, and $S_{(D,w)}(z_l) > 2k - 2$.*

Proof. By our assumption the in-weight of p is minimum. Thus, $S_{(D,w)}(p) \leq S_{(D,w)}(o_i)$ for all $i \in [r]$, and $S_{(D,w)}(p) \leq S_{(D,w)}(z_l)$. On the other hand, the function $S_{(D,w)}$ is a proper colouring, so $S_{(D,w)}(p) < S_{(D,w)}(o_i)$ for all $i \in [r]$, and $S_{(D,w)}(p) < S_{(D,w)}(z_l)$. Now, by Claim 3.4, $S_{(D,w)}(p) \geq 2k - 2$. Hence $S_{(D,w)}(o_i) > 2k - 2$ for all $i \in [r]$, and $S_{(D,w)}(z_l) > 2k - 2$. ■

Now, we are ready to obtain a contradiction for Case 2. Let q be such that $k - 1 \leq q \leq 2k - 2$ and $q \notin \{S_{(D,w)}(u_i) : 2 \leq i \leq k\}$. For each arc u_iv, $2 \leq i \leq k$, we choose $Var(u_iv)$ in $\{1, 2\}$ such that $\sum_{i=1}^{k} Var(u_iv) = q$. This is clearly possible.

Let D' be the orientation of G obtained from D by reversing the arcs of $\mathcal{P}$ and let w' be the arc-weighting of D' defined as follows:

$$w'(a) = \begin{cases} w(uv) - 1, & \text{if } a = uv, \\ 1, & \text{if } a = z_1v, \\ w(vz_1), & \text{if } a = z_2z_1, \\ w(z_{i-1}z_i), & \text{if } a = z_{i+1}z_i,\ 2 \leq i < l, \\ w(z_{l-1}z_l), & \text{if } a = pz_l, \\ Var(u_iv), & \text{if } a = u_iv,\ 2 \leq i \leq k, \\ w(a), & \text{otherwise.} \end{cases}$$

By Claim 3.5, and the way that we choose q, it is clear that (D', w') is an optimal semi-proper orientation. Moreover it has at most t bad arcs and the sum of the weights of the bad arcs is at most $b - 1$, a contradiction.

This completes the proof of Theorem 1.4. □

4 Monotonicity of the semi-proper orientation number

Lemma 4.1. *If H is a subgraph of G, then $\overrightarrow{\chi_s}(H) \leq \overrightarrow{\chi_s}(G)$.*

Proof. Let (D, w) be an optimal semi-proper orientation of G. Set D' be the orientation of H which agrees with D on every edge of H. Let X be the set of vertices of H that have an in-neighbour in D'. Note that $Y = V(D') \setminus X$ is the set of sources of D'. For every vertex $x \in X$ choose an arc a_x in $A(D')$ with head x. Let $T(x)$ be the sum of the weights of the arcs of $A(D) \setminus A(D')$ with head x. ($T(x) = 0$ if there is no such arc.) Let us now define a weight function w' on the arcs of D' as follows: $w'(a_x) = w(a_x) + T(x)$ for all $x \in X$, and $w'(a) = w(a)$ if $a \notin \{a_x \mid x \in X\}$.

Now consider (D', w'). By construction, if $x \in X$ then $S_{(D',w')}(x) = S_{(D,w)}(x)$ and $S_{(D',w')}(y) = 0$ for all $y \in Y$. Now, because two sources are not adjacent in a digraph, (D', w') is a semi-proper orientation of H. Moreover, we clearly have $\max_{v\in V(H)} S_{(D',w')}(v) \leq \max_{v\in V(H)} S_{(D,w)}(v) \leq \max_{v\in V(G)} S_{(D,w)}(v) = \overrightarrow{\chi_s}(G)$. □

5 Bounds on the semi-proper orientation number

5.1 Bounds in terms of maximum average degree and chromatic number

The aim of this subsection is to establish Theorem 1.5.

Theorem 5.1. *$\Delta^-(D) \leq \mu(D) \leq \Delta^-(D) + \chi(D) - 1$ for all digraph D.*

Proof. Let D be a digraph. By definition, $d^-_D(v) \leq S_{(D,w)}(v)$ for any arc-weighting w. Thus $\Delta^-(D) \leq \mu(D)$.

Let $(S_1, \dots, S_{\chi(D)})$ be a $\chi(D)$-colouring of D. For $i \in [\chi(D)]$ and $v \in S_i$, let α_v be the integer in $[\chi(D)]$ such that $d^-_D(v) + \alpha_v - 1 \equiv i \mod \chi(D)$. Let w be the arc-weighting of D defined as follows. For every vertex v with in-degree at least 1 choose an arc a_v with head v; now if a is an arc with head v, set $w(a) = \alpha_v$ if $a = a_v$ and $w(a) = 1$ otherwise.

Observe that for every vertex $v \in S_i$, either v is a source and $S_{(D,w)}(v) = 0$, or $S_{(D,w)}(v) = d^-_D(v) + \alpha_v - 1$ so $S_{(D,w)}(v) > 0$ and $S_{(D,w)}(v) \equiv i \mod \chi(D)$. This implies that (D, w) is semi-proper. Moreover, for every vertex v, $S_{(D,w)}(v) \leq d^-_D(v) + \alpha_v - 1 \leq \Delta^-(D) + \chi(D) - 1$. □

For a graph G, we denote by $m^-(G)$ the minimum of the maximum in-degree over its orientations. In symbols, $m^-(G) = \min\{\Delta^-(D) \mid D \text{ is an orientation of } G\}$.

Theorem 5.1 immediately implies the following.

Corollary 5.2. *$m^-(G) \leq \overrightarrow{\chi_s}(G) \leq m^-(G) + \chi(G) - 1$ for all graph G.*

Proposition 5.3. *$m^-(G) = \left\lceil \frac{1}{2} \operatorname{Mad}(G) \right\rceil$ for all graph G.*

Proof. Let G be a graph. Let H be a subgraph of G such that $\operatorname{Ad}(H) = \operatorname{Mad}(G)$. Then H has $\frac{1}{2} \operatorname{Ad}(H)|V(H)|$ edges. Hence every orientation of H has a vertex with in-degree at least $\lceil \frac{1}{2} \operatorname{Ad}(H) \rceil = \lceil \frac{1}{2} \operatorname{Mad}(G) \rceil$. Since every orientation of G contains an orientation of H, we have $m^-(G) \geq \lceil \frac{1}{2} \operatorname{Mad}(G) \rceil$.

Consider now an orientation D of G such that $\Delta^-(D) = m^-(G)$ and such that the number of vertices with in-degree $m^-(G)$ is minimum. Let Z (resp. Y, X) be the set of vertices with in-degree $m^-(G)$ (resp. $m^-(G) - 1$, at most $m^-(G) - 2$) in D. Then (X, Y, Z) is a partition of $V(G)$. Observe that, in D, there is no directed path from X to Z for otherwise reversing the arcs of such a path would result in an orientation of G with maximum in-degree $m^-(G)$ and less vertices with in-degree $m^-(G)$, a contradiction. Let R be the set of vertices for which there is a directed path to a vertex of Z. By definition $Z \subseteq R$ and by the above observation $R \subseteq Y \cup Z$. By definition there is no arcs entering R in D, so $|E(G\langle R\rangle)| = \sum_{v\in R} d^-_D(v)$. Since $R \subseteq Y \cup Z$ and Z is non-empty, we have $\sum_{v\in R} d^-_D(v) > |R| \cdot (m^-(G) - 1)$. Thus $\frac{1}{2} \operatorname{Mad}(G) \geq |E(G\langle R\rangle)|/|R| > m^-(G) - 1$, so $\lceil \frac{1}{2} \operatorname{Mad}(G) \rceil \geq m^-(G)$. □

Proposition 5.3 and Corollary 5.2 immediately imply Theorem 1.5.

5.2 Cacti

For every nonnegative integer i, an *i-vertex* is a vertex of degree i.

Lemma 5.4. *Let $k \geq 2$ be an integer, G a graph, and v a 1-vertex in G. $\overrightarrow{\chi_s}(G) \leq k$ if and only if $\overrightarrow{\chi_s}(G-v) \leq k$.*

Proof. By Lemma 4.1, if $\overrightarrow{\chi_s}(G) \leq k$ then $\overrightarrow{\chi_s}(G-v) \leq k$.

