k-Leibniz algebras from lower order ones: from Lie triple to Lie ℓ-ple systems

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Abstract

Two types of higher order Lie ℓ-ple systems are introduced in this paper. They are defined by brackets with $\ell > 3$ arguments satisfying certain conditions, and generalize the well known Lie triple systems. One of the generalizations uses a construction that allows us to associate a $(2n-3)$-Leibniz algebra $\mathfrak{L}$ with a metric n-Leibniz algebra $\tilde{\mathfrak{L}}$ by using a $2(n-1)$-linear Kasymov trace form for $\tilde{\mathfrak{L}}$. Some specific types of k-Leibniz algebras, relevant in the construction, are introduced as well. Both higher order Lie ℓ-ple generalizations reduce to the standard Lie triple systems for $\ell = 3$.

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1 Introduction

It is natural to generalize Lie algebras and Poisson structures when their defining brackets have more than two arguments. The quest for $(n > 2)$-ary algebras of various types has arisen repeatedly both in mathematics and physics, there motivated by their possible physical applications; this was the case of Nambu mechanics [1] introduced long ago, when the quark statistics was being discussed.

A possible higher order Lie algebra structure is provided by the *generalized* or higher order Lie algebras (GLAs) $\mathcal{G}$ [2, 3] [4, 5], based on an antisymmetric multibracket with even n entries. The characteristic relation for $\mathcal{G}$ is the *generalized Jacobi identity* (GJI), which expresses that the total antisymmetrization of two nested multibrackets vanishes. The GLAs $\mathcal{G}$ and the GJI reduce to the ordinary Lie algebras $\mathfrak{g}$ and the JI, respectively, when $n = 2$. Like Lie algebras, GLAs have their higher order Poisson counterparts or *generalized Poisson structures* (GPS) [2].

Another generalization is provided by the *n-Lie* or *Filippov algebras*[1] (FAs) $\mathfrak{G}$ [7–9] and, further, by the n-Leibniz algebras $\mathfrak{L}$ [10], which will be of our concern here. FAs have a fully skewsymmetric bracket with an arbitrary number $n \geq 2$ entries, and their characteristic identity is the *Filippov identity* (FI), which expresses that the adjoint action is a derivation of $\mathfrak{G}$; again, $\mathfrak{G} = \mathfrak{g}$ for $n = 2$. The FAs Poisson counterparts are the Nambu-Poisson structures (NPS) [11], their precedent being n=3 Nambu mechanics [1]. The NPS also satisfy the FI (called 'five-point identity' for $n = 3$ in [12] and later 'fundamental identity' in [11]). The problem of the quantization of both n-ary Poisson structures, GPS and NPS, has been the subject of intense study and discussion (see [12, 11, 13, 14] and the first ref. in [15] for an outlook and further references).

Filippov and related algebras have attracted considerable attention in the last few years due to their appearance in three-dimensional superconformal Chern-Simons theories. This is the case of the metric 3-Lie FAs in the original $\mathcal{N} = 8$ manifestly supersymetric Bagger-Lambert-Gustavsson [16, 17] (BLG) model describing (actually two) coincident membranes, although a $\mathcal{N} = 6$ ABJM [18] superconformal Chern-Simons theory not requiring 3-FAs was soon pointed out. The key ingredient for the appearance of three-algebras in BLG models is the close connection between 3-Lie (or 3-Leibniz) algebras and

[1] There is some unfortunate confusion concerning the terminology. We call these algebras n-Lie (Filippov's original name) or Filippov algebras indistinctly but *not* 'Lie n-algebras' (as sometimes the very different FAs and GLAs are *both* referred to). See Sec. 1.1 of [15] for the terminology of the various n-ary algebras. Intermediate generalizations between FAs and GLAs also exist; see [6].

their associated Lie (gauge) ones, the properties of which are encoded in the former (eq. (2.6) and Sec. 2.2). The positivity restrictions of the first $\mathcal{N} = 8$ BLG model, that limit its 3-Lie algebra to be [20, 21] a sum of copies of the A_4 FA plus central ideals, led to consider other possibilities (see [19] for a detailed review of developments in the theory of multiple, parallel membranes in M-theory). The new algebras were obtained by relaxing the positive definiteness of the metric [22] or the complete antisymmetry of the FA 3-bracket. Structures of this last type were already known in the mathematical literature as n-Leibniz algebras [10]; they also satisfy the Filippov identity, which for an $\mathfrak{L}$ may be called n-Leibniz identity [10]. Relaxing the anticommutativity led, in particular, to the 'real generalized metric 3-algebras' of Cherkis and Sämann (CS) [23]. These are metric 3-Leibniz algebras with a 'symmetry property' (Sec. 2.2) and give rise to an $\mathcal{N} = 2$ BLG-type action. Similarly, in the complex case the Bagger and Lambert hermitian three-algebras [24] are used in their $\mathcal{N} = 6$ theory, which incorporates the ABJM one [18] in a three-algebra approach.

A class of 3-Leibniz algebras is provided by the *Lie triple systems* (Sec. 3), introduced in mathematics long ago [25–28] (see further [29]). They may be defined (Sec. 3) as 3-Leibniz algebras with brackets that are antisymmetric in the first two arguments that, besides the FI, satisfy an additional cyclic property (there are also Leibniz triple systems [30] of which Lie triple systems are a particular case, but these will not be considered here). The aim of this paper is to generalize Lie triple systems, which have brackets with ℓ=3 entries, to higher $\ell \geq 3$; the resulting structure will define the *Lie ℓ-ple systems*. As a preliminary example, we recall the connection between the simple $n = 3$ Filippov algebra A_4, the Lie algebra $\mathrm{Lie}\, A_4 = so(4)$ and its associated triple system. We will then show that it is possible to obtain an ℓ-Leibniz algebra $\mathfrak{L}$ from two Leibniz algebras $\mathfrak{L}^1$ and $\mathfrak{L}^2$ satisfying certain conditions. The analysis will lead us to two possible higher order generalizations of the Lie triple systems, the Lie n-ple and the Lie ℓ-ple systems with $\ell = 2n-3, n > 3$, defined by special types of k-Leibniz algebras. For n=3=ℓ, both Lie 3-ple systems coincide and reproduce the standard Lie triple ones.

The plan of the paper is the following. Sec. 2 summarizes the properties of Filippov $\mathfrak{G}$ and n-Leibniz $\mathfrak{L}$ algebras needed here; Sec. 2.3 extends the CS 3-algebras [23] to $k = 2n - 3$. Sec. 3 relates Lie triple systems to a specific type of 3-Leibniz algebras. Sec. 4 provides a method to obtain a metric k-Leibniz algebra, $k = n + m - 3$, from two n- and m-Leibniz algebras with certain requirements, and considers some particular cases of later interest. Sec. 5 introduces the two k-ple generalizations of the Lie triple systems and Sec. 6 contains an outlook.

Only finite dimensional real algebras are considered in this paper.

2 Filippov $\mathfrak{G}$ and n-Leibniz algebras $\mathfrak{L}$

2.1 Filippov or n-Lie algebras

Definition 1. A *Filippov* or *n-Lie algebra* [7] $\mathfrak{G}$ is a vector space (also denoted by $\mathfrak{G}$) endowed with an antisymmetric n-linear bracket $\mathfrak{G} \times \overset{n}{\cdots} \times \mathfrak{G} \to \mathfrak{G}$, $[X_1, \ldots, X_n]$, that obeys the *Filippov identity* (FI),

$$[X_1, \ldots, X_{n-1}, [Y_1, \ldots, Y_n]] = \sum_{r=1}^{n} [Y_1, \ldots, Y_{r-1}, [X_1, \ldots, X_{n-1}, Y_r], Y_{r+1}, \ldots, Y_n] \ , \ X, Y \in \mathfrak{G}. \tag{2.1}$$

