Spatially dependent decoherence and anomalous diffussion of quantum walks

A. Pérez

Departament de Física Teòrica and IFIC, Universitat de València-CSIC

Dr. Moliner 50, 46100-Burjassot, Spain

A. Romanelli

Instituto de Física, Facultad de Ingeniería, Universidad de la República,

C.C. 30, C.P. 11300, Montevideo, Uruguay

Abstract

We analyze the long time behavior of a discrete time quantum walk subject to decoherence with a strong spatial dependence, acting on one half of the lattice. We show that, except for limiting cases on the decoherence parameter, the quantum walk at late times behaves sub-ballistically, meaning that the characteristic features of the quantum walk are not completely spoiled. Contrarily to expectations, the asymptotic behavior is non Markovian, and depends on the amount of decoherence. This feature can be clearly shown on the long time value of the Generalized Chiral Distribution (GCD).

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I. INTRODUCTION

Quantum Walks (QW) constitute the quantum analogy to classical Random Walks. The latter are an important piece in the design of classical algorithms and are used, for example, to efficiently explore the parameter space of some model. Here we will consider the special case of a QW on a line [1–7]. As in the classical case, QWs have been proposed as an element to design quantum algorithms [8–12]. The importance of the QW has increased with the discovery that it can be used for universal quantum computation [13, 14], and several experimental setups have been proposed or realized to implement it [15–24].

An important point to be discussed, as in all implementations of quantum algorithms, is the possible effect of decoherence due to the interaction of the experimental setup with the environment. This interaction will change the problem of an isolated quantum device to the one corresponding to an *open quantum system* [25, 26]. In most cases, the consequence will be that the quantum algorithm will loose its quantum advantages and, therefore, its outer-performance (as compared to classical algorithms) will be ruined. The effect of decoherence on the QW has been investigated in a number of papers [6, 27–34]. As a general conclusion, it seems clear that decoherence in the coin space of the walker spoils the performance of the QW more effectively. It must be noticed, however, that a small amount of decoherence both on the coin and position of the one dimensional lattice QW might produce some benefits [29], as it produces more uniform distributions. On the other hand, purely spatial decoherence (i.e., decoherence introduced by some kind of defects on the sites of the lattice) may have a distinctive behavior. For example, the authors in [31] study the effects of tunneling on the spatial sites. They find that the characteristic quadratic dependency of the variance on time is not ruined, even for maximal noise. Also, one obtains a smooth probability distribution, except for very strong decoherence.

From the above considerations, it is clear that the effects of noise on the QW are worth to study in order to design practical quantum algorithms which use the properties of the QW. Most papers on this subject have considered the case when the noise appears uniformly distributed on the lattice. This assumption allows the effects of decoherence to be treated within a translationally-invariant formalism, so that some analytical results can be obtained regarding the characteristic properties of the QW distribution [31, 35]. From the experimental point of view, it is true that one can force the setup so as to mimic a spatially

uniform noise acting on the coin and reproduce the expected Anderson localization [36]. But the question that arises is what happens if, in a given experiment, decoherence appears in an uncontrollable, spatially dependent, way. More precisely, decoherence might appear only on a given region of the lattice, or it may act stronger in some parts of the system than in others. Are results developed under the hypothesis of translational invariance still valid under these circumstances, at least approximately? Another possibility is that new phenomena, which were not present for translationally invariant systems, may appear when this restriction does not apply. We think that examining such a possibility can be useful for the design of new experiments, and for the general understanding of decoherence in the QW.

In this paper, motivated by the above discussion, we study a simple model of non-translationally invariant noise in the QW, in which decoherence acts on the coin degree of freedom with some probability p, but only when the walker moves on one half of the lattice. As we will show, even such a simple model will give rise to interesting phenomena, which were not shown (at least to our knowledge) in previous models. At first sight, given the interference properties of the problem under study, one might expect that the characteristic properties of the QW, such as the quadratic growth of the variance with time, should be completely destroyed. We show that, in fact, this is not the case, and we analyze the long time behavior in connection with the given initial conditions. We observe that, for large time steps, the variance grows as a power law. Surprisingly, the walk remains subballistic even for strong values of p.