Let us now prove the reciprocal. Assume that $\overrightarrow{\chi_s}(G-v) \leq k$. Let (D,w) be an optimal semi-proper orientation of $G-v$. Let u be the neighbour of v in G. We extend it by orienting uv from u to v and choosing $w(uv)$ in $\{1,2\} \setminus \{S_{(D,w)}(u)\}$. One easily checks that this yields a semi-proper orientation of G. Hence $\overrightarrow{\chi_s}(G) \leq k$. $\square$

Remark 5.5. Note that Lemma 5.4 and an easy induction yields that $\overrightarrow{\chi_s}(T) \leq 2$ for every tree T.

Lemma 5.6. *Let $k \geq 3$ be an integer, G a graph, and u, v two adjacent 2-vertices in G. $\overrightarrow{\chi_s}(G) \leq k$ if and only if $\overrightarrow{\chi_s}(G - \{u,v\}) \leq k$.*

Proof. By Lemma 4.1, if $\overrightarrow{\chi_s}(G) \leq k$ then $\overrightarrow{\chi_s}(G - \{u,v\}) \leq k$.

Let us now prove the reciprocal. Let u' (resp. v') be the neighbour of u (resp. v) distinct from v (resp. u). Let (D,w) be a semi-proper orientation of $G - \{u,v\}$. We shall extend it into a semi-proper orientation of G as follows.

If $S_{(D,w)}(u') = S_{(D,w)}(v') = 1$, then orient $u'u$ from u' to u, $v'v$ from v' to v, and uv from u to v and set $w(u'u) = w(v'v) = 2$ and $w(uv) = 1$ to obtain a semi-proper orientation of G.

Henceforth, without loss of generality, we may assume $S_{(D,w)}(u') \neq 1$. Orient $u'u$ from u' to u, $v'v$ from v' to v, and uv from u to v and set $w(u'u) = w(v'v) = 1$ and $w(uv) = 1$ if $S_{(D,w)}(v') \neq 2$ and $w(uv) = 2$ otherwise. One easily checks that this yields a semi-proper orientation of G. Hence $\overrightarrow{\chi_s}(G) \leq k$. $\square$

Theorem 5.7. *If G is a cactus, then $\overrightarrow{\chi_s}(G) \leq 3$.*

Proof. By induction on the number of vertices of G, the result holding trivially if G has one vertex.

Assume now that G is a cactus on more than one vertex. Then it has either a 1-vertex or two adjacent 2-vertices. The induction hypothesis and Lemmas 5.4 and 5.6 yield the result. $\square$

5.3 Bounds in terms of degeneracy

The aim of this subsection is to establish Theorem 1.7 which says that for every graph G, we have $\left\lceil \frac{\delta^*(G)+1}{2} \right\rceil \leq \overrightarrow{\chi_s}(G) \leq 2\delta^*(G)$.

Proof of Theorem 1.7. Let G be a graph. Let us first prove $\left\lceil \frac{\delta^*(G)+1}{2} \right\rceil \leq \overrightarrow{\chi_s}(G)$. Let (D,w) be an optimal semi-proper orientation of G. Let H be a subgraph of G such that $\delta(H) = \delta^*(G)$. H has at least $\frac{1}{2}\delta(H)|V(H)| = \frac{1}{2}\delta^*(G)|V(H)|$ edges. Thus $D\langle V(H)\rangle$ has at least $\frac{1}{2}\delta^*(G)|V(H)|$ arcs.

If there is a vertex v of $V(H)$ with in-degree greater than $\lceil \delta^*(G)/2 \rceil$ in $D\langle V(H)\rangle$, that is at least $\left\lceil \frac{\delta^*(G)+1}{2} \right\rceil$, then $\overrightarrow{\chi_s}(G) \geq S_{(D,w)}(v) \geq d^-(v) \geq \left\lceil \frac{\delta^*(G)+1}{2} \right\rceil$.

If no vertex of $V(H)$ has in-degree greater than $\lceil \delta^*(G)/2 \rceil$ in $D\langle V(H)\rangle$, then every vertex v of $V(H)$ has in-degree exactly $\lceil \delta^*(G)/2 \rceil$, and so every for every vertex $v \in V(H)$, we have $S_{(D,w)}(v) \geq \lceil \delta^*(G)/2 \rceil$. Let x and y be two adjacent vertices in H. Without loss of generality, we may assume that $S_{(D,w)}(x) > S_{(D,w)}(y)$. Hence $S_{(D,w)}(x) > \lceil \delta^*(G)/2 \rceil$, so $S_{(D,w)}(x) \geq \left\lceil \frac{\delta^*(G)+1}{2} \right\rceil$. Therefore $\overrightarrow{\chi_s}(G) \geq \left\lceil \frac{\delta^*(G)+1}{2} \right\rceil$.

Next, we focus on the second part of the equation. Since $\chi(G) \leq \delta^*(G) + 1$ and $\mathrm{Mad}(G) \leq 2\delta^*(G)$, Theorem 5.1 yields $\overrightarrow{\chi_s}(G) \leq 2\delta^*(G)$. However, we give here a simple and algorithmic proof of $\overrightarrow{\chi_s}(G) \leq 2\delta^*(G)$

which shows that one can find a semi-proper orientation of G with maximum in-weight $2\delta^*(G)$ using a greedy procedure.

This procedure is the following:

1. We first compute an ordering $(v_1, \dots, v_n)$ of the vertices of G such that v_i has at most $\delta^*(G)$ neighbours in $\{v_1, \dots, v_{i_1}\}$ for all $2 \le i \le n$. It is well-known that such an ordering can be easily computed by taking a vertex of minimum degree for v_n and apply recursively the procedure on $G - v_n$.

2. Let D be the orientation of G in which every edge is oriented from its lower-indexed endvertex to its higher indexed vertex (i.e. if $v_i v_j i$ is an edge and $i < j$, then $v_i v_j \in A(D)$).

3. For $1 \le k \le n$, we construct inductively an arc-weighting w_k of semi-proper orientation of $D_k = D\langle\{v_1, \dots, v_k\}\rangle$ with maximum in-weight $2\delta^*(G)$.

   (a) For $k = 1$, the empty arc-weighting w_1 is fine.

   (b) For $k > 2$, then set $w_k(a) = w_{k-1}(a)$ for every arc of $a \in A(D_{k-1})$. Let $v_{i_1}, \dots, v_{i_p}$ be the in-neighbours of v_k in $\{v_1, \dots, v_{k-1}\}$. By construction, we know that $p \le \delta^*(G)$. One can choose $w_k(v_{i_1} v_k), \dots, w_k(v_{i_p} v_k)$ in $\{1, 2\}$ such that $S(D_k, w_k)(v_k) = \sum_{\ell=1}^{p} w(v_{i_\ell} v_k) \neq S_{(D_{k-1}, w_{k-1})}(v_{i_j}) = S_{(D_k, w_k)}(v_{i_j})$ for all $1 \le j \le p$. This is possible because there are $p + 1$ possible values for $S(D_k, w_k)(v_k)$ (all integers between p and $2p$) and only p forbidden ones. Moreover $S(D_k, w_k)(v_k) \le 2p \le 2\delta^*(G)$.

4. Return (D, w_n).

$\square$

6 Tightness of the bounds

Proposition 6.1. *There exists a planar graph G such that $\overrightarrow{\chi_s}(G) \le 6$.*

Proof. For every triangulation T, we define by T^+ the triangulation obtained by adding a vertex in each face and joining it to the three vertices incident to the face.

Let T_0 be the triangle. For every $i \in \mathbb{N}$, let $T_i = T_{i-1}^+$. Note that if $i < j$, then T_i is a subgraph of T_j.

We shall prove that $\overrightarrow{\chi_s}(T_{44}) = 6$.

Assume for a contradiction that there is a semi-proper orientation (D, w) of T_{44} with $\Sigma(D, w) \le 5$.

The *minimum weight* of an edge uv is $m(uv) = \min\{S_{(D,w)}(u), S_{(D,w)}(v)\}$.

Claim 6.2. *Let (a, b) be an edge in T_i. If $m(ab) \le 3$, then T_{i+8} has an edge $a'b'$ such that $m(a'b') > m(ab)$.*

Proof. Assume that $m(ab) \le 3$. Without loss of generality, we may assume that $m(ab) = S_{(D,w)}(a) < S_{(D,w)}(b)$.