It is also possible to define a right FI using a right action *i.e.*, starting from $[[Y_1, \ldots, Y_n], X_1, \ldots, X_{n-1}]$ at the *l.h.s.* of eq. (2.1) but, due to the skewsymmetry, the right and left FIs coincide. Given a basis $\{\mathbf{e}_a\}$ in $\mathfrak{G}$, the n-bracket

$$[\mathbf{e}_{a_1}, \ldots, \mathbf{e}_{a_n}] = f_{a_1 \ldots a_n}{}^b \, \mathbf{e}_b \quad , \; a = 1, \ldots, \dim \mathfrak{G} \ , \tag{2.2}$$

gives the *structure constants* $f_{a_1 \ldots a_n}{}^b$ of $\mathfrak{G}$, in terms of which the FI reads

$$f_{b_1 \ldots b_n}{}^l \, f_{a_1 \ldots a_{n-1} l}{}^s = \sum_{k=1}^{n} f_{a_1 \ldots a_{n-1} b_k}{}^l \, f_{b_1 \ldots b_{k-1} l b_{k+1} \ldots b_n}{}^s \quad , \tag{2.3}$$

Defining the adjoint action $ad_{\mathscr{X}} \in \mathrm{End}\mathfrak{G}$ of $\mathscr{X} = (X_1, \ldots, X_{n-1}) \in \wedge^{n-1}\mathfrak{G}$ by $ad_{\mathscr{X}} X \equiv \mathscr{X} \cdot X := [X_1, \ldots, X_{n-1}, X]$, the FI may be rewritten as

$$ad_{\mathscr{X}} [Y_1, \ldots, Y_n] = \sum_{r=1}^{n} [Y_1, \ldots, Y_{r-1}, ad_{\mathscr{X}} Y_r, Y_{r+1}, \ldots, Y_n] \ , \tag{2.4}$$

which expresses that $ad_{\mathscr{X}}$ is an *(inner) derivation* of $\mathfrak{G}$. Clearly,

$$ad_{\mathscr{X}_{a_1 \ldots a_{n-1}}} \equiv [\mathbf{e}_{a_1}, ..., \mathbf{e}_{a_{n-1}}, \cdot\,] \quad , \quad ad_{\mathscr{X}_{a_1 \ldots a_{n-1}}} \mathbf{e}_{a_n} = f_{a_1 \ldots a_n}{}^b \, \mathbf{e}_b \quad . \tag{2.5}$$

A linear transformation $\rho_{\mathscr{X}}$ of a vector space V satisfying the analogue of (2.4) is called [8] a representation ρ of $\mathfrak{G}$ (although it is really of $\mathscr{X} \in \wedge^{n-1}\mathfrak{G}$); we shall only consider here $\rho = ad$ (the regular representation).

The $\{ad_{\mathscr{X}}\}$ are closed under the commutator and generate the *Lie algebra* $\mathrm{Lie}\,\mathfrak{G}$ *associated with the FA* $\mathfrak{G}$. Indeed, the FI determines the $\mathrm{End}\,\mathfrak{G}$ relation

$$[ad_{\mathscr{X}}, ad_{\mathscr{Y}}] = ad_{\mathscr{X} \cdot \mathscr{Y}} \ (= -ad_{\mathscr{Y} \cdot \mathscr{X}}) \tag{2.6}$$

where, with $\mathscr{Y} = (Y_1, \ldots, Y_{n-1}) \in \wedge^{n-1}\mathfrak{G}$, $\mathscr{X} \cdot \mathscr{Y} \in \wedge^{n-1}\mathfrak{G}$ is given by

$$\mathscr{X} \cdot \mathscr{Y} := \sum_{r=1}^{n-1} (Y_1, \ldots, Y_{r-1}, ad_{\mathscr{X}} Y_r, Y_{r+1}, \ldots, Y_{n-1}) \quad (\neq -\mathscr{Y} \cdot \mathscr{X}) \quad . \tag{2.7}$$

Eq. (2.7) defines the composition law [31] of $\mathscr{X}$ and $\mathscr{Y}$ (see also [7,8]). It is non-associative since $\mathscr{X}\cdot(\mathscr{Y}\cdot\mathscr{Z})-(\mathscr{X}\cdot\mathscr{Y})\cdot\mathscr{Z}=\mathscr{Y}\cdot(\mathscr{X}\cdot\mathscr{Z})$, which follows from the FI. For $\mathfrak{g}$, this is the JI, $[X,[Y,Z]]-[[X,Y],Z]=[Y,[X,Z]]$ but, since $\mathscr{X}\cdot\mathscr{Y}\neq-\mathscr{Y}\cdot\mathscr{X}$ (*cf.* (2.6)), the $\mathscr{X}$'s generate a Leibniz (Sec. 2.2) rather than a Lie algebra. Due to the importance of the $\mathscr{X}\in\wedge^{n-1}\mathfrak{G}$, we refer to them as *fundamental objects*; in BLG models, Lie $\mathfrak{G}$ is relevant for the gauge transformations. For $n=2$, Lie $\mathfrak{g}\equiv ad\mathfrak{g}$ is generated by $\{ad_{X_a}\}$ and, obviously, eq. (2.6) is $[ad_X,ad_Y]=ad_{[X,Y]}$; Lie $\mathfrak{g}=\mathfrak{g}/Z(\mathfrak{g})$, where $Z(\mathfrak{g})=\{Y|\,ad_XY=0\,,\forall X\in\mathfrak{g}\}=\{Y|\,ad_YX=0\,\forall X\in\mathfrak{g}\}$ is the centre of $\mathfrak{g}$.

The study of FAs follows closely (but not fully) that of ordinary Lie algebras [7–9] (see [15] for further references). For instance, a FA is simple if $[\mathfrak{G},\ldots,\mathfrak{G}]\neq 0$ and does not have non-trivial ideals ($I\subset\mathfrak{G}$ is an ideal if $[\mathfrak{G},\ldots,\mathfrak{G},Y]\in I\ \forall Y\in I$); semisimple FAs are direct sums of simple ones. The centre of $\mathfrak{G}$ is the ideal $Z(\mathfrak{G})=\{Y\in\mathfrak{G}|ad_{\mathscr{X}}Y=0\ \forall\mathscr{X}\}$; the regular representation is called faithful if $Z(\mathfrak{G})=0$. Thus, ad is faithful for the n-Lie algebra $\mathfrak{G}/Z(\mathfrak{G})$ [8]; if $\mathfrak{G}$ is semisimple, ad is faithful. If it is further simple, the $ad_{\mathscr{X}}\in$ Lie $\mathfrak{G}$ act irreducibly on $\mathfrak{G}$.

In contrast with the $n=2$ (Lie) case, the only real simple ($n>2$)-Lie algebras are $(n+1)$-dimensional [7,9] and given by

$$[\mathbf{e}_{a_1},\ldots,\mathbf{e}_{a_n}]=\epsilon_{a_1\ldots a_n}{}^b\mathbf{e}_b\ , \tag{2.8}$$

where, in terms of the Levi-Civita symbol, $\epsilon_{a_1\ldots a_{n+1}}$, $\epsilon_{a_1\ldots a_n}{}^b=\eta^{ba_{n+1}}\epsilon_{a_1\ldots a_na_{n+1}}$ and η is the (euclidean or pseudoeuclidean) metric on the $\mathfrak{G}$ vector space. The real euclidean simple n-Lie algebras are labelled A_{n+1} [7]; the simple lorentzian FAs may be denoted A_{p+q}, with $p+q=n+1$ (these inequivalent real $(n+1)$-dimensional simple algebras are the same as complex FAs).