A characteristic property of the QW is the evolution of the Global Chirality Distribution (GCD) [33, 34, 37]. This distribution gives information about the chirality of the walk, independently of the position. It has been found, for example, that the GCD possesses in some cases an asymptotic limit, which can be related to the initial conditions of the coin. This result showed an unexpected behavior of the QW's dynamics, that is more characteristic of Markov processes. It shows that, watching only the degrees of freedom associated with the chirality, it would be very hard to appreciate the unitary character of the quantum evolution. In more generic words, the simple observation of variables that belong to only one of the simplest sub-spaces can camouflage the unitary character of the evolution [34, 38]. Therefore, it is clear that the study of the GCD is necessary in order to understand the equilibrium between degrees of freedom of the QW. We will study the

evolution of the chirality, as given by the GCD, and specially its long-time behavior for a QW subject to decoherence for the model described above.

The paper is organized as follows. In Sect. II we briefly review the discrete-time QW on a line. In Sect. III we introduce our model for decoherence, and derive the recursion formulae obeyed by the left and right components of the GCD. We discuss the asymptotic expressions for these magnitudes. Numerical calculations that illustrate the behavior of these magnitudes are shown on Sect. IV. Sect. V summarizes our main results.

II. DISCRETE TIME QW WALK ON A LINE

The QW may be defined using either its discrete-time or continuous time version. In this paper, we concentrate on the discrete time version. The discrete time QW on the line corresponds to the evolution of a one-dimensional quantum system in a direction which depends on an additional degree of freedom, the chirality, with two possible states: "left" $|L\rangle$ or "right" $|R\rangle$. The global Hilbert space of the system is the tensor product $H_s \otimes H_c$ where H_s is the Hilbert space associated to the motion on the line, and H_c is the chirality Hilbert space. Let us call T_- (T_+) the operators in H_s that move the walker one site to the left (right) of a unidimensional lattice, and $|L\rangle\langle L|$, $|R\rangle\langle R|$ the chirality projector operators in H_c. We consider the unitary transformations

$$U(\gamma) = T_- \otimes |L\rangle\langle L| \ K_c(\gamma) + T_+ \otimes |R\rangle\langle R| \ K_c(\gamma), \tag{1}$$

where $K_c(\gamma) = \sigma_z \cos\gamma + \sigma_x \sin\gamma$, and σ_z, σ_x are Pauli matrices acting in H_c. For $\gamma = \pi/4$ the Hadamard coin is obtained. When decoherence is not present, the unitary operator $U(\gamma)$ evolves the state in one time step as

$$|\Psi(t+1)\rangle = U(\gamma)|\Psi(t)\rangle, \tag{2}$$

and the state at time t can be expressed as the spinor

$$|\Psi(t)\rangle = \sum_{x=-\infty}^{\infty} \begin{bmatrix} a_x(t) \\ b_x(t) \end{bmatrix} |x\rangle, \tag{3}$$

where the upper (lower) component is associated to the left (right) chirality.

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III. DECOHERENCE IN THE QW

We assume a simple model in which decoherence in the QW appears as a consequence of the *additional* action of σ_z on the coin space with a given probability. Moreover, this operator is assumed to act only on the semipositive line, with a characteristic probability p. Therefore, the simple dynamics described in the previous section has to be modified: Instead of Eq. (2), one has to deal with a density operator ρ describing the state of the QW. The dynamics can be described by the action of given Kraus operators. In our case, we have two operators E_1 and E_2, defined by

$$E_1 = \sqrt{1 - p\theta(x)}\, U(\gamma), \tag{4}$$

$$E_2 = \sqrt{p}\,\theta(x)\sigma_z U(\gamma), \tag{5}$$

where $\theta(x)$ is the Heaviside step function. In this way, the operator E_2 only acts on a position eigenstate $|x\rangle$ when $x \geq 0$, and E_1 reproduces the ordinary QW, Eq. (2) whenever $x < 0$. In other words, one has the usual QW on the left side of the lattice, whereas on the right side an additional dephasing operation σ_z appears with some probability p, where $p \in [0, 1]$ is a real number. One can readily check that the necessary condition for trace-preserving Kraus operators

$$E_1 E_1^\dagger + E_2 E_2^\dagger = 1, \tag{6}$$

is fulfilled. The time evolution for the density matrix of the quantum walk is then given by the map

$$\rho(t + 1) = E_1\rho(t)E_1^\dagger + E_2\rho(t)E_2^\dagger. \tag{7}$$

Later on, we will also consider the map produced on the GCD, which can be obtained from $\rho(t)$ by tracing out the spatial degrees of freedom. In this way, we define the reduced density operator for the chirality evolution

$$\rho_c(t) \equiv Tr_s\{\rho(t)\} = \sum_x \langle x|\rho(t)|x\rangle. \tag{8}$$