Set $j = i + 8$. The common neighbourhood of a and b in T_j is the union of two disjoint paths P and Q of order $j - i = 8$. There are at most $S_{(D,w)}(a) + S_{(D,w)}(b) - 1 \le 7$ vertices which are not dominated by both a and b. Hence, one of the two paths, say P, contains at most three vertices which are not dominated by both a and b. Since P has order 8, there are at least two consecutive vertices on P, say c and d, which are both dominated by a and b. Both c and d have in-weight at least 2. Without loss of generality, $S_{(D,w)}(c) < S_{(D,w)}(d)$, so $S_{(D,w)}(d) \ge 3 \ge S_{(D,w)}(a)$. Since a and d are adjacent, $S_{(D,w)}(d) > S_{(D,w)}(a)$. Hence the edge db satisfies $m(db) > S_{(D,w)}(a) = m(ab)$. ■

Now T_0 has an edge with minimum weight at least 1, so by Claim 6.2, T_8 has an edge with minimum weight at least 2, T_{16} has an edge with minimum weight at least 3, and T_{24} has an edge ab with minimum weight 4. Without loss of generality, we have $S_{(D,w)}(a) = 4$ and $S_{(D,w)}(b) = 5$. The common neighbourhood of a and b in T_{44} is the union of two disjoint paths P and Q of order 20. There are at most $S_{(D,w)}(a) + S_{(D,w)}(b) - 1 \le 8$ vertices which are not dominated by both a and b. Hence, one of the two paths, say P, contains at most four vertices which are not dominated by both a and b. Since P has order 20, there are four consecutive vertices v_1, v_2, v_3, v_4 of P that are all dominated by both a and b. Those four vertices must have in-weight in $\{2, 3\}$. Without loss of generality, $S_{(D,w)}(v_1) = S_{(D,w)}(v_3) = 3$ and $S_{(D,w)}(v_2) = S_{(D,w)}(v_4) = 2$. Vertex v_2 is dominated by a and b and has in-weight 2, so $v_2 \to v_3$. Now v_3 is dominated by a, b and v_2 and has in-weight 3 so so $v_3 \to v_4$. Therefore v_4 has in-degree at least 3, but has in-weight at most 2, a contradiction. □

The bound 4 of Corollary 1.6 (ii) is best possible, as shown by the following proposition.

Proposition 6.3. *There exist outerplanar graphs with semi-proper orientation number* 4.

Proof. Let G be a 2-connected outerplanar graph with outer cycle $C = x_1 \dots x_n x_1$. The *budding* of G, denoted by $B(G)$, is the outerplanar graph obtained from G by adding for every edge $e = uv$ of C a new vertex z_e (in the outer face) connected to both u and v. The outer cycle of $B(G)$ is then $x_1 z_{x_1x_2} x_2 z_{x_2x_3} \dots x_n z_{x_nx_1} x_1$.

Let B_1 be the triangle and for each $i > 1$ let $B_i = B(B_{i-1})$. See Figure 1. For every positive integer i, we denote by C_i the outer cycle of B_i.

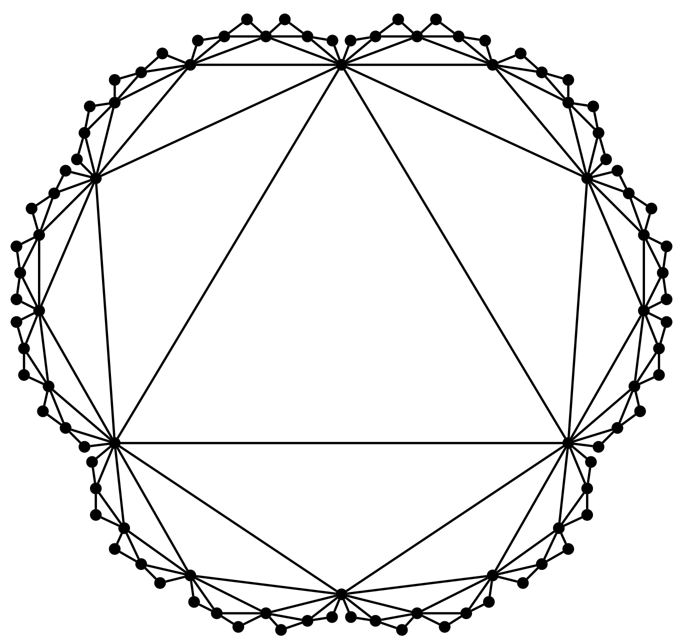


Figure 1: B_6, an outerplanar graph with semi-proper orientation number 4.

We shall prove that $\overrightarrow{\chi_s}(B_6) \geq 4$. For every interior edge $e = xy$ (i.e. which is not in C_6), let P_6 be the subpath of C_6 with end-vertices x and y and containing z_{xy}. The *bud of* e, denoted by $B(e)$, is the subgraph of B_6 induced by $V(P_6)$ that is the outerplanar graph with outer face $P_6 \cup \{e\}$.

Suppose for a contradiction that B_6 admits a semi proper orientation (D, w) with $\Sigma(D, w) \leq 3$. An interior edge $e = xy$ of C_i (for some $i \in [5]$) is *dominating* if there is no arc from $V(B(e)) \setminus \{x, y\}$ into $\{x, y\}$ in D. An edge xy is (k_1, k_2)-*dominating* if it is dominating and $S_{(D,w)}(x) = k_1$ and $S_{(D,w)}(y) = k_2$.

A (k_1, k_2, k_3)-*triangle* xyz is a set of vertices $\{x, y, z\}$ such that $G\langle\{x, y, z\}\rangle$ is a complete subgraph and $S_{(D,w)}(x) = k_1$, $S_{(D,w)}(y) = k_2$, and $S_{(D,w)}(z) = k_3$.

Claim 6.4. (i) *There is no* $(2, 3)$-*dominating edge in* B_5.

(ii) *There is no* $(0, 3)$- *or* $(1, 3)$-*dominating edge in* B_4.

(iii) *There is no* $(0, 2)$- *or* $(1, 2)$-*dominating edge in* B_3.

(iv) *There is no* $(1,2,3)$*-triangle in* B_2.

Proof. (i) Assume for a contradiction that an edge $xy \in E(B_5)$ is $(2,3)$-dominating. Then z_{xy} is dominated by both x and y. So $S_{(D,w)}(z_{xy}) \geq 2$. This is a contradiction because it is adjacent to a vertex with in-weight 2 (namely x) and to a vertex with in-weight 3 (namely y).

(ii) Assume for a contradiction that an edge $xy \in E(B_4)$ is $(0,3)$- or $(1,3)$-dominating. Then z_{xy} is dominated by both x and y and so $S_{(D,w)}(z_{xy}) = 2$, because $S_{(D,w)}(y) = 3$. But then the edge $z_{xy}y$ is $(2,3)$-dominating and in $E(B_5)$, a contradiction with (i).

(iii) Assume for a contradiction that an edge $xy \in E(B_3)$ is $(0,2)$- or $(1,2)$-dominating. Then z_{xy} is dominated by both x and y and so $S_{(D,w)}(z_{xy}) = 3$, because $S_{(D,w)}(y) = 2$. Now z_{xy} has at most one in-neighbour distinct from x and y. Since $B(xz_{xy})$ and $B(yz_{xy})$ only intersect in z_{xy}, this out-neighbour is only is one of those two buds. Therefore either xz_{xy} is $(0,3)$- or $(1,3)$-dominating, a contradiction to (ii) or yz_{xy} is $(2,3)$-dominating, a contradiction to (i).

(iv) Assume for a contradiction that there is a $(1,2,3)$-triangle abc in B_2.

There are exactly three arcs from $V(B_6) \setminus \{a,b,c\}$ to $\{a,b,c\}$. But by (iii), (resp. (ii), (i)) the edge ab, (resp. ac, bc) is not dominating, and so there is an arc from $V(B(ab) \setminus \{a,b\}$ to $\{a,b\}$ (resp. from $V(B(ac) \setminus \{a,c\}$ to $\{a,c\}$, from $V(B(bc) \setminus \{b,c\}$ to $\{b,c\}$). Thus there is exactly one arc from $V(B(ab) \setminus \{a,b\}$ to $\{a,b\}$, one arc from $V(B(ac) \setminus \{a,c\}$ to $\{a,c\}$, and one arc from $V(B(bc) \setminus \{b,c\}$ to $\{b,c\}$. Consequently, z_{bc} dominates at most one vertex in $\{b,c\}$. Now $S_{(D,w)}(z_{bc}) \in \{0,1\}$, so $d^-(z_{bc}) \leq 1$. Hence $d^-(z_{bc}) \leq 1$, and so z_{bc} is dominated by at most one vertex in $\{b,c\}$. Therefore, either b is the unique in-neighbour of z_{bc} and $z_{bc} \rightarrow c$, or c is the unique in-neighbour of z_{bc} and $z_{bc} \rightarrow c$. In both case bz_{bc} is a $(1,2)$-dominating edge, a contradiction to (iii).