There is an analogue of the Cartan criterion that applies to general n-Lie algebras: a FA $\mathfrak{G}$ is semisimple [32] iff the $2(n-1)$-linear *Kasymov trace form* k [8,32] $k:\wedge^{n-1}\mathfrak{G}\times\wedge^{n-1}\mathfrak{G}\to\mathbb{R}$, defined for ad as

$$k(\mathscr{X},\mathscr{Y})=k(X_1,\ldots,X_{n-1},Y_1,\ldots,Y_{n-1}):=\mathrm{Tr}(ad_{\mathscr{X}}ad_{\mathscr{Y}})\ , \tag{2.9}$$

is non-degenerate *i.e.*, $k(X,\mathfrak{G},\overset{n-2}{\cdots},\mathfrak{G},\mathfrak{G},\overset{n-1}{\cdots},\mathfrak{G})=0\Rightarrow X=0$ (actually, k may be defined for other representations ρ of $\mathfrak{G}$). For the regular representation ad, k was called [32] the Killing form for $\mathfrak{G}$ (to which k reduces for $n=2$); nevertheless, we shall refer to k as the Kasymov trace form for $\mathfrak{G}$. Since $ad_{\mathscr{X}_{a_1\ldots a_{n-1}}}$ is given by the $\dim\mathfrak{G}\times\dim\mathfrak{G}$ matrix $(ad_{\mathscr{X}_{a_1\ldots a_{n-1}}})_b{}^c=f_{a_1\ldots a_{n-1}b}{}^c$, the coordinates of k are

$$k_{a_1\ldots a_{n-1}b_1\ldots b_{n-1}}\equiv k(\mathbf{e}_{a_1},\ldots,\mathbf{e}_{a_{n-1}},\mathbf{e}_{b_1},\ldots,\mathbf{e}_{b_{n-1}})=f_{a_1\ldots a_{n-1}b}{}^cf_{b_1\ldots b_{n-1}c}{}^b\ . \tag{2.10}$$

In general, $\mathfrak{G}$ is a *metric Filippov algebra* when it is endowed with a non-degenerate bilinear metric $<,>$: $\mathfrak{G}\times\mathfrak{G}\to\mathbb{R}$, $<X,Y>=<Y,X>$, which is Lie $\mathfrak{G}$-invariant, $\mathscr{X}.\eta(Y,Z)\equiv\mathscr{X}.<Y,Z>=0$ *i.e.*,

$$<ad_{\mathscr{X}}Y_1,Y_2>+<Y_1,ad_{\mathscr{X}}Y_2>=0\ ,\quad \forall Y\in\mathfrak{G}\ ,\ \forall\mathscr{X}\in\wedge^{n-1}\mathfrak{G}\ ; \quad (2.11)$$

we shall refer to (2.11) as the *metricity property.* If $<,>$ is euclidean, Lie $\mathfrak{G}\subset so(\dim\mathfrak{G})$; Lie $A_{n+1}=so(n+1)$. For a 3-Lie algebra eq (2.11) simply reads

$$<[X_1,X_2,Y_1],Y_2>+<Y_1,[X_1,X_2,Y_2]>=0\ . \quad (2.12)$$

It follows that the structure constants of a metric FA with all indices down, $f_{a_1\dots a_{n+1}}:=f_{a_1\dots a_n}{}^b\langle\mathbf{e}_b,\mathbf{e}_{a_{n+1}}\rangle$, are completely antisymmetric. When $\mathfrak{G}$ is semisimple it is also metric, Lie $\mathfrak{G}$ is semisimple and the $\mathfrak{G}$ Kasymov trace form may be also looked at as the non-singular Lie $\mathfrak{G}$ Killing metric.

2.2 n-Leibniz algebras

n-Leibniz algebras [10] $\mathfrak{L}$ result from relaxing the requirement of full skewsymmetry in Def. 1; thus, n-FAs are a particular case of n-Leibniz algebras. The FI (or n-Leibniz identity [10] for $\mathfrak{L}$) now depends on whether the adjoint derivative of the n-Leibniz bracket is a left or a right one. The n-Leibniz algebras for which (2.1) is satisfied are then *left* n-Leibniz algebras; for definiteness sake, we shall consider these henceforth.

Since the n-Leibniz bracket needs not being anticommutative, the fundamental objects for $\mathfrak{L}$ are now $\mathscr{X}\in\otimes^{n-1}\mathfrak{L}$. But, since expressions such as (2.6) and (2.7) only depend on the FI, the composition law (2.7) for fundamental objects, $\mathscr{X}\cdot\mathscr{Y}\in\otimes^{n-1}\mathfrak{L}$, defines again a Leibniz algebra, and there is [31,8] still an associated *Lie* algebra Lie$\mathfrak{L}$ (relevant in BLG-type models). Lie$\mathfrak{L}$ is defined [33] for the quotient space $\otimes^{n-1}\mathfrak{L}/K$, where K is the kernel of the adjoint map ad, $K=\{\mathscr{X}\in\otimes^{n-1}\mathfrak{L}\mid ad_{\mathscr{X}}=0\}$ and $ad_{\mathscr{X}}=0$ obviously means that $ad_{\mathscr{X}}Y=[X_1,\dots,X_{n-1},Y]=0\ \forall Y\in\mathfrak{L}$ (a similar consideration also holds for Lie$\mathfrak{G}$). An *n-Leibniz algebra $\mathfrak{L}$ is metric* when it is endowed with a Lie$\mathfrak{L}$-invariant scalar product $<,>$. Then, condition (2.11) $\forall\mathscr{X}\in\otimes^{n-1}\mathfrak{L}$ is expressed in terms of the structure constants by $f_{a_1\dots a_{n-1}bc}=-f_{a_1\dots a_{n-1}cb}$.

An example is provided by

Definition 2. *The 'real generalized 3-algebras' (CS 3-algebras)* [23]

These are metric (eq. (2.12)) 3-Leibniz algebras that satisfy the additional 'symmetry property'

$$<[X_1,X_2,Y_1],Y_2>=<[Y_1,Y_2,X_1],X_2>\ , \quad (2.13)$$

which implies $f_{a_1a_2b_1b_2} = f_{b_1b_2a_1a_2}$ for its structure constants. An obvious particular case of CS 3-algebras is the simple 3-Lie algebra A_4 since $\langle[\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{a_3}], \mathbf{e}_{a_4}\rangle = \epsilon_{a_1a_2a_3a_4}$ satisfies both eqs. (2.12) and (2.13).

The symmetry condition (2.13), together with the metricity property (2.12), implies that the Leibniz 3-bracket of these CS algebras is antisymmetric in its first two arguments. Similarly, it is easy to check that when the 3-Leibniz bracket is antisymmetric in its first two arguments and satisfies the symmetry property $f_{a_1a_2b_1b_2} = f_{b_1b_2a_1a_2}$ (eq. (2.13)) for a certain metric, the 3-Leibinz algebra is also metric, $f_{a_1a_2b_1b_2} = -f_{a_1a_2b_2b_1}$ (in which case $f_{a_1a_2b_1b_2} = f_{b_2b_1a_2a_1}$).

2.3 Higher order Leibniz algebras of CS type

The 3-Leibniz algebras of Def. 2 admit the following higher order generalization:

Definition 3. A 'generalized metric ℓ-algebra' is an odd euclidean ℓ-Leibniz algebra defined by an ℓ-bracket $[X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-3}, Y_{n-2}]$, with $\ell = 2n-3$, $n \geq 3$, that is antisymmetric in the $(n-1)$ X's and in $(n-2)$ the Y's and satisfies
a) the metricity property (2.11),

$$\begin{aligned}\langle[X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-3}, Y_{n-2}], Y_{n-1}\rangle = \\ -\langle Y_{n-2}, [X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-3}, Y_{n-1}]\rangle \ ,\end{aligned} \tag{2.14}$$

b) and the symmetry condition

$$< [X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-2}], Y_{n-1} >=< [Y_1, \ldots, Y_{n-1}, X_1, \ldots, X_{n-2}], X_{n-1} > \ , \tag{2.15}$$

which reduce to eqs. (2.12), (2.13) for $\ell = 3 = n$.

The reason of the numbering and the origin of the symmetry property will become apparent later (eq. (4.36)). Note that, without any specific assumption and as in the CS $\ell = 3$ case above, the symmetry condition (2.15) plus the metricity (2.14) imply (X_{n-1}, X_{n-2})-skewsymmetry; also, (X_{n-1}, X_{n-2})-skewsymmetry plus the symmetry condition (2.15) suffice to imply metricity.

In terms of the structure constants $g_{a_1\ldots a_{n-1}b_1\ldots b_{m-2}}{}^c$ of the ℓ-Leibniz algebra, the above relations read

$$\begin{aligned} g_{a_1\ldots a_{n-1}b_1\ldots b_{n-2}b_{n-1}} &= -g_{a_1\ldots a_{n-1}b_1\ldots b_{n-3}b_{n-1}b_{n-2}} \quad \text{(metricity)} \quad &(2.16)\\ g_{a_1\ldots a_{n-1}b_1\ldots b_{n-1}} &= g_{b_1\ldots b_{n-1}a_1\ldots a_{n-1}} \quad \text{(symm. property)}\ . &(2.17)\end{aligned}$$

3 Lie triple Systems

A particular case of 3-Leibniz algebras is that of *Lie triple systems* [25–29]. They have also appeared in physics as *e.g.*, in connection with parastatistics [34] or the Yang-Baxter equation [35–37]. Further triple (and supertriple) system generalizations may be found in [38, 39] and references therein.

