Generalizations of Maxwell (super)algebras by the expansion method

J. A. de Azcárraga[a], J. M. Izquierdo[b],
J. Lukierski[c] and M. Woronowicz[c]

[a] *Dept. Theor. Phys. and IFIC (CSIC-UVEG), Valencia Univ. 46100-Burjassot (Valencia), Spain*

[b] *Dept. Theor. Phys., Valladolid Univ., 47011-Valladolid, Spain*

[c] *Institute of Theoretical Physics, Wrocław Univ., 50-204 Wrocław, Poland*

October 3, 2012

Abstract

The Lie algebras expansion method is used to show that the four-dimensional spacetime Maxwell (super)algebras and some of their generalizations can be derived in a simple way as particular expansions of $o(3,2)$ and $osp(N|4)$.

1 Introduction

There are four methods of obtaining new Lie (super)algebras from a given one: contractions, deformations, extensions and expansions. Contractions and deformations lead to new algebras of the same dimension as the original one. The same can be said of an extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by another one $\mathcal{A}$ (see *e.g.* [1]) in the sense that $\dim\widetilde{\mathcal{G}} = \dim\mathcal{G} + \dim\mathcal{A}$ since $\widetilde{\mathcal{G}} = \mathcal{G}/\mathcal{A}$. A fourth way of obtaining new Lie (super)algebras from a given $\mathcal{G}$ ($s\mathcal{G}$) is the *expansion of algebras*, first used in [2] and studied in general in [3, 4];

see [5] for further developments. In contrast with the first three procedures, expanded algebras have, in general, higher dimension than the original $\mathcal{G}$ because expansions give rise to additional generators (expansions also include contractions as a particular case [3], in which they are dimension-preserving). In this paper we shall consider some basic aspects of the expansion procedure to derive Maxwell (super)algebras and other new generalizations as expanded algebras.

The main idea of the Lie algebras expansion method is to promote the standard Maurer-Cartan (MC) forms ω^i of the Lie algebra $\mathcal{G}$ of a group G (resp. superalgebra and supergoup),

$$\theta(g) = g^{-1}\, dg = \omega^i X_i \ , \ \ i = 1, \ldots, \dim \mathcal{G} \ , \ \ X_i \in \mathcal{G} \ , \ \ g \in G \ , \tag{1.1}$$

to functions $\omega^i(\xi)$ of a real parameter ξ. We shall consider here expansions of the ω's with the generic form [3]

$$\omega^{i_p}(\xi) = \sum_{\alpha_p = p}^{M_p} \xi^{\alpha_p} \omega^{i_p,\alpha_p} = \xi^p \omega^{i_p,p} + \xi^{p+1}\omega^{i_p,p+1} + \ldots \omega^{i_p,M_p} \ , \tag{1.2}$$

where p refers to the subspace V_p in the splitting $\mathcal{G} = V_0 \oplus V_1 \oplus \ldots V_p \oplus \ldots$ of the Lie algebra vector space (thus, $i_p = 1, \ldots, \mathrm{dim}V_p$, $\Sigma_p \mathrm{dim} V_p = \mathrm{dim}\mathcal{G}$); α_p is the power of ξ in the series expansion that accompanies a given ω^{i_p,α_p}, which is therefore characterized by the values of the index i_p and the order α_p in the expansion of ω^{i_p}. Depending on the problem, ξ will be expressed as different powers of λ with dimensions $[\lambda]$=$L^{-\frac{1}{2}}$. When $M_p = 0$, $\omega^{i_p,0} = \omega^{i_p}$ *i.e.*, the ω's are the original MC forms for the various subspaces $V_p \subset \mathcal{G}$. Certain terms of the expansion may be absent in (1.2), as we will see later.

After inserting the expansions (1.2) into the MC equations for $\mathcal{G}$ we get

$$d\omega^i(\xi) = -\tfrac{1}{2}\, c^i_{jk}\, \omega^j(\xi) \wedge \omega^k(\xi) \, . \tag{1.3}$$

Then, equating the coefficients of equal powers α_p of ξ at the right and left hand sides, a set of equations for the differentials $d\omega^{i_p,\alpha_p}$ is obtained. In principle, the various M_p in (1.2) could be arbitrarily large, but the idea of the expansion method is to cut the series consistently *i.e.*, in such a way that the retained $\{\omega^{i_p,\alpha_p}\}$ ($\alpha_p \leq M_p$) become the MC forms of a new, *expanded Lie (super)algebra* characterized by the MC equations satisfied by these ω^{i_p,α_p}'s. The closure of d on the set $\{\omega^{i_p,\alpha_p}\}$ requires that the highest M_p's satisfy certain relations to guarantee that the various expressions for $d\omega^{i_p,\alpha_p}$ define the MC equations of a new Lie (super)algebra, the *expansion of* $\mathcal{G}$, denoted $\mathcal{G}(M_0,\ldots,M_p)$. In this paper we shall only consider splittings with $p = 0,1,2$; only even or only odd powers of ξ will appear in the expansions.

It is well known that the $D = 4$ Poincaré algebra $\mathcal{P}(3,1) = t_4 \niplus o(3,1)$ and the $D = 4$ Poincaré superalgebras $s\mathcal{P}(3,1|N)$ $(N = 0,1,2...)$ can be obtained, respectively, as Wigner-İnönü (WI) contraction of the $D = 4$ adS algebra $o(3,2)$ and of the $D = 4$ adS superalgebras $osp(N|4)$. As shown in [2, 3] the expansion method goes beyond the original IW contraction of algebras or its generalizations (as [6]), all contractions being characterized by the equality of the dimensions of the original and the contracted algebras. But, as it is the case with IW contractions (where the algebra with respect to which the contraction is made is preserved in the process), expanded algebras also keep memory of the original algebra (through eqs. (2.1),(2.2) below). This fact is particularly useful as a guiding principle to identify known algebras as expansions or to generate new ones with some desired properties. Moreover, since as mentioned contractions constitute a particular case [3] of expansions, it is natural to use the expansion method by taking the $o(3,2)$ algebra and $osp(N|4)$ superalgebras as starting points in order to obtain various enlargements of the $D = 4$ Poincaré algebra and the $D = 4$ Poincaré superalgebras. We shall show that the expansion procedure leads, in particular, to a 16-dimensional enlargement of $D = 4$ Poincaré algebra containing six additional tensorial Abelian charges, the Maxwell algebra[1] (see e.g. [7–11]) and, further, to its generalizations. Analogously, the expansion

[1]The name 'Maxwell' was proposed in [8] for a sixteen generators algebra larger than the Bacry-Combe-Richard (BCR) [7] one, which has six generators plus two central charges. The BCR algebra describes the symmetry of a relativistic particle in a constant e.m. field.

of supersymmetric adS algebra $osp(N|4)$ will produce known as well as new supersymmetric generalizations of Maxwell algebra.

The plan of this paper is the following. Sect. 2 briefly recalls some general aspects of the expansion procedure, with particular attention to Lie algebras (and superalgebras) with a symmetric coset structure. Sect. 3 considers the expansions of the $o(3,1)$ Lorentz algebra; further, using $o(3,2)$ and the splitting $o(3,2) = o(3,1) \oplus \frac{o(3,2)}{o(3,1)}$, the Maxwell algebra and its generalization are obtained as specific expansions of $D = 4$ adS algebra $o(3,2)$. In Sect. 4 the expansion method is applied to a suitable coset decomposition of the $D = 4$ adS superalgebra $osp(N|4)$; this will lead to a new supersymmetric version of Maxwell algebra. We conclude here by stressing that the expansion method is general, and provides an effective algebraic scheme to derive larger (super)algebras from a given one.