■

Let us now consider the triangle $B_1 = abc$. Without loss of generality, we may assume $S_{(D,w)}(a) < S_{(D,w)}(b) < S_{(D,w)}(c)$. By Claim 6.4 (iv), it is not a $(1,2,3)$-triangle. Henceforth, abc is either a $(0,1,3)$- or a $(0,2,3)$-triangle. In particular, a dominates b and c.

If B_1 is a $(0,1,3)$-triangle, then b dominates z_{bc} which must have in-weight 2. So $bz_{bc}c$ is a $(1,2,3)$-triangle, a contradiction to Claim 6.4 (iv).

Henceforth B_1 is a $(0,2,3)$-triangle. By Claim 6.4 (iv), there is no $(1,2,3)$-triangle, hence z_{bc} must have in-weight 0. In particular z_{bc} dominates b. But then ab is a $(0,2)$-dominating edge, a contradiction to (iii). □

The bound 3 of Theorem 5.7 is best possible as shown by the following proposition.

Proposition 6.5. *There exist cacti with semi-proper orientation number* 3.

Proof. Consider the cactus C obtained from a triangle $T = x_1x_2x_3$ by adding three triangles $T_i = x_iy_iz_i$, $i \in [3]$, intersecting T in x_i. See Figure 2.

Let (D,w) be a semi-proper orientation of C. We shall prove that there is a vertex of C with in-weight at least 3.

If $D\langle T\rangle$ is a directed cycle, then all vertices of T have in-degree and so in-weight at least 1. Since T is a clique, the vertices of T have different in-weights. Thus at least one of them has in-weight at least 3.

If $D\langle T\rangle$ is a transitive tournament, then one vertex of T, say x_1 has in-degree at least 2 in D. If x_1 has in-weight at least 3, then we are done, so we may assume that $S_{(D,w)}(x_1) = 2$. Necessarily, x_1 dominates y_1 and z_1. Hence those two vertices have in-weight at least 1. Moreover since T_1 is a clique, x_1, y_1 and z_1 have different in-weights. Thus one of y_1, z_1 has in-weight at least 3.

Therefore $\overrightarrow{\chi_s}(C) \geq 3$, and so $\overrightarrow{\chi_s}(C) = 3$ by Theorem 5.7. □

The lower bound of Theorem 1.7 is tight. Indeed if G is a k-regular bipartite graph, then $\delta^*(G) = k$ and $\overrightarrow{\chi_s}(G) \leq \lceil \frac{k+1}{2} \rceil$ by Theorem 1.5. The upper bound of Theorem 1.7 is tight as shown by the following proposition.

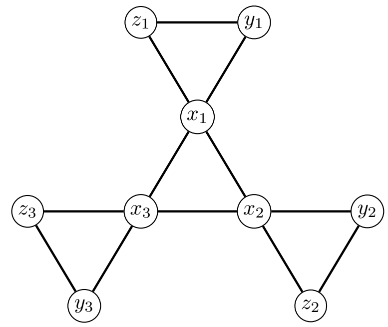


Figure 2: Cactus C with semi-proper orientation number 3.

Proposition 6.6. *For every k, there is a k-tree H_k such that $\overrightarrow{\chi_s}(H_k) = 2k$.*

Proof. Let G be a graph. We denote by $G^{\#k}$ the supergraph of G obtained as follows. For each clique C of size k in G, we create a set Z_C of $2k^2 + 1$ new vertices which is complete to C. Observe that if G is a k-tree then $G^{\#k}$ is also a k-tree.

Let H_0 be the complete graph on $k+1$ vertices. For $i = 1$ to $k+1$, let $H_i = H_{i-1}^{\#k}$. We shall prove that $\overrightarrow{\chi_s}(H_{k+1}) = 2k$.

Let (D, w) be an optimal semi-proper orientation of H_{k+1}. By Theorem 1.7, $\overrightarrow{\chi_s}(H_{k+1}) \leq 2k$, so for every vertex $v \in V(H_{k+1})$, we have $d_D^-(v) \leq S_{(D,w)}(v) \leq 2k$. We say that a vertex v is *big* if $S_{(D,w)}(v) \geq k$. We shall prove the following by induction on i, $0 \leq i \leq k+1$.

($\star$) *There is a $(k+1)$-clique $C_i \subseteq V(H_i)$ with at least i big vertices.*

For $i = 0$, ($\star$) holds trivially. Assume now that $i > 0$. By the induction hypothesis, there is a $(k+1)$-clique $C_{i-1} \subseteq V(H_i)$ with at least $i-1$ big vertices. If C_{i-1} has at least i big vertices, then ($\star$) holds for i with $C_i = C_{i-1}$. Assume now that C_{i-1} has exactly $i-1$ big vertices. Let y be a vertex of C_{i-1} which is not big and let $B_i = C_{i-1} \setminus \{y\}$. By construction, there is a set Z_{B_i} in H_i which is complete to B_i. Moreover the vertices of B_i have in-degree at most $2k$ and $|Z_{B_i}| \geq 2k^2 + 1$, so there is a vertex z_i in Z_{B_i} which is dominated in D by all vertices of B_i. Hence $S_{(D,w)}(z_i) \geq k$. Now $C_i = B_i \cup \{z_i\}$ shows that ($\star$) holds for i. This completes the proof of ($\star$).

Now ($\star$) for $k+1$ states there is a $(k+1)$-clique C with at least $k+1$ big vertices. All vertices of C must have different values of $S_{(D,w)}$, all at least k. Hence there is a vertex v such that $S_{(D,w)}(v) \geq 2k$. □

Remark 6.7. Since $2k = \overrightarrow{\chi_s}(H_k) \leq \left\lceil \frac{\mathrm{Mad}(H_k)}{2} \right\rceil + \chi(H_k) - 1 \leq 2\delta^*(H_k) = 2k$, the graph H_k also shows that the upper bound of Theorem 1.5 is tight.

7 Hardness results

7.1 Proof of Theorem 1.8

In this subsection, we prove Theorem 1.8 which states that it is NP-complete to determine whether a given planar graph G with $\overrightarrow{\chi}_s(G) = 2$ also satisfies $\vec{\chi}(G) = 2$.

Our proof is a polynomial-time reduction from Planar 3-SAT that we now define.
Let Φ be a 3-SAT formula with the set of clauses $\mathcal{C} = \{C_1, \dots, C_k\}$ and the set of variables $X = \{x_1, \dots, x_n\}$. Let G_Φ be the graph with vertex set $\mathcal{C} \cup X \cup \bar{X}$, where $\bar{X}$ is the set of negated literals $\{\bar{x}_1, \dots, \bar{x}_n\}$, in which each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ is adjacent to the vertices ℓ_1, ℓ_2 and ℓ_3 and $x_i \in X$ is adjacent to $\bar{x}_i$ for all $1 \leq i \leq n$. The formula Φ is *planar* if the graph G_Φ is a planar graph.

Problem: Planar 3-SAT.
Input: A planar 3-SAT formula Φ.
Question: Is there a truth assignment for Φ that satisfies all the clauses?

Planar 3-SAT has been proved NP-complete [10].

Let Φ be a planar 3-SAT formula with set of variables $X = \{x_1, \dots, x_n\}$ and set of clauses $\mathcal{C} = \{C_1, \dots, C_m\}$. Let us transform this formula into a planar graph H_Φ such that $\overrightarrow{\chi_s}(H_\Phi) = 2$. The graph H_Φ is constructed as follows.

- For each $1 \leq i \leq n$, we create a *variable gadget* VG_i as shown in Figure 3.
- For each $1 \leq j \leq m$, we create a *clause gadget* CG_j as shown in Figure 3.
- For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, we connect the vertex c_1^j to ℓ_1, the vertex c_2^j to ℓ_2, and the vertex c_3^j to ℓ_3. Note that here ℓ_i refers to one of $\bar{x}_i$ and x_i, as appropriate.