Definition 4. A *Lie triple system* is a (left) 3-Leibniz algebra $\mathfrak{L}$ such that its 3-bracket satisfies, besides the FI, the conditions

(a) $[X_1, X_2, Y] = -[X_2, X_1, Y] \quad \forall X_1, X_2, Y \in \mathfrak{L}$

(b) $[X_1, X_2, X_3] + [X_2, X_3, X_1] + [X_3, X_1, X_2] = 0 \quad \forall X_1, X_2, X_3 \in \mathfrak{L}$

Note that the cyclic property (b), together with (a), is equivalent to saying that the full antisymmetrization of the arguments in $[X_1, X_2, X_3]$ vanishes.

A generic 3-bracket with the property (a) above is a map $[\ ,\ ,\ ] : \wedge^2\mathfrak{L}\times\mathfrak{L} \to \mathfrak{L}$ and hence its symmetry properties correspond to those of

$$\begin{array}{c}\square\\\square\end{array} \otimes \square = \begin{array}{c}\square\\\square\\\square\end{array} \oplus \begin{array}{l}\square\square\\\square\end{array} \quad . \tag{3.18}$$

Thus, as a $GL(\dim\mathfrak{L})$-tensor, the irreducible symmetry components of a 3-bracket antisymmetric in its first two arguments corresponds to a fully antisymmetric one or to a 3-bracket with the mixed symmetry of $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$. When the 3-Leibniz algebra bracket has the symmetry of $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$, the cyclic property (b) in Def. 4 is also satisfied and it defines a Lie triple system. When the $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ part is absent, the 3-Leibniz algebra $\mathfrak{L}$ is actually a FA $\mathfrak{G}$.

As mentioned, an euclidean (say) n-Leibniz $\mathfrak{L}$ algebra has an associated *Lie* algebra, Lie$\mathfrak{L} \subset so(\dim\mathfrak{L})$; thus, the vector space $\mathfrak{L}$ carries a representation of Lie$\mathfrak{L}$. There is a canonical procedure [26–28, 36, 38, 39] that also goes backwards, from a metric $\mathfrak{g}$ to $\mathfrak{L}$. It starts from a Lie algebra $\mathfrak{g}$ endowed with a $\mathfrak{g}$-invariant scalar product $(\ ,\ )$, which preserves the euclidean metric $<,>$ of a vector space $\mathfrak{L}$ ($\mathfrak{g} \subset so(\dim\mathfrak{L})$) on which $\mathfrak{g}$ acts faithfully; the construction endows $\mathfrak{L}$ with a Leibniz algebra structure.

Let $\mathfrak{g} = so(4)$ with basis given by the $L_{a_1a_2}$ that generate the rotations in $\mathfrak{L} = \mathbb{R}^4$ vector space. Then, the 3-Leibniz algebra is defined by

$$(L_{a_1a_2}, L_{b_1b_2}) =< ad_{a,a_2}\mathbf{e}_{b_1}, \mathbf{e}_{b_2} >=< [\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_1}], \mathbf{e}_{b_2} > , \ a, b = 1, \ldots, 4 \ . \tag{3.19}$$

Clearly, the symmetry property (2.13) is satisfied and eq. (2.11) follows from $L_{b_1b_2} = -L_{b_2b_1}$. Since the 3-bracket also fulfills the FI (this follows from

Prop. 5 below as a particular case), the construction leads to CS 3-algebras [23], as shown in [40], Th. 11 (for the hermitian algebras in [24] see [40,41]).

As a first example of (3.19), let (,) be the $so(4)$ Killing metric $\frac{1}{2}\mathrm{Tr}(ad_{a_1a_2}ad_{a_3b}) = \frac{1}{2}\epsilon_{a_1a_2b}{}^{c}\epsilon_{a_3a_4c}{}^{b}$. This defines the 3-Leibniz algebra

$$[\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_1}] = -(\delta_{a_1b_1}\mathbf{e}_{a_2} - \delta_{a_2b_1}\mathbf{e}_{a_1}) = -\sum_{\sigma\in S_2} \delta_{a_{\sigma(1)}b_1}\delta_{a_{\sigma(2)}}{}^{c}\mathbf{e}_c \ . \qquad (3.20)$$

Reciprocally, we see that the $ad_{\mathscr{X}_{a_1a_2}}$ generate the original $so(4)$ rotations since

$$ad_{\mathscr{X}_{a_1a_2}}\mathbf{e}_b = [\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_b] = -(\delta_{a_1b}\mathbf{e}_{a_2} - \delta_{a_2b}\mathbf{e}_{a_1}) = L_{a_1a_2}\mathbf{e}_b \ . \qquad (3.21)$$

Further, the above 3-bracket does not have a $\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ component. Thus, it defines a (rather old [25]) Lie triple system which is also an example of the CS 3-algebras of Def. 2. In general, however, Lie triple systems are not metric.

Let (,) now be the invariant metric $k^{(2)}(L_{a_1a_2}, L_{b_1b_2}) = \epsilon_{a_1a_2b_1b_2}$ on $so(4)$, which is symmetric and *semi*definite. Then, eq. (3.19) for $k^{(2)}$ gives $\epsilon_{a_1a_2b_1b_2} = \langle[\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_1}], \mathbf{e}_{b_2}\rangle$, which corresponds to $\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ and to the A_4 3-Lie algebra, an obvious example of Def. 2. The existence of $k^{(2)}$ is a fortunate accident for the Chern-Simons part of the lagrangian of the original A_4-BLG model: Lie$A_{n+1} = so(n+1)$ is simple but for $n = 3$ and the fully skewsymmetric $k^{(2)}$ does not generalize to $n > 3$ (see the footnote in Sec. 14.4 of [15]).

Remark. The Lie triple system in eq. (3.20) also follows from the Kasymov trace form k for the FA A_4, $k : \wedge^2 A_4 \times \wedge^2 A_4 \to \mathbb{R}$; since A_4 is simple k is, in fact, the Killing metric for $so(4)$. Let $< , >$ denote the euclidean metric on $\mathbb{R}^4 = A_4$. Then, Kasymov's k ($f_{a_1a_2b}{}^{c} = \epsilon_{a_1a_2b}{}^{c}$ in (2.10))

$$k_{a_1a_2a_3b} = \frac{1}{2}\mathrm{Tr}(ad_{a_1a_2}ad_{a_3b}) = \frac{1}{2}\epsilon_{a_1a_2d}{}^{c}\epsilon_{a_3bc}{}^{d} = \langle[\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{a_3}], \mathbf{e}_b\rangle \qquad (3.22)$$

also reproduces the metric Lie triple system defined by (3.20).

We shall use the Kasymov form and introduce other 'mixed metric' generalizations below to obtain higher order k-Leibniz algebras from m- and n-Lie algebras and, later, to introduce Lie ℓ-ple systems (Sec. 5).

