arXiv:1212.0303v1 [hep-th] 3 Dec 2012

Black-hole solutions of $N = 2$, $d = 4$ supergravity with a quantum correction, in the H-FGK formalism

Pietro Galli,[†a] *Tomás Ortín,*[◇b] *Jan Perz*[*c] *and Carlos S. Shahbazi*[◇d]

[†]*Departament de Física Teòrica and IFIC (CSIC-UVEG), Universitat de València,*
C/ Dr. Moliner, 50, 46100 Burjassot (València), Spain

[◇]*Instituto de Física Teórica UAM/CSIC*
C/ Nicolás Cabrera, 13–15, 28049 Madrid, Spain

[*]*Istituto Nazionale di Fisica Nucleare, Sezione di Padova*
Via Marzolo, 8, 35131 Padova, Italy

Abstract

We apply the H-FGK formalism to the study of some properties of the general class of black holes in $N = 2$ supergravity in four dimensions that correspond to the harmonic and hyperbolic ansätze and obtain explicit extremal and non-extremal solutions for the t^3 model with and without a quantum correction. Not all solutions of the corrected model (quantum black holes), including in particular a solution with a single q_1 charge, have a regular classical limit.

[a]E-mail: `Pietro.Galli [at] ific.uv.es`
[b]E-mail: `Tomas.Ortin [at] csic.es`
[c]E-mail: `Jan.Perz [at] pd.infn.it`
[d]E-mail: `Carlos.Shabazi [at] uam.es`

Contents

1 Introduction

In [1, 2] a new formalism for constructing single-center, static, spherically-symmetric black-hole solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets was proposed.[1] It is based on rewriting the effective FGK action [8] in terms of a set of functions ("H-variables") of the original dynamical fields, chosen in such a way that they are real and transform linearly under duality. The appropriate choice, which significantly simplifies the equations of motion, can be made with the same algorithm for all supergravity prepotentials and for both extremal and non-extremal black holes. Substituting an ansatz for the H-variables (in [9] taken to be harmonic and hyperbolic for, respectively, extremal and non-extremal solutions[2]) transforms the equations of motion into a system of ordinary equations on the parameters of the ansatz in many examples.

This new formalism should simplify considerably the construction of new black-hole solutions and their systematic study, as it has been shown in the $N = 2, d = 5$ case [3, 4, 7]. So far, the construction of black-hole solutions demanded the use of a specific ansatz for each type of solution which had to be plugged into the equations of motion and checked in detail with considerable effort and, in general, with meaningful loss of generality, although, eventually, very general ansätze were

---

[1]An analogous formalism exists for $N = 2, d = 5$ supergravity theories [3, 4, 2, 5] and can be extended to black-string solutions as well [6, 7].

[2]The same ansatz has been exploited also in five dimensions [3, 4, 10, 2].

proposed. The supersymmetric solutions of ungauged $N = 2, d = 4$ supergravity coupled to vector supermultiplets, which are the only theories we are going to study and discuss here, were constructed in this way in a long series of papers [11, 12, 13, 14, 15, 16]. The effect of the inclusion of R^2 corrections was studied in ref. [17]. The outcome of all this work was a very general recipe that allows the systematic construction of supersymmetric black-hole solutions from harmonic functions. The same general class of solutions was eventually shown to contain regular stationary multicenter black holes [18, 19]. The complete generality of the construction has been proven by the use of supersymmetry methods in [20]. For extremal non-supersymmetric black holes [21] no completely general construction procedure is known, although some general solutions of families of theories have been found such as for the almost-BPS ones [22, 23], those of the cubic models [24] which originate from more particular examples [25, 26, 27, 28] and the interacting non-BPS solutions of ref. [29]. In the non-extremal case, the situation is much worse: only a few examples of general non-extremal black-hole solutions are known [9]. The H-FGK formalism can improve this situation.

Our goals in this paper are similar to those of ref. [7] in the 5-dimensional context: firstly to derive useful model-independent relationships between the quantities appearing in the H-FGK formalism and the physical characteristics of the solutions, in sec. 2, and secondly to use them in sec. 3 for finding explicit examples of black holes in the t^3 model with a quadratic correction to the prepotential, whose string-theoretical origin we recall in appendix A. We restrict ourselves to solutions described by harmonic and hyperbolic functions (for the discussion of generality of these ansätze see refs. [30, 31]). Sec. 4 contains our conclusions.

2   The H-FGK formalism for $N = 2, d = 4$ supergravity

In this section we briefly review the H-FGK formalism for theories of $N = 2, d = 4$ supergravity coupled to n vector multiplets, following [2], whose conventions we use.

As shown in [1, 2], searching for single-center, static, spherically symmetric black-hole solutions of an $N = 2, d = 4$ supergravity coupled to n vector multiplets (and, correspondingly, including n complex scalars Z^i and $n+1$ Abelian vector fields $A^\Lambda{}_\mu$) with electric (q_Λ) and magnetic (p^Λ) charges described by the $2(n + 1)$-dimensional symplectic vector $(\mathcal{Q}^M) \equiv (p^\Lambda, q_\Lambda)^{\mathrm{T}}$ is equivalent to solving the following equations of motion for $2(n + 1)$ dynamical variables that we denote by $H^M(\tau)$ and identify below with a certain combination of physical fields:

$$\left(\partial_M \partial_N \log \mathsf{W} - 2\frac{H_M H_N}{\mathsf{W}^2} \right) \ddot{H}^N + \tfrac{1}{2}\partial_M \partial_N \partial_P \log \mathsf{W} \left(\dot{H}^N \dot{H}^P - \tfrac{1}{2}\mathcal{Q}^N \mathcal{Q}^P \right)$$

$$-4\dot{H}_M \frac{\dot{H}^N H_N}{\mathsf{W}^2} + 8H_M \frac{\dot{H}^P \tilde{H}_P \, \dot{H}^N H_N}{\mathsf{W}^3} + 2\mathcal{Q}_M \frac{H^N \mathcal{Q}_N}{\mathsf{W}^2}$$

$$-4\tilde{H}_M \frac{(H^N \dot{H}_N)^2}{\mathsf{W}^3} - 4\tilde{H}_M \frac{(H^N \mathcal{Q}_N)^2}{\mathsf{W}^3} = 0, \quad (2.1)$$

$$-\tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right) + \left(\frac{\dot{H}^M H_M}{\mathsf{W}} \right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2 = r_0^2. \quad (2.2)$$

In these equations r_0 is the non-extremality parameter, we use the symplectic form $(\Omega_{MN}) \equiv \left(\begin{smallmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{smallmatrix} \right)$ and $\Omega^{MN} = \Omega_{MN}$ to lower and raise the symplectic indices according to the convention

$$H_M = \Omega_{MN} H^N, \qquad H^M = H_N \Omega^{NM}, \quad (2.3)$$

and $\mathsf{W}(H)$ is the *Hesse potential*.[3] For a theory defined by the covariantly holomorphic symplectic section $\mathcal{V}^M$, the Hesse potential can be found as follows: introducing a complex variable X with the same Kähler weight as $\mathcal{V}^M$, we can define the Kähler-neutral real symplectic vectors

$$\mathcal{R}^M = \Re \, \mathcal{V}^M / X \,, \qquad \mathcal{I}^M = \Im \, \mathcal{V}^M / X \,. \tag{2.4}$$

The components $\mathcal{R}^M$ can be expressed in terms of the $\mathcal{I}^M$, to which process we refer later as solving Freudenthal duality equations.[4] Then, the Hesse potential, as a function of the components $\mathcal{I}^M$ is given by

$$\mathsf{W}(\mathcal{I}) \equiv \langle \, \mathcal{R}(\mathcal{I}) \, | \, \mathcal{I} \, \rangle \equiv \mathcal{R}_M(\mathcal{I})\mathcal{I}^M \,, \tag{2.5}$$

and identifying $\mathcal{I}^M = H^M$ we get $\mathsf{W}(H)$. We can use $\mathcal{R}^M$ to define dual variables:

$$\tilde{H}^M(H) \equiv \mathcal{R}^M(H) \,. \tag{2.6}$$

Given a solution $H^M(\tau)$ of the equations (2.1) and (2.2), the warp factor e^{2U} of the spacetime metric

$$ds^2 = e^{2U}dt^2 - e^{-2U} \left(\frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega_{(2)}^2 \right) \,, \tag{2.7}$$

takes the form

$$e^{-2U} = \mathsf{W}(H) \tag{2.8}$$

and the scalar fields are given by

$$Z^i = \frac{\tilde{H}^i + iH^i}{\tilde{H}^0 + iH^0} \,. \tag{2.9}$$

The equations of motion (2.1) can be derived from the effective action

$$-I_{\text{eff}}[H] = \int d\tau \left[\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N + \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right) - \left(\frac{\dot{H}^M H_M}{\mathsf{W}} \right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2 \right] \,. \tag{2.10}$$

Then, eq. (2.2) is nothing but the Hamiltonian constraint associated with the τ-independence of the action, with a particular value of the integration constant, which we cannot change because it is part of the transverse metric ansatz.

If we contract the equations of motion (2.1) with H^P and use the homogeneity properties of the different terms and the Hamiltonian constraint eq. (2.2), we find a useful equation

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) + \frac{(\dot{H}^M H_M)^2}{\mathsf{W}} = 0 \,, \tag{2.11}$$

which corresponds to that of the variable U minus the Hamiltonian constraint in the standard formulation.[5]

In what follows we shall impose on the variables H^M the constraint

$$\dot{H}^M H_M = 0 \,. \tag{2.12}$$

In the supersymmetric (hence, extremal) case it has been shown [33] that this constraint enforces the absence of NUT charge: a non-zero NUT charge would lead to a non-static metric with string-like

---

[3]For a historical perspective on the real formulation of special Kähler geometry and the Hesse potential see e.g. [1, 32].
[4]In earlier papers sometimes called "stabilization equations".
[5]This equation in the extremal limit agrees with the special static case of eq. (3.31) of ref. [28].

singularities. Here, this condition is nothing but a possible simplifying assumption which does not imply non-staticity since staticity has been assumed in this formalism form the onset. Here we take it as a convenient ansatz and leave the possibility and implications of violating this constraint to be studied elsewhere [30, 31].

The above constraint simplifies eq. (2.11)

$$\tilde{H}_M \left(\ddot{H}^M - r_0^2 H^M \right) = 0 \,, \tag{2.13}$$

which can be solved by harmonic (in the extremal $r_0 = 0$ case) or hyperbolic (in the non-extremal $r_0 \neq 0$ case) ansätze for the variables H^M, satisfying

$$\ddot{H}^M - r_0^2 H^M = 0 \,. \tag{2.14}$$

These are the ansätze that we will use in the rest of the paper, bearing in mind that they are adapted to the additional constraint (2.12) that we impose by hand. Taking into account this constraint, the equations that need to be solved are:

$$\partial_P \partial_M \log \mathsf{W} \, \ddot{H}^M + \tfrac{1}{2} \partial_P \partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N - \tfrac{1}{2} \mathcal{Q}^M \mathcal{Q}^N \right) + \partial_P \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2 = 0 \,, \tag{2.15}$$

$$-\tfrac{1}{2} \partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N - \tfrac{1}{2} \mathcal{Q}^M \mathcal{Q}^N \right) - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}} \right)^2 = r_0^2 \,, \tag{2.16}$$

$$\dot{H}^M H_M = 0 \,. \tag{2.17}$$

It is also useful to have the expression of the black-hole potential as a zeroth-degree homogeneous function of the variables H^M:

$$- V_{\mathrm{bh}}(H, \mathcal{Q}) = -\tfrac{1}{4} \mathsf{W} \left(\partial_M \partial_N \log \mathsf{W} - 4 \mathsf{W}^{-2} H_M H_N \right) \mathcal{Q}^M \mathcal{Q}^N \,. \tag{2.18}$$

2.1 Extremal black holes

As explained above, for extremal black holes we take $H^M(\tau)$ to be harmonic in Euclidean $\mathbb{R}^3$, i.e. linear in τ:[6]

$$H^M = A^M - \tfrac{1}{\sqrt{2}} B^M \tau \,, \tag{2.19}$$

where A^M and B^M are integration constants to be determined as functions of the independent physical constants (namely, the charges $\mathcal{Q}^M$ and the values of the scalars at spatial infinity Z_∞^i) by using the equations of motion (2.15)–(2.17) and the asymptotic conditions.

---

[6]Known non-supersymmetric extremal solutions that do not conform to this ansatz do not satisfy constraint (2.12) either [28, 30]. On the other hand, the representation of a solution in terms of the H^M may not be unique and the harmonicity or the fact that the constraint eq. (2.17) is satisfied may not always be a characteristic feature of a solution [31].

The equations of motion for the above ansatz can be written in a simple and suggestive form[7]

$$\partial_P \left[V_{\text{bh}}(H, \mathcal{Q}) - V_{\text{bh}}(H, B) \right] = 0 \,, \tag{2.20}$$

$$V_{\text{bh}}(H, \mathcal{Q}) - V_{\text{bh}}(H, B) = 0 \,, \tag{2.21}$$

$$A^M B_M = 0 \,. \tag{2.22}$$

Observe that the first two equations are automatically solved for $B^M = \mathcal{Q}^M$, which corresponds to the supersymmetric case. The third equation then takes the form $A^M \mathcal{Q}_M$ and still has to be solved, which can be done generically [18, 19] as we are going to show.