From this reduced operator, one can calculate the diagonal components of the GCD, defined simply as the corresponding elements in the chiral $\{|L\rangle, |R\rangle\}$ basis

$$\Pi_L(t) \equiv \langle L \mid \rho_c(t) \mid L\rangle, \tag{9}$$

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$$\Pi_R(t) \equiv \langle R \mid \rho_c(t) \mid R \rangle, \tag{10}$$

whereas the interference term is given by

$$Q(t) = \frac{1}{2} \left(\langle L \mid \rho_c(t) \mid R \rangle + \langle R \mid \rho_c(t) \mid L \rangle \right). \tag{11}$$

In order to obtain a recursive formula for the GCD, let us expand the density matrix $\rho(t)$ in the basis $\{|x\rangle \otimes |i\rangle, x \in \mathbb{Z}, i = L, R\}$ of the whole Hilbert space, such as

$$\rho(t) = \sum_{x,y \in \mathbb{Z}} \sum_{i,j=L,R} R_{x,y;i,j}(t) \, |x\rangle\langle y| \otimes |i\rangle\langle j| \tag{12}$$

After substitution in Eq. (8), using Eqs. (4,5,7) we obtain, with the help of some algebra,

$$\begin{aligned}
Tr_s\{E_1\rho(t)E_1^\dagger\} = &\sum_x \{[1 - p\theta(x)][M_R R_{x,x}(t)M_R^\dagger + M_L R_{x,x}(t)M_L^\dagger] \\
&+ \sum_x \{\sqrt{1 - p\theta(x+1)}\sqrt{1 - p\theta(x-1)}[M_R R_{x-1,x+1}(t)M_L^\dagger \\
&+ M_L R_{x+1,x-1}(t)M_R^\dagger]\}
\end{aligned} \tag{13}$$

where $M_L = (|L\rangle\langle L|)K_c$ and $M_L = (|R\rangle\langle R|)K_c$. On the other hand,

$$\begin{aligned}
Tr_s\{E_2\rho(t)E_2^\dagger\} = &\sigma_z \sum_x \{p\theta(x)[M_R R_{x,x}(t)M_R^\dagger + M_L R_{x,x}(t)M_L^\dagger] \\
&+ \sum_x \{p\theta(x+1)\theta(x-1)[M_R R_{x-1,x+1}(t)M_L^\dagger \\
&+ M_L R_{x+1,x-1}(t)M_R^\dagger]\}\sigma_z.
\end{aligned} \tag{14}$$

In the above equations, we have introduced

$$R_{x,y}(t) \equiv \sum_{i,j=L,R} R_{x,y;i,j}(t) \, |i\rangle\langle j|, \tag{15}$$

which is an operator defined on the coin space. The corresponding magnitudes at time $t+1$ can be derived from Eqs. (13,14,15). After a lengthy, but straightforward calculation, we arrive to the following equations relating the diagonal elements of the GCD at time $t+1$ to those at time t:

$$\Pi_L(t+1) = \cos^2\gamma \, \Pi_L(t) + \sin^2\gamma \, \Pi_R(t) + \sin 2\gamma \, Q(t) \tag{16}$$

$$\Pi_R(t+1) = \sin^2\gamma \, \Pi_L(t) + \cos^2\gamma \, \Pi_R(t) - \sin 2\gamma \, Q(t) \tag{17}$$

The latter results agree with similar expressions obtained in Refs. [33, 34, 37] for the GCD dynamics, but there a handmade technique [39, 40] was used to separate the Markovian evolution from the interference term. It can also be obtained [34] from Eqs. (16,17) that, for $p = 0$, $Q(t)$, $\Pi_L(t)$ and $\Pi_R(t)$ have long-time limits whose values are determined by the initial conditions. In the next section, it is shown that, when $p \neq 0$, Eqs. (16,17) also have stationary solutions.
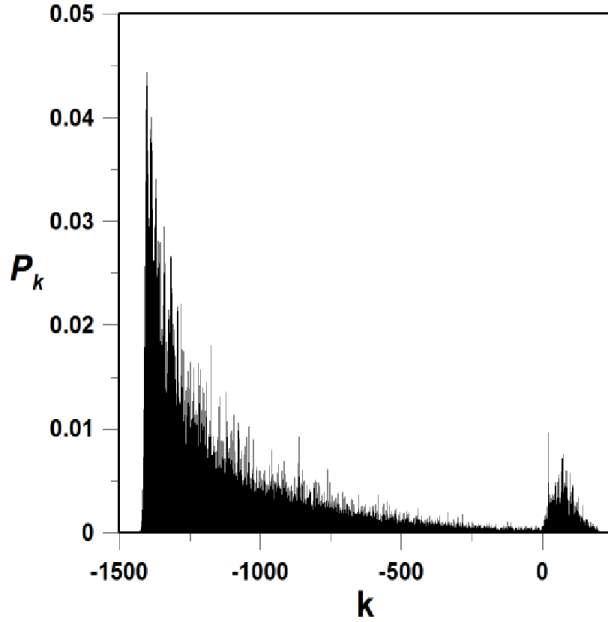