2 Expansions of Lie (super)algebras: general considerations

In a rather general framework [3], the MC equations for the expansion $\mathcal{G}(M_0, \ldots, M_p)$ follow from eqs. (1.2) and (1.3). They have the form

$$\begin{aligned} d\omega^{k_s,\alpha_s} &= -\tfrac{1}{2}\, C^{k_s,\alpha_s}_{i_p,\beta_p\;j_q,\gamma_q}\, \omega^{i_p,\beta_p} \wedge \omega^{j_q,\gamma_q} \ , \qquad (2.1) \\ i_{p,q,s} &= 1, \ldots, \dim V_{p,q,s} \ , \ \alpha_p, \beta_p, \gamma_p = p, p+1, p+2, \ldots, M_p \ , \end{aligned}$$

where the M_p have to satisfy certain conditions and the structure constants of the expansion $\mathcal{G}(M_0, \ldots, M_p)$ are given in terms of those of $\mathcal{G}$ by

$$C^{k_s,\alpha_s}_{i_p,\beta_p\;j_q,\gamma_q} = \begin{cases} 0, & \beta_p + \gamma_q \neq \alpha_s \\ c^{k_s}{}_{i_p j_q}, & \beta_p + \gamma_q = \alpha_s \end{cases} \ . \qquad (2.2)$$

We shall consider the following cases of the above general structure:

1) All the MC forms ω^i in eq. (1.2) are expanded similarly,

$$\omega^i(\xi) = \sum_{\alpha=0}^{M} \xi^\alpha\, \omega^{i,\alpha} \ , \ i = 1, \ldots, \dim\mathcal{G} \ , \qquad (2.3)$$

i.e. $\mathcal{G} = V_0$, $i_0 = i$. Eqs. (2.3) in (2.1) give

$$d\omega^{i,\alpha} = -\tfrac{1}{2}\, c^i_{\;jk} \sum_{\beta=0}^{\alpha} \omega^{j,\beta} \wedge \omega^{k,\alpha-\beta} \quad , \quad \alpha{=}0,1\ldots M \ . \tag{2.4}$$

The resulting Lie algebra expansions, denoted $\mathcal{G}(M)$, have generators $\{X_{j,\beta}\} = (X_{j,0}, X_{j,1} \ldots X_{j,M})$ dual to the MC forms $\{\omega^{i,\alpha}\} = (\omega^{i,0}, \omega^{i,1}, \ldots, \omega^{i,M})$ that satisfy the MC equations (2.4). Consistency requires $d(d\omega^{i,\alpha}) \equiv 0$; this follows using (2.4) repeatedly for $d\omega^{j,\beta}$ etc. in the *r.h.s.* of $d(d\omega^{i,\alpha}) = 0$ and the Jacobi identity (JI) for $\mathcal{G}$. Alternatively, $d(d\omega^{i,\alpha}) \equiv 0$ follows as a consequence of the JI for $\mathcal{G}(M)$. The dimension of the expansions $\mathcal{G}(M)$ is $\dim \mathcal{G}(M){=}(M+1) \times \dim \mathcal{G}$. Eq. (2.4) implies

$$[X_{j,\alpha}, X_{k,\beta}] = 0 \qquad \text{if} \quad \alpha + \beta > M \ . \tag{2.5}$$

Therefore $\mathcal{G}\,(M)$ contains ℓ sets of $(\dim \mathcal{G})$-dimensional abelian subalgebras of generators $\{X_{j,\ell+1}\} \ldots \{X_{j,M}\}$ when $M = 2\ell$ even, and ℓ sets $\{X_{j,\ell}\} \ldots \{X_{j,M}\}$ in the odd $M{=}2\ell - 1$ case.

To be consistent later with the notation in the supersymmetric case it will prove convenient to set $\xi = \lambda^2$, $2M = N$ and relabel the expansion as $\mathcal{G}(N)$. Then, eq. (2.3) reads

$$\omega^i(\lambda) = \sum_{\alpha=0,\,\alpha\,\text{even}}^{N} \lambda^\alpha\, \omega^{i,\alpha} \ , i = 1, \ldots, \dim\mathcal{G} \ . \tag{2.6}$$

2) Let us assume that the algebra has a symmetric coset structure, $\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$, with generators $H_l \in \mathcal{H}$, $K_r \in \mathcal{K}$ so that the indices in $\mathcal{H}$ ($\mathcal{K}$) take the values $l, m, n{=}1 \ldots \dim\mathcal{H}$ ($r, s{=}1 \ldots \dim\mathcal{K}$). Then,

$$\begin{gathered} [H_l, H_m] = c^n_{\;lm} H_n \quad , \quad [H_l, K_r] = c^s_{\;lr} K_s \quad , \\ [K_r, K_s] = c^l_{\;rs} H_l \ . \end{gathered} \tag{2.7}$$

If we denote the dual MC forms of the subalgebra $\mathcal{H}$ by ω^l ($\omega^l(H_m) = \delta^l_{\;m}$) and e^r are those of $\mathcal{K}$ ($e^r(K_s) = \delta^r_s$) the algebra (2.7) is equally characterized by its MC equations,

$$
\begin{aligned}
d\omega^l &= -\tfrac{1}{2}(\, c^l{}_{mn}\, \omega^m \wedge \omega^n + c^l_{rs} e^r \wedge e^s) \ , \\
de^r &= -c^r{}_{ms}\, \omega^m \wedge e^s \ .
\end{aligned}
\tag{2.8}
$$

Then, due to the symmetric coset structure, the expansions of the MC forms take the form [3]

$$
\omega^l(\xi) = \sum_{\alpha_0=0,\, \alpha_0 \text{ even}}^{M_0} \xi^{\alpha_0}\, \omega^{l,\alpha_0} \quad , \quad e^r(\xi) = \sum_{\alpha_1=1,\, \alpha_1 \text{ odd}}^{M_1} \xi^{\alpha_1}\, e^{r,\alpha_1} \quad , \tag{2.9}
$$

i.e. $p = 0$, $i_0 = l$ and $p = 1$, $i_1 = r$ in eq. (1.2) respectively. When $M_1 = M_0 + 1$ or $M_1 = M_0 - 1$ [3] the retained forms determine a Lie algebra. To compare these expressions with the supersymmetric ones avoiding the odd powers in (2.9), we take as mentioned $\xi = \lambda^2$, $2M_0 = N_0$, $2M_1 = N_2$. Then, eqs. (2.9) read

$$
\omega^l(\lambda) = \sum_{\alpha_0=0,\, \mathrm{mod}\, 4}^{N_0} \lambda^{\alpha_0}\, \omega^{l,\alpha_0} \quad , \quad e^r(\lambda) = \sum_{\alpha_2=2,\, \mathrm{mod}\, 4}^{N_2} \lambda^{\alpha_2}\, e^{r,\alpha_2} \ . \tag{2.10}
$$

We shall call $\mathcal{H} = V_0$ and $\mathcal{K} = V_2$ referring to the first powers in λ (rather than ξ) that appear in the expansion of the corresponding MC forms, and denote the expansions $\mathcal{G}(N_0, N_2)$. In Sect. 3 we will consider the case $M_0 = 2$, $M_1 = 1$, *i.e.* N_0=4 and N_2=2; this will lead to the Maxwell algebra as the expansion $o(3,2)(N_0 = 4, N_2 = 2)$