Furthermore, observe that the Hamiltonian constraint (2.21) is equivalent to the requirement that the black-hole potential *evaluated on the solutions* has the same form in terms of the *fake central charge* which we can define for any symplectic (*fake* or not *fake*) charge vector B^M by

$$\tilde{\mathcal{Z}}(Z, Z^*, B) \equiv \langle \, \mathcal{V} \mid B \, \rangle \tag{2.23}$$

as in terms of the actual central charge $\mathcal{Z}(Z, Z^*, Q) \equiv \langle \, \mathcal{V} \mid Q \, \rangle = \tilde{\mathcal{Z}}(Z, Z^*, \mathcal{Q})$, that is

$$- V_{\text{bh}}(Z, Z^*, \mathcal{Q}) = |\tilde{\mathcal{Z}}|^2 + \mathcal{G}^{ij^*} \mathcal{D}_i \tilde{\mathcal{Z}} \, \mathcal{D}_{j^*} \tilde{\mathcal{Z}}^* \,. \tag{2.24}$$

The asymptotic conditions take the form

$$\mathsf{W}(A) = 1 \,, \tag{2.25}$$

$$Z^i_\infty = \frac{\tilde{H}^i(A) + iA^i}{\tilde{H}^0(A) + iA^0} \,, \tag{2.26}$$

but can always be solved, together with (2.22), as follows: if we write X as

$$X = \tfrac{1}{\sqrt{2}} e^{U + i\alpha} \,, \tag{2.27}$$

then, from the definition (2.4) of $\mathcal{I}^M$ we get

$$H^M = \sqrt{2} e^{-U} \, \Im\mathfrak{m}(e^{-i\alpha} \mathcal{V}^M) \,, \tag{2.28}$$

and, at spatial infinity $\tau = 0$, using asymptotic flatness (2.25)

$$A^M = \sqrt{2} \, \Im\mathfrak{m}(e^{-i\alpha_\infty} \mathcal{V}^M_\infty) \,. \tag{2.29}$$

Now, to determine α_∞ we can use (2.22) and the definition of fake central charge (2.23). Observe that

$$A_M B^M = \langle \, H \mid B \, \rangle = \Im\mathfrak{m}\langle \, \mathcal{V}/X \mid B \, \rangle = \Im\mathfrak{m}(\tilde{\mathcal{Z}}/X) = \sqrt{2} e^{-U} \Im\mathfrak{m}(e^{-i\alpha} \tilde{\mathcal{Z}}) = 0 \,, \tag{2.30}$$

from which one first obtains the relation

$$e^{i\alpha} = \pm \tilde{\mathcal{Z}}/|\tilde{\mathcal{Z}}| \tag{2.31}$$

---

[7] It is worth stressing that, even though the first equation is the derivative of the second with respect to H^P, solving the second for some functions H^M does not imply having solved the first. Only if we find a B^M such that the second equation is satisfied identically for any H^M will the first equation be satisfied as well. The number of B^M with this property and their value depend on the particular theory under consideration, but their existence is quite a general phenomenon.

and then the general expression for the A^M as a function of the B^M and the Z^i_∞:

$$A^M = \pm\sqrt{2}\,\Im\mathfrak{m}\left(\frac{\tilde{\mathcal{Z}}^*_\infty}{|\tilde{\mathcal{Z}}_\infty|}\mathcal{V}^M_\infty\right). \tag{2.32}$$

The sign of A^M should be chosen to make H^M finite (and, generically, the metric non-singular) in the range $\tau \in (-\infty, 0)$. The positivity of the mass is a physical condition that eliminates some singularities of the metric. As we shall see in eq. (2.40), this requirement singles out the upper sign in the above formula.

Having reduced the problem of finding a complete solution to the determination of the constants B^M that must satisfy equations (2.20) and (2.21) as functions of the physical parameters $\mathcal{Q}^M, Z^i_\infty$, it is useful to analyze the near-horizon and spatial-infinity limits of these two equations. The near-horizon limit of (2.21) plus the definition of the fake central charge lead to the following chain of relations[8]

$$S/\pi = \tfrac{1}{2}\mathsf{W}(B) = -V_{\mathrm{bh}}(B, \mathcal{Q}) = |\tilde{\mathcal{Z}}(B, B)|^2, \tag{2.33}$$

where S is the Bekenstein–Hawking black hole entropy and $\tilde{\mathcal{Z}}(B, B)$ is the near-horizon value of the fake central charge. The last of these relations, together with the condition (2.24) imply that, on the horizon, the fake central charge reaches an extremum

$$\partial_i|\tilde{\mathcal{Z}}(Z_{\mathrm{h}}, Z^*_{\mathrm{h}}, B)| = 0. \tag{2.34}$$

The near-horizon limit of (2.20) leads to

$$\partial_M V_{\mathrm{bh}}(B, \mathcal{Q}) = 0, \tag{2.35}$$

which says that the B^M extremize the value of the black-hole potential on the horizon. Since the black-hole potential is invariant under a global rescaling of the H^M, the solutions (that we generically call *attractors* B^M) of these equations are determined up to a global rescaling, which can be fixed by imposing eq. (2.21).

The B^M must transform under the duality group of the theory (embedded in $Sp(2n + 2, \mathbb{R})$) in the same representation as the H^M, the charges $\mathcal{Q}^M$ and the constants A^M. In certain cases this poses strong constraints on the possible solutions, since building from $\mathcal{Q}^M$ and Z^i_∞ an object that transforms in the right representation of the duality group and has dimensions of length squared may be far from trivial. A possibility that is always available is the Freudenthal dual defined in ref. [34], generalizing the definition made in ref. [35]. Freudenthal duality in $N = 2, d = 4$ theories can be understood as the transformation from the H^M to the $\tilde{H}_M(H)$ variables. The same transformation can be applied to any symplectic vector, such as the charge vector. Then, in our notation and conventions, the Freudenthal dual of the charge vector, $\tilde{\mathcal{Q}}_M$, is defined by

$$\tilde{\mathcal{Q}}_M = \frac{1}{2}\frac{\partial\mathsf{W}(\mathcal{Q})}{\partial\mathcal{Q}^M}. \tag{2.36}$$

It is not difficult to prove that this duality transformation is an antiinvolution

$$\tilde{\tilde{\mathcal{Q}}}_M = -\mathcal{Q}_M, \tag{2.37}$$

---

[8]In this and other equations, the expression $V_{\mathrm{bh}}(B, \mathcal{Q})$ stands for the standard black-hole potential with the functions $H^M(\tau)$ replaced by the constants B^M.

and using eq. (2.5) to show that

$$W(\tilde{\mathcal{Q}}) = W(\mathcal{Q}).$$ (2.38)

With more effort one can also show that the critical points of the black-hole potential are invariant under Freudenthal duality [34]. Therefore, as $B^M = \mathcal{Q}^M$ is always an attractor (the supersymmetric one),

$$B^M = \tilde{\mathcal{Q}}^M$$ (2.39)

will always be another attractor.

Let us now consider the spatial-infinity limit, taking into account the definition of the mass in these spacetimes and the definition of the fake central charge

$$M = \dot{U}(0) = \tfrac{1}{\sqrt{2}} \langle\, \tilde{A} \mid B \,\rangle = \pm |\tilde{\mathcal{Z}}(A,B)|.$$ (2.40)

As mentioned before, to have a positive mass we must use exclusively the upper sign in (2.31) and (2.32) and we do so from now onwards. In the supersymmetric case, when $B^M = \mathcal{Q}^M$ and the fake central charge becomes the true one, this is the supersymmetric BPS relation.

The asymptotic limit of (2.21) plus (2.24) and the above relation give

$$M^2 + \left[\mathcal{G}^{ij^*} \mathcal{D}_i \tilde{\mathcal{Z}} \, \mathcal{D}_{j^*} \tilde{\mathcal{Z}}^* \right]_\infty + V_{\mathrm{bh}\,\infty} = 0,$$ (2.41)

which, when compared with the general BPS bound [8], leads to the identification of the scalar charges Σ_i with the values of the covariant derivatives of the fake central charges at spatial infinity

$$\Sigma_i = \mathcal{D}_i \tilde{\mathcal{Z}} \Big|_\infty.$$ (2.42)

2.1.1 First-order flow equations

First-order flow equations for extremal BPS and non-BPS black holes can be easily found following [36] but using the generic harmonic functions (2.19): let us consider the Kähler-covariant derivative of the inverse of the auxiliary function

$$
\begin{aligned}
\mathcal{D}X^{-1} &= i\langle \mathcal{V} \mid \mathcal{V}^* \rangle \mathcal{D}X^{-1} = i\langle \mathcal{D}(\mathcal{V}/X) \mid \mathcal{V}^* \rangle = i\langle d(\mathcal{V}/X) \mid \mathcal{V}^* \rangle \\[2mm]
&= i\langle d(\mathcal{V}/X) - d(\mathcal{V}/X)^* \mid \mathcal{V}^* \rangle = -2\langle dH \mid \mathcal{V}^* \rangle \\[2mm]
&= -\sqrt{2}\, \tilde{\mathcal{Z}}^*(Z,Z^*,B)\, d\tau,
\end{aligned}
$$ (2.43)

where we have used the normalization of the symplectic section in the first step, the property $\langle\, \mathcal{D}\mathcal{V} \mid \mathcal{V}^* \,\rangle = 0$ in the second, the Kähler-neutrality of $\mathcal{V}/X$ in the third, $\langle\, \mathcal{D}\mathcal{V}^* \mid \mathcal{V}^* \,\rangle = \langle\, \mathcal{V}^* \mid \mathcal{V}^* \,\rangle = 0$ in the fourth, the definition of $\mathcal{I} = H$ in the fifth, and the ansatz (2.19) and the definition of the fake central charge (2.23) in the sixth.

From this equation, eqs. (2.27) and (2.31) and the relation (cf. eqs. (3.8), (3.28) in ref. [28])

$$\dot{\alpha} = -\mathcal{Q}_\star, \quad \text{where} \quad \mathcal{Q}_\star = \tfrac{1}{2i}\dot{Z}^i \partial_i \mathcal{K} + \text{c.c.}$$ (2.44)

is the pullback of the Kähler connection 1-form, we find the standard first-order equation for the metric function U:

$$\frac{de^{-U}}{d\tau} = -|\tilde{\mathcal{Z}}(Z,Z^*,B)|.$$ (2.45)

8

Let us now consider the differential of the complex scalar fields:

$$
\begin{aligned}
dZ^i &= i\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k\mathcal{V}\rangle dZ^k = iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{D}_k(\mathcal{V}/X)\rangle dZ^k \\
&= iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid \partial_k(\mathcal{V}/X)\rangle dZ^k = iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X)\rangle \\
&= iX\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid d(\mathcal{V}/X) - d(\mathcal{V}/X)^*\rangle = -2X\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid dH\rangle \\
&= +\sqrt{2}\,X\mathcal{G}^{ij^*}\langle \mathcal{D}_{j^*}\mathcal{V}^* \mid B\rangle d\tau = \sqrt{2}\,X\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^*(Z,Z^*,B)\,d\tau\,,
\end{aligned}
\tag{2.46}
$$

where we have used the same properties as before. To put this expression in a more conventional form we can use the covariant holomorphicity of $\tilde{\mathcal{Z}}$ writing

$$
\mathcal{D}_{j^*}\tilde{\mathcal{Z}}^* = \mathcal{D}_{j^*}\frac{|\tilde{\mathcal{Z}}|^2}{\tilde{\mathcal{Z}}} = \frac{2|\tilde{\mathcal{Z}}|\partial_{j^*}|\tilde{\mathcal{Z}}|}{\tilde{\mathcal{Z}}} = 2e^{-i\alpha}\partial_{j^*}|\tilde{\mathcal{Z}}|\,,
\tag{2.47}
$$

and plugging this result in the expression above:

$$
\frac{dZ^i}{d\tau} = 2e^U\mathcal{G}^{ij^*}\partial_{j^*}|\tilde{\mathcal{Z}}|\,.
\tag{2.48}
$$

It is easy to check that these first order equations imply the second-order equations of motion

$$
\ddot{U} + e^{2U}V_{\text{bh}}(Z,Z^*,B) = 0\,,
\tag{2.49}
$$

$$
\ddot{Z}^i + \Gamma_{jk}{}^i\dot{Z}^j\dot{Z}^k + e^{2U}\partial^i V_{\text{bh}}(Z,Z^*,B) = 0\,,
\tag{2.50}
$$

with $\Gamma_{jk}{}^i = \mathcal{G}^{il^*}\partial_j\mathcal{G}_{kl^*}$, which coincide with the original ones if

$$
V_{\text{bh}}(Z,Z^*,B) = V_{\text{bh}}(Z,Z^*,\mathcal{Q})
\tag{2.51}
$$

for any Z^i (not just for the solution; see the remark in footnote 7).

2.2 Non-extremal black holes

Previous experience [9] (see also [1] and, further, [10, 7] for 5-dimensional examples) suggests that a suitable ansatz for the variables H^M for non-extremal black holes of $N = 2, d = 4$ supergravity, compatible with the constraint (2.12), is

$$
H^M(\tau) = A^M\cosh(r_0\tau) + \frac{B^M}{r_0}\sinh(r_0\tau)\,,
\tag{2.52}
$$

for some integration constants A^M and B^M that, as in the extremal case, have to be determined by solving the equations of motion and by imposing the standard normalization of the physical fields at spatial infinity.

Using this ansatz, the equations of motion (2.15)–(2.17) take the form

$$
\tfrac{1}{2}\partial_P\partial_M\partial_N\log\mathsf{W}\left(B^M B^N - r_0^2 A^M A^N\right) - \partial_P\left[V_{\text{bh}}(Z,Z^*,\mathcal{Q})/\mathsf{W}\right] = 0\,,
\tag{2.53}
$$

$$
-\tfrac{1}{2}\partial_M\partial_N\log\mathsf{W}\left(B^M B^N - r_0^2 A^M A^N\right) - V_{\text{bh}}(Z,Z^*,\mathcal{Q})/\mathsf{W} = 0\,,
\tag{2.54}
$$

$$
A^M B_M = 0\,,
\tag{2.55}
$$

9

where we have used the third equation and the homogeneity properties of the Hesse potential W in order to simplify the first two.

In the non-extremal case we can define several fake central charges:

$$\tilde{\mathcal{Z}}(Z, Z^*, B) \equiv \langle \mathcal{V} \mid B \rangle , \qquad \tilde{\mathcal{Z}}(Z, Z^*, B_\pm) \equiv \langle \mathcal{V} \mid B_\pm \rangle , \tag{2.56}$$

with the shifted coefficients

$$B_\pm^M \equiv \lim_{\tau \to \mp \infty} \frac{r_0 H^M(\tau)}{\sinh(r_0 \tau)} = B^M \mp r_0 A^M . \tag{2.57}$$

Imposing the same asymptotic conditions on the fields as in the extremal case and the condition (2.55), we arrive again at (2.32). Left to be determined from the equations of motion are then only the constants B^M and the non-extremality parameter r_0.

The mass is given again by eq. (2.40) and the expressions for the event horizon area $(+)$ and the Cauchy horizon area $(-)$ are

$$\frac{A_{\text{h}\pm}}{4\pi} = \mathsf{W}(B_\pm) . \tag{2.58}$$

In the near-horizon limit, the equations of motion, upon use of the above formulae for the area of the event horizon, lead to the following relations

$$\frac{A_{\text{h}\pm}}{4\pi} = -V_{\text{bh}}(B_\pm) \pm 2r_0 \mathcal{M}_{MN}[\mathcal{F}(B_\pm)] A^M B_\pm^N = \mathsf{W}(B_\pm) , \tag{2.59}$$

$$\partial_P V_{\text{bh}}(B_\pm) = \pm 2r_0 \partial_P \mathcal{M}_{MN}[\mathcal{F}(B)] A^M B_\pm^N = -2r_0^2 \partial_P \mathcal{M}_{MN}[\mathcal{F}(B)] A^M A^N , \tag{2.60}$$

which generalize eqs. (2.33) and (2.35) to the non-extremal case. In the last relation we have used the identity

$$H^M \partial_P \mathcal{M}_{MN}(\mathcal{F}) = 0 . \tag{2.61}$$

The right-hand side of eq. (2.60) vanishes if $A^M \propto B^M$. This is a special case that we study in section 2.2.2. Another possibility is that $\mathcal{F}_{\Lambda\Sigma}$ and hence also $\mathcal{M}_{MN}(\mathcal{F})$ are constant, as happens in quadratic models. In general, however, $\partial_P V_{\text{bh}}(B_\pm) \neq 0$ and we conclude that the values of the scalars on the horizon of a non-extremal black hole do not necessarily extremize the black-hole potential.