4 The k-Leibniz algebra associated with two n- and m-Leibniz algebras

Let $\mathfrak{L}^1$ and $\mathfrak{L}^2$ be, respectively, n- and m-Leibniz algebras defined on the same vector space V, and let $\mathfrak{L}^2$ be metric with respect to $\langle\ ,\ \rangle$ so that

$ad^2_{\mathscr{X}} \in so(\mathrm{dim}V)$. Assume now that
a) the m-bracket of $\mathfrak{L}^2$ satisfies

$$< [Z_1, \ldots, Z_{m-3}, X_1, X_2, Y_1]_{\mathfrak{L}^2}, Y_2 >=< [Z_1, \ldots, Z_{m-3}, Y_1, Y_2, X_1]_{\mathfrak{L}^2}, X_2 > , \tag{4.23}$$

(a condition satisfied by all metric m-FAs and that, for $m = 3$, reduces to the symmetry property eq. (2.13)). Further,
b) $ad^2_{\mathscr{Y}}$, $\mathscr{Y} \in \otimes^{m-1}\mathfrak{L}^2$, is a derivation of $\mathfrak{L}^1$,

$$ad^2_{\mathscr{Y}}[X_1, \ldots, X_n]_{\mathfrak{L}^1} = \sum_{r=1}^{n} \left[X_1, \ldots, ad^2_{\mathscr{Y}} X_r, \ldots X_n\right]_{\mathfrak{L}^1} , \tag{4.24}$$

where $ad^2_{\mathscr{X}} X_r$ is the m-bracket in $\mathfrak{L}^2$. Then, the following proposition follows:

Proposition 5. *Let $\mathfrak{L}^1$ and $\mathfrak{L}^2$ be as above satisfying conditions* (4.23), (4.24)*. Let the generalization of the Kasymov trace form $k : (\mathscr{X}, \mathscr{Y}) \to \mathbb{R}$, where now $\mathscr{X} \in \otimes^{n-1}\mathfrak{L}^1$, $\mathscr{Y} \in \otimes^{m-1}\mathfrak{L}^2$, be defined by $k(\mathscr{X}, \mathscr{Y}) = \mathrm{Tr}(ad^1_{\mathscr{X}} ad^2_{\mathscr{Y}}) = k(\mathscr{Y}, \mathscr{X})$; clearly, it reduces to Kasymov's k for $\mathfrak{L}^1 = \mathfrak{L}^2$. Then, the $(n + m - 3)$-bracket defined on V by*

$$\langle [X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{m-2}]_{\mathfrak{L}}, Y_{m-1}\rangle = \mathrm{Tr}(ad^1_{X_1 \ldots X_{n-1}} ad^2_{Y_1 \ldots Y_{m-1}}) , \tag{4.25}$$

satisfies the FI and therefore defines a k-Leibniz algebra $\mathfrak{L}$, $k = n + m - 3$. Moreover, this k-Leibniz algebra is metric w.r.t. $\langle\, ,\, \rangle$.

Proof. Let $\{\mathbf{e}_a\}$ be a basis of the common underlying vector space V of the $\mathfrak{L}^1$ and $\mathfrak{L}^2$ algebras and let $f_{a_1 \cdots a_n}{}^b$ and $h_{b_1 \cdots b_m}{}^c$ be, respectively, their structure constants in that basis,

$$[\mathbf{e}_{a_1}, \cdots, \mathbf{e}_{a_n}]_{\mathfrak{L}^1} = f_{a_1 \cdots a_n}{}^b \mathbf{e}_b \quad , \quad [\mathbf{e}_{b_1}, \cdots, \mathbf{e}_{b_m}]_{\mathfrak{L}^2} = h_{b_1 \cdots b_m}{}^c \mathbf{e}_c , \tag{4.26}$$

where, since $\mathfrak{L}^2$ is metric,

$$h_{b_1 \cdots b_{m-1} u_1 u_2} = -h_{b_1 \cdots b_{m-1} u_2 u_1} . \tag{4.27}$$

It follows from eq. (4.25) that the structure constants $g_{a_1 \cdots a_{n-1} b_1 \cdots b_{m-2}}{}^d$

$$[\mathbf{e}_{a_1}, \cdots, \mathbf{e}_{a_{n-1}}, \mathbf{e}_{b_1}, \cdots, \mathbf{e}_{b_{m-2}}]_{\mathfrak{L}} = g_{a_1 \cdots a_{n-1} b_1 \cdots b_{m-2}}{}^d \mathbf{e}_d \tag{4.28}$$

of the $(n + m - 3)$-bracket defining $\mathfrak{L}$ are expressed in terms of those of $\mathfrak{L}^1$ and $\mathfrak{L}^2$ as

$$g_{a_1 \cdots a_{n-1} b_1 \cdots b_{m-2} d} = f_{a_1 \cdots a_{n-1}}{}^{uv} h_{b_1 \cdots b_{m-2} d v u} , \tag{4.29}$$

where indices are raised and lowered by the metric $\langle\ ,\ \rangle$ on V.

To prove that the $k = (n-m-3)$-bracket (4.25) defines a k-Leibniz algebra $\mathfrak{L}$ it suffices to check the FI (eq. (2.3)),

$$\begin{aligned}
&g_{a_1\cdots a_{n-1}b_1\cdots b_{m-2}}{}^{l} g_{c_1\cdots c_{n-1}d_1\cdots d_{m-3}l}{}^{s} \\
&-\sum_{r=1}^{n-1} g_{c_1\cdots c_{n-1}d_1\cdots d_{m-3}a_r}{}^{l} g_{a_1\cdots a_{r-1}la_{r+1}\cdot a_{n-1}b_1\cdots b_{m-2}}{}^{s} \\
&-\sum_{r=1}^{m-2} g_{c_1\cdots c_{n-1}d_1\cdots d_{m-3}b_r}{}^{l} g_{a_1\cdots a_{n-1}b_1\cdots b_{r-1}lb_{r+1}\cdots b_{m-2}}{}^{s} = 0\ . \qquad (4.30)
\end{aligned}$$

The FI for $\mathfrak{L}^2$ and the derivation property (4.24) read, respectively,

$$\begin{aligned}
h_{b_1\cdots b_m}{}^{l} h_{a_1\cdots a_{m-1}l}{}^{s} &= \sum_{r=1}^{m} h_{a_1\cdots a_{m-1}b_r}{}^{l} h_{b_1\ldots b_{r-1}lb_{r+1}\cdots b_m}{}^{s}\ , \\
f_{a_1\cdots a_n}{}^{l} h_{b_1\cdots b_{m-1}l}{}^{s} &= \sum_{r=1}^{n} h_{b_1\cdots b_{m-1}a_r}{}^{l} f_{a_1\ldots a_{r-1}la_{r+1}\cdots a_n}{}^{s}\ . \qquad (4.31)
\end{aligned}$$

Using eq. (4.29) to express the g's in terms of the f's and h's in the FI (4.30) for $\mathfrak{L}$, and property (4.23) of $\mathfrak{L}^2$,

$$h_{b_1\cdots b_{m-3}u_1u_2v_1v_2} = h_{b_1\cdots b_{m-3}v_1v_2u_1u_2}\ , \qquad (4.32)$$

the $l.h.s.$ of (4.30) becomes

$$\begin{aligned}
&f_{a_1\cdots a_{n-1}uv} h_{b_1\cdots b_{m-2}}{}^{lvu} f_{c_1\cdots c_{n-1}}{}^{wt} h_{d_1\cdots d_{m-3}l}{}^{s}{}_{tw} \\
&-f_{c_1\cdots c_{n-1}uv} h_{b_1\cdots b_{m-2}}{}^{s}{}_{tw} \sum_{r=1}^{n-1} h_{d_1\cdots d_{m-3}vua_r}{}^{l} f_{a_1\cdots a_{r-1}la_{r+1}\cdots a_{n-1}}{}^{wt} \\
&-f_{c_1\cdots c_{n-1}uv} f_{a_1\cdots a_{n-1}}{}^{wt} \sum_{r=1}^{m-2} h_{d_1\cdots d_{m-3}vub_r}{}^{l} h_{b_1\cdots b_{r-1}lb_{r+1}\cdots b_{m-2}}{}^{s}{}_{tw}\ . \qquad (4.33)
\end{aligned}$$

Now, using eqs.(4.31), the sums in (4.33) can be rewritten as

$$\begin{aligned}
&-\sum_{r=1}^{n-1} h_{d_1\cdots d_{m-3}vua_r}{}^{l} f_{a_1\cdots a_{r-1}la_{r+1}\cdots a_{n+1}}{}^{wt} \\
&= -f_{a_1\cdots a_{n-1}}{}^{wl} h_{d_1\cdots d_{m-3}vul}{}^{t} + h_{d_1\cdots d_{m-3}vu}{}^{wl} f_{a_1\cdots a_{n-1}l}{}^{t}\ , \qquad (4.34)
\end{aligned}$$

and

$$-\sum_{r=1}^{m-2} h_{d_1\cdots d_{m-3}b_r}{}^{lvu} h_{b_1\cdots b_{r-1}lb_{r+1}\cdots b_{m-2}}{}^{s}{}_{tw} = -h_{b_1\cdots b_{m-2}}{}^{s}{}_{t}{}^{l} h_{d_1\cdots d_{m-3}}{}^{vu}{}_{lw}$$

$$+h_{d_1\cdots d_{m-3}}{}^{vusl}h_{b_1\cdots b_{m-2}ltw}+h_{d_1\cdots d_{m-3}}{}^{vu}{}_t{}^l h_{b_1\cdots b_{m-2}}{}^s{}_{lw}\ . \tag{4.35}$$

Inserting (4.34) and (4.35) into (4.33) and using again property (4.23), it is found that (4.33) vanishes. Hence, the FI (4.30) is satisfied and the k-bracket (eqs. (4.25), (4.28)) defines a k-Leibniz algebra $\mathfrak{L}$ associated with $\mathfrak{L}^1$ and $\mathfrak{L}^2$. Further, $\mathfrak{L}$ is metric with respect to $\langle\ ,\ \rangle$, as can be easily seen from eq. (4.29) by using the assumed metricity of $\mathfrak{L}^2$ (eq. (4.27)) together with property (4.32). □

We shall use below two particular Corolaries of Prop. 5.