3) The case of superalgebras corresponds to adding the fermionic sector V_1 of generators $F_\alpha \in V_1$ to the bosonic one $\mathcal{G} = V_0 \oplus V_2$, for which we still assume eqs. (2.7). Thus, $s\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ and the commutation relations of $s\mathcal{G}$ are given by eqs. (2.7) plus

$$
\begin{aligned}
&\{F_\alpha, F_\beta\} = c^n{}_{\alpha\beta} H_n + c^r{}_{\alpha\beta} K_r \ , \\
&[H_l, F_\alpha] = c^\beta{}_{l\alpha} F_\beta \quad , \quad [K_r, F_\alpha] = c^\beta{}_{r\alpha} F_\beta \ ,
\end{aligned}
\tag{2.11}
$$

where $\alpha, \beta{=}1 \ldots \dim V_1$ above refer to the spinorial index of the fermionic generator. Introducing fermionic MC forms ψ^α, $\psi^\alpha(F_\beta) = \delta^\alpha_\beta$, the MC equations for the Lie superalgebra $s\mathcal{G}$ follow from (2.7), (2.11),

$$\begin{aligned} d\omega^l &= -\tfrac{1}{2}(c^l_{mn}\omega^m \wedge \omega^n + c^l_{rs} e^r \wedge e^s + c^l_{\alpha\beta}\psi^\alpha \wedge \psi^\beta) \ , \\ de^r &= -\tfrac{1}{2}(2c^r_{ms}\omega^m \wedge e^s + c^r_{\alpha\beta}\psi^\alpha \wedge \psi^\beta) \ , \\ d\psi^\alpha &= -(c^\alpha_{\beta l}\psi^\beta \wedge \omega^l + c^\alpha_{\beta r}\psi^\beta \wedge e^r) \ . \end{aligned} \tag{2.12}$$

Given the coset structure of the superalgebra $s\mathcal{G}$ ($p = 0, 1, 2$), we take $\xi = \lambda$ so that λ accompanies the first term in the expansion

$$\psi^\alpha(\lambda) = \sum_{\alpha_1=1,\, \alpha_1 \text{ odd}}^{N_1} \lambda^{\alpha_1} \psi^{\alpha,\alpha_1} \ . \tag{2.13}$$

Note that the first index α in ψ^{α,α_1} is the usual spinorial one and that the second index α_1 refers to the order (power) of the expansion of the fermionic MC form ψ^α of $s\mathcal{G}$. The expansions of the bosonic MC forms are accordingly

$$\omega^l(\lambda) = \sum_{\alpha_0=0,\, \alpha_0 \text{ even}}^{N_0} \lambda^{\alpha_0}\, \omega^{l,\alpha_0} \quad , \quad e^r(\lambda) = \sum_{\alpha_2=2,\, \alpha_2 \text{ even}}^{N_2} \lambda^{\alpha_2}\, e^{r,\alpha_2} \ . \tag{2.14}$$

The powers of λ in eqs. (2.13), (2.14), where $[\lambda] = L^{-\frac{1}{2}}$, will determine later suitable physical dimensions for the MC forms in the expansion and for their dual Lie (super)algebra generators. In fact, if the MC forms are identified with the one-form fields of a physical theory, the lower orders will lead to the standard physical dimensions of the bosonic and fermionic fields in geometrized units (*e.g.*, the one-form $\psi^{\alpha,1}$ in eq. (2.13) has dimension $[\psi^{\alpha,1}] = L^{\frac{1}{2}}$, which gives $[Q_\alpha] = L^{-\frac{1}{2}}$ for its dual generator in the superalgebra, etc). It is also seen that supersymmetry makes the expansion of the bosonic part to be of the form (2.14) rather than (2.10). If we now set the fermionic sector (eq. (2.13)) in the expansion of the superalgebra $s\mathcal{G}$ equal to zero, the result gives a possible expansion of the bosonic subalgebra $V_0 \oplus V_2 \subset s\mathcal{G}$, which is consistent and different from the one in eq. (2.10).

In Sec. 3 we shall consider the bosonic expansion given by eq. (2.14) for $N_0 = 4 = N_2$ to derive a new generalization of Maxwell algebra, and in Sect. 4 we shall consider the supersymmetric expansion (eqs. (2.13), (2.14)) with $N_0 = 4{=}N_2$ and $N_1 = 3$ (case (a) below) to obtain new supersymmetrizations of the Maxwell algebra. Note that a set of integers (N_0, N_1, N_2) will not lead to an expansion $s\mathcal{G}(N_0, N_1, N_2)$ unless one of the following conditions is satisfied: (a) $N_0 = N_1 + 1 = N_2$, (b) $N_0 = N_1 - 1 = N_2$, (c) $N_0 = N_1 - 1 = N_2 - 2$ or (d) $N_0 - 2 = N_1 - 1 = N_2$. Case (d) would be absent if we allowed for an H component in the second commutator of (2.7) (see [3,2] for details).

3 Generalized D = 4 Lorentz and Maxwell algebras as expansions

1) *Expansions of the D=4 Lorentz algebra.*

The $o(3,1)$ Lorentz algebra is given by the relations $(\mu, \nu = 0, 1, 2, 3)$.

$$[M_{\mu\nu}, M_{\rho\sigma}] = (\eta_{\rho\nu} M_{\mu\sigma} - \eta_{\sigma\nu} M_{\mu\rho}) - (\mu \leftrightarrow \nu) \tag{3.1}$$

or, alternatively, by the MC equations for the Lorentz algebra which, with $\omega^{\mu\nu}(M_{\rho\tau}) = \delta^{\mu\nu}_{\rho\tau}$, are

$$d\omega^{\mu\nu} = -\,\omega^{\mu}{}_{\rho} \wedge \omega^{\rho\nu} \quad . \tag{3.2}$$

We now use the expansion formula (2.6) with $N{=}4$,

$$\omega^{\mu\nu}(\lambda) = \omega^{\mu\nu,0} + \lambda^2\, \omega^{\mu\nu,2} + \lambda^4 \omega^{\mu\nu,4} \ . \tag{3.3}$$

From (2.4) and taking into account that only even orders of α and β appear, we obtain the relations

$$d\,\omega^{\mu\nu,0} = -\omega^{\mu}{}_{\rho}{}^{,0} \wedge \omega^{\rho\nu,0} \ , \tag{3.4a}$$

$$d\,\omega^{\mu\nu,2} = -(\omega^{\mu}{}_{\rho}{}^{,0} \wedge \omega^{\rho\nu,2} + \omega^{\mu}{}_{\rho}{}^{,2} \wedge \omega^{\rho\nu,0}) \ , \tag{3.4b}$$

$$d\,\omega^{\mu\nu,4} = -(\omega^{\mu}{}_{\rho}{}^{,0} \wedge \omega^{\rho\nu,4} + \omega^{\mu}{}_{\rho}{}^{,2} \wedge \omega^{\rho\nu,2} + \omega^{\mu}{}_{\rho}{}^{,4} \wedge \omega^{\rho\nu,0}) \ . \tag{3.4c}$$