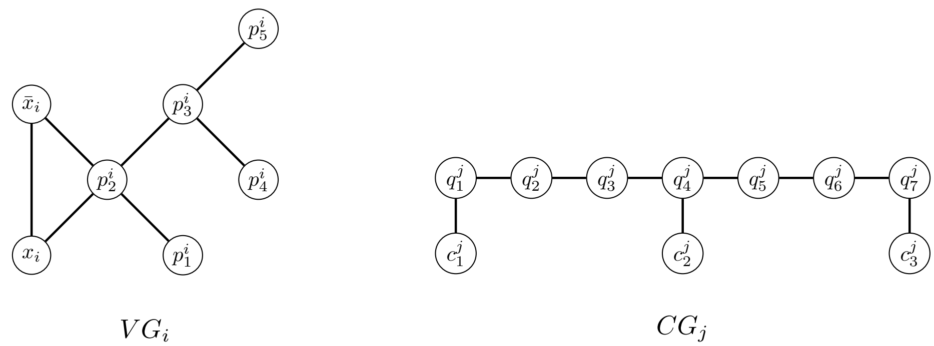


Figure 3: The variable gadget VG_i and the clause gadget CG_j.

One can observe that it is possible to construct H_Φ from G_Φ by applying edge subdivisions and adding planar subgraphs pending at some edges. Hence H_Φ is also planar. Moreover $\overrightarrow{\chi_s}(H_\Phi) \geq 2$. Indeed since its contains 3-cycle (one in each variable gadget), $\overrightarrow{\chi_s}(H_\Phi) \geq \chi(H_\phi) - 1 \geq 2$. On the other hand, one can easily check that the weighted orientation defined as follows is semi-proper and has maximum in-weight 2.

- For every $1 \leq i \leq n$, orient the variable gadget VG_i and assign weights to the arcs as shown in Figure 4;
- For every $1 \leq j \leq m$, orient the clause gadget CG_j as in Figure 4 and assign weight 1 to each of its arcs;
- Orient each edge between a variable gadget and a clause gadget from the vertex in the variable gadget to the vertex in the clause gadget and assign this arc a weight of 1.

Let us now prove that $\vec{\chi}(H_\Phi) = 2$ if and only if Φ is satisfiable. Assume first that there is an assignment $\Gamma : X \to \{\mathsf{true}, \mathsf{false}\}$ satisfying Φ. Let us construct a proper orientation of H_Φ with maximum in-degree 2.

- If e is an edge that connects a vertex of a clause gadget to a vertex of a variable gadget, then orient e from the vertex of the variable gadget to the vertex of the clause gadget.
- For every $1 \leq i \leq n$, orient the variable gadget VG_i as in Figure 5 (a) if $\Gamma(x_i) = \mathsf{true}$, and as in Figure 5 (b) if $\Gamma(x_i) = \mathsf{false}$. Observe that a literal (x_i or $\bar{x}_i$) has in-degree 1 in this orientation if its true, and in-degree 2 if it is false.

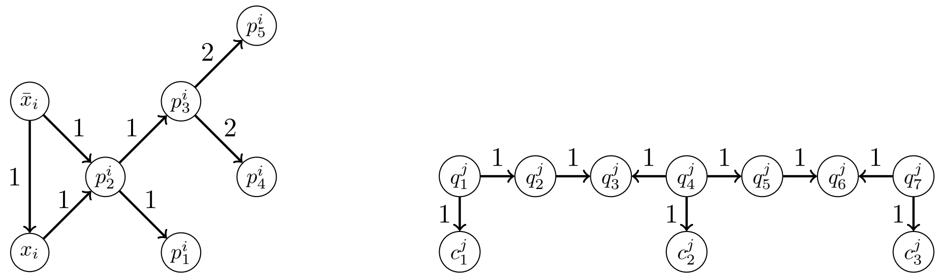
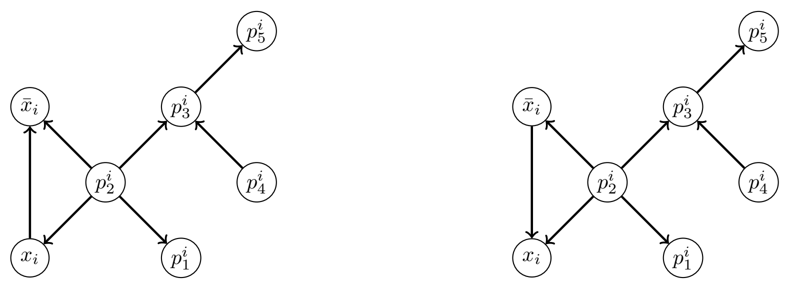


Figure 4: Orientations and arc-weightings of the variable and clause gadgets.



(a) (b)

Figure 5: Orientations of the variable gadget VG_i when $\Gamma(x_i) = \mathsf{true}$ (a) and when $\Gamma(x_i) = \mathsf{false}$ (b).

- For every clause $C_j = (\ell_1 \vee \ell_1 \vee \ell_3)$, we want to orient CG_j so that, for all $k \in [3]$, the vertex c_k^j has in-degree 1 (resp. 2) if ℓ_k has in-degree 2 (resp. 1) in its variable gadget (and thus in the orientation we are building). Since Γ satisfies Φ, at least one of the literals of C_j is assigned true and so has in-degree 1 in its variable gadget. Therefore we want at least one of the c_k^j to have in-degree 2. Now for any possible triple of desired in-degrees, we orient CG_j has shown in Figure 6.

It is simple matter to check that the resulting orientation is proper and has maximum in-degree 2. Hence $\vec{\chi}(H_\Phi) = 2$.

Assume now that $\vec{\chi}(H_\Phi) = 2$. Let D be a proper orientation of H_Φ with maximum in-degree 2. For each $1 \leq i \leq n$, $D\langle\{p_2^i, x_i, \bar{x}_i\}\rangle$ forms a cycle of length 3. So $\{d_D^-(p_2^i), d_D^-(x_i), d_D^-(\bar{x}_i)\} = \{0,1,2\}$. Therefore $d_D^-(p_2^i) + d_D^-(x_i) + d_D^-(\bar{x}_i) = 3 = |A(D\langle\{p_2^i, x_i, \bar{x}_i\}\rangle)|$. Hence D has no arc with tail in $V(D) \setminus \{p_2^i, x_i, \bar{x}_i\}$ and head in $\{p_2^i, x_i, \bar{x}_i\}$. In particular, the following holds:

Claim 7.1. *All the arcs between variable gadgets and clause gadgets are oriented from the variable gadget towards the clause gadget.*

Claim 7.2. *For every $1 \leq i \leq n$, $\{d_D^-(x_i), d_D^-(\bar{x}_i)\} = \{1,2\}$.*

Proof. Recall that $\{d_D^-(p_2^i), d_D^-(x_i), d_D^-(\bar{x}_i)\} = \{0,1,2\}$ and $p_2^i p_1^i$ and $p_2^i p_3^i$ are arcs of D. Therefore, $d_D^-(p_1^i) = 1$ and $d_D^-(p_3^i) \in \{1,2\}$.

Suppose for a contradiction that $d_D^-(p_3^i) = 1$. Then $p_3^i p_4^i$ and $p_3^i p_5^i$ are arcs of D. Thus, $d_D^-(p_4^i) = d_D^-(p_5^i) = 1$. Now the two adjacent vertices p_3^i and p_4^i have same in-degree, which is impossible because D is proper. Hence $d_D^-(p_3^i) = 2$. Consequently, $\{d_D^-(p_1^i), d_D^-(p_3^i)\} = \{1,2\}$. Thus, $d_D^-(p_2^i) = 0$. Hence $\{d_D^-(x_i), d_D^-(\bar{x}_i)\} = \{1,2\}$. ■

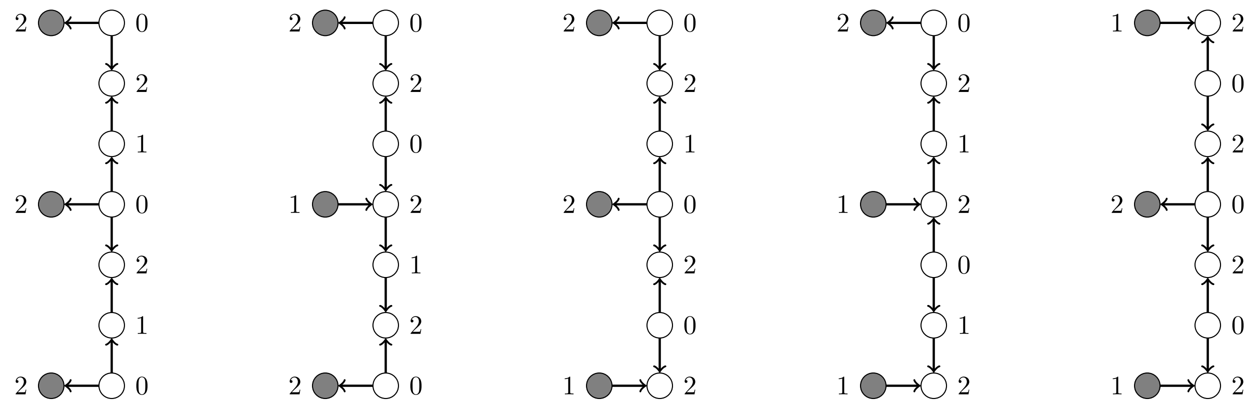


Figure 6: Orientations of CG_j depending on the desired in-degrees for the c_k^j (in gray). The in-degree of each vertex is indicated by the side of it.