2.2.1 First-order flow equations

The derivation carried out for extremal black holes in section 2.1.1 can be straightforwardly extended to the non-extremal case. As in the 5-dimensional case studied in ref. [7], one defines a new coordinate ρ and a function $f(\rho)$

$$\rho \equiv \frac{\sinh(r_0\tau)}{r_0 \cosh(r_0\tau)} , \qquad f(\rho) \equiv \frac{1}{\sqrt{1 - r_0^2 \rho^2}} = \cosh(r_0\tau) , \tag{2.62}$$

so that the hyperbolic ansatz (2.52) for H^M can be rewritten in the "almost extremal form":

$$H^M = f(\rho)(A^M + B^M \rho) \equiv f(\rho)\hat{H}^M . \tag{2.63}$$

Then, following the same steps that led to eqs. (2.45) and (2.65), one can obtain the first-order flow equations:

$$\frac{de^{-\hat{U}}}{d\rho} = \sqrt{2}|\tilde{\mathcal{Z}}(Z, Z^*, B)|,\tag{2.64}$$

$$\frac{dZ^i}{d\rho} = -2\sqrt{2}\, e^{\hat{U}} \mathcal{G}^{ij^*} \partial_{j^*} |\tilde{\mathcal{Z}}(Z, Z^*, B)|,\tag{2.65}$$

where we have introduced the hatted warp factor $\hat{U} = U + \log f$.

Similarly to the extremal case, it is not difficult to show that this first-order flow implies the second-order equations:

$$\frac{d^2\hat{U}}{d\rho^2} + e^{2\hat{U}} V_{\mathrm{bh}}(Z, Z^*, \sqrt{2}B) = 0,\tag{2.66}$$

$$\frac{d^2 Z^i}{d\rho^2} + \Gamma_{kl}{}^i \frac{dZ^k}{d\rho} \frac{dZ^l}{d\rho} + e^{2\hat{U}} \mathcal{G}^{ij^*} \partial_{j^*} V_{\mathrm{bh}}(Z, Z^*, \sqrt{2}B) = 0,\tag{2.67}$$

plus the constraint[9]

$$\left(\frac{d\hat{U}}{d\rho}\right)^2 + \mathcal{G}_{ij^*} \frac{dZ^i}{d\rho} \frac{dZ^{*j^*}}{d\rho} + e^{2\hat{U}} V_{\mathrm{bh}}(Z, Z^*, \sqrt{2}B) = 0,\tag{2.68}$$

but now with respect to the new variable ρ and the new function $\hat{U}$.

In order to compare these equations with the actual second-order equations for the warp factor and the scalars we have to rewrite them in terms of the variable τ and rescale $\hat{U}$ to U. For the former, by using $d/d\rho = f^2 d/d\tau$ and eq. (2.64), one finds:

$$\ddot{U} - \frac{2\sqrt{2}\rho}{f} e^U |\mathcal{Z}(Z, Z^*, \sqrt{2}B)| + \frac{r_0^2}{f^2} + \frac{e^{2U}}{f^2} V_{\mathrm{bh}}(Z, Z^*, \sqrt{2}B),\tag{2.69}$$

from which follows the relation between the true and the fake black hole potential that must hold for the above second-order equations to imply the equations of motion:

$$e^{2U} V_{\mathrm{bh}}(Z, Z^*, \mathcal{Q}) = \frac{e^{2U}}{f^2} V_{\mathrm{bh}}(Z, Z^*, \sqrt{2}B) - \frac{2\sqrt{2}r_0^2\rho}{f} e^U |\mathcal{Z}(Z, Z^*, \sqrt{2}B)| + \frac{r_0^2}{f^2}.\tag{2.70}$$

The same condition ensures that the constraint eq. (2.68) implies the standard Hamiltonian constraint. For the scalar equations we find the condition

$$\partial_i \left(e^{2U} V_{\mathrm{bh}}(Z, Z^*, \mathcal{Q}) - \frac{e^{2U}}{f^2} V_{\mathrm{bh}}(Z, Z^*, \sqrt{2}B) + \frac{4\sqrt{2}r_0^2\rho}{f} e^U |\mathcal{Z}(Z, Z^*, \sqrt{2}B)| \right) = 0.\tag{2.71}$$

No other conditions need to be satisfied for the first-order equations to imply all the second-order equations of motion. Taking the derivative with respect to ρ of eq. (2.70) we find that, if this relation is satisfied for any Z^i (or any H^M), then the last equation is also satisfied, as are all the second-order equations.

Evaluating eq. (2.70) at spatial infinity ($\tau = 0$, which corresponds to $\rho = 0$) we find the following relation between the charges, the fake charges, the asymptotic values of the moduli and the non-extremality parameter:

$$V_{\mathrm{bh}}(Z_\infty, Z_\infty^*, \mathcal{Q}) - V_{\mathrm{bh}}(Z_\infty, Z_\infty^*, \sqrt{2}B) = r_0^2.\tag{2.72}$$

---

[9]Observe that the right-hand side of this equation is not r_0^2.

2.2.2 Non-extremal generalization of doubly-extremal black holes

For non-extremal black holes whose scalars are constant over the whole spacetime, it is possible to solve the equations of motion of the H-FGK system with the hyperbolic ansatz (2.52) in a model-independent way, i.e. for any theory of $N = 2, d = 4$ supergravity. Given the constancy of the scalars we assume

$$Z^i_\infty = Z^i_{\mathrm{h}}, \qquad (2.73)$$

which requires

$$B^M \propto A^M, \qquad (2.74)$$

where the constants A^M are given by eq. (2.32).

Using the proportionality of the B^M and A^M in the $\tau \to 0^-$ or $\tau \to \pm\infty$ limit of eq. (2.53) we get

$$\partial_K V_{\mathrm{bh}}(Z_\infty, Z^*_\infty, \mathcal{Q}) = 0, \qquad (2.75)$$

which proves that the scalars must assume attractor values $Z^i_\infty = Z^i_{\mathrm{att}}$ that are a stationary point of the black hole potential, just as in the extremal case. We can thus use eq. (2.33), which gives the value of the black-hole potential at the horizons in terms of the fake central charge there $\tilde{\mathcal{Z}}(B, B)$ (not $\tilde{\mathcal{Z}}(Z, Z^*, B_\pm)$):

$$- V_{\mathrm{bh}}(Z_\infty, Z^*_\infty, \mathcal{Q}) = |\tilde{\mathcal{Z}}(B, B)|^2. \qquad (2.76)$$

The proportionality constant between B^M and A^M is easily determined to be $-\mathsf{W}^{1/2}(B)$ by using the normalization at infinity $\mathsf{W}(A) = 1$ and choosing the sign so as to make the functions $H^M \neq 0$ for $\tau \in (-\infty, 0)$. Then we can write

$$H^M(\tau) = A^M \left(\cosh (r_0\tau) - \mathsf{W}^{1/2}(B)\frac{\sinh (r_0\tau)}{r_0} \right). \qquad (2.77)$$

The values of $B^M_\pm$ are

$$B^M_\pm = - \left(\mathsf{W}^{1/2}(B) \pm r_0 \right) A^M, \qquad (2.78)$$

and

$$\mathsf{W}(B_\pm) = \left(\mathsf{W}^{1/2}(B) \pm r_0 \right)^2. \qquad (2.79)$$

A relation between the value of $\mathsf{W}^{1/2}(B)$ and physical parameters and r_0 can be found by taking the $\tau \to 0^-$ limit of eq. (2.54):

$$\mathsf{W}(B) = r_0^2 - V_{\mathrm{bh}}(Z_\infty, Z^*_\infty, \mathcal{Q}). \qquad (2.80)$$

Another relation comes from the definition of mass $M = \dot{U}(0)$, which gives $M = -\tilde{H}_M(A)B^M$. Using the proportionality between A^M and B^M we find that

$$M = \mathsf{W}^{1/2}(B). \qquad (2.81)$$

The final expression for the functions $H^M(\tau)$ is, regardless of the details of the model:

$$H^M(\tau) = A^M \left(\cosh (r_0\tau) - M\frac{\sinh (r_0\tau)}{r_0} \right), \qquad (2.82)$$

$$S_\pm = \pi \left(M \pm r_0 \right)^2, \qquad (2.83)$$

where the non-extremality parameter, upon use of eq. (2.76), is given by

$$r_0 = \sqrt{M^2 - |\tilde{\mathcal{Z}}(B, B)|^2}. \qquad (2.84)$$

12

3 One-modulus quantum-corrected geometries

We shall now use the formalism developed in the last section to explore the black-hole solutions of one-modulus quantum-corrected models that typically appear as one-modulus Calabi–Yau compactification of type II string theory. For one-modulus models of this kind the perturbative prepotential $\mathcal{F}_{\text{pert}}$ can be brought to the form:

$$\mathcal{F}_{\text{IIA}}^{\text{pert}} = -\frac{\kappa_{1,1,1}^0}{6} \frac{(\hat{\mathcal{X}}^1)^3}{\hat{\mathcal{X}}^0} - \frac{i}{2} c (\hat{\mathcal{X}}^0)^2 , \tag{3.1}$$

where the *correction* is encoded in the model-dependent positive constant c, $\kappa_{1,1,1}^0$ is the triple intersection number and the hat indicates that we are working in a possibly rotated (by a symplectic matrix) frame of the homogeneous coordinates $\{\mathcal{X}^0, \mathcal{X}^i\}$ of the moduli space. In what follows we take the explicit example of the type IIA superstring compactified on the quintic Calabi–Yau manifold ($\kappa_{1,1,1}^0 = 5$), which we review in the appendix.

For the sake of simplicity and in order to be able to make a comparison, in the following we first study the *uncorrected* model corresponding to the prepotential $\mathcal{F}_{\text{IIA}}^0 \equiv \mathcal{F}_{\text{IIA}}^{\text{pert}}(c = 0)$ and only afterwards the general case of eq. (3.1).

3.1 Uncorrected case: the t^3 model

In this section we consider the tree-level prepotential:

$$\mathcal{F}_{\text{pert}}^0(\mathcal{X}) = -\frac{5}{6} \frac{(\mathcal{X}^1)^3}{\mathcal{X}^0} . \tag{3.2}$$

In terms of the coordinate $t = \mathcal{X}^1/\mathcal{X}^0$ the Kähler potential and metric are given by:

$$e^{-\mathcal{K}^0} = \tfrac{20}{3}(\Im\mathfrak{m}\, t)^3 , \qquad \mathcal{G}_{tt^*}^0 = \tfrac{3}{4}(\Im\mathfrak{m}\, t)^{-2} , \tag{3.3}$$

whereas the covariantly holomorphic symplectic section is

$$\mathcal{V}^0(t, t^*) = e^{\mathcal{K}^0/2} \begin{pmatrix} 1 \\ t \\ \frac{5}{6}t^3 \\ -\frac{5}{2}t^2 \end{pmatrix} \tag{3.4}$$

and the central charge, its covariant derivative, the black-hole potential and its partial derivative read:

$$\mathcal{Z} \equiv e^{\mathcal{K}^0/2} \hat{\mathcal{Z}} , \tag{3.5}$$

$$\mathcal{D}_t \mathcal{Z} = \frac{i}{2} \frac{e^{\mathcal{K}^0/2}}{\Im\mathfrak{m}\, t} \hat{\mathcal{W}} , \tag{3.6}$$

$$-V_{\text{bh}} = e^{\mathcal{K}^0} \left(|\hat{\mathcal{Z}}|^2 + \tfrac{1}{3}|\hat{\mathcal{W}}|^2 \right) , \tag{3.7}$$

$$-\partial_t V_{\text{bh}} = \tfrac{i}{20}(\Im\mathfrak{m}\, t)^{-4} \left((\hat{\mathcal{W}}^*)^2 + 3\hat{\mathcal{W}}\hat{\mathcal{Z}}^* \right) . \tag{3.8}$$

13

In the above:

$$\hat{\mathcal{Z}} = \tfrac{5}{6}p^0 t^3 - \tfrac{5}{2}p^1 t^2 - q_1 t - q_0 \,, \tag{3.9}$$

$$\hat{\mathcal{W}} = \tfrac{5}{2}p^0 t^2 t^* - \tfrac{5}{2}p^1 t(t + 2t^*) - q^1(2t + t^*) - 3q^0 \,. \tag{3.10}$$

Notice all these objects are well defined only for $\Im\mathfrak{m}\, t > 0$. Furthermore, it must be taken into account that the theory given by the tree-level prepotential is a good approximation to the full theory only when $|t| \gg 1$.

3.1.1 Extremal solutions

Extremal solutions are associated with the critical points of the black-hole potential. Following from eqs. (3.6) and (3.8), there are two kinds of critical points:

1. Supersymmetric, when

$$\hat{\mathcal{W}} = 0 \,. \tag{3.11}$$

   For generic (non-vanishing) values of the charges, there exist three complex solutions for the critical values t_{att}, but at most two can be physical ($\Im\mathfrak{m}\, t > 0$). Their expressions are complicated and will be recovered below by taking the appropriate limits in the solutions.

2. Non-supersymmetric [37, 38], when $\hat{\mathcal{W}} \neq 0$ and

$$3\hat{\mathcal{Z}}\hat{\mathcal{W}}^* + \hat{\mathcal{W}}^2 = 0 \,. \tag{3.12}$$

The extremal BPS solutions can be constructed by the procedure explained in section 2.1. The Freudenthal duality equations can be solved in a general way [39] and the metric function and scalar field read:

$$
\begin{aligned}
e^{-2U} &= \mathsf{W}(H) = \tfrac{2}{\sqrt{3}}\sqrt{\tfrac{8}{15}H^0(H_1)^3 + (H^1 H_1)^2 - 3(H^0 H_0)^2 - 6H^0 H_0 H^1 H_1 - 10(H^1)^3 H_0} \,, \\
t &= -\frac{3H^0 H_0 + H^1 H_1}{5(H^1)^2 + 2H^0 H_1} + i\frac{3e^{-2U}}{2\left[5(H^1)^2 + 2H^0 H_1\right]} \,.
\end{aligned}
\tag{3.13}
$$

The harmonic functions $(H^M) = (H^0, H^1, H_0, H_1)$ are given by eq. (2.19) with $B^M = \mathcal{Q}^M$ and the A^M are given by eq. (2.32) (with the upper sign), where now the asymptotic values of the symplectic section (3.4) and the central charge (3.5) have to be used. This guarantees the absence of NUT charge (necessary for the consistency of the solution) and the correct asymptotic behavior of the above fields: $e^{-2U(0)} = 1$, $t(0) = t_\infty$.