Eqs. (3.4a–c) constitute the MC equations for the $(6\times(2+1))$-dimensional Lie-algebra expansion $o(3,1)(N=4)$. Introducing the generators $M_{\mu\nu}$, $\widetilde{Z}_{\mu\nu}$, $Z_{\mu\nu}$ dual respectively to $\omega^{\mu\nu,0}$, $\omega^{\mu\nu,2}$, $\omega^{\mu\nu,4}$ we obtain, besides (3.1), the following $o(3,1)(4)$ commutators (we usually omit the vanishing ones)

$$[M_{\mu\nu}, \widetilde{Z}_{\rho\sigma}] = (\eta_{\rho\nu}\, \widetilde{Z}_{\mu\sigma} - \eta_{\sigma\nu}\widetilde{Z}_{\mu\rho}) - (\mu \leftrightarrow \nu)\ , \tag{3.5a}$$

$$[M_{\mu\nu}, Z_{\rho\sigma}] = (\eta_{\rho\nu}\, Z_{\mu\sigma} - \eta_{\sigma\nu} Z_{\mu\rho}) - (\mu \leftrightarrow \nu)\ , \tag{3.5b}$$

$$[\widetilde{Z}_{\mu\nu}, \widetilde{Z}_{\rho\sigma}] = (\eta_{\rho\nu}\, Z_{\mu\sigma} - \eta_{\sigma\nu} Z_{\mu\rho}) - (\mu \leftrightarrow \nu)\ ; \tag{3.5c}$$

in agreement with the general relations (2.5) we also get $[Z_{\mu\nu}, Z_{\rho\tau}] = 0$. Therefore, $o(3,1)(4)$ is the semidirect product of the Lorentz algebra and the ideal generated by $(\widetilde{Z}_{\mu\nu}, Z_{\mu\nu})$ in which the generators $Z_{\mu\nu}$ are central.

2) $D=4$ *Maxwell algebra.*

Let us now consider $D=4$ adS algebra $o(3,2)$ with the splitting (2.7), namely $o(3,2){=}o(3,1) \oplus \frac{o(3,2)}{o(3,1)} \equiv V_0 \oplus V_2$. Denoting the generators in the coset $\frac{o(3,2)}{o(3,1)}$ describing the curved translations in $D{=}4$ AdS space by $\mathcal{P}_\mu$, the algebra $o(3,2)$ is obtained supplementing the Lorentz algebra (3.1) with

$$[M_{\mu\nu}, \mathcal{P}_\rho] = 2(\mathcal{P}_\mu \eta_{\nu\rho} - \mathcal{P}_\nu \eta_{\mu\rho}) \quad , \quad [\mathcal{P}_\mu, \mathcal{P}_\nu] = M_{\mu\nu} \quad . \tag{3.6}$$

The $o(3,2)$ algebra MC equations are

$$d\omega^{\mu\nu} = -\,\omega^\mu{}_\rho \wedge \omega^{\rho\nu} - e^\mu \wedge e^\nu \quad , \quad d\, e^\mu = -\omega^\mu \rho \wedge e^\rho\ , \tag{3.7}$$

where all the generators in (3.6) and the forms in (3.7) are dimensionless. The well known contraction of $o(3,2)$ algebra to the Poincaré algebra is obtained as an expansion if we use expression (2.9) and retain there only the first terms ($M_0 = 0, M_1 = 1$).

Let us further consider the expansion using the form (2.10) for the coset structure (2.7) (with $V_0 = o(3,1)$ and $V_2 = \frac{o(3,2)}{o(3,1)}$) with $N_0 = 4$, $N_2 = 2$,

$$\omega^{\mu\nu}(\lambda) = \omega^{\mu\nu,0} + \lambda^4\, \omega^{\mu\nu,4} \quad , \quad e^\mu(\lambda) = \lambda^2\, e^{\mu,2}\ . \tag{3.8}$$

Inserting these expressions in the $o(3,2)$ MC equations above, and identifying the coefficients of equal powers of λ, we obtain

$$
\begin{aligned}
d\,\omega^{\mu\nu,0} &= -\omega^{\mu}{}_{\rho}{}^{,0} \wedge \omega^{\rho\nu,0}\;, \quad d\,e^{\mu,2} = -\omega^{\mu}{}_{\nu}{}^{,0} \wedge e^{\nu,2}\;, \\
d\,\omega^{\mu\nu,4} &= -(\omega^{\mu}{}_{\rho}{}^{,0} \wedge \omega^{\rho\nu,4} + \omega^{\mu}{}_{\rho}{}^{,4} \wedge \omega^{\rho\nu,0} + e^{\mu,2} \wedge e^{\nu,2})\;. \qquad (3.9)
\end{aligned}
$$

These are the MC equations of the *Maxwell algebra* as the expansion $o(3,2)(N_0 = 4, N_2 = 2)$. In terms of the generators $M_{\mu\nu}, P_\mu$ and $Z_{\mu\nu}$ dual to $\omega^{\mu\nu,0}, e^{\mu,2}$ and $\omega^{\mu\nu,4}$ respectively, the commutators of the algebra are

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= (\eta_{\rho\nu} M_{\mu\sigma} - \eta_{\sigma\nu} M_{\mu\rho}) - (\mu \leftrightarrow \nu)\;, \\
[P_\mu, P_\nu] &= Z_{\mu\nu}\;, \\
[M_{\mu\nu}, P_\rho] &= 2(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho})\;, \qquad (3.10)
\end{aligned}
$$

plus eq. (3.5b).

3) *The generalized $D = 4$ Maxwell algebra.*

To obtain it, now expand the MC forms $(\omega^{\mu\nu}, e^\mu)$ dual to $(M_{\mu\nu}, \mathcal{P}_\mu)$ as in eq. (2.14) with N_0=4 and N_2=4 (these expansions follow, as shown in case 3 in Sec. 2, from those of $s\mathcal{G}$). Besides eq. (3.3) we have from eq. (2.14)

$$
e^\mu(\lambda) = \lambda^2\, e^{\mu,2} + \lambda^4\, e^{\mu,4}\;. \qquad (3.11)
$$

Using the expansion (3.3) for $\omega^{\mu\nu}$ and (3.11) for e^μ in the $o(3,2)$ MC equations (3.7), we obtain eqs. (3.4a–b), the following modified eq. (3.4c)

$$
d\,\omega^{\mu\nu,4} = -(\omega^{\mu}{}_{\rho}{}^{,0} \wedge \omega^{\rho\nu,4} + \omega^{\mu}{}_{\rho}{}^{,2} \wedge \omega^{\rho\nu,2} + \omega^{\mu}{}_{\rho}{}^{,4} \wedge \omega^{\rho\nu,0} + e^{\mu,2} \wedge e^{\nu,2}) \quad (3.12)
$$

and

$$
d\,e^{\mu,2} = -\omega^{\mu}{}_{\nu}{}^{,0} \wedge e^{\nu,2} \quad , \quad d\,e^{\mu,4} = -\,\omega^{\mu}{}_{\nu}{}^{,0} \wedge e^{\nu,4} - \,\omega^{\mu}{}_{\nu}{}^{,2} \wedge e^{\nu,2}\;. \qquad (3.13)
$$