Claim 7.3. *For every* $1 \leq j \leq m$*, we have* $2 \in \{d_D^-(c_1^j), d_D^-(c_2^j), d_D^-(c_3^j)\}$.

Proof. To the contrary suppose that $d_D^-(c_1^j) = d_D^-(c_2^j) = d_D^-(c_3^j) = 1$. By Claim 7.1, this implies that the arcs $c_1^j q_1^j$, $c_2^j q_4^j$ and $c_3^j q_7^j$ must be in D. Thus, since D is proper with $\Delta^-(D) = 2$, we have $d_D^-(q_1^j) = d_D^-(q_4^j) = d_D^-(q_7^j) = 2$. In turn, this implies that $\{d_D^-(q_2^j), d_D^-(q_3^j)\} = \{0, 1\}$ and $\{d_D^-(q_5^j), d_D^-(q_6^j)\} = \{0, 1\}$. Thus $q_3^j q_4^j$ and $q_5^j q_4^j$ are arcs of D. But this implies that q_4^j has in-degree 3, a contradiction. ■

Let $\Gamma : X \rightarrow \{\mathsf{true}, \mathsf{false}\}$ be the assignment defined by $\Gamma(x_i) = \mathsf{true}$ if $d_D^-(x_i) = 1$, and $\Gamma(x_i) = \mathsf{false}$ if $d_D^-(x_i) = 2$. Next, we prove that Γ satisfies Φ. Consider a clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$. By Claim 7.3, $2 \in \{d_D^-(c_1^j), d_D^-(c_2^j), d_D^-(c_3^j)\}$. Thus, by Claim 7.2, we have $1 \in \{d_D^-(\ell_1), d_D^-(\ell_2), d_D^-(\ell_3)\}$. Hence by our definition of Γ at least one literal of C_j is assigned true, i.e., Γ satisfies Φ.

This completes the proof of Theorem 1.8.

7.2 Proof of Theorem 1.9

In this subsection, we prove Theorem 1.9.

Let us first prove Part (1) of this theorem which states that it is NP-complete to determine whether a given planar bipartite graph G satisfies $\overrightarrow{\chi_s}(G) \leq 2$. Our prove is a reduction from the problem defined as follows. Let Φ be a 3-SAT formula with clause set $\mathcal{C}$ and variable set X. A *1-in-3 assignment* for Φ is a truth assignment such that each clause of Φ has exactly one true literal.

Problem: Cubic Planar 1-in-3 SAT.
Instance: A planar 3-SAT formula Φ such that every variable appears in exactly three clauses, and there is no negated literals in the clauses.
Question: Is there a 1-in-3 assignment for Φ ?

Moore and Robson proved [13] that Cubic planar 1-in-3 SAT is NP-complete.

Let Φ be a planar 3-SAT formula such that every variable appears in exactly three clauses, and there is no negated literals in the clauses. We shall transform this into a planar bipartite graph F_Φ such that $\vec{\chi}(F_\Phi) = 2$ if and only if Φ has a 1-in-3 assignment.

Let us first describe some gadgets and their properties. The gadgets $\mathcal{F}_1$ and $\mathcal{F}_2$ are depicted in Figure 7. The variable gadget $\mathcal{H}_i$ is depicted in Figure 8.

Lemma 7.4. *Let G be graph such that $\overrightarrow{\chi}_s(G) \leq 2$ and let (D, w) be one of its optimal semi-proper orientations.*

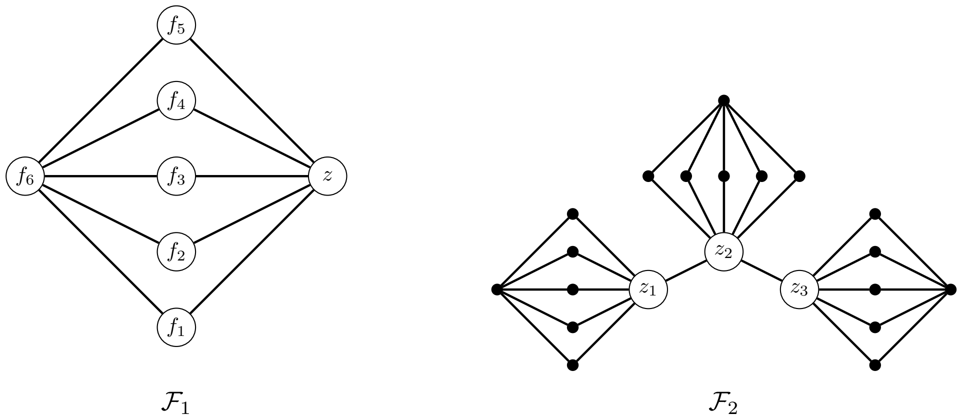


Figure 7: The gadgets $\mathcal{F}_1$ and $\mathcal{F}_2$.

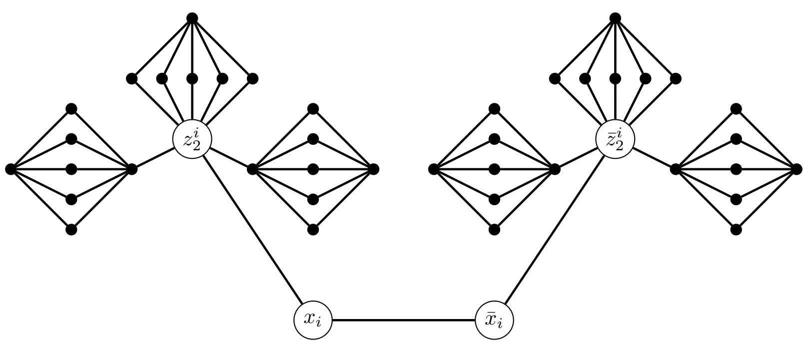


Figure 8: The variable gadget $\mathcal{H}_i$.

(i) If G contains the gadget $\mathcal{F}_1$ as an induced subgraph, then $S_{(D,w)}(z) \in \{0,1\}$.

(ii) If G contains the gadget $\mathcal{F}_2$ as an induced subgraph, then $S_{(D,w)}(z_2) = 0$.

(iii) If G contains the variable gadget $\mathcal{H}_i$ as an induced subgraph, then $\{S_{(D,w)}(x_i), S_{(D,w)}(\bar{x}_i)\} = \{1,2\}$.

Proof. (i) To the contrary suppose that $S_{(D,w)}(z) = 2$. By the symmetry of $\mathcal{F}_1$, we may assume that zf_1, zf_2, zf_3 are arcs of D. Since $S_{(D,w)}$ is a 2-colouring, we have $S_{(D,w)}(f_1) = S_{(D,w)}(f_2) = S_{(D,w)}(f_3) = 1$. Thus f_1f_6, f_2f_6, f_3f_6 must be arcs of D. Hence $S_{(D,w)}(f_6) \geq 3$, a contradiction.

(ii) By (i), we have $S_{(D,w)}(z_1), S_{(D,w)}(z_2), S_{(D,w)}(z_3) \in \{0,1\}$. To the contrary suppose that $S_{(D,w)}(z_2) = 1$. Then at least one of the edges z_1z_2, z_2z_3 is oriented away from z_2. Without loss of generality, assume that z_2z_3 is an arc of D. Then $S_{(D,w)}(z_2) = S_{(D,w)}(z_3) = 1$, a contradiction.

(iii) By (ii), we have $S_{(D,w)}(z_2^i) = S_{(D,w)}(\bar{z}_2^i) = 0$. Thus $z_2^i x_i$ and $\bar{z}_2^i \bar{x}_i$ are arcs of D. Consequently, $\{S_{(D,w)}(x_i), S_{(D,w)}(\bar{x}_i)\} = \{1,2\}$. □

The gadget $\mathcal{Q}$ is depicted in Figure 9, the gadget $\mathcal{L}$ is depicted in Figure 10, and the clause gadget $\mathcal{S}_j$ is depicted in Figure 11.