On the horizon, the values taken by these fields can be found by replacing the harmonic functions H^M by $-\mathcal{Q}^M/\sqrt{2}$, that is

$$
\begin{aligned}
S_{\text{e}}/\pi &= \tfrac{1}{2}\mathsf{W}(\mathcal{Q}) = \tfrac{1}{\sqrt{3}}\sqrt{\tfrac{8}{15}p^0(q_1)^3 + (p^1 q_1)^2 - 3(p^0 q_0)^2 - 6p^0 q_0 p^1 q_1 - 10(p^1)^3 q_0} \,, \\
t_{\text{att}} &= -\frac{3p^0 q_0 + p^1 q_1}{5(p^1)^2 + 2p^0 q_1} + i\frac{3\mathsf{W}(\mathcal{Q})}{2[5(p^1)^2 + 2p^0 q_1]} \,.
\end{aligned}
\tag{3.14}
$$

The values of the fields on the horizon are well defined only if the charges are such that the entropy and, hence, $\mathsf{W}(\mathcal{Q})$ is real and non-vanishing and if $\Im\mathfrak{m}\, t > 0$. Furthermore, in order to be able to write

the above expressions we have assumed that $p^0 > 0$. Then, the conditions that the charges must satisfy are

$$p^0 > 0 \,, \tag{3.15}$$

$$5(p^1)^2 + 2p^0 q_1 > 0 \,, \tag{3.16}$$

$$\tfrac{8}{15} p^0 (q_1)^3 + (p^1 q_1)^2 - 3(p^0 q_0)^2 - 6p^0 q_0 p^1 q_1 - 10(p^1)^3 q_0 > 0 \,. \tag{3.17}$$

The analysis of the possible values of the charges in the most general case is complicated and unilluminating, so we will not attempt it here. The inequalities (3.15)–(3.17) must be extended to the H^M in order to guarantee the regularity of the solution. The first-order flow equations imply that the metric function grows monotonically from spatial infinity to the event horizon, therefore it is enough to give it admissible values there to ensure that it does not vanish for any value of $\tau \in (-\infty, 0)$. A similar argument applies to the scalar field.[10]

Because the general supersymmetric solution turns out to be very difficult to deform into the general non-extremal solution, we consider a simpler three-charge case with $p^0 = 0$. The supersymmetric solution (with $H^0_\infty = 0$ as well) takes the form:

$$
\begin{aligned}
e^{-2U} &= \tfrac{2}{\sqrt{3}} |H^1| \sqrt{(H_1)^2 - 10 H^1 H_0} \,, \\[2mm]
t &= -\frac{H_1}{5 H^1} + i \frac{\sqrt{3}}{5} \frac{\sqrt{(H_1)^2 - 10 H^1 H_0}}{|H^1|} \,,
\end{aligned}
\tag{3.18}
$$

For this simpler charge configuration it is also possible to directly study the stationary points of the black hole potential to find a non-supersymmetric critical point given by:

$$t_{\text{att}} = -\frac{q_1}{5 p^1} + i \frac{\sqrt{3}}{5} \frac{\sqrt{-[(q_1)^2 - 10 p^1 q_0]}}{|p^1|} \tag{3.19}$$

and the corresponding entropy:

$$S_{\text{e}}/\pi = \frac{1}{\sqrt{3}} |p^1| \sqrt{-[(q_1)^2 - 10 p^1 q_0]} \,. \tag{3.20}$$

They differ from the supersymmetric case by the sign of the discriminant

$$\Lambda = -p^1 q_0 + \frac{(q_1)^2}{10} \,. \tag{3.21}$$

Rather than trying to construct the corresponding solutions directly, we shall obtain them as a limit of the non-extremal solution that we construct using the general procedure discussed in the previous section.

3.1.2   Non-extremal solution with $p^0 = 0$

As we showed in section 2.2, by using the ansatz

$$H^M(\tau) = A^M \cosh(r_0 \tau) + \frac{B^M}{r_0} \sinh(r_0 \tau) \,. \tag{3.22}$$

---

[10]With more scalar fields and non-diagonal metrics it would be more complicated to argue the same.

valid for non-extremal black holes satisfying $H^M \dot{H}_M = 0$, one can reduce the differential equations of motion to the algebraic equations (2.53)–(2.55) and solve them for the coefficients B^M. For a non-extremal black hole in the t^3 model with charges p^1, q_0 and q_1 one finds:

$$B_0 = s^1 \left(\sqrt{\frac{\Lambda^2}{2(p^1)^2} + \frac{5r_0^2(\Im\mathfrak{m}\, t_\infty)^3}{24}} - \frac{q_1^2}{10(p^1)^2}\sqrt{\frac{(p^1)^2}{2} + \frac{3r_0^2}{10(\Im\mathfrak{m}\, t_\infty)^3}} \right), \qquad (3.23)$$

$$B^1 = -s^1 \sqrt{\frac{3r_0^2}{10\Im\mathfrak{m}\, t_\infty} + \frac{1}{2}(p^1)^2}\,, \qquad (3.24)$$

$$B_1 = -s^1 \frac{q_1}{p^1} \sqrt{\frac{3r_0^2}{10\Im\mathfrak{m}\, t_\infty} + \frac{1}{2}(p^1)^2}\,, \qquad (3.25)$$

where we have defined

$$s^1 \equiv \mathrm{sgn}(p^1)\,, \qquad (3.26)$$

The coefficients A^M can be determined by using the general expression (2.32) and in our case turn out to be:

$$A_0 = s^1 \frac{\sqrt{3}}{10\sqrt{10}\sqrt{\Im\mathfrak{m}\, t_\infty}} \left[\left(\frac{q_1}{p^1}\right)^2 - \frac{25}{3}(\Im\mathfrak{m}\, t_\infty)^2 \right]\,, \qquad (3.27)$$

$$A^1 = s^1 \sqrt{\frac{3}{10\Im\mathfrak{m}\, t_\infty}}\,, \qquad (3.28)$$

$$A_1 = s^1 \frac{q_1}{p^1} \sqrt{\frac{3}{10\Im\mathfrak{m}\, t_\infty}}\,. \qquad (3.29)$$

From the relation $M = \dot{U}(0)$ the mass is found to be

$$M = \frac{1}{4} \left(\sqrt{\frac{-60p^1q_0(q_1)^2 + 3(q_1)^4 + 25(p^1)^2[12(q_0)^2 + 5r_0^2(\Im\mathfrak{m}\, t_\infty)^3]}{125(p^1)^2(\Im\mathfrak{m}\, t_\infty)^3}} + \sqrt{9r_0^2 + 15(p^1)^2(\Im\mathfrak{m}\, t_\infty)^3} \right). \qquad (3.30)$$

One can invert this expression to obtain r_0 in terms of the physical parameters M, $\Im\mathfrak{m}\, t_\infty$, p^1, q_0:

$$
\begin{aligned}
r_0^2 = \frac{1}{1000(p^1)^4\Im\mathfrak{m}\, t_\infty^6} \Big(& -60(p^1)^3 q_0(q_1)^2\Im\mathfrak{m}\, t_\infty^3 + 3(p^1)^2(q_1)^4\Im\mathfrak{m}\, t_\infty^3 \\
& - 1875(p^1)^6\Im\mathfrak{m}\, t_\infty^7 + 100(p^1)^4 \left[3(q_0)^2\Im\mathfrak{m}\, t_\infty^3 + 25M^2\Im\mathfrak{m}\, t_\infty^6\right] \\
& + 10\sqrt{30M^2(p^1)^6\Im\mathfrak{m}\, t_\infty^9} \\
& \overline{\sqrt{9(q_1)^2\left[(q_1)^2 - 20p^1 q_0\right] + 25(p^1)^2\left[36(q_0)^2 - 25(p^1)^2\Im\mathfrak{m}\, t_\infty^4 + 30M^2\Im\mathfrak{m}\, t_\infty^3\right]}} \Big).
\end{aligned} \qquad (3.31)
$$

This result allows one to obtain the expression for the mass in the extremal limit $r_0 \to 0$, namely:

$$M = \sqrt{\frac{3}{5}} \frac{25(p^1)^2\Im\mathfrak{m}\, t_\infty^2 + 10|\Lambda|}{20|p^1|\Im\mathfrak{m}\, t_\infty^{3/2}}\,. \qquad (3.32)$$

16

s^1	s_0	s_Λ
+	−	+
−	+	+
+	+	+
−	−	+
+	+	−
−	−	−

Table 1: The extremal limits depend on s^1 and s_Λ. Here $s_0 = \text{sgn}(q_0)$, $s^1 = \text{sgn}(p^1)$ and $(s_\Lambda = \text{sgn}(\Lambda)$ where the discriminant Λ has been defined in eq. (3.21). There are 6 possible cases : the first 4 possibilities $(s_\Lambda = +1)$ would produce a supersymmetric extremal black hole while the others $(s_\Lambda = -1)$ a non-supersymmetric one.

It is easy to check that $M > 0$. As mentioned at the end of the previous section, when $s_\Lambda = \text{sgn}(\Lambda)$ is positive, the solution is supersymmetric (see table 1), in which case the anharmonic function $H_0 = A_0 \cosh(r_0\tau) + \frac{B_0}{r_0} \sinh(r_0\tau)$ becomes for $r_0 \to 0$:

$$H_0 = s^1 \frac{\sqrt{3}}{10\sqrt{10}\sqrt{\Im\mathfrak{m}\, t_\infty}} \left[\left(\frac{q_1}{p^1}\right)^2 - \frac{25}{3} (\Im\mathfrak{m}\, t_\infty)^2 \right] - \frac{1}{\sqrt{2}} q_0\tau \,, \tag{3.33}$$

whereas in the non-supersymmetric case:

$$H_0 = s^1 \frac{\sqrt{3}}{10\sqrt{10}\sqrt{\Im\mathfrak{m}\, t_\infty}} \left[\left(\frac{q_1}{p^1}\right)^2 - \frac{25}{3} (\Im\mathfrak{m}\, t_\infty)^2 \right] + \frac{1}{\sqrt{2}} \left(q_0 - 2\frac{q_1^2}{10p^1} \right) \tau \,. \tag{3.34}$$

The extremal limit for $H^1 = \frac{p^1}{q_1} H_1$ is in turn:

$$H^1 = s^1 \sqrt{\frac{3}{10\Im\mathfrak{m}\, t_\infty}} - \frac{1}{\sqrt{2}} p^1\tau \,. \tag{3.35}$$

Accordingly, for the warp factor after some simplification one obtains

$$e^{-2U} = \tfrac{2}{\sqrt{3}} \sqrt{\pm\,[-10(H^1)^3 H_0 + (H^1 H_1)^2]} \,, \tag{3.36}$$

where the plus holds for supersymmetric solutions and the minus for non-supersymmetric.

The entropies associated with the outer $(\tau \to -\infty)$ and inner $(\tau \to +\infty)$ horizon can be computed to be respectively:

$$\frac{S_+}{\pi} = \frac{1}{15^{3/4}} \left[\frac{\left(\sqrt{3}r_0 + \sqrt{3r_0^2 + 5(p^1)^2\Im\mathfrak{m}\, t_\infty}\right)^3}{\Im\mathfrak{m}\, t_\infty^2} \right. \tag{3.37}$$

$$\left. \left(5\sqrt{5}r_0 + \sqrt{\frac{300(q_0)^2}{\Im\mathfrak{m}\, t_\infty^3} - \frac{60q_0(q_1)^2}{p^1\Im\mathfrak{m}\, t_\infty^3} + \frac{3(q_1)^4}{\Im\mathfrak{m}\, t_\infty^3(p^1)^2} + 125r_0^2}\right) \right]^{1/2} \,,$$

$$\frac{S_-}{\pi} = \frac{1}{15^{3/4}} \left[\frac{\left(-\sqrt{3}r_0 + \sqrt{3r_0^2 + 5(p^1)^2\Im\mathfrak{m}\, t_\infty}\right)^3}{\Im\mathfrak{m}\, t_\infty^2} \right. \tag{3.38}$$

$$\left. \left(5\sqrt{5}r_0 - \sqrt{\frac{300(q_0)^2}{\Im\mathfrak{m}\, t_\infty^3} - \frac{60q_0(q_1)^2}{p^1\Im\mathfrak{m}\, t_\infty^3} + \frac{3(q_1)^4}{\Im\mathfrak{m}\, t_\infty^3(p^1)^2} + 125r_0^2}\right) \right]^{1/2} \,.$$

By taking the limit $r_0 \to 0$ the extremal black hole entropy is recovered from both S_+ and S_- and their product satisfies the geometric mean property $S_+ S_- = \frac{\pi^2}{3}(p^1)^2 \left[-10 p^1 q_0 + (q_1)^2\right] = S_e^2$.

3.2 Quantum-corrected case

For the quantum-corrected model of type IIA superstring on the quintic, whose prepotential can be brought to the form (3.1) by a symplectic rotation of the coordinate frame (see the appendix), the covariantly holomorphic period vector reads:

$$
\mathcal{V}_{\text{pert}} = e^{\mathcal{K}_{\text{pert}}/2}
\begin{pmatrix}
1 \\
t \\
\frac{5}{6}t^3 - ic \\
-\frac{5}{2}t^2
\end{pmatrix},
\tag{3.39}
$$

where (in the compactification we are considering) $c = \frac{25}{\pi^3}\zeta(3) \approx 0.969204$. Because the general case is very complicated, we deal only with two-charge and three-charge black holes.