Introducing the generators P_μ, Z_μ dual to $e^{\mu,2}, e^{\mu,4}$, it follows from eqs. (3.12), (3.13) that the expanded algebra, denoted $o(3,2)(4,4)$, provides the generalization of the Maxwell algebra given by the eqs. (3.10), (3.5a–c), plus

$$[M_{\mu\nu}, Z_\rho] = 2(Z_\mu \eta_{\nu\rho} - Z_\nu \eta_{\mu\rho}) \ , \quad (3.14a)$$

$$[\widetilde{Z}_{\mu\nu}, P_\rho] = 2(Z_\mu \eta_{\nu\rho} - Z_\nu \eta_{\mu\rho}) \ , \quad (3.14b)$$

$$0 = [Z_{\mu\nu}, Z_{\rho\tau}] = [Z_{\mu\nu}, \widetilde{Z}_{\rho\tau}] = [Z_\mu, Z_\nu] = [Z_\rho, \widetilde{Z}_{\mu\nu}] = [Z_\rho, Z_{\mu\nu}] \quad (3.14c)$$

Thus, the expansion $o(3,2)(4,4)$ contains the Maxwell algebra as the subalgebra generated by $(M_{\mu\nu}, P_\nu, Z_{\mu\nu})$. The addition of $\widetilde{Z}_{\mu\nu}$ provides, through eq. (3.5c), the 'bosonic roots' of the abelian charges $Z_{\mu\nu}$ appearing in the Maxwell algebra. The abelian vector charges Z_ρ, dual to $e^{\mu,4}$ in eq. (3.11), are central but for their Lorentz vector character.

4 N-extended Maxwell D = 4 superalgebras as expansions of $osp(N|4)$

To obtain the N-extended $D = 4$ Maxwell superalgebras we now expand the $D = 4$ adS superalgebra $osp(N|4)$ with the coset splitting of case 3) in Sec. 2. Explicitly, $osp(N|4) = V_0 \oplus V_2 \oplus V_1$ is given by $(N = 1, 2, 3, \dots)$

$$osp(N|4) = [o(1,3) \oplus o(N)] \oplus \frac{sp(4)}{o(1,3)} \oplus \frac{osp(N|4)}{sp(4) \oplus o(N)} \quad . \quad (4.1)$$

Since $sp(4) \simeq o(3,2)$ the algebra (4.1) is the supersymmetric counterpart of the D=4 AdS algebra obtained by adding the $\frac{N(N-1)}{2}$ generators T^{ab} of $o(N)$ and N real D=4 Majorana spinor fermionic generators $\mathcal{Q}^a_\alpha$ $(a = 1 \dots N)$. They satisfy the relations ($C = \gamma_0$ in the Majorana realization)

$$\{\mathcal{Q}^a_\alpha, \mathcal{Q}^b_\beta\} = \delta^{ab}(C\gamma^\mu)_{\alpha\beta}\mathcal{P}_\mu - \frac{1}{2}\delta^{ab}(C\gamma^{\mu\nu})_{\alpha\beta}\, M_{\mu\nu} + C_{\alpha\beta}\, T^{ab} \ , \quad (4.2)$$

$$[M_{\mu\nu}, \mathcal{Q}^a_\alpha] = (\mathcal{Q}^a\, \gamma_{\mu\nu})_\alpha \ , \ [\mathcal{P}_\mu, \mathcal{Q}^a_\alpha] = \tfrac{1}{2}(\mathcal{Q}^a\, \gamma_\mu)_\alpha \ , \ [T^{ab}, \mathcal{Q}^c_\alpha] = 2(\mathcal{Q}^a_\alpha \delta^{bc} - \mathcal{Q}^b_\alpha \delta^{ac}) \ , \quad (4.3)$$

where T_{ab} are the internal symmetry $o(N)$ generators

$$[T_{ab}, T_{cd}] = (\delta_{cb}T_{ad} - \delta_{db}T_{ac}) - (a \leftrightarrow b)) \quad , \quad a, b = 1 \dots, N \ . \quad (4.4)$$

Let ω^{ab} be the one-forms dual to T_{ab} (the indices a,b, being euclidean, can be placed up or down) and ψ^α_a the fermionic MC forms dual to Q^a_α. The splitting (4.1) corresponds to the following assignments of the MC one-forms in the generic MC equations (2.12)

$$\omega^l \to (\omega^{\mu\nu}, \omega^{ab}) \quad , \quad e^r \to (e^\mu) \quad , \quad \psi^\alpha \to (\psi^\alpha_a) \quad . \tag{4.5}$$

For $d\omega^{\mu\nu}$, de^μ we obtain ($\overline{\psi} = \psi^T C$)

$$\begin{aligned} d\,\omega^{\mu\nu} &= -\,\omega^\mu{}_\rho \wedge \omega^{\rho\nu} - e^\mu \wedge e^\nu + \frac{1}{2}\overline{\psi}^\alpha_a(\gamma^{\mu\nu})_{\alpha\beta}\psi^\beta_a \quad , \\ d\,e^\mu &= -\omega^\mu{}_\rho \wedge e^\rho - \frac{1}{2}\,\overline{\psi}^\alpha_a(\gamma^\mu)_{\alpha\beta} \wedge \psi^\beta_a \quad . \end{aligned} \tag{4.6}$$

The $osp(N|4)$ MC equations require adding to (4.6) those for $d\omega_{ab}$ and $d\psi^\alpha{}_a$,

$$d\,\omega_{ab} = -\omega_{ac} \wedge \omega^c{}_b - \overline{\psi}^\alpha{}_a \wedge \psi_{\alpha b} \quad , \tag{4.7}$$

$$d\,\psi^\alpha{}_a = -\frac{1}{4}(\omega^{\mu\nu}\gamma_{\mu\nu})^\alpha{}_\beta \wedge \psi^\beta{}_a - \frac{1}{2}e^\mu \gamma_\mu{}^\alpha{}_\beta \wedge \psi^\beta{}_a + \omega^b{}_a \wedge \psi^\alpha{}_b \quad . \tag{4.8}$$

At this stage, all the above generators and MC forms are dimensionless.

To expand now $osp(N|4)$ we use the splitting (4.1) and eqs. (2.14), (2.13) for $\omega^{\mu\nu}$, ω^{ab}, e^ν and $\psi^\alpha{}_i$, and choose N_0=4=N_2 in (2.14) and N_1=3 in (2.13) (case a) at the end of Sec. 2). In all, we have

$$\begin{aligned} \omega^{\mu\nu}(\lambda) &= \omega^{\mu\nu,0} + \lambda^2\,\omega^{\mu\nu,2} + \lambda^4\,\omega^{\mu\nu,4} \; , \\ e^\mu(\lambda) &= \lambda^2\,e^{\mu,2} + \lambda^4\,e^{\mu,4} \; , \end{aligned} \tag{4.9}$$

$$\omega^{ab}(\lambda) = \omega^{ab,0} + \lambda^2\,\omega^{ab,2} + \lambda^4\,\omega^{ab,4} \; , \tag{4.10}$$

$$\psi^\alpha{}_a(\lambda) = \lambda\,\psi^\alpha{}_a{}^{,1} + \lambda^3\,\psi^\alpha{}_a{}^{,3} \; . \tag{4.11}$$