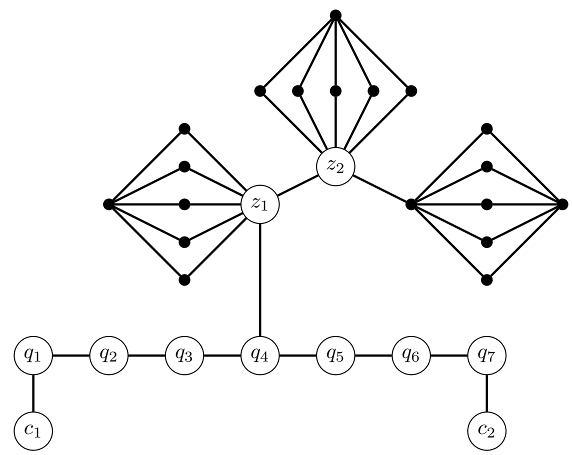


Figure 9: The gadget $\mathcal{Q}$.

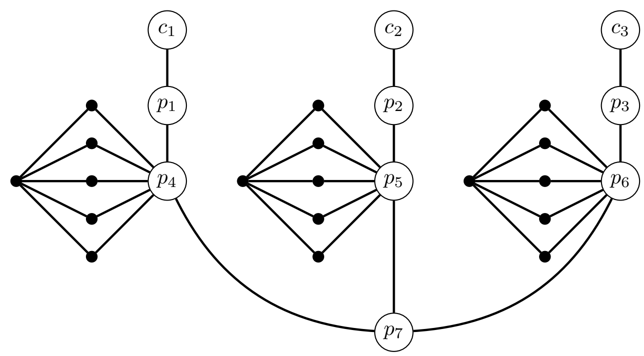


Figure 10: The gadget $\mathcal{L}$.

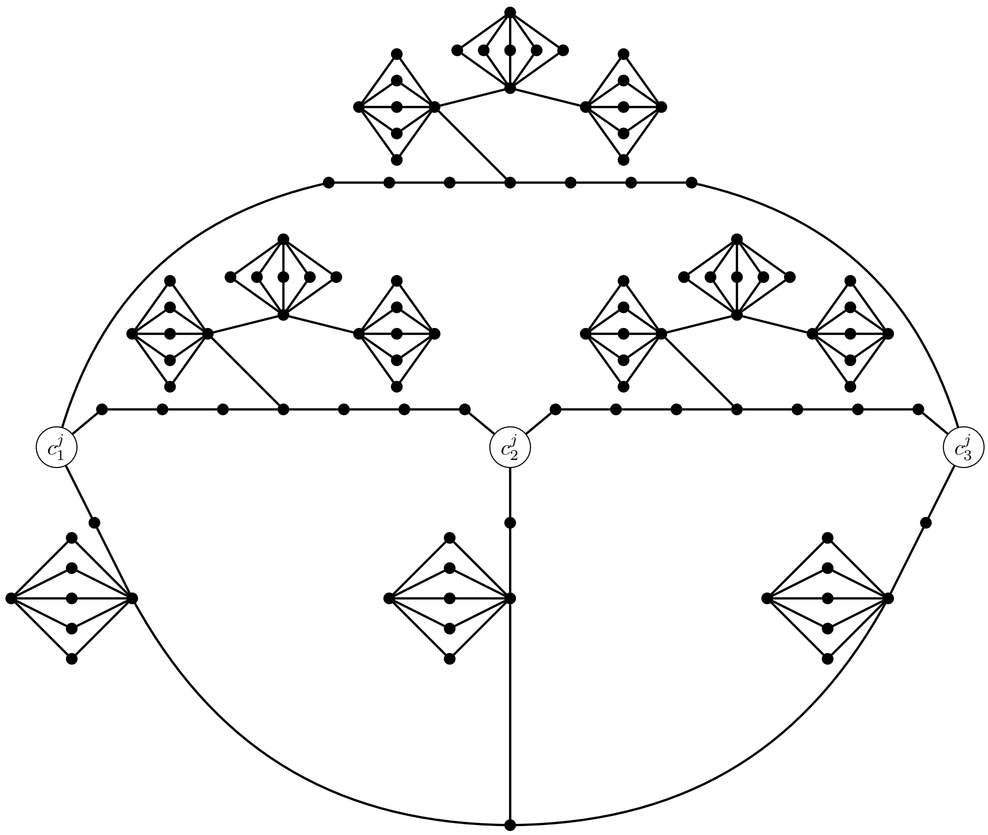


Figure 11: The clause gadget $\mathcal{S}_j$.

Lemma 7.5. *Let G be graph such that $\overrightarrow{\chi}_s(G) \leq 2$ and let (D, w) be one of its optimal semi-proper orientations.*

(i) If G contains the gadget $\mathcal{Q}$ as an induced subgraph and there are vertices x, y not in $V(\mathcal{Q})$ such that $x \to c_1$ and $y \to c_2$ in D, then $2 \in \{S_{(D,w)}(c_1), S_{(D,w)}(c_2)\}$.

(ii) If G contains the gadget $\mathcal{L}$ as an induced subgraph, then $\{S_{(D,w)}(c_1), S_{(D,w)}(c_2), S_{(D,w)}(c_3)\} \neq \{2, 2, 2\}$.

(iii) If G contains the variable gadget $\mathcal{S}_j$ as an induced subgraph, then two vertices in $\{c_1^j, c_2^j, c_3^j\}$ have in-weight 2 and the third one has in-weight 1.

Proof. (i) By Lemma 7.4-(ii), we have $S_{(D,w)}(z_2) = 0$, so $z_2z_1 \in A(D)$. But by Lemma 7.4-(i), $S_{(D,w)}(z_1) \in \{0, 1\}$, thus $S_{(D,w)}(z_1) = 1$ and z_2z_1 is the unique arc entering z_1. In particular $z_1 \to q_4$. Assume for a contradiction that $2 \notin \{S_{(D,w)}(c_1), S_{(D,w)}(c_2)\}$. Then $S_{(D,w)}(c_1) = 1$ and xc_1 is the only arc entering c_1 and $S_{(D,w)}(c_2) = 1$ and yc_2 is the only arcs entering c_2. Hence $c_1 \to q_1$ and $c_2 \to q_7$. Thus, since D is semi-proper with maximum in-weight 2, we have $S_{(D,w)}(q_1) = S_{(D,w)}(q_4) = S_{(D,w)}(q_7) = 2$. In turn, this implies that $\{S_{(D,w)}(q_2), S_{(D,w)}(q_3)\} = \{0, 1\}$ and $\{S_{(D,w)}(q_5), S_{(D,w)}(q_6)\} = \{0, 1\}$. Thus $q_3 \to q_4$ and $q_5 \to q_4$. But this implies that q_4 has in-degree 3 and so in-weight at least 3, a contradiction.

(ii) To the contrary assume that $S_{(D,w)}(c_k) = 2$ for all $k \in [3]$. The semi-proper orientation number of G is 2, so at least one of the three edges p_7p_4, p_7p_5, p_7p_6 is oriented away from p_7. By the symmetry, we may assume that $p_7 \to p_4$. On the other hand, by Lemma 7.4-(i), we have $S_{(D,w)}(p_4) \in \{0, 1\}$. Thus, $S_{(D,w)}(p_4) = 1$ and p_7p_4 is the unique arc entering p_4. Thus $p_4 \to p_1$. Thus $S_{(D,w)}(p_1) \geq 1$. Since (D, w) is semi-proper $S_{(D,w)}(p_1) \neq S_{(D,w)}(p_4)$ so $S_{(D,w)}(p_1) = 2$. But this is a contradiction with $S_{(D,w)}(c_1) = 2$.

(iii) follows directly from (i) and (ii). ☐

Let F_Φ be the graph constructed as follows.

- We first take the disjoint union of the variable gadgets $\mathcal{H}_i$, $1 \leq i \leq n$, and the clause gadgets $\mathcal{S}_j$,$1 \leq j \leq m$.
- For every clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$, add the edges $\ell_1c_1^j$, $\ell_2c_2^j$ and $\ell_3c_3^j$.

The graph G_Φ is planar, so the graph F_Φ is also planar. Moreover since Φ has no negated literals, one can easily check that F_Φ is bipartite.

Let us show that $\overrightarrow{\chi_s}(F_\Phi) = 2$ if and only if there is a 1-in-3 satisfying assignment for Φ.