3.2.1 Supersymmetric solution with $\hat{\mathcal{Q}} = (\hat{p}^0, 0, 0, \hat{q}_1)^{\text{T}}$, $\mathcal{Q} = (p^0, 0, 0, q_1)^{\text{T}}$

The relations between the two pairs of charges in the rotated frame and in the original one are:

$$
\hat{p}^0 = p^0, \qquad \hat{q}_1 = q_1 - \frac{25}{12}p^0.
\tag{3.40}
$$

By solving the equation for the extremal supersymmetric case one finds:[11]

$$
t = s_i i \sqrt{\frac{2}{5}\frac{H_1}{H^0}},
\tag{3.41}
$$

$$
e^{-2U_e} = s_i \frac{4}{3}\sqrt{\frac{2}{5}H^0(H_1)^3} + c(H^0)^2,
\tag{3.42}
$$

with $H^M = A^M - \frac{1}{\sqrt{2}}\hat{\mathcal{Q}}^M \tau$ and $s_i = +1$ when

$$
\sqrt{\frac{2}{5}\frac{\hat{q}_1}{\hat{p}^0}} \in \left(\left(\frac{3c}{5}\right)^{1/3}, \infty\right),
\tag{3.43}
$$

while $s_i = -1$ for

$$
\sqrt{\frac{2}{5}\frac{\hat{q}_1}{\hat{p}^0}} \in \left(0, \left(\frac{3c}{10}\right)^{1/3}\right),
\tag{3.44}
$$

so that $\Im m\, t$ lies in the allowed domain (A.48) (for other values of the charges the supersymmetric solution simply does not exist). By using (2.32) one can determine the constant part of the harmonic functions:

$$
A^0 = s_{\mathcal{Q}} \sqrt{\frac{3}{10\, s_i\, \Im m\, t_\infty^3 + 3c}}, \qquad A_1 = s_{\mathcal{Q}} \frac{5}{2}\Im m\, t_\infty^2 \sqrt{\frac{3}{10\, s_i\, \Im m\, t_\infty^3 + 3c}}.
\tag{3.45}
$$

---

[11]As the H^M in the original frame do not appear (and $\hat{H}^M$ has been already used with a different meaning in eq. (2.62)), we suppress the hats on the rotated H^M.

Notice that two disconnected branches of supersymmetric solutions appear and only one of them, the case (3.43), survives when $c = 0$. For both supersymmetric possibilities $\text{sgn}(\hat{p}^0) = s_{\mathcal{Q}} = \text{sgn}(\hat{q}_1)$ and depending on the charges, the scalar at infinity is bound to a certain set of possible values. If the charges, for example, satisfy (3.43) also $\Im m\, t_\infty$ must belong to this interval and all the flow of the scalar in the moduli space takes place inside this confined region. By looking at the explicit form of the solutions it is possible to convince oneself that the two distinct branches of solutions cannot be connected smoothly by changing the value of the charges.

The entropy and the mass, once computed, can be written in the form:

$$\frac{S_e}{\pi} = \frac{\frac{45}{4}c^2(\hat{p}^0)^3 + 8(\hat{q}_1)^3 + s_i\, 6\sqrt{10}\, c\, \sqrt{(\hat{p}^0\hat{q}_1)^3}}{\frac{45}{2}c\,\hat{p}^0 + s_i\, 6\sqrt{10}\,\hat{q}_1\sqrt{\frac{\hat{q}_1}{\hat{p}^0}}}\,, \tag{3.46}$$

$$M_e = \frac{\left|6c\,\hat{p}^0 + 6\,\hat{q}_1\Im m\, t_\infty + 5\,\hat{p}^0\Im m\, t_\infty^3\right|}{4\sqrt{\frac{9}{2}c + 15\Im m\, t_\infty^3}}\,. \tag{3.47}$$

The positivity of both the entropy and the mass is guaranteed by the fact that the charges are confined to the intervals (3.43), (3.44).

The study of this two-charge configuration in the rotated symplectic frame allows the analysis of the single charge configurations $\mathcal{Q} = (p^0, 0, 0, 0)^{\mathrm{T}}$ and $\mathcal{Q} = (0, 0, 0, q_1)^{\mathrm{T}}$ in the original frame. For the former one should substitute in the formulae above $\hat{q}_1 = -\frac{25}{12}\hat{p}^0$ but already here an inconsistency occurs due to the requirement $\text{sgn}(\hat{p}^0) = \text{sgn}(\hat{q}_1)$ that would not be respected. Also for the other single-charge configuration, by setting $\hat{p}^0 = p^0 = 0$, it is easy to realize that the expressions become ill-defined. This suggests that no physical BPS solutions exist for the single-charge case at hand.

Before passing to non-extremal black holes, it is worth mentioning that the Freudenthal duality equations also admit a solution that cannot be accepted, namely:

$$t = \frac{3c}{2H_1}\sqrt{\frac{8}{45c^2}\frac{H_1^3}{H^0} - (H^0)^2} + ic\frac{3H^0}{2H_1}\,, \tag{3.48}$$

$$e^{-2U_e} = 2(H^0)^2 c + \frac{2}{45c}\frac{(H_1)^3}{H^0}\,. \tag{3.49}$$

These expressions would be well defined only for charges that violate the constraint (A.48), which leads to invalid Kähler metric and Kähler potential.

3.2.2 Supersymmetric solution with $\hat{\mathcal{Q}} = (0, \hat{p}^1, \hat{q}_0, \hat{q}_1)^{\mathrm{T}}$, $\mathcal{Q} = (0, p^1, q_0, q_1)^{\mathrm{T}}$

This configuration corresponds to a three-charge black hole also in the original frame, according to the relations:

$$\hat{p}^1 = p^1\,, \qquad \hat{q}_0 = q_0 - \frac{25}{12}p^1\,, \qquad \hat{q}_1 = q_1 + \frac{11}{2}p^1\,. \tag{3.50}$$

We solve the Freudenthal duality equations with the harmonic function H^0 set to zero. This yields:

$$\hat{\mathcal{X}}^0 = \frac{\rho^2 + \rho\alpha^{1/3} + \alpha^{2/3}}{30c\,H^1\,\alpha^{1/3}}\,, \tag{3.51}$$

$$\hat{\mathcal{X}}^1 = iH^1 - \frac{H_1}{5H^1}\hat{\mathcal{X}}^0\,, \tag{3.52}$$

$$U = -\frac{1}{2}\log\left(\frac{\alpha^{1/3}\left(\beta + \gamma\alpha^{1/3}\right) + \delta}{100\,c\,(H^1)^2\,\alpha^{2/3}\left(\alpha^{2/3} + \rho\,\alpha^{1/3} + \rho^2\right)}\right), \tag{3.53}$$

where

$$\rho = -10H^1H_0 + (H_1)^2\,,$$

$$\alpha = \rho^3 - 11250c^2\,(H^1)^6 + 150\,\sqrt{(H^1)^6c^2[5625c^2(H^1)^6 - \rho^3]}\,,$$

$$\beta = \rho^2\left(\rho^3 - 7500c^2\,(H^1)^6 + 100\,\sqrt{(H^1)^6c^2[5625c^2(H^1)^6 - \rho^3]}\right), \tag{3.54}$$

$$\gamma = \rho\left(\rho^3 - 3750c^2\,(H^1)^6 + 50\,\sqrt{(H^1)^6c^2[5625c^2(H^1)^6 - \rho^3]}\right),$$

$$\delta = \left(\rho^3 + 7500c^2\,(H^1)^6\right)\alpha\,.$$

The expression for the physical scalar then becomes

$$t = \frac{\hat{\mathcal{X}}^1}{\hat{\mathcal{X}}^0} = -\frac{\hat{q}_1}{5\hat{p}^1} + i\,\frac{30c\,(H^1)^2\alpha^{1/3}}{\alpha^{2/3} + \rho\,\alpha^{1/3} + \rho^2}\,. \tag{3.55}$$

The constant parts of the harmonic functions turn out to be:

$$A^1 = s^1\,\frac{\sqrt{3}\,\Im\mathfrak{m}\,t_\infty}{\sqrt{3c + 10\Im\mathfrak{m}\,t_\infty^3}}\,, \qquad A_1 = s_1\,\frac{\sqrt{3}\hat{q}_1\,\Im\mathfrak{m}\,t_\infty}{\hat{p}^1\sqrt{3c + 10\Im\mathfrak{m}\,t_\infty^3}}\,, \tag{3.56}$$

$$A_0 = s_0\,\frac{3(\hat{q}_1)^2\Im\mathfrak{m}\,t_\infty - 25(\hat{p}^1)^2\Im\mathfrak{m}\,t_\infty^3 - 30c\,(\hat{p}^1)^2}{10(\hat{p}^1)^2\,\sqrt{9c + 30\Im\mathfrak{m}\,t_\infty^3}}\,, \tag{3.57}$$

where $s^M = \mathrm{sgn}(\hat{\mathcal{Q}}^M)$.

The solution just displayed is a purely "quantum black hole": it diverges when c is put to zero and it is well defined only for a restricted set of values of the parameters $\{\hat{p}^1,\,\hat{q}_0,\,\hat{q}_1,\,\Im\mathfrak{m}\,t_\infty\}$. By looking at the expressions of the scalar and the warp factor we realize that the problematic part is the square root

$$\sqrt{(H^1)^6c^2[5625c^2(H^1)^6 - \rho^3]} \tag{3.58}$$

that, in order to be real, needs the radicand to be bigger than or equal to zero. This condition must be considered besides the requirement that the imaginary part of the scalar should belong to the intervals (A.48) and the positivity of the warp factor. Then the allowed values of the charges can be determined

20

by studying the behavior of solutions on the horizon whereas the allowed values for $\Im\mathfrak{m}\, t_\infty$ are given by the limit at infinity ($\tau \to 0^-$). In the end one obtains the following restrictions:

$$\Im\mathfrak{m}\, t_\infty \in \left(-\left(\frac{3c}{10}\right)^{1/3}, 0\right) \approx \left(-0.662489\,, 0\right), \tag{3.59}$$

$$\hat{q}_0 > \frac{(\frac{75}{2}\, c)^{2/3}\, (\hat{p}^1)^2 + (\hat{q}_1)^2}{10\hat{p}^1} \quad \text{if} \quad \hat{p}^1 > 0\,, \tag{3.60}$$

$$\hat{q}_0 < \frac{(\frac{75}{2}\, c)^{2/3}\, (\hat{p}^1)^2 + (\hat{q}_1)^2}{10\hat{p}^1} \quad \text{if} \quad \hat{p}^1 < 0\,. \tag{3.61}$$

It is not difficult to see that the conditions (3.60), (3.61) would be violated by the charge configuration $\hat{\mathcal{Q}} = (0, \hat{p}^1, 0, \hat{q}_1)^{\mathrm{T}}$, which would produce a black hole with singular metric (differently from the uncorrected t^3 model). Similarly one can exclude the existence of black holes with the charge vector $\hat{\mathcal{Q}} = (0, \hat{p}^1, -\frac{25}{12}\hat{p}^1, \frac{11}{2}\hat{p}^1)^{\mathrm{T}}$, corresponding in the original frame to $\mathcal{Q} = (0, p^1, 0, 0)$: when $\hat{p}^1 = p^1 = 0$ the expression for the scalar would diverge. This last observation, together with the discussion in the previous subsection, indicates that this model does not admit regular supersymmetric single-charge black holes.

On the other hand, solutions with $H_1 = 0$ (corresponding to the charge configuration $\hat{\mathcal{Q}} = (0, \hat{p}_1, \hat{q}_0, 0)^{\mathrm{T}}$, $\mathcal{Q} = (0, p^1, q_0, -\frac{11}{2}p^1)^{\mathrm{T}}$) or with $q_1 = 0$ (two-charge in the unrotated frame), are physical. In the former case the scalar becomes purely imaginary

$$t = -\,3i\,\frac{(H^1)^2 c\,\lambda^{1/3}}{(H^1)^2 (H_0)^2 + H^1 H_0 \lambda^{1/3} + \lambda^{2/3}}\,,$$

$$\lambda = \frac{45}{4}(H^1)^6 c^2 + (H^1)^3 H_0^3 - 3\sqrt{\frac{5}{4}(H^1)^9 c^2 \left(\frac{45}{4}(H^1)^3 c^2 + 2(H_0)^3\right)} \tag{3.62}$$

and in line with eqs. (3.60), (3.61) the charges must satisfy $\mathrm{sgn}\,\hat{p}^1 = \mathrm{sgn}\,\hat{q}_0$ and $|\hat{q}_0| > \left(\frac{(75/2\, c)^{2/3}}{10}\right)|\hat{p}^1|$. When instead $q_1 = 0$, the real part of the scalar takes a fixed value independent of parameters, namely $\Re\mathfrak{e}\, t = -\frac{11}{10}$, and the restrictions on the allowed charges become $\mathrm{sgn}\,\hat{q}_0 = \mathrm{sgn}\,\hat{p}^1$ and $|\hat{q}_0| > \frac{4(75/2\, c)^{2/3}+121}{40}|\hat{p}^1|$.

The entropy and the mass for the black holes in this section can be calculated as usual, but due to the complexity of the expressions, we do not display them.

3.2.3 Non-extremal solutions

The expressions for the scalar and warp factor are in general very involved and this turns out to make the pursuit of non-supersymmetric black holes cumbersome. The difficulty resides in the fact that the equations for the coefficients turn out to be polynomials of a very high degree, which cannot be solved analytically.

The only non-extremal black holes that can be quite straightforwardly studied are those with the scalar assuming a constant value that extremizes the black hole potential. From the general treatment in 2.2.2 we know that for such non-extremal solutions

$$B^M = -\,A^M M = -\,A^M \sqrt{|\mathcal{Z}(Z_\infty, Z_\infty^*, \mathcal{Q})|^2 + r_0^2}\,, \tag{3.63}$$

and the only quantity to calculate is the absolute value of the central charge in the stationary points of the black hole potential. In the current case it reads:

$$|\mathcal{Z}(Z_\infty, Z_\infty^*, \mathcal{Q})| = \frac{|6\hat{q}_0 + 6\hat{q}_1 t_{\text{att}} + 15\hat{p}^1 t_{\text{att}}^2 - 5\hat{p}^0 t_{\text{att}}^3 + 6ic\hat{p}^0|}{\sqrt{6(12c + 5i\Im\mathfrak{m}\, t_{\text{att}}^3)}}\,,$$ (3.64)

where t_{att} is the constant value of the scalar all along the flow.

So far no analytic expressions for non-supersymmetric stationary points of $V_{\text{bh}}(Z, Z^*, \mathcal{Q})$ have been obtained for a general charge configuration.[12] We study the non-extremal version of (some of) the supersymmetric black holes of the previous subsections and present an example of a constant-scalar non-extremal black hole built from a non-supersymmetric critical point of a system with a particular charge vector.

Configuration $\hat{\mathcal{Q}} = (\hat{p}^0, 0, 0, \hat{q}_1)^{\text{T}}$**:**   When $\hat{q}_1 \hat{p}^0 > 0$, we read off from eq. (3.41) that

$$t_{\text{att}} = i\, s_i \sqrt{\frac{2\hat{q}_1}{5\hat{p}^0}}$$ (3.65)

and by plugging it in (3.45) and (3.64) we find:

$$B^0 = -s_{\mathcal{Q}} \sqrt{\frac{15\, M^2}{15\, c + s_i\, 4\sqrt{10\left(\frac{\hat{q}_1}{\hat{p}^0}\right)^3}}}\,, \qquad B_1 = \frac{\hat{q}_1}{\hat{p}^0} B^0\,.$$ (3.66)

where the mass M is equal to:

$$M = \sqrt{\frac{c}{2}\,(\hat{p}^0)^2 + s_i \sqrt{\frac{8}{45}\,\hat{p}^0(\hat{q}_1)^3} + r_0^2}\,.$$ (3.67)

With this last expression the outer and the inner entropy follow from eq. (2.83). It is worth noticing that all these formulae reduce to the extremal counterparts in the limit $r_0 \to 0$ and for the entropy it holds $S_+ S_- = S_{\text{e}}^2$.