Now, inserting expressions (4.9)-(4.11) into the $osp(N|4)$ MC equations (4.6)-(4.8), we obtain the MC equations of the expansion $osp(N|4)(N_0, N_1, N_2) = osp(4|N)(4,3,4)$ plus other new superalgebras through the following consecutive steps:

i) The MC equations for $\omega^{\mu\nu,0}$, $e^{\mu,2}$, $\omega^{ab,0}$, $\psi^{\alpha,1}$ already determine a superalgebra, the expansion $osp(N|4)(0,1,2)$. This is [3], in fact, a contracted algebra (in the generalized sense of Weimar-Woods [6]) and hence has the same dimension as $osp(N|4)$. Its MC equations are given by

$$
\begin{aligned}
d\,\omega^{\mu\nu,0} &= -\omega^{\mu}{}_{\rho}{}^{,0} \wedge \omega^{\rho\nu,0} \,, \\
d\,e^{\mu,2} &= -\omega^{\mu}{}_{\rho}{}^{,0} \wedge e^{\rho,2} - \frac{1}{2}\,\overline{\psi}^{\alpha,1}_{a}(\gamma^{\mu})_{\alpha\beta}\psi^{\beta,1}_{a} \,, \\
d\,\omega_{ab,0} &= -\omega_{ac,0} \wedge \omega^{c}{}_{b}{}^{,0} \,, \\
d\,\psi^{\alpha,1}{}_{a} &= -\frac{1}{4}(\omega^{\mu\nu,0}\gamma_{\mu\nu})^{\alpha}{}_{\beta}\,\psi^{\beta,1}_{a} + \omega^{b,0}_{a}\,\psi^{\alpha,1}_{b} \,.
\end{aligned}
\tag{4.12}
$$

After using the duality relations for $\omega^{\mu\nu}$, e^{μ} from Sect. 3 and $\omega^{ab,0}(T_{cd}) = \delta^{ab}_{cd}$, $\psi^{\alpha,1}{}_{a}(Q^{b}{}_{\beta}) = \delta^{\alpha}_{\beta}\delta^{a}_{b}$ one gets from (4.12) that $osp(N|4)(0,1,2) = s\mathcal{P}^{(N)}$ *i.e.*, the standard N-extended Poincaré superalgebra generated by $(M_{\mu\nu}, Q_{\alpha}{}^{a}, P_{\mu}, T^{ab})$ with $[Q_{\alpha}{}^{a}] = L^{-\frac{1}{2}}$ and $[P_{\mu}] = L^{-1}$ as usual.

ii) The remaining equations in the expansion of $osp(N|4)$ provide an enlargement of $s\mathcal{P}^{(N)}$ superalgebra, with new generators $\widetilde{Z}_{\mu\nu}$, $Z_{\mu\nu}$, Z_{μ}, $\widetilde{Y}_{ab}$, Y_{ab} and $\Sigma_{\alpha}{}^{b}$ dual, respectively, to the one-forms $\omega^{\mu\nu,2}$, $\omega^{\mu\nu,4}$, $e^{\mu,4}$, $\omega^{ab,2}$, $\omega^{ab,4}$ and $\psi^{\alpha,3}{}_{a}$. These MC forms have dimensions $[\omega^{\mu\nu,2}] = L$, $[\omega^{\mu\nu,4}] = L^2$, $[e^{\mu,4}] = L^2$, $[\omega^{ab,2}] = L$, $[\omega^{ab,4}] = L^2$ and $[\psi^{\alpha,3}{}_{a}] = L^{\frac{3}{2}}$, inverse, respectively, to those of the corresponding expanded Lie algebra generators.

From (4.6)-(4.8) the $osp(N|4)(4,3,4)$ MC equations follow:

$$\begin{aligned}
d\,\omega^{\mu\nu,2} &= -\omega^{\mu}{}_{\rho}{}^{,0}\wedge\omega^{\rho\nu,2}-\omega^{\mu}{}_{\rho}{}^{,2}\wedge\omega^{\rho\nu,0}+\frac{1}{2}\overline{\psi}^{\alpha,1}_{a}(\gamma^{\mu\nu})_{\alpha\beta}\wedge\psi^{\beta,1}_{a} && (4.13)\\
d\,\omega^{\mu\nu,4} &= -\omega^{\mu}{}_{\rho}{}^{,0}\wedge\omega^{\rho\nu,4}-\omega^{\mu}{}_{\rho}{}^{,4}\wedge\omega^{\rho\nu,0}-\omega^{\mu}{}_{\rho}{}^{,2}\wedge\omega^{\rho\nu,2}-e^{\mu,2}\wedge e^{\nu,2}\\
&\quad+\frac{1}{2}\overline{\psi}^{\alpha,1}_{a}(\gamma^{\mu\nu})_{\alpha\beta}\wedge\psi^{\beta,3}_{a}+\frac{1}{2}\overline{\psi}^{\alpha,3}_{a}(\gamma^{\mu\nu})_{\alpha\beta}\wedge\psi^{\beta,1}_{a} && (4.14)\\
d\,e^{\mu,4} &= -\omega^{\mu}{}_{\rho}{}^{,0}\wedge e^{\rho,4}-\omega^{\mu}{}_{\rho}{}^{,2}\wedge e^{\rho,2}\\
&\quad-\frac{1}{2}\,\overline{\psi}^{\alpha,1}_{a}(\gamma^{\mu})_{\alpha\beta}\wedge\psi^{\beta,3}_{a}-\frac{1}{2}\,\overline{\psi}^{\alpha,3}_{a}(\gamma^{\mu})_{\alpha\beta}\wedge\psi^{\beta,1}_{a} && (4.15)\\
d\,\omega_{ab}{}^{,2} &= -\omega^{,0}_{ac}\wedge\omega^{c}{}_{b}{}^{,2}-\omega^{,2}_{ac}\wedge\omega^{c}{}_{b}{}^{,0}-\overline{\psi}^{\alpha,1}{}_{a}\wedge\psi_{\alpha b}{}^{,1} && (4.16)\\
d\,\omega_{ab}{}^{,4} &= -\omega^{,0}_{ac}\wedge\omega^{c}{}_{b}{}^{,4}-\omega^{,4}_{ac}\wedge\omega^{c}{}_{b}{}^{,0}-\omega_{ac,2}\wedge\omega^{c}{}_{b}{}^{,2}\\
&\quad-\overline{\psi}^{\alpha,1}{}_{a}\wedge\psi_{\alpha b}{}^{,3}-\overline{\psi}^{\alpha,3}{}_{a}\wedge\psi_{\alpha b}{}^{,1} && (4.17)\\
d\,\psi^{\alpha,3}{}_{a} &= -\frac{1}{4}(\omega^{\mu\nu,0}\gamma_{\mu\nu})^{\alpha}{}_{\beta}\wedge\psi^{\beta,3}{}_{a}-\frac{1}{4}(\omega^{\mu\nu,2}\gamma_{\mu\nu})^{\alpha}{}_{\beta}\wedge\psi^{\beta,1}{}_{a}-\frac{1}{2}(e^{\mu,2}\gamma_{\mu})^{\alpha}{}_{\beta}\wedge\psi^{\beta,1}{}_{a}\\
&\quad+\omega^{b}{}_{a}{}^{,0}\wedge\psi^{\alpha,3}_{b}+\omega^{b}{}_{a}{}^{,2}\wedge\psi^{\alpha,1}{}_{b} && (4.18)
\end{aligned}$$

Some terms in the above expressions may be added up, but we have preferred to exhibit the way they are generated by the expansion.