Assume first that $\overrightarrow{\chi_s}(F_\Phi) = 2$. Let (D, w) be an optimal semi-proper orientation of G. Let $\Gamma : X \to \{\mathsf{true}, \mathsf{false}\}$ be the assignment defined by $\Gamma(x_i) = \mathsf{true}$ if $S_{(D,w)}(x_i) = 2$, and $\Gamma(x_i) = \mathsf{false}$ if $S_{(D,w)}(x_i) = 1$. For each clause $C_j = (\ell_1 \vee \ell_2 \vee \ell_3)$, by Lemma 7.5-(iii), two vertices of $\{c_1^j, c_2^j, c_3^j\}$ have in-weight 2 and the third one has in-weight 1. Moreover, by Lemma 7.4-(iii), the vertices of ℓ_1, ℓ_2, ℓ_3 have in-weight 1 or 2. Thus, because (D, w) is semi-proper, two vertices of $\{\ell_1, \ell_2, \ell_3\}$ have in-weight 1 and the third one has in-weight 2. Hence, Γ is a 1-in-3 satisfying assignment for Φ.

Assume now that Φ has a 1-in-3 assignment $\Gamma : X \to \{\mathsf{true}, \mathsf{false}\}$. We shall construct a proper orientation D of F_Φ with maximum in-degree 2. With the arc-weighting w that assigns weight 1 to all the arcs, this is a semi-proper orientation with maximum in-weight 2. For every $1 \leq i \leq n$, orient the edges of $E(\mathcal{H}_i) \setminus \{x_i\bar{x}_i\}$ as shown in Figure 12, and orient $x_i\bar{x}_i$ from $\bar{x}_i$ to x_i if $\Gamma(x_i) = \mathsf{true}$ and from x_i to $\bar{x}_i$ if $\Gamma(x_i) = \mathsf{false}$. Orient all the edges linking a vertex of a clause gadget to a vertex of a variable gadget from the vertex of the variable gadget to the vertex of the clause gadget. Doing so, for every $1 \leq i \leq n$, $d^-(x_i) = 2$ if $\Gamma(x_i) = \mathsf{true}$ and $d^-(x_i) = 1$ if $\Gamma(x_i) = \mathsf{false}$.

Finally we orient the clause gadgets. Consider a clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$. Since Γ is a 1-in-3 assignment for Φ, exactly one literal among ℓ_1, ℓ_2, ℓ_3 is assigned true. Hence one vertex in ℓ_1, ℓ_2, ℓ_3 has in-degree 2 and the two others have in-degree 1. We shall orient the clause gadget S_j such that each c_k^j, $k \in [3]$, has in-degree 1 if $\Gamma(\ell_k) = \mathsf{true}$ and has in-degree 2 if $\Gamma(\ell_k) = \mathsf{false}$. Hence we need to orient S_j such that two of the c_k^j

have in-degree 2 and the third one has in-degree 1. We do it as shown in Figure 13. It is simple matter that the resulting orientation is a proper orientation of F_Φ with maximum in-degree 2. This completes the proof of Part (1).

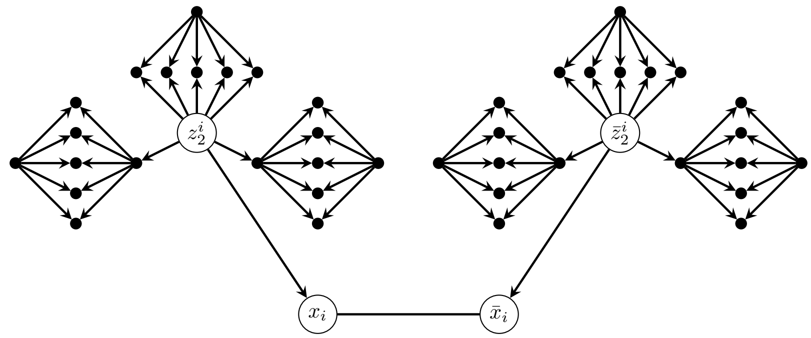


Figure 12: Orientation of the variable gadget $\mathcal{H}_i$.

The proof of Part (2) is exactly the same as the one of Part (1). Indeed, when we wanted to have a semi-proper orientation with maximum in-weight 2, we in fact gave a proper orientation with maximum in-degree 2.

This completes the proof of Theorem 1.9.

8 Conclusions and future research

In this work, we introduced semi-proper orientations and the semi-proper orientation number of a graph, and established some properties of those. In particular, we gave positive answers to the analogues of Problem 1.2 and Problem 1.3. However, these two original problems are still open. An affirmative answer to them would follow from the following conjecture, which is interesting in its own.

Conjecture 8.1. There is a function f such that $\vec{\chi}(G) \leq f(\overrightarrow{\chi_s}(G))$ for any graph G?

Since the graphs with semi-proper orientation number 1 are the ones with proper orientation 1, that are the union of stars we have $f(1) = 1$. Araujo *et al.* [4] showed trees with proper orientation number 4, so $f(2) \geq 4$. Araujo *et al.* [5] showed a cactus with proper orientation number 7, so $f(3) \geq 7$.

Theorem 1.9 states that determining the semi-proper orientation number of planar bipartite graphs is NP-hard. On the other hand, deciding whether the proper orientation number of a given 4-regular graph is 3 is NP-complete [1]. What can we say about the complexity of computing the semi-proper orientation number of regular graphs?

Problem 8.2. What is the computational complexity of computing the semi-proper orientation number of regular graphs?

We proved that the semi-proper orientation number can be 4-approximated. A natural question is whether it could be better approximated and how better?

Problem 8.3. Does there exists $\alpha < 4$ such that there exists a polynomial-time algorithm yielding an α-approximation of the semi-proper orientation number?

Problem 8.4. Does there exists a polynomial-time approximation scheme for the semi-proper orientation number?

Finally, another interesting question for future work is the problem of finding an optimal weighting for a given directed graph. Is there a polynomial time algorithm to solve it?

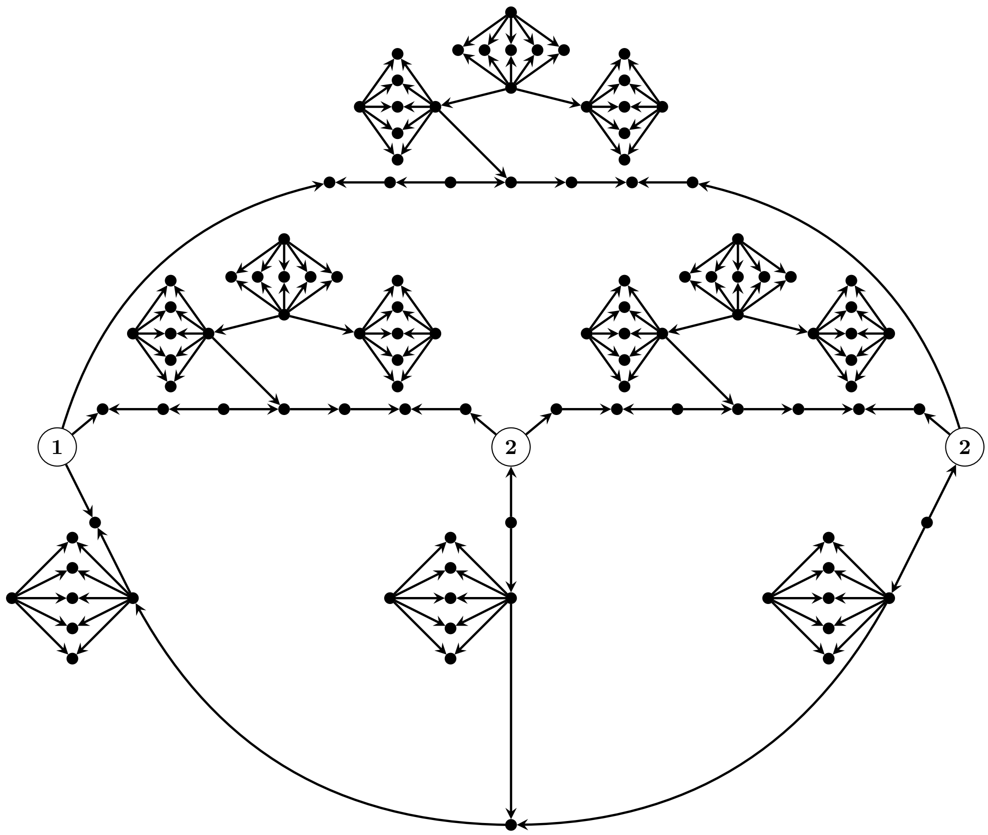


Figure 13: Orientation of the clause gadget $\mathcal{S}_j$. White vertices correspond to c_1^j, c_2^j, c_3^j. The number written in such a vertex corresponds to the desired in-degree.

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