Configuration $\hat{\mathcal{Q}} = (0, \hat{p}^1, 2\hat{p}^1, 0)^{\text{T}}$**:**   For the sake of simplicity we take $\hat{q}_0 = 2\hat{p}^1$. The black hole potential has a charge-independent critical point (corresponding to a supersymmetric attractor) at:

$$t_{\text{att}} = -6\,i\,c\, \frac{\left(64 + 90\,c^2 - 6c\sqrt{5\left(64 + 45c^2\right)}\right)^{1/3}}{12 + \left(2 + \left(64 + 90c^2 - 6c\sqrt{5\left(64 + 45c^2\right)}\right)^{1/3}\right)^2} \equiv -6\,i\,c\,\xi$$

$$\approx -0.447310\,i$$

and the coefficients of the hyperbolic functions are:

$$B^1 = s^1\, \frac{6\,c\,M\,\xi}{\sqrt{c - 720\,c^3\,\xi^3}}\,, \qquad B_0 = \frac{1 - 180\,c^2\,\xi^3}{6\,\xi}\, B^1\,.$$ (3.68)

---

[12]An accurate numerical study has been carried out in [40].

For the mass one finds:

$$M = \sqrt{(p^1)^2 \frac{2\,(1 - 45\,c^2\,\xi^2)^2}{c\,(1 - 720\,c^2\,\xi^3)} + r_0^2}\,. \tag{3.69}$$

From these expressions it is easy to see by setting $c = 0$ that this black hole does not reduce to a regular solution of the t^3 model.

Configuration $\hat{\mathcal{Q}} = (\hat{p}^0, 0, 0, -\frac{35}{2}(\frac{3}{2}c)^{2/3}\hat{p}^0)^{\mathrm{T}}$: Also in this case the stationary point of the black-hole potential does not depend on the value of $\hat{p}^0$ (although this time it corresponds to a non-supersymmetric attractor):

$$t_{\mathrm{att}} = i\,\left(\tfrac{3}{2}c\right)^{1/3} \approx 1.13284\,i\,. \tag{3.70}$$

The non-extremal solution with a constant scalar is then completely characterized by

$$B^0 = -\,s^0\,\frac{M}{\sqrt{6\,c}}\,, \qquad B_1 = -\,s^0\,\frac{5}{4}\left(\frac{3}{2}\,c\right)^{1/6} M\,, \tag{3.71}$$

with

$$M = \sqrt{48\,c\,(\hat{p}^0)^2 + r_0^2}\,. \tag{3.72}$$

The limit $r_0 \to 0$ gives a doubly-extremal non-supersymmetric black hole. Setting $c = 0$ again does not lead to a regular solution.

Configurations $\hat{\mathcal{Q}} = (0, \hat{p}^1, 0, 0)^{\mathrm{T}}$ **and** $\hat{\mathcal{Q}} = (0, 0, 0, \hat{q}_1)^{\mathrm{T}}$: Of these two configurations that are both single-charge in the rotated frame, the second is one-charge also in the original frame, $\mathcal{Q} = (0, 0, 0, \hat{q}_1)^{\mathrm{T}} = (0, 0, 0, q_1)^{\mathrm{T}}$. The admissible critical points of the black hole potential $-V_{\mathrm{bh}}$ give in each case one non-supersymmetric attractor,

$$t_{\mathrm{att}} = i\,\sqrt[3]{\left(6 + \sqrt[3]{206 - 6\sqrt{87}} + \frac{17\sqrt[3]{4}}{\sqrt[3]{103 - 3\sqrt{87}}}\right)\frac{3c}{10}} \approx 1.37065\,i \tag{3.73}$$

or

$$t_{\mathrm{att}} = -i\,\sqrt[3]{(3\sqrt{2} - 4)\frac{3c}{10}} \approx -0.327962\,i\,, \tag{3.74}$$

which (by the analysis of eigenvalues of the Hessian matrix of $-V_{\mathrm{bh}}$ with respect to t and t^*, in a real basis [41, 42]) is found to be stable.[13] Neither depends on the value of the charge.

The metric function of non-extremal solutions with the constant scalar, fixed to one of the above values,

$$e^{-U} = e^{-r_0\tau}\left(\frac{-V_{\mathrm{bh}}|_{\mathrm{att}}}{-2r_0^2 \pm 2\sqrt{r_0^2(r_0^2 - V_{\mathrm{bh}}|_{\mathrm{att}})}}(e^{2r_0\tau} - 1) + 1\right), \tag{3.75}$$

has the extremal ($r_0 \to 0$) limit:

$$\lim_{r_0\to 0} e^{-U} = -\sqrt{-V_{\mathrm{bh}}}\,\Big|_{\mathrm{att}}\,\tau + 1\,, \tag{3.76}$$

---

[13]In each case there are in addition multiple stationary points outside of the allowed domain. For q_1 there is also one admissible saddle point of the black hole potential at $t = i[(3\sqrt{2} + 4)\frac{3c}{10}]^{1/3} \approx 1.06216\,i$.

with the minus sign due to the negative τ in our conventions and the constant 1 for asymptotic flatness. The respective stationary values of the black hole potential read

$$- V_{\text{bh}}\Big|_{\text{att}} = -\frac{5 t_{\text{att}} \left(144 c^2 + 30 c t_{\text{att}}^3 + 100 t_{\text{att}}^6\right)}{8 \left(36 c^2 + 30 c t_{\text{att}}^3 - 50 t_{\text{att}}^6\right)} (\hat{p}^1)^2 \approx 2.20225 (\hat{p}^1)^2 \qquad (3.77)$$

and

$$- V_{\text{bh}}\Big|_{\text{att}} = \frac{\sqrt{2}}{2} \left(\frac{3\sqrt{2} + 4}{75 c}\right)^{1/3} (\hat{q}_1)^2 \approx 0.431213 (\hat{q}_1)^2 \, , \qquad (3.78)$$

the second of which does not have a finite $c \to 0$ limit.

4   Conclusions

The use of the H-FGK approach has enabled us, apart from studying some model-independent properties of black-holes in four-dimensional $N = 2$ supergravity, to find extremal and non-extremal solutions for the t^3 model without and, for the first time analytically, with a quadratic quantum correction to the prepotential. We study the solutions for the corrected model in a symplectically rotated frame of homogenous coordinates on the scalar manifold, which simplifies the prepotential (and allows one to interpret the results as pairs of solutions for two closely related, but not mutually dual prepotentials with quadratic corrections).

The formalism itself can be applied with equal ease to any charge configuration of either model, but the polynomial equations that determine the parameters make the explicit solutions unfeasible except when some charges vanish and, in the non-extremal case, when the scalar is constant.

We find that the correction leads to the appearance of solutions, which one might call quantum black holes, that do not possess a regular classical limit. Perhaps surprisingly, we find in particular that the quantum correction is sufficient to render the otherwise divergent solution with only one charge, q_1, regular. (The other solution that is single-charge in the rotated frame, but which is not single-charge in the original frame, without the quantum correction reduces to the empty Minkowski spacetime.)

In contrast to the solutions in ref. [43], the truncations ($H^M = 0$ for some M) of the H-functions corresponding to our quantum black holes are non-singular in the classical limit. This means that in our case we can construct the classical counterpart to a corrected solution with no regular $c \to 0$ limit by simply considering the theory with $c = 0$, imposing the same constraints on H^M and $\mathcal{Q}^M$, and then solving the Freudhental duality equations.

Acknowledgments

A Type II Calabi–Yau compactifications

In this appendix we review the compactification of the type IIA theory on the quintic manifold $\mathcal{M}$ and of the type IIB on the mirror quintic manifold $\mathcal{W}$, following refs. [44, 45, 46, 47, 48, 49, 50, 51, 52]. It is well known that the low-energy limit of type II superstring theory compactified on a Calabi–Yau manifold is an $N = 2, d = 4$ supergravity with a number of vector multiplets and hypermultiplets that depend on the Hodge numbers of the Calabi–Yau manifold. Only the vector multiplets moduli space is relevant for the construction of black-hole solutions in these theories: black-hole-type solutions with non-trivial hyperscalars in ungauged $N = 2, d = 4$ theories are expected to be generically singular since they would have primary scalar hair [53]. On the other hand, in the unguaged theories, the only bosonic field the hyperscalars couple to in the ungauged theories is the metric, and, therefore, they can always be consistently truncated or, equivalently, set to some constant value.

A.1 Type IIB on the mirror quintic $\mathcal{W}$

Let $\mathcal{M}$ be the family of manifolds associated with the vanishing of a quintic polynomial in $\mathbb{CP}_4$. An element of $\mathcal{M}$ has $h^{(2,1)} = 101$ degrees of freedom describing the complex structure of the manifold, that can be associated with the coefficients of the defining polynomial.[14] Furthermore, $h^{(1,1)} = 1$ and the only independent harmonic $(1, 1)$-form can identified with the Kähler form of the manifold: any other harmonic $(1, 1)$-form is the Kähler form multiplied by a real number, which corresponds to the freedom to adjust the overall scale of the manifold. The Euler number of a quintic manifold is $\chi = -200$.

Let us consider the family of quintic polynomials [47, 48]

$$p_\psi = \sum_{k=1}^{5} x_k^5 - 5\psi \prod_{k=1}^{5} x_k, \qquad \psi \in \mathbb{C}, \tag{A.1}$$

parametrized by the complex modulus ψ, M_ψ the manifold described by $p_\psi = 0$ and $\mathcal{M}_0 \subset \mathcal{M}$ the family of all manifolds M_ψ for $\psi \in \mathbb{C}$ The family of quintic polynomials $(p_\psi, \psi \in \mathbb{C})$ is invariant under the group generated by:

$$
\begin{aligned}
g_0 &= (1, 0, 0, 0, 4) \,, \\
g_1 &= (0, 1, 0, 0, 4) \,, \\
g_2 &= (0, 0, 1, 0, 4) \,, \\
g_3 &= (0, 0, 0, 1, 4) \,,
\end{aligned}
\tag{A.2}
$$

where g_i, $i = 0, \ldots, 3$, acts on $(x_1, \ldots, x_5)$ by multiplying the $(i + 1)$-th entry by the phase $\alpha = e^{2\pi i/5}$ and the last entry by α^4, so $g_i^5 = 1$ for all i. The transformation $g_0 g_1 g_2 g_3$ leaves each p_ψ invariant because it multiplies the homogeneous coordinates by a common phase, hence only three of the g_i are independent, say g_1, g_2 and g_3. These three elements generate the group $\mathbb{Z}_5^3$.

It turns out that the mirror family $\mathcal{W}$ is $\mathcal{W} = W_\psi \equiv M_\psi / \mathbb{Z}_5^3, \psi \in \mathbb{C}$. It can be shown that the elements of $\mathcal{W}$ have $h^{(2,1)} = 1, h^{(1,1)} = 101$ and $\chi = 200$, as they must.

Since the transformation $\psi \to \alpha\psi$ can be undone by a coordinate transformation, we have that $\psi \sim \alpha\psi$, thus it is ψ^5 that plays the role of the modulus that parametrizes the complex-structure moduli space of $\mathcal{W}$ that we denote by $C_{\mathrm{IIB}}^{(2,1)}$. This is in agreement with $h^{(2,1)} = 1$. There are two values of ψ^5 for which M_ψ (and, correspondingly, W_ψ) is singular: $\psi^5 = 1$ and $\psi = \infty$.

---

[14]A quintic polynomial has 126 possible terms and complex coefficients. However, 25 of them can be eliminated by linear transformations of the 5 complex coordinates.

W_1 has a single singular point given by the equivalence class $[(1,1,1,1)]$ and W_∞ is given by the quotient by $\mathbb{Z}_5^3$ of the singular quintic

$$p_\infty = \prod_{k=1}^{5} x_k = 0 \,. \tag{A.3}$$

W_∞ is the large complex structure limit of $\mathcal{W}$: we will see in the following section that it is the mirror of the large-radius limit of $\mathcal{M}$.

We are interested in the compactification of the type IIB theory on $\mathcal{W}$. The low-energy effective field theory is an ungauged $N = 2, d = 4$ supergravity coupled to $h^{(2,1)} = 1$ vector multiplets and $h^{(1,1)} + 1 = 102$ hypermultiplets that can be consistently ignored (set to some constant value). We will thus be dealing with just one complex scalar parametrizing the special Kähler manifold $C_{\text{IIB}}^{(2,1)}$.