The expansion procedure leads to the possible generalizations of $s\mathcal{P}(N)$ described below:

1) *An extension of the $\{Q^a_\alpha, Q^b_\beta\}$ anticommutator of the N-extended superPoincaré algebra by the abelian $SO(1,3)$ tensorial generators $\widetilde{Z}_{\mu\nu}$ and $SO(N)$-tensorial ones $\widetilde{Y}_{ab}$.*

This is the expansion $osp(N|4)(2,1,2)$, which is obtained by taking eqs. (4.9)-(4.11) up to order two. From (4.13) and (4.16) it follows that

$$\{Q^a_\alpha, Q^b_\beta\} = \delta^{ab}(C\gamma^\mu)_{\alpha\beta}P_\mu - \frac{1}{2}\delta^{ab}(C\gamma^{\mu\nu})_{\alpha\beta}\,\widetilde{Z}_{\mu\nu} + C_{\alpha\beta}\widetilde{Y}^{ab}\ , \qquad (4.19)$$

plus

$$\begin{aligned}
[M_{\mu\nu}, \widetilde{Z}_{\rho\sigma}] &= (\eta_{\rho\nu}\widetilde{Z}_{\mu\sigma} - \eta_{\sigma\nu}\widetilde{Z}_{\mu\rho}) - (\mu\leftrightarrow\nu)\ ,\\
[T^{ab}, \widetilde{Y}^{cd}] &= (\delta^{cb}\widetilde{Y}^{ad} - \delta^{db}\,\widetilde{Y}^{ac}) - (a\leftrightarrow b)\ . \qquad (4.20)
\end{aligned}$$

The new abelian generators ($\widetilde{Z}_{\mu\nu}$, $\widetilde{Y}_{ab}$) are the tensorial and isotensorial central charges that are added to N-extended super Poincaré algebra $s\mathcal{P}^{(N)}$.

The $osp(N|4)(2,1,2)$ superalgebra (4.19)-(4.20) constitutes another example of case a) at the end of Sec. 2.

2) *Minimal enlargement of $D = 4$ N-superPoincaré algebra including $Z_{\mu\nu}$.*

Looking at (3.10), one might simply think of making the replacement

$$[P_\mu, P_\nu] = 0 \quad \longrightarrow \quad [P_\mu, P_\nu] = Z_{\mu\nu} \tag{4.21}$$

in the $s\mathcal{P}^{(N)}$ algebra. Nevertheless, this would not lead to a superalgebra since, when checking that $dd \equiv 0$ on $\omega^{\mu\nu,4}$, we first obtain $d\,\omega^{\mu\nu,4} = -e^{\mu,2} \wedge e^{\nu,2}$ (*i.e.* the second commutator in eq. (4.21)) and then $dd\omega^{\mu\nu,4} \simeq e^{[\mu,2} \wedge \overline{\psi}_a^{\alpha,1}(\gamma^{\nu]})_{\alpha\beta}\psi_a^{\beta,1} \neq 0$, reflecting that the JI is not satisfied for $(Z_{\mu\nu}, Q_\alpha^a, Q_\beta^b)$. However, $dd\omega^{\mu\nu,4}$ will vanish if the MC equation for $d\,\omega^{\mu\nu,4}$ is replaced (see (4.14)) by $d\,\omega^{\mu\nu,4} = -e^{\mu,2} \wedge e^{\nu,2} + \overline{\psi}_a^{\alpha,1}(\gamma^{\mu\nu})_{\alpha\beta}\psi_a^{\beta,3}$, which shows that an additional fermionic generator is required. Thus, the inconsistency can be removed if the one-forms $(\omega^{\mu\nu,0}, e^{\mu,2}, \psi_a^{\alpha,1}, \omega^{\mu\nu,2})$ are supplemented by a new fermionic one, $\psi_a^{\beta,3}$, dual to the additional set of fermionic generators Σ_β^i. Then, the odd-odd sector of the N-superPoincaré algebra is completed with a non-trivial additional relation

$$\{Q_\alpha^a, Q_\beta^b\} = \delta^{ab}(C\gamma^\mu)_{\alpha\beta}P_\mu \quad , \quad \{Q_\alpha^a, \Sigma_\beta^b\} = -\frac{1}{2}\delta^{ab}(C\gamma^{\mu\nu})_{\alpha\beta}Z_{\mu\nu} \,. \tag{4.22}$$

This was referred to as the minimal supersymmetrization of the Maxwell algebra in [12–15].

The new spinorial generators Σ_a^β were originally added by Green [16] on supersting theory grounds (see further [17] and [2] in an expansions context) in the commutator

$$[P_\mu, Q_\alpha^a] = \gamma_{\mu\alpha}^{\beta}\,\Sigma_\beta^a \quad , \tag{4.23}$$

which here is a consequence of eq. (4.18). Since (eqs. (4.9)-(4.11)) our MC forms expansions do not contain sixth powers of λ, we obtain as in [16] that

$$\{\Sigma_\alpha^a, \Sigma_\beta^b\} = 0 \,. \tag{4.24}$$

Eq. (4.18) for $d\psi_a^{\alpha,3}$ also gives the commutators expressing the covariance properties of Σ_α^a,

$$[M^{\mu\nu}, \Sigma_\alpha^a] = \frac{1}{4}(\gamma^{\mu\nu})_\alpha{}^\beta\,\Sigma_\alpha^a \quad , \quad [T^{ab}, \Sigma_\alpha^c] = 2(\Sigma_\alpha^a\delta^{bc} - \Sigma_\alpha^b\delta^{ac}) \quad . \tag{4.25}$$

3) $D = 4$ N*-extended Maxwell superalgebras with additional bosonic charges*

Let us write explicitly the main commutators of $osp(N|4)(4,3,4)$ that follow from the MC equations given before. In the expansions (4.9), (4.10), the forms $(\omega^{\mu\nu,4}, e^{\mu,4}, \omega^{ab,4})$ correspond to the highest powers in λ and, hence, their dual generators $(Z_{\mu\nu}, Z_\mu, Y_{ab})$ are abelian. The last two modify the $\{Q, \Sigma\}$ anticommutator of the minimal superMaxwell algebra (4.22), which becomes

$$\{Q^a_\alpha, \Sigma^b_\beta\} = \delta^{ab}\left[(C\gamma^\mu)_{\alpha\beta}\, Z_\mu - \frac{1}{2}(C\gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu}\right] + C_{\alpha\beta}\, Y^{ab}\ . \qquad (4.26)$$

Besides the commutators expressing the Lorentz ($SO(N)$) covariance properties of $\widetilde{Z}_{\mu\nu}, Z_{\mu\nu}, Z_\mu$, $(\widetilde{Y}_{ab}, Y_{ab})$, we have the non-trivial relations

$$[\widetilde{Y}^{ab}, \widetilde{Y}^{cd}] = \left(\delta^{cb} Y^{ad} - \delta^{db} Y^{ac}\right) - (a \leftrightarrow b)\ , \qquad (4.27)$$

$$[\widetilde{Z}^{\mu\nu}, Q^a_\alpha] = (\gamma^{\mu\nu})_\alpha{}^\beta\, \Sigma^a_\beta \quad , \quad [\widetilde{Y}^{ab}, Q^c_\alpha] = 2(\Sigma^a_\alpha \delta^{bc} - \Sigma^b_\alpha \delta^{ac})\ . \qquad (4.28)$$

For $N = 1$, the superalgebra relations (4.22)-(4.24) with $Z_\mu = 0 = \widetilde{Z}_{\mu\nu}$ were proposed in [12] as the simplest supersymmetrization of the $D = 4$ Maxwell algebra (for N=1 the generators $T_{ab}, Y_{ab}, \widetilde{Y}_{ab}$ are clearly absent). Obviously, setting some generators equal to zero in an algebra, as done above for Z_μ and $\widetilde{Z}_{\mu\nu}$, does not lead in general to a subalgebra, but here all the resulting commutators satisfy the JI by virtue of the D=4 Fierz identity $(C\gamma^\mu)_{(\alpha\beta}(C\gamma_\mu)_{\gamma\delta)} = 0$, where the bracket means symmetrization.