Corollary 6. *Let $\tilde{\mathfrak{L}}$ be a metric n-Leibniz algebra with an n-bracket skewsymmetric in its first $n-1$ arguments that satisfies condition* (4.23)*. Obviously, condition* (4.24) *holds. Then, the ℓ-bracket with $\ell = 2n-3$, defined by eq.* (4.25) *with $\mathfrak{L}^1 = \mathfrak{L}^2 = \tilde{\mathfrak{L}}$,*

$$\langle [X_1,\cdots,X_{n-1},Y_1,\cdots,Y_{n-2}]_{\mathfrak{L}},Y_{n-1}\rangle = Tr(ad_{(X_1,\cdots,X_{n-1})}ad_{(Y_1,\cdots,Y_{n-1})})\ , \tag{4.36}$$

where $\langle\ ,\ \rangle$ is the invariant metric on the common vector space, defines a metric ℓ-Leibniz algebra $\mathfrak{L}$ with a bracket that is antisymmetric under both the first $n-1$ and last $n-2$ arguments. Further, it satisfies (2.15) *by construction and hence $\mathfrak{L}$ is a generalized metric ℓ-algebra in the sense of Def. 3.*

Clearly, the conditions in this Corollary are met when $\tilde{\mathfrak{L}}$ is in particular a metric n-Lie algebra. This is the case of

Example 7. Let $\tilde{\mathfrak{L}} = A_{n+1}$ and $<\ ,\ >$ euclidean in Cor. 6. Then, eqs. (2.5) and (2.8) give

$$ad_{a_1\ldots a_{n-1}}.\mathbf{e}_{a_n} = \epsilon_{a_1\ldots a_{n-1}a_n}{}^{a_{n+1}}\mathbf{e}_{a_{n+1}}\ ,\ a=1,\ldots,n+1 \tag{4.37}$$

and, by eq. (4.36), the Kasymov trace form leads to

$$\begin{aligned} k_{a_1,\ldots,a_{n-1},b_1,\ldots,b_{n-1}} &= \frac{1}{2}\mathrm{Tr}(ad_{a_1,\ldots,a_{n-1}}ad_{b_1,\ldots,b_{n-1}}) = \\ \frac{1}{2}\epsilon_{a_1\ldots a_{n-1}b}{}^c\epsilon_{b_1\ldots b_{n-1}c}{}^b &= -\sum_{\sigma\in S_{n-1}}\delta_{a_{\sigma(1)b_1}}\ldots\delta_{a_{\sigma(n-2)b_{n-2}}}\delta_{a_{\sigma(n-1)b_{n-1}}} \\ &= \langle[\mathbf{e}_{a_1},\ldots,\mathbf{e}_{a_{n-1}},\mathbf{e}_{b_1},\ldots,\mathbf{e}_{b_{n-2}}],\mathbf{e}_{b_{n-1}}\rangle\ , \end{aligned} \tag{4.38}$$

which defines a $(2n-3)$-bracket antisymmetric in its first $(n-1)$ and second $(n-2)$ indices separately. This $(2n-3)$-Leibniz algebra will be used to define the Lie ℓ-ple system in Sec. 5.2.

It is sufficient to replace δ_{ab} by the Minkowskian η_{ab} to account for the case of the Lorentzian algebras A_{p+q}. Notice that we may also follow the procedure in Sec. 3 (eq. (3.19)) to obtain the above ($\ell = 2n-3$)-Leibniz algebra $\mathfrak{L}$ from $\mathfrak{g} = so(n+1)$ and its Killing metric. Characterizing the $so(n+1)$ generators by $n-1$ indices[2] $(a_1, \ldots, a_{n-1})$ ($\binom{n+1}{n-1} = \binom{n+1}{2}$), the Killing metric (,) on $so(n+1)$ leads to the k in (4.38) and the ℓ-Leibniz algebra defined there.

Corollary 8. *Let $\mathfrak{L}^2$ be a CS 3-algebra (Def. 2). Thus, condition (4.23) holds. Let $\mathfrak{L}^1$ be an n-Leibniz algebra on the same vector space V endowed with an n-bracket skewsymmetric in its first $n-1$ arguments and let $ad^2_{(X_1,X_2)}$ be a derivation of $\mathfrak{L}^1$. Then, the n-bracket ($h = n+3-3$)*

$$\langle [X_1, \cdots, X_{n-1}, X_n], Y\rangle = Tr(ad^1_{(X_1,\cdots,X_{n-1})} ad^2_{(X_n,Y)}) \qquad (4.39)$$

is skewsymmetric under the interchange of its first $n-1$ arguments and defines by Prop. 5 a metric n-Leibniz algebra $\mathfrak{L}$.

For $n=3$, the metric 3-Filippov algebras obtained from Cor. 6 and 8 are both CS 3-algebras.

Example 9. Consider first a metric m-Leibniz algebra $\mathfrak{L}^2$ defined on the $(n+1)$- dimensional space of $\mathfrak{L}^1 = \mathfrak{G}^1 = A_{n+1}$. It follows that the adjoint action of $\mathfrak{L}^2$ is a derivation of A_{n+1} *i.e.*, the second equation in (4.31) for $f_{a_1\cdots a_n}{}^l = \epsilon_{a_1\cdots a_n}{}^l$,

$$\epsilon_{a_1\cdots a_n}{}^l h_{b_1\cdots b_{m-1}l}{}^s = \sum_{r=1}^{n} h_{b_1\cdots b_{m-1}a_r}{}^l \epsilon_{a_1\ldots a_{r-1}la_{r+1}\cdots a_n}{}^r \ , \qquad (4.40)$$

holds. To see it, consider the Schouten-type identity

$$h_{b_1\cdots b_{m-1}}{}^l{}_{[s}\epsilon_{a_1\cdots a_n l]} \equiv 0 \quad , \quad a,b,l,s = 1\ldots, n+1 \ . \qquad (4.41)$$

This reduces to the sum of the $n+2$ cyclic permutations

$$h_{b_1\cdots b_{m-1}}{}^l{}_s\epsilon_{a_1\cdots a_n l} = \sum_{r=1}^{n} h_{b_1\cdots b_{m-1}}{}^l{}_{a_r}\epsilon_{a_1\cdots a_{r-1}s a_{r+1}\cdots a_n l} + h_{b_1\cdots b_{m-1}}{}^l{}_l\epsilon_{a_1\cdots a_n s} \ . \qquad (4.42)$$

Since the last term vanishes by the complete antisymmetry of the h's with all indices down (recall that $\mathfrak{L}^2$ is metric), what remains reproduces (4.40).

Now, let $\mathfrak{L}^2$ be a metric CS-Lie algebra. Then, all the conditions of Cor. 8 are met and eq. (4.39) defines an n-Leibniz algebra $\mathfrak{L}$ which is skewsymmetric in its first $n-1$ arguments.

[2]The action of $so(n+1)$ on $\mathbb{R}^{n+1}$ is given (*cf.* eq. (3.21)) by eq. (4.37). This leads, using the dual as $L_{b_1b_2} = \frac{-1}{(n-1)!}\epsilon_{b_1b_2}{}^{a_1\ldots a_{n-1}} ad_{a_1\ldots a_{n-1}}$, to the familiar expression $L_{b_1b_2}\mathbf{e}_{a_n} = \frac{-1}{(n-1)!}\epsilon_{b_1b_2}{}^{a_1\ldots a_{n-1}}\epsilon_{a_1\ldots a_{n-1}a_n}{}^{a_{n+1}}\mathbf{e}_{a_{n+1}} = -(\delta_{b_1a_n}\mathbf{e}_{b_2} - \delta_{b_2a_n}\mathbf{e}_{b_1})$.