Following ref. [51], we can describe the complex-structure moduli space $C_{\text{IIB}}^{(2,1)}$ by the periods of the holomorphic three-form Ω over a canonical basis of $H_3(W_\psi, \mathbb{Z})$, which in our case, since $b_3 = 4$, can be taken to be $(\gamma^M) = \left(A^0, A^1, B_0, B_1\right)^{\text{T}}$ with the intersections

$$A^\Lambda \cap B_\Gamma = \delta_\Gamma^\Lambda \,, \qquad A^\Lambda \cap A^\Gamma = 0 \,, \qquad B_\Lambda \cap B_\Gamma = 0 \,. \tag{A.4}$$

The dual cohomology basis is denoted by $\left(\alpha_\Lambda, \beta^\Gamma\right)$ and obeys

$$\int_{A^\Lambda} \alpha_\Gamma = \delta^\Lambda_{\ \Gamma} \,, \qquad \int_{B_\Lambda} \beta^\Gamma = -\delta^\Gamma_{\ \Lambda} \,, \qquad \int_{A^\Lambda} \beta^\Gamma = \int_{B_\Lambda} \alpha_\Gamma = 0 \,. \tag{A.5}$$

The holomorphic 3-form Ω is given by

$$\Omega = \mathcal{X}^\Lambda \alpha_\Lambda - \mathcal{F}_{\text{IIB},\Lambda} \beta^\Lambda \,, \tag{A.6}$$

where $\mathcal{X}^\Lambda$ and $\mathcal{F}_{\text{IIB}\,\Lambda}$, which will be identified as the components of the holomorphic symplectic section

$$\Pi_{\text{IIB}}(\psi) = \begin{pmatrix} \mathcal{X}^0 \\ \mathcal{X}^1 \\ \mathcal{F}_{\text{IIB}\,0} \\ \mathcal{F}_{\text{IIB}\,1} \end{pmatrix} \,, \tag{A.7}$$

are the periods of the holomorphic 3-form with respect to the canonical homology basis

$$\mathcal{X}^\Lambda = \int_{A^\Lambda} \Omega \,, \qquad \mathcal{F}_\Lambda = \int_{B_\Lambda} \Omega \,. \tag{A.8}$$

There are 4 periods, but the complex-structure manifold is one-dimensional and hence we can take the $\mathcal{F}_\Lambda$ to be holomorphic functions of the $\mathcal{X}^\Lambda$. Since Ω is defined up to rescalings $\Omega \to g(\psi)\Omega$, where $g(\psi)$ is a holomorphic function of the modulus ψ, we can take the $\mathcal{X}^\Lambda$ to be projective coordinates of the scalar manifold, and hence we end up with one complex coordinate, which is what we need in order to parametrize $C_{\text{IIB}}^{(2,1)}$. Different choices of $g(\psi)$ can be understood as different gauge choices. In addition, the periods $\mathcal{F}_{\text{IIB}\,\Lambda}$ can be expressed as derivatives of a single function $\mathcal{F}_{\text{IIB}}$ of the $\mathcal{X}^\Lambda$:

$$\mathcal{F}_{\text{IIB}\,\Lambda} = \frac{\partial \mathcal{F}_{\text{IIB}}}{\partial \mathcal{X}^\Lambda} \,. \tag{A.9}$$

We will find later on that it is more natural to consider $\mathcal{F}_{\text{IIB}\,\Lambda}$ as the projective coordinates and the $\mathcal{X}^{\Lambda}$ given in terms of them. A good special coordinate in the large complex-structure limit is therefore provided by:

$$Z(\psi) = \frac{\mathcal{F}_{\text{IIB}\,0}(\psi)}{\mathcal{F}_{\text{IIB}\,1}(\psi)}. \tag{A.10}$$

It can be shown [54, 55] that the components of the holomorphic symplectic section of an $N = 2, d = 4$ supergravity theory have to obey a set of differential identities due to the properties of the special Kähler geometry. When the theory originates from a Calabi–Yau compactification, these identities are the Picard–Fuchs equations. In our case, there is only one fourth-order Picard–Fuchs equation associated with $\mathcal{W}$ [54, 56]

$$(1 - \psi^5)\omega^{iv} - 10\psi^4\omega''' - 25\psi^3\omega'' - 15\psi^2\omega' - \psi\,\omega = 0. \tag{A.11}$$

and its 4 independent solutions $\omega_0, \omega_1, \omega_2, \omega_3$ can be identified with the 4 periods [52].

Eq. (A.11) is an ordinary differential equation with regular singular points at $\psi^5 = 0, 1, \infty$ and, hence, a system of solutions may be obtained following the method of Froebenius for such equations. At $\psi^5 = \infty$ one solution, ω_0, is given as a pure power series and the other three solutions $\omega_1, \omega_2, \omega_3$ contain logarithms, with powers 1, 2 and 3, respectively. At $\psi^5 = 0$ all four solutions are pure power series. We will not need the solutions at $\psi^5 = 1$.

The pure power series solution around $\psi^5 = \infty$ is

$$\omega_0(\psi) = \frac{1}{5\psi} \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5\,(5\psi)^{5n}}, \qquad |\psi| > 1, \qquad 0 \leq \text{Arg}(\psi) < \frac{2\pi}{5}. \tag{A.12}$$

This expression has been obtained with the choice of $g(\psi)$ normally used to study the (mirror) Landau–Ginzburg or Fermat limit $\psi \to 0$. An expression for ω_0 in the large complex structure limit can be obtained from the one above by a gauge transformation with $g(\psi) = 5\psi$ [56] that gets rid of the overall factor $(5\psi)^{-1}$. We will use this new gauge for both limits, since we have found no complications in using it in the Fermat limit $\psi \to 0$. In conclusion, we take ω_0 to be

$$\omega_0(\psi) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5\,(5\psi)^{5n}}, \qquad |\psi| > 1, \qquad 0 \leq \text{Arg}(\psi) < \frac{2\pi}{5}. \tag{A.13}$$

The solution around $\psi = 0$ can be obtained by analytical continuation of eq. (A.13):

$$\omega_0(\psi) = -\frac{1}{5} \sum_{m=1}^{\infty} \frac{\alpha^{2m}\Gamma\left(m/5\right)(5\psi)^m}{\Gamma\left(m\right)\Gamma^4\left(1 - m/4\right)}, \qquad |\psi| < 1. \tag{A.14}$$

The 5 functions

$$\omega_k(\psi) \equiv \omega_0(\alpha^k\psi), \qquad k = 0, \ldots, 4, \tag{A.15}$$

are also solutions, but one of them cannot be linearly independent: the ω_k obey a linear relation which turns out to be

$$\sum_{k=0}^{4} \omega_k = 0. \tag{A.16}$$

The expressions for the ω_k, $k = 1, \ldots, 4$ for $|\psi| > 1, 0 < \text{Arg}(\psi) < \frac{2\pi}{5}$ are quite involved and can be found in appendix A.3.

To construct the holomorphic symplectic section Π_{IIB} we choose a set of four linearly independent solutions, that we combine into a vector $\hat{\omega}$ (also called the period vector on the Picard–Fuchs basis)

$$
\hat{\omega} = - \left(\frac{2\pi i}{5} \right)^3 \begin{pmatrix} \omega_2 \\ \omega_1 \\ \omega_0 \\ \omega_4 \end{pmatrix} ,
\tag{A.17}
$$

and then define $\Pi_{IIB}(\psi)$ by

$$
\Pi_{\text{IIB}}(\psi) = M\,\hat{\omega} \qquad M = \begin{pmatrix} -1 & 0 & 8 & 3 \\ 0 & 1 & -1 & 0 \\ -3/5 & -1/5 & 21/5 & 8/5 \\ 0 & 0 & -1 & 0 \end{pmatrix} .
\tag{A.18}
$$

The Kähler potential is given by

$$
e^{-\mathcal{K}} = i \left(\mathcal{X}^{*\Sigma} \mathcal{F}_{\text{IIB}\,\Sigma} - \mathcal{X}^{\Sigma} \mathcal{F}^*_{\text{IIB}\,\Sigma} \right) = \omega^\dagger \sigma\, \omega \, ,
\tag{A.19}
$$

where

$$
\sigma \equiv \frac{1}{5} \begin{pmatrix} 0 & 1 & 3 & 1 \\ -1 & 0 & 3 & 3 \\ -3 & -3 & 0 & 1 \\ -1 & -3 & -1 & 0 \end{pmatrix} .
\tag{A.20}
$$

Eq. (A.19) is a very complicated function of ψ, hence some simplification limit is in order. It can be shown that in the large complex-structure limit (given by eq. (A.40)) $\psi \to \infty$ the Kähler potential is given by:

$$
e^{-\mathcal{K}} = \left(\frac{2\pi}{5} \right)^3 \left(\frac{20}{3} \log^3 |5\psi| + \frac{16}{5}\zeta(3) \right) .
\tag{A.21}
$$

From (A.21) we can compute the Kähler metric

$$
\mathcal{G}_{\psi\psi^*} = \frac{15 \left(-24\zeta(3) \log|5\psi| + 5 \log^4|5\psi| \right)}{|\psi|^2 \left(24\zeta(3) + 10 \log^3|5\psi| \right)} .
\tag{A.22}
$$

We can expand (A.22) as to obtain:

$$
\mathcal{G}_{\psi\psi^*} = \frac{3}{4|\psi|^2 \log^2|5\psi|} \left(1 - \frac{48\zeta(3)}{25 \log^3|5\psi|} + \cdots \right) .
\tag{A.23}
$$

We perform the change of variable

$$
t \equiv -\frac{5}{2\pi i} \log(5\psi)
\tag{A.24}
$$

in order to make easier the comparison with the metric of the large-radius limit of type IIA on $\mathcal{M}$, which is obtained in the following section. The leading term of (A.23) becomes

$$
\mathcal{G}_{tt^*} = \tfrac{3}{4}(\Im\mathfrak{m}\, t)^{-2} ,
\tag{A.25}
$$

which is, as we will see, the large-radius limit metric of the Kähler-structure moduli space, the scalar manifold of type IIA on $\mathcal{M}$.

A.2 Type IIA on the quintic $\mathcal{M}$ and mirror map

The low-energy effective theory of of type IIA superstring theory compactified on a Calabi–Yau manifold is $N = 2$ supergravity coupled to $h^{(1,1)}$ vector multiplets and $h^{(2,1)} + 1$ hypermultiplets. The prepotential in the large compactification radius limit is given by [51]

$$\mathcal{F}^0_{\text{IIA}}(\mathcal{X}) = -\frac{1}{3!} \frac{\kappa^0_{ijk} \mathcal{X}^i \mathcal{X}^j \mathcal{X}^k}{\mathcal{X}^0}, \qquad i, j, k = 1, \dots, h^{(1,1)}. \tag{A.26}$$

where κ^0_{ijk} are the triple intersection numbers.

We take the compactification manifold to be quintic $\mathcal{M}$, hence $h^{(1,1)} = 1$ and $h^{(2,1)} = 101$. Since, as in the type IIB case, we are only interested in the vector multiplet moduli space, we set the hypermultiplets to zero and deal solely with the complex Kähler-structure moduli space $C^{(1,1)}_{\text{IIA}}$, which is a complex one-dimensional special Kähler manifold.

If we denote by e the generator of $H^2(\mathcal{M}, \mathbb{Z})$, the only non-vanishing triple intersection number at tree level is

$$\kappa^0_{1,1,1} = \int_{\mathcal{M}} e \wedge e \wedge e = 5. \tag{A.27}$$

Then, in terms of the coordinate

$$t \equiv \mathcal{X}^1/\mathcal{X}^0 \tag{A.28}$$

and in the Kähler gauge $\mathcal{X}^0 = 1$, the Kähler potential is given by

$$\mathcal{K}^0_{\text{IIA}} = -\log\left[\frac{20}{3}(\Im\mathfrak{m}\, t)^3\right]. \tag{A.29}$$

The Kähler metric reads

$$\mathcal{G}^0_{tt^*} = \frac{3}{4}\left(\Im\mathfrak{m}\, t\right)^{-2}. \tag{A.30}$$

Comparing eqs. (A.25) and (A.30) we can see that the large complex-structure limit of the metric of $C^{(2,1)}_{\text{IIB}}$ agrees with the corresponding bare (uncorrected) quantities for $C^{(1,1)}_{\text{IIA}}$.

We are interested in how the loop corrections and worldsheet instanton corrections (we restrict ourselves to a two-derivative action) to eq. (A.26) affect non-extremal black-hole solutions. One can write the corrected prepotential [52] in the form $\mathcal{F}_{\text{IIA}} = \mathcal{F}^{\text{pert}}_{\text{IIA}} + \mathcal{F}^{\text{npert}}_{\text{IIA}}$, where $\mathcal{F}^{\text{pert}}_{\text{IIA}}$ denotes the perturbatively-corrected prepotential and $\mathcal{F}^{\text{npert}}_{\text{IIA}}$ denotes the exponentially small terms due to instanton corrections. They are given by:

$$\mathcal{F}^{\text{pert}}_{\text{IIA}} = \mathcal{F}^0_{\text{IIA}} + \mathcal{F}^{\text{loop}}_{\text{IIA}} = -\frac{5}{6}\frac{\left(\mathcal{X}^1\right)^3}{\mathcal{X}^0} - \frac{11}{4}(\mathcal{X}^1)^2 + \frac{25}{12}\mathcal{X}^0\mathcal{X}^1 - ik(\mathcal{X}^0)^2, \tag{A.31}$$

$$\mathcal{F}^{\text{npert}}_{\text{IIA}} = \sum_l n_l \, \text{Li}_3\left(e^{2\pi il\mathcal{X}^1/\mathcal{X}^0}\right), \tag{A.32}$$

where

$$\text{Li}_3(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^3}, \tag{A.33}$$

and n_k is the number of rational curves of degree k, and where we have defined the real numerical constant

$$k \equiv \frac{c}{2} \equiv \frac{25}{2\pi^3}\zeta(3). \tag{A.34}$$

For large values of the quintic radius $\Im m\, t \gg 1$, the non-perturbative contribution to the prepotential are exponentially small and can be ignored.

The type IIB theory compactified on $\mathcal{W}$ is related to the type IIA one compactified on $\mathcal{M}$ through the mirror map, which can be expressed as a symplectic transformation of the holomorphic symplectic section with matrix N given by [52]

$$\Pi_{\mathrm{IIA}} = \frac{\mathcal{F}_{\mathrm{IIB}\,1}}{\mathcal{X}^0} N \Pi_{\mathrm{IIB}}\,, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}\,, \tag{A.35}$$

and the coordinate transformation

$$t = \frac{2\,(\omega_1 - \omega_0) + \omega_2 - \omega_4}{5\omega_0}\,, \tag{A.36}$$

where we are denoting the holomorphic symplectic section of the type IIA theory compactified on $\mathcal{W}$ by

$$\Pi_{\mathrm{IIA}}(\psi) = \begin{pmatrix} \mathcal{X}^0 \\ \mathcal{X}^1 \\ \mathcal{F}_{\mathrm{IIA}\,0} \\ \mathcal{F}_{\mathrm{IIA}\,1} \end{pmatrix}\,. \tag{A.37}$$

Consequently, at the supergravity level, both theories are the same theory in different coordinates and symplectic frames.

A.3   Large complex-structure limit

In this section we give the explicit expressions for the periods in region $|\psi| > 1, 0 \leq \mathrm{Arg}(\psi) < \frac{2\pi}{5}$, and we also obtain the large complex-structure limit [52]. The periods are given by:

$$\omega_j(\psi) = \sum_{r=0}^{3} \log^r(5\psi) \sum_{n=0}^{\infty} b_{jrn} \frac{(5\psi)!}{(n!)^5 (5\psi)^{5n}}\,, \qquad |\psi| > 1\,, \tag{A.38}$$

where the coefficients are given by lengthy expressions that can be found in [52]. In the large complex-structure limit $\psi \to \infty$ we keep the first term in the pure power expansion of eq. (A.38). We can then write a vector of coefficients:

$$b_r = -\left(\frac{2\pi i}{5}\right)^3 \begin{pmatrix} b_{2r0} \\ b_{1r0} \\ b_{0r0} \\ b_{4r0} \end{pmatrix}\,, \tag{A.39}$$

in terms of which the large complex-structure limit of the period vector in Picard–Fuchs basis is written as:

$$\hat{\omega} \sim \sum_{r=0}^{3} b_r \log^r(5\psi)\,. \tag{A.40}$$

Eq. (A.40) is the starting point for obtaining the relevant quantities of the model in the limit $\psi \to \infty$.