Other $D = 4$ N-extended Maxwell superalgebras, with supersymmetrized tensorial charges $Z_{\mu\nu}$, were considered in [14, 15]. In [14] the following coset decomposition of $D = 4$ SUSY adS superalgebra $osp(N|4)$, different from the one given by (4.1), was introduced for even N=$2n$,

$$osp(2n;4) = (sl(2;C) \oplus u(n)) \oplus \frac{sp(4)}{sl(2;C)} \oplus \frac{o(2n)}{u(n)} \oplus \frac{osp(2n;4)}{sp(4) \oplus o(2n)} \qquad (4.29)$$

To recover the algebras of [14] as expansions, the coset part $\frac{o(2n)}{u(n)}$ of the internal symmetry generators should be expanded in powers of λ in the same way as the vierbein e^μ. In [15] the $D = 4$ N-extended Maxwell algebras were obtained as a particular contraction of the direct sum of two real superalgebras, describing respectively the supersymmetrization of $o(3,1) \simeq sl(2,\mathbb{C})$ ($sl(k|2;\mathbb{C})$, $0 \leq k \leq 2N$) and the supersymmetrized $o(3,2) \simeq sp(4)$ algebra (the D=4 extended AdS superalgebra $osp(2N-k|4)$).

5 Final Remarks

The main aim of this paper was to provide further examples showing that quite complicated (super)algebras can be derived easily as expansions of a basic (super)algebra which encodes some essential features (as reflected by a certain coset decomposition). Previous studies of the expansion method [2,3] were used to derive [3, 4] the D=11 full M-algebra (including the Lorentz part) as a particular expansion of $osp(1|32)$ and to look at Chern-Simons supergravities. Also, the (p, q)-Poincaré superalgebras [18] governing the extended D=3 supergravities were found [19] to be expansions of $osp(p+q|2)$.

Here we have applied the expansion method to (super)algebras to obtain various Maxwell algebras and new generalizations of the Green algebra [16]. These (super)Maxwell algebras, all characterized by the presence of the four-momenta commutator $[P_\mu, P_\nu] = Z_{\mu\nu}$, may be considered as symmetries of an enlarged spacetime with additional bosonic coordinates. Recently, it has been shown [20] that the quantization of a free particle in such a ten-dimensional enlarged D=4 spacetime describes a Lorentz covariant extension of the planar Landau problem (the non-relativistic particle in a constant magnetic field background).

We have found, in particular, that the λ^4 term in the expansion of the Lorentz MC one-forms (see eq. (3.8)) generates the tensorial 'central' charges $Z_{\mu\nu}$ of the $D = 4$ Maxwell algebra (3.10), which turns out to be the expansion o(3,2)(4,2) of the adS algebra $o(3,2)$. These generators $Z_{\mu\nu}$ also appear, as they should, in all the generalized Maxwell algebras described in Sec. 3. The inclusion of the generators $Z_{\mu\nu}$ in a superPoincaré algebra to obtain a

Maxwell superalgebra requires further the addition of the Green fermionic generator. The N-extended Maxwell superalgebras are discussed in Sec. 4. The most general one considered in detail in this paper is the expansion $osp(N|4)(4,3,4)$; it includes two sets of bosonic generators, a Lorentz vector Z_μ and the usual $Z_{\mu\nu}$ tensor, and two other sets $\widetilde{Y}_{ab}, Y_{ab}$ that are $SO(N)$ tensors. The 'minimal' N-extended Maxwell superalgebra, with generators $\{M_{\mu\nu}, P_\mu, Z_{\mu\nu}, Q^a_\alpha, \Sigma^a_\alpha, T_{ab}\}$, may be obtained by a suitable reduction of the general case, as shown in Sec. 4.

We have not considered other (super)algebras as those of the type given in [15] from the expansion method point of view. It is unclear whether they can be obtained by this procedure, but we recall here that contractions do appear as a particular case of expansions [3] (see also [2, 4, 19]).

Acknowledgments. This paper has been supported by research grants from the Spanish MINECO (FIS2008-01980, FIS2009-09002, CONSOLIDER CPAN-CSD2007-00042), from the Polish Ministry of Science and Education (202332139) and from the Polish National Science Center (project 2011/01/B/ST2/0335).

References

[1] J.A. de Azcárraga, J.M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics*, Camb. Univ. Press, 1995.

[2] M. Hatsuda and M. Sakaguchi, Progr. Theor. Phys. **109**, 853-867 (2003) [hep-th/0106114].

[3] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Nucl. Phys. **B662**, 185-219 (2003) [hep-th/0212347].

[4] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Class. Quant. Grav. **21**, S1375-S1384 (2004) [hep-th/0401033]; Int. J. Theor. Phys. **46** (2007) 2738-2752 (2007) [hep-th/0703017 [hep-th]].

[5] F. Izaurieta, E. Rodríguez and P. Salgado, J. Math. Phys. **47**, 123512-1-28 (2006) [hep-th/0606215].

[6] E. Weimar-Woods, J. Math. Phys. **36**, 4519-4548 (1995); Rev. Math. Phys. **12**, 1505-1529 (2000).

[7] H. Bacry, P. Combe, J.L. Richard, Nuovo Cim. **A67**, 267-299 (1970).

[8] R. Schrader, Fortschr. Phys. **20**, 701-734 (1972).

[9] J. Negro and M.A. del Olmo, J. Math. Phys **31**, 568-578, 2811-2821 (1990).

[10] S. Bonanos, J. Gomis, J. Phys. **A43**, 015201 (2010) [arXiv:0812.4140 [hep-th]].

[11] J. Gomis, K. Kamimura, J. Lukierski, JHEP08, 039 (2009). [arXiv:1005.3714 [hep-th]].

[12] S. Bonanos, J. Gomis, K. Kamimura, J. Lukierski, Phys. Rev. Lett. **104**, 090401 (2010) [arXiv:0911.5072 [hep-th]].

[13] S. Bonanos, J. Gomis, K. Kamimura, J. Lukierski, J. Math. Phys. **51**, 102301 (2010) [arXiv:1005.3714 [hep-th]].

[14] J. Lukierski, Proc. Stekl. Inst. Math. **272**, 1-8 (2011) [arXiv:1007.3405 [hep-th]].

[15] K. Kamimura, J. Lukierski, Phys. Lett. B **707**, 292-297 (2012) [arXiv:1111.3598 [math-ph]].

[16] M. B. Green, Phys. Lett. B **223**, 157-164 (1989).

[17] C. Chryssomalakos, J. A. de Azcárraga, J. M. Izquierdo and J. C. Pérez Bueno, Nucl. Phys. **B567**, 293 (2000) [hep-th/9904137].

[18] A. Achúcarro and P.K. Townsend, Phys. Lett. B 229, 383-387 (1989).

[19] J. A. de Azcárraga and J. M. Izquierdo, Nucl. Phys. **B854**, 276-291 (2012) [arXiv:1107.2569 [hep-th]].

[20] S. Fedoruk and J. Lukierski, Phys. Lett. **B718**, 646-652 (2012) [arXiv:1207.5683 [hep-th]].