5 Higher order Lie k-ple systems

5.1 Lie n-ple systems: a first generalization

There is a higher-order generalization of the Lie triple system that is very close to Def. 4. For it, it is sufficient to look at the symmetry pattern of an n-Leibniz algebra with skewsymmetric fundamental objects. The symmetry pattern decomposition is determined by

$$n\text{-}1\left\{\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right. \otimes \square = n\left\{\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right. \oplus \; n\text{-}1\left\{\begin{array}{|c|c|}\hline & \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right. \tag{5.43}$$

Clearly, the mixed symmetry pattern in the *r.h.s.* suggests

Definition 10. A *Lie n-ple system* is given by an n-Leibniz algebra such that its bracket is
a) skewsymmetric in its first $n-1$ arguments and
b) satisfies the cyclic property,

$$\sum_{cyclic} [X_1, X_2, \ldots, X_n] = 0 \; . \tag{5.44}$$

Example 11. Let $\mathfrak{L}^1 = \mathfrak{G}^1 = A_8$ and $\mathfrak{L}^2 = \mathfrak{G}^2 = A_4 \oplus A_4$. Let the basis of the common vector space $V = \mathbb{R}^8$ be $\{\mathbf{e}_c\}$, $c, d = 1, \ldots, 8$. By Ex. 9, $\mathfrak{G}^2$ is a derivation of $\mathfrak{G}^1$. Let the indices of $\mathfrak{G}^2$ be denoted a, b when they refer, respectively, to the first and second ideals A_4; we may set *e.g.* $a = 1, \ldots, 4$ and $b = 5, \ldots, 8$. Then, the structure constants of $\mathfrak{G}^2$ satisfy $f_{abcd} = 0 \, \forall c, d$. Let ϵ be the $\mathbb{R}^8$ Levi-Civita tensor, and $\bar{\epsilon}$ that on the A_4 ideals. The structure constants of the 7-Leibniz algebra $\mathfrak{L}$ constructed as in Cor. 8 are given by (see eq. (4.29))

$$g_{c_1 \cdots c_6 c_7 c_8} = \epsilon_{c_1 \cdots c_6}{}^{d_1}{}_{d_2} f_{c_7 c_8}{}^{d_2}{}_{d_1} \; ; \tag{5.45}$$

they are antisymmetric in $c_1 \ldots c_6$ and $c_7 c_8$ separately. Since c_7 and c_8 cannot take values in different ideals without $f_{c_7 c_8}{}^{d_2}{}_{d_1}$ being zero, we are left with two possibilities $g_{c_1 \cdots c_6 a_1 a_2}$ and $g_{c_1 \cdots c_6 b_1 b_2}$. Let us consider the first case so that f has only indices in the first A_4,

$$g_{c_1 \cdots c_6 a_1 a_2} = -\epsilon_{c_1 \cdots c_6 a_3 a_4} f_{a_1 a_2}{}^{a_3 a_4} \; . \tag{5.46}$$

It is clear that among the $c_1 \ldots c_6$ of the above expression there must be four b's and two a'2 in different orders, so $g_{c_1 \cdots c_6 a_1 a_2}$ may be written as

$$g_{c_1 \cdots c_6 a_1 a_2} \quad \sim \quad \delta^{b_1 \cdots b_4 a_5 a_6}_{c_1 \cdots \cdots c_6} \bar{\epsilon}_{b_1 \cdots b_4} \bar{\epsilon}_{a_5 a_6 a_3 a_4} \bar{\epsilon}_{a_1 a_2}{}^{a_4 a_3}$$

$$\sim \quad \delta^{b_1\cdots b_4 a_5 a_6}_{c_1\cdots\cdots c_6} \bar{\epsilon}_{b_1\cdots b_4} \delta_{a_5 a_1} \delta_{a_6 a_2} \ ; \tag{5.47}$$

there is a similar expression for $g_{c_1\cdots c_6 b_1 b_2}$, obtained by substituting b's for a's.

We may check that the 7-Leibniz algebra $\mathfrak{L}$ determined by eq. (5.45) satisfies condition (5.44), which is equivalent to requiring that the full anti-symmetrization of its 7-bracket vanishes. Actually, this is always the case for n-Leibniz algebras obtained as in Ex. 9 when $\mathfrak{L}^2 = \mathfrak{G}^2$ is a metric 3-Filippov algebra. Indeed, the structure constants of the n-Leibniz algebra $\mathfrak{L}$ are given by

$$g_{a_1\cdots a_{n-1} a_n a_{n+1}} = \epsilon_{a_1\cdots a_{n-1} uv} f_{a_n a_{n+1}}{}^{vu} \ , \tag{5.48}$$

where the f's are the structure constants of the 3-Lie algebra $\mathfrak{G}^2$. The full antisymmetrization of the n entries of the bracket of $\mathfrak{L}$ corresponds to

$$g_{[a_1\cdots a_{n-1} a_n] a_{n+1}} = \epsilon_{uv[a_1\cdots a_{n-1}} f_{a_n] a_{n+1}}{}^{vu} \ . \tag{5.49}$$

But this is zero, as can be seen by dualizing the *r.h.s.* of (5.49), which gives

$$\epsilon_{uva_1\cdots a_{n-1}} f_{a_n a_{n+1}}{}^{vu} \epsilon^{a_1\cdots a_n b} = (n-1)! \delta^{a_n b}_{uv} f_{a_n a_{n+1}}{}^{vu} = 0 \ , \tag{5.50}$$

the last equality being a consequence of the complete antisymmetry of the structure constants of $\mathfrak{G}^2$.

This last fact allows us to construct other examples of Lie n-ple systems based on a simple n-Lie algebra $\mathfrak{G}^1 = A_{n+1}$ and a metric 3-Lie algebra $\mathfrak{G}^2$.

5.2 Lie ℓ-ple systems, $\ell = 2n-3$

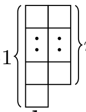
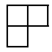
This second generalization uses ℓ-Leibniz algebras with and ℓ-bracket anti-symmetric in its first $(n-1)$ and last $(n-2)$ arguments (as in Cor. 6). To introduce the ℓ-ple Lie systems we have to look for the property that replaces (b) in Def. 4 when ℓ=$(2n-3)$ or, equivalently, for the symmetry pattern of the generic ℓ-Leibniz bracket that generalizes $\ydiagram{2,1}$ in eq. (4.28) when $\ell > 3$ and reduces to it for $\ell = 3$.

Let $\ell = (2n-3) > 3$. An ℓ-bracket skewsymmetric in its first $n-1$ and last $n-2$ entries has the generic symmetry of $n\text{-}1\left\{\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right. \otimes \left.\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right\} n\text{-}2$. Its decomposition in terms of irreducible Young patterns is given by

$$n\text{-}1\left\{\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right. \otimes \left.\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\right\} n\text{-}2 \ = \bigoplus_{r=0}^{n-2} \ \ell\text{-}r\left\{\begin{array}{|c|c|}\hline & \\ \hline \vdots & \vdots \\ \hline & \\ \cline{2-2} \vdots \\ \cline{1-1} \\ \hline \end{array}\right. \Big\} r \ , \tag{5.51}$$

where the dimensions of those at the $r.h.s.$ are

$$\begin{pmatrix} \dim\mathfrak{G}+1 \\ r \end{pmatrix} \begin{pmatrix} \dim\mathfrak{G} \\ \ell-r \end{pmatrix} \frac{\ell-2r+1}{\ell-r+1} \; ; \tag{5.52}$$

note that in all terms above $\ell - r \leq \dim\mathfrak{G}$. Since the first (longest) column of the Young patterns above may have $\dim\mathfrak{G}$ boxes at the most, $2n-3-r \leq \dim\mathfrak{G}$ or $\ell-r \leq \dim\mathfrak{G}$. The above decomposition of the outer product at the $l.h.s.$ of (5.51) in representations of the S_{2n-3} symmetric group determines the possible 'elementary' or $(GL(\dim\mathfrak{G}))$-irreducible symmetry patterns of the ℓ-bracket. In particular, the $r = 0$ component above would correspond to a fully skewsymmetric $(2n-3)$-bracket and hence to a $(2n-3)$-Lie algebra. We now argue that in this context the adequate generalization of the Lie triple system requires that the bracket of the Lie ℓ-ple system has the symmetry of the $r = n-2$ Young pattern in the sum (5.51) $i.e.$, that the bracket has the symmetry determined by n-1  n-2 , which indeed reduces to  for $n = 3$.