A.4 A simpler prepotential

As already mentioned, for large values of the quintic radius $\Im m\, t \gg 1$, the non-perturbative contributions to the prepotential are exponentially small, so $\mathcal{F}_{\mathrm{IIA}}^{\mathrm{npert}}$ of eq. (A.32) can be neglected. Taking into account just eq. (A.31), the holomorphic symplectic section is given by

$$
\Pi_{\mathrm{pert}} = \begin{pmatrix} \mathcal{X}^0 \\[6pt] \mathcal{X}^1 \\[6pt] \dfrac{5}{6}\dfrac{(\mathcal{X}^1)^3}{(\mathcal{X}^0)^2} + \dfrac{25}{12}\mathcal{X}^1 - ic\mathcal{X}^0 \\[12pt] -\dfrac{5}{2}\dfrac{(\mathcal{X}^1)^2}{\mathcal{X}^0} - \dfrac{11}{2}\mathcal{X}^1 + \dfrac{25}{12}\mathcal{X}^0 \end{pmatrix} , \tag{A.41}
$$

In the spirit of ref. [57], the symplectic (Peccei–Quinn) transformation

$$
\hat{\mathcal{S}} \equiv \left(\begin{array}{c|c} \mathbb{I} & 0 \\ \hline \begin{matrix} 0 & -\frac{25}{12} \\ -\frac{25}{12} & \frac{11}{2} \end{matrix} & \mathbb{I} \end{array} \right) , \tag{A.42}
$$

brings the section to the simpler form

$$
\hat{\Pi}_{\mathrm{pert}} = \begin{pmatrix} \hat{\mathcal{X}}^0 \\[6pt] \hat{\mathcal{X}}^1 \\[6pt] \dfrac{5}{6}\dfrac{(\hat{\mathcal{X}}^1)^3}{(\hat{\mathcal{X}}^0)^2} - ic\hat{\mathcal{X}}^0 \\[12pt] -\dfrac{5}{2}\dfrac{(\hat{\mathcal{X}}^1)^2}{\hat{\mathcal{X}}^0} \end{pmatrix} , \tag{A.43}
$$

which can be derived from the prepotential

$$
\hat{\mathcal{F}}_{\mathrm{quintic}}^{\mathrm{pert}} = -\frac{5}{6}\frac{(\hat{\mathcal{X}}^1)^3}{\hat{\mathcal{X}}^0} - \frac{i}{2}c(\hat{\mathcal{X}}^0)^2 . \tag{A.44}
$$

The geometry of the scalar manifold in the corrected case is quite different from $SL(2,\mathbb{R})/U(1)$ of the pure t^3 model. It is not a homogeneous space and the conditions that $\Im m\, t$ has to satisfy are also different: the Kähler potential is given by

$$
e^{-\mathcal{K}_{\mathrm{pert}}} = \tfrac{20}{3}(\Im m\, t)^3 + 2c \tag{A.45}
$$

and the fact that $\mathcal{K}$ must be real implies

$$
\Im m\, t > -\left(\tfrac{3}{10}c\right)^{1/3} . \tag{A.46}
$$

The Kähler metric is given by

$$\mathcal{G}_{t\bar{t}} = \frac{15 \,\Im\mathfrak{m}\, t \left[-3c + 5(\Im\mathfrak{m}\, t)^3\right]}{\left[3c + 10(\Im\mathfrak{m}\, t)^3\right]^2}\,. \tag{A.47}$$

For it to be positive definite, we need to demand $\Im\mathfrak{m}\, t \left[-3\epsilon c + 5(\Im\mathfrak{m}\, t)^3\right] > 0$. This condition, together with eq. (A.46), gives the domain of definition for $\Im\mathfrak{m}\, t$:

$$\Im\mathfrak{m}\, t \in \left(-\left(\frac{3c}{10}\right)^{1/3}, 0\right) \cup \left(\left(\frac{3c}{5}\right)^{1/3}, \infty\right)\,. \tag{A.48}$$

From the point of view of the supergravity theory, this is the only condition that the scalar needs to satisfy for the solution to be well defined. If, however, this supergravity is to be seen as an effective description of the underlying superstring theory, there are more conditions to be met by t. In particular, the prepotential (A.44) is an expansion around $t \to \infty$, valid only inside the radius of convergence:

$$\Im\mathfrak{m}\, t > \Im\mathfrak{m}\, t(1)\,, \tag{A.49}$$

where $t(\psi)$ is the mirror map, ψ is the modulus of the mirror related theory, and the conifold point is assumed to be at $\psi = 1$.

References

[1] T. Mohaupt and O. Vaughan, *The Hesse potential, the c-map and black hole solutions*, *JHEP* **1207** (2012) 163, [`arXiv:1112.2876`].

[2] P. Meessen, T. Ortín, J. Perz, and C. S. Shahbazi, *H-FGK formalism for black-hole solutions of N=2, d=4 and d=5 supergravity*, *Phys.Lett.* **B709** (2012) 260–265, [`arXiv:1112.3332`].

[3] T. Mohaupt and K. Waite, *Instantons, black holes and harmonic functions*, *JHEP* **0910** (2009) 058, [`arXiv:0906.3451`].

[4] T. Mohaupt and O. Vaughan, *Non-extremal Black Holes, Harmonic Functions, and Attractor Equations*, *Class.Quant.Grav.* **27** (2010) 235008, [`arXiv:1006.3439`].

[5] T. Mohaupt and O. Vaughan, *Non-extremal black holes from the generalised r-map*, `arXiv:1208.4302`.

[6] A. de Antonio Martín, T. Ortín, and C. S. Shahbazi, *The FGK formalism for black p-branes in d dimensions*, *JHEP* **1205** (2012) 045, [`arXiv:1203.0260`].

[7] P. Meessen, T. Ortín, J. Perz, and C. S. Shahbazi, *Black holes and black strings of N=2, d=5 supergravity in the H-FGK formalism*, *JHEP* **1209** (2012) 001, [`arXiv:1204.0507`].

[8] S. Ferrara, G. W. Gibbons, and R. Kalloshh, *Black holes and critical points in moduli space*, *Nucl.Phys.* **B500** (1997) 75–93, [`hep-th/9702103`].

[9] P. Galli, T. Ortín, J. Perz, and C. S. Shahbazi, *Non-extremal black holes of N=2, d=4 supergravity*, *JHEP* **1107** (2011) 041, [`arXiv:1105.3311`].

[10] P. Meessen and T. Ortín, *Non-Extremal Black Holes of N=2,d=5 Supergravity*, *Phys.Lett.* **B707** (2012) 178–183, [`arXiv:1107.5454`].

[11] S. Ferrara, R. Kallosh, and A. Strominger, *N=2 extremal black holes*, *Phys.Rev.* **D52** (1995) 5412–5416, [`hep-th/9508072`].

[12] A. Strominger, *Macroscopic entropy of N=2 extremal black holes*, *Phys.Lett.* **B383** (1996) 39–43, [`hep-th/9602111`].

[13] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lüst, et al., *Classical and quantum N=2 supersymmetric black holes*, *Nucl.Phys.* **B488** (1997) 236–260, [`hep-th/9610105`].

[14] W. Sabra, *General static N=2 black holes*, *Mod.Phys.Lett.* **A12** (1997) 2585–2590, [`hep-th/9703101`].

[15] W. Sabra, *Black holes in N=2 supergravity theories and harmonic functions*, *Nucl.Phys.* **B510** (1998) 247–263, [`hep-th/9704147`].

[16] K. Behrndt, D. Lüst, and W. A. Sabra, *Stationary solutions of N=2 supergravity*, *Nucl.Phys.* **B510** (1998) 264–288, [`hep-th/9705169`].

[17] G. L. Cardoso, B. de Wit, J. Käppeli, and T. Mohaupt, *Stationary BPS solutions in N=2 supergravity with R**2 interactions*, *JHEP* **0012** (2000) 019, [`hep-th/0009234`].

[18] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **0008** (2000) 050, [`hep-th/0005049`].

[19] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, *JHEP* **1111** (2011) 127, [`hep-th/0304094`].

[20] P. Meessen and T. Ortín, *The Supersymmetric configurations of N=2, D=4 supergravity coupled to vector supermultiplets*, *Nucl.Phys.* **B749** (2006) 291–324, [`hep-th/0603099`].

[21] R. R. Khuri and T. Ortín, *A Nonsupersymmetric dyonic extreme Reissner–Nordström black hole*, *Phys.Lett.* **B373** (1996) 56–60, [`hep-th/9512178`].

[22] T. Ortín, *Extremality versus supersymmetry in stringy black holes*, *Phys.Lett.* **B422** (1998) 93–100, [`hep-th/9612142`].

[23] K. Goldstein and S. Katmadas, *Almost BPS black holes*, *JHEP* **0905** (2009) 058, [`arXiv:0812.4183`].

[24] G. Bossard and S. Katmadas, *Duality covariant non-BPS first order systems*, *JHEP* **1209** (2012) 100, [`arXiv:1205.5461`].

[25] G. L. Cardoso, A. Ceresole, G. Dall'Agata, J. M. Oberreuter, and J. Perz, *First-order flow equations for extremal black holes in very special geometry*, *JHEP* **0710** (2007) 063, [`arXiv:0706.3373`].

[26] E. G. Gimon, F. Larsen, and J. Simon, *Constituent Model of Extremal non-BPS Black Holes*, *JHEP* **0907** (2009) 052, [`arXiv:0903.0719`].

[27] I. Bena, G. Dall'Agata, S. Giusto, C. Ruef, and N. P. Warner, *Non-BPS Black Rings and Black Holes in Taub-NUT*, *JHEP* **0906** (2009) 015, [`arXiv:0902.4526`].

[28] P. Galli, K. Goldstein, S. Katmadas, and J. Perz, *First-order flows and stabilisation equations for non-BPS extremal black holes*, *JHEP* **1106** (2011) 070, [`arXiv:1012.4020`].

[29] G. Bossard and C. Ruef, *Interacting non-BPS black holes*, *Gen.Rel.Grav.* **44** (2012) 21–66, [`arXiv:1106.5806`].

[30] P. Galli, K. Goldstein, and J. Perz, *On anharmonic stabilisation equations for black holes*, 2012. To appear.

[31] P. Galli, P. Meessen, and T. Ortín, *The Freudenthal gauge symmetry of the black holes of N=2,d=4 supergravity*, 2012. To appear.

[32] G. L. Cardoso, B. de Wit, and S. Mahapatra, *Non-holomorphic deformations of special geometry and their applications*, `arXiv:1206.0577`.

[33] J. Bellorín, P. Meessen, and T. Ortín, *Supersymmetry, attractors and cosmic censorship*, *Nucl.Phys.* **B762** (2007) 229–255, [`hep-th/0606201`].

[34] S. Ferrara, A. Marrani, and A. Yeranyan, *Freudenthal Duality and Generalized Special Geometry*, *Phys.Lett.* **B701** (2011) 640–645, [`arXiv:1102.4857`].

[35] L. Borsten, D. Dahanayake, M. Duff, and W. Rubens, *Black holes admitting a Freudenthal dual*, *Phys.Rev.* **D80** (2009) 026003, [`arXiv:0903.5517`].

[36] T. Ortín, *A Simple Derivation of Supersymmetric Extremal Black Hole Attractors*, *Phys.Lett.* **B700** (2011) 261–264, [`arXiv:1103.2738`].

[37] P. K. Tripathy and S. P. Trivedi, *Non-supersymmetric attractors in string theory*, *JHEP* **0603** (2006) 022, [`hep-th/0511117`].

[38] K. Saraikin and C. Vafa, *Non-supersymmetric black holes and topological strings*, *Class.Quant.Grav.* **25** (2008) 095007, [`hep-th/0703214`].

[39] M. Shmakova, *Calabi-Yau black holes*, *Phys.Rev.* **D56** (1997) 540–544, [`hep-th/9612076`].

[40] S. Bellucci, S. Ferrara, A. Marrani, and A. Shcherbakov, *Splitting of Attractors in 1-modulus Quantum Corrected Special Geometry*, *JHEP* **0802** (2008) 088, [`arXiv:0710.3559`].

[41] S. Bellucci, S. Ferrara, and A. Marrani, *On some properties of the attractor equations*, *Phys.Lett.* **B635** (2006) 172–179, [`hep-th/0602161`].

[42] G. L. Cardoso, V. Grass, D. Lüst, and J. Perz, *Extremal non-BPS black holes and entropy extremization*, *JHEP* **0609** (2006) 078, [`hep-th/0607202`].

[43] P. Bueno, R. Davies, and C. S. Shahbazi, *Quantum Black Holes in Type-IIA String Theory*, `arXiv:1210.2817`.

[44] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, *Vacuum Configurations for Superstrings*, *Nucl.Phys.* **B258** (1985) 46–74.

[45] A. Strominger and E. Witten, *New Manifolds for Superstring Compactification*, *Commun.Math.Phys.* **101** (1985) 341.

[46] P. Candelas, *Yukawa Couplings Between (2,1) Forms*, *Nucl.Phys.* **B298** (1988) 458.

[47] B. R. Greene and M. Plesser, *(2,2) and (2,0) superconformal orbifolds*, .

[48] B. R. Greene and M. Plesser, *Duality in Calabi–Yau moduli space*, *Nucl.Phys.* **B338** (1990) 15–37.

[49] P. Candelas, P. S. Green, and T. Hübsch, *Rolling Among Calabi-Yau Vacua*, *Nucl.Phys.* **B330** (1990) 49.

[50] P. Candelas and X. C. de la Ossa, *Comments on Conifolds*, *Nucl.Phys.* **B342** (1990) 246–268.

[51] P. Candelas and X. de la Ossa, *Moduli Space of Calabi–Yau Manifolds*, *Nucl.Phys.* **B355** (1991) 455–481.

[52] P. Candelas, X. C. De la Ossa, P. S. Green, and L. Parkes, *An Exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds*, *Phys.Lett.* **B258** (1991) 118–126.

[53] M. Hübscher, P. Meessen, and T. Ortín, *Supersymmetric solutions of N=2 D=4 sugra: The Whole ungauged shebang*, *Nucl.Phys.* **B759** (2006) 228–248, [hep-th/0606281].

[54] S. Ferrara and J. Louis, *Flat holomorphic connections and Picard-Fuchs identities from N=2 supergravity*, *Phys.Lett.* **B278** (1992) 240–245, [hep-th/9112049].

[55] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche, and J. Louis, *Picard-Fuchs equations and special geometry*, *Int.J.Mod.Phys.* **A8** (1993) 79–114, [hep-th/9204035].

[56] A. Klemm and S. Theisen, *Considerations of one modulus Calabi-Yau compactifications: Picard-Fuchs equations, Kahler potentials and mirror maps*, *Nucl.Phys.* **B389** (1993) 153–180, [hep-th/9205041].

[57] S. Bellucci, A. Marrani, and R. Roychowdhury, *Topics in Cubic Special Geometry*, *J.Math.Phys.* **52** (2011) 082302, [arXiv:1011.0705].