Figure 1:   The position distribution P_k as a function of the dimensionless position at $t = 2000$ with $p = 0.2$.

According to this, it is expected that the asymptotic GCD will satisfy the following equations

$$\Pi_L(\infty) = \cos^2 \gamma \, \Pi_L(\infty) + \sin^2 \gamma \, \Pi_R(\infty) + \sin 2\gamma \, Q(\infty), \qquad (18)$$

$$\Pi_R(\infty) = \sin^2 \gamma \, \Pi_L(\infty) + \cos^2 \gamma \, \Pi_R(\infty) - \sin 2\gamma \, Q(\infty), \qquad (19)$$

where we have defined $\Pi_L(\infty) \equiv \Pi_L(t \to \infty)$, $\Pi_R(\infty) \equiv \Pi_R(t \to \infty)$ and $Q(\infty) \equiv Q(t \to \infty)$. The asymptotic behavior of the GCD has no explicit dependence on the parameter p, but the asymptotic values of $\Pi_L(\infty)$, $\Pi_R(\infty)$ and $Q(\infty)$ do have an implicit dependence.
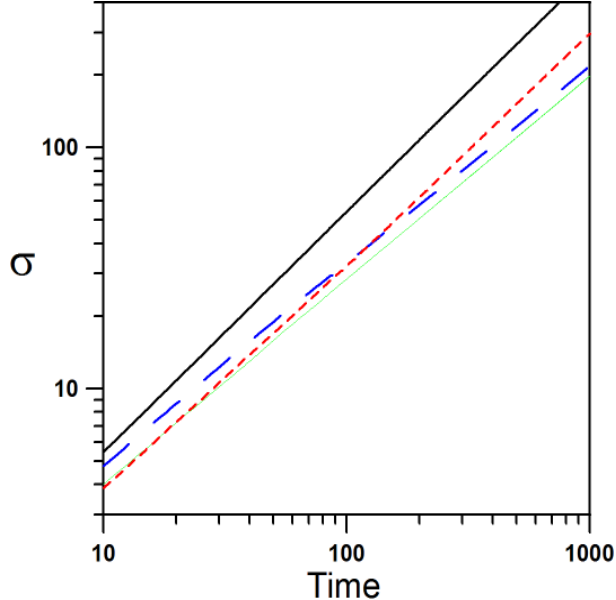
Figure 2: (Color online): The standard deviation, σ obtained from all points on the lattice (positive and negative), as a function of the time step in log-log scales, for different values of p, corresponding to $p = 0$ (black, thick solid line), $p = 0.9$ (red, short-dashed line), $p = 0.5$ (green, thin line) and $p = 0.1$ (blue, long dashed line).

Using Eqs. (18,19), the stationary solution for the GCD is, then

$$\begin{bmatrix} \Pi_L(\infty) \\ \Pi_R(\infty) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + 2Q(\infty)/\tan\gamma \\ 1 - 2Q(\infty)/\tan\gamma \end{bmatrix}. \tag{20}$$

These solutions are the same as those obtained in Ref. [34] for a more complicated decoherence. Therefore, the dynamical evolution of the GCD is non Markovian, in the sense that asymptotic magnitudes depend on the initial conditions (as well as on the probability p). As we show in the next Section, the interference term $Q(\infty)$ is not negligible: The behavior of this composite QW is at first sight unexpected, since usually decoherence destroys the unitary correlation, providing a route towards a classical-like behavior described by a Markov process.

IV. NUMERICAL CALCULATIONS

The global evolution of the system depends on the application of the two operators Eqs. (4, 5) in the Hilbert space. Each operator acts on both the chirality and position spaces, and