To this aim, let us go back to the metric ℓ-Leibniz algebra in Ex. 7 as it follows from Cor. 6. Since $\dim\mathfrak{G} = n+1$, there is a restriction since $\ell - r \leq n+1$. Thus, $r \geq n-4$ and, therefore, $n-4 \leq r \leq n-2$. Consider now the $(2n-3)$-commutators as defined by (4.38)

$$[\mathbf{e}_{a_1}, \ldots, \mathbf{e}_{a_{n-1}}, \mathbf{e}_{b_1}, \ldots, \mathbf{e}_{b_{n-2}}] = -\sum_{\sigma\in S_{n-1}} \delta_{a_{\sigma(1)}b_1} \ldots \delta_{a_{\sigma(n-2)}b_{n-2}} \delta_{a_{\sigma(n-1)}}{}^{c}\mathbf{e}_c \; . \tag{5.53}$$

To see how (5.53) selects a specific symmetry among the irreducible components in the $r.h.s.$ of (5.51), let us look at the symmetry of a generic pattern. The $2n-3$ indices of the Young tableau are split into two sets with $2n-3-r$ and r indices respectively. The primitive projector associated to the Young tableau symmetrizes r pairs of indices, where each pair contains an index of the first set and another of the second one and, then, it antisymmetrizes the indices of both sets separately. This projector, applied to the $r.h.s.$ of eq. (5.53), gives zero due to the δ_{ab} factors unless the indices of the first set are the $(n-1)$ a's all placed in the first column of the Young tableau (and thus the indices of the second set are the $n-2$ b's in the second column), since otherwise there will be a δ with antisymmetrized indices. Thus, $\ell - r = n-1$, $r = n-2$, select in (5.51) the pattern that corresponds to the ℓ-bracket (5.53).

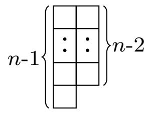
This motivates our second generalization of Lie triple systems:

Definition 12. A *Lie ℓ-ple system*, $\ell = (2n-3)$, is a real vector space $\mathfrak{L}$ endowed with a bracket given by a ℓ-linear map $\mathfrak{L} \times \overset{2n-3}{\cdots} \times \mathfrak{L} \to \mathfrak{L}$, $(X_1, \ldots, X_{2n-3}) \mapsto [X_1, \ldots, X_{2n-3}]$ such that

(a) it is antisymmetric in the first $n-1$ and in the last $n-2$ indices;

(b) it satisfies the (left) FI;

(c) its overall symmetry structure is given by the Young pattern



Properties (a) and (b) above define a particular (left) ℓ-Leibniz algebra structure $\mathfrak{L}$, ℓ odd; (c) makes of $\mathfrak{L}$ an ℓ-ple system. Note that, strictly speaking, (a) above is included in (c) and that, due to the properties of the projectors that determine the ($GL(\dim\mathfrak{L})$-) irreducible symmetries associated with the different patterns, (c) automatically implies that the symmetrizations and subsequent antisymmetrizations implied by any of the other ($r \neq n-2$) Young patterns in the *r.h.s.* of (5.51) give zero necessarily. When $n = 3 = l$, the resulting Lie triple system is the standard one (Def. 4).

6 Concluding remarks

In this paper we have introduced two Lie ℓ-ple generalizations of the Lie 3-ple, or triple, systems; they appear as special cases of k-Leibniz algebras, themselves a generalization of k-Lie algebras. As mentioned in the Introduction, 3-Lie algebras underlie the BLG model; they are also behind the Basu-Harvey (BH) [42] equation, which is naturally recovered as a BPS condition of the BLG theory (the BH equation was originally given in terms of a GLA four-bracket with a fixed entry; see [15] for n-Lie algebras given in terms of $(n+1)$-multibrackets of GLAs defined by the fully antisymmetrized *associative* products of its entries). It is natural to think of physical applications for larger $k > 3$ algebras. In fact, higher order FAs appeared in suspersymmetric physics before the advent of the BLG model: their FIs may be thought of as generalized Plücker relations [43], and these arise naturally in the classification of maximally supersymmetric solutions of supergravity theories. Thus, from this point of view, 4-Lie algebras are relevant [43] for the maximally supersymmetric backgrounds in IIB supergravity.

Let us go back to the BH equation for M2 branes ending on a M5 brane. This relation may be considered as a generalization of the Nahm equation [44] for D1 branes ending on a D3 brane, which involves an ordinary Lie bracket. The Nahm and the BH BPS equations have, respectively, the schematic form $\dot{X}(s) \sim [X, X]$, $\dot{X}(s) \sim [X, X, X]$, where s is some direction in the D1 or M2 branes along which they extend apart from the D3 and the M5 ones (*e.g.*, the D3 brane is located at s=0). The structure of the Nahm and BH equations immediately suggest moving to a general n-Lie bracket to write [45] $\frac{dX}{ds} \sim [X, \overset{n}{\ldots}, X]$; in fact, all these expressions have the appearance of Maurer-Cartan equations for FAs (see [15]).

To see the effect of a possible n-Lie generalization, let us first recall how the D=11 M2-M5 system, with coordinates

$$\begin{array}{lcccccccc} M2: & 0 & 1 & 2 & & & & & \\ M5: & 0 & 1 & & 3 & 4 & 5 & 6 & , \end{array} \tag{6.54}$$

is described. From the M2 worldvolume point of view, the M5 brane is given by four transverse 3-Lie algebra-valued functions $X^{\mathcal{I}}(s)$, $\mathcal{I} = 3, 4, 5, 6$, where s corresponds to the spatial M2 worldvolume coordinate transverse to the M5 brane (the second one), and which obey the BH equation for a 3-Lie bracket. From the (dual) point of view of the M5 brane, the coordinate s becomes a field depending on the transverse $X^{\mathcal{I}}$ coordinates of the M5 brane, $s = s(X^3, X^4, X^5, X^6)$. In the general case, $\frac{dX}{ds} \sim [X, \overset{n}{\ldots}, X]$, we may think of a generic solution with the behaviour $X(s) \sim \frac{1}{s^{\frac{1}{n-1}}}$, $s \sim \frac{1}{X^{n-1}}$, where the exponent is determined by the number n of entries of the n-bracket. We would expect s, as field, to be a harmonic function in d-dimensional transverse space, for which we would need $n = d-1$ since a harmonic function in d-dimensions depends on the radius as $1/R^{d-2}$. Thus, the Nahm (BH) equations correspond to $n = 2$ (3) since in the D1-D3 (M2-M5) systems the D1 (M2) branes appear, from the point of view of the D3 (M5) ones, as a scalar field with the behaviour $s \sim \frac{1}{R}$ ($s \sim \frac{1}{R^2}$). Thus we may speculate, for *e.g.* D=10, whether other (supersymmetric) brane systems determined by suitable 'brane-boundary rules' [46, 47] could be described by a generalized BH equation involving other n-Lie algebra brackets (we thank Neil Lambert on this point).

Further, there is also the question of moving from n-Lie to the more general n-Leibniz algebras with non fully anticommuting brackets; in particular, n-Leibniz algebras which retain fully skewsymetric fundamental objects appear often as an important subclass (see [15, 48]), as we have also seen in this paper. It turns out here that there is also a BPS relation [49] that is the BH-like equation for a ($\mathcal{N} = 2$)-supersymmetric BLG-type model [23], which uses CS algebras rather than 3-*Lie* ones. Thus, all the above considerations provide a motivation for considering, setting aside their mathematical interest, the various higher order k-Leibniz and, in particular, Lie k-ple algebras introduced here, and raises the issue of their possible applications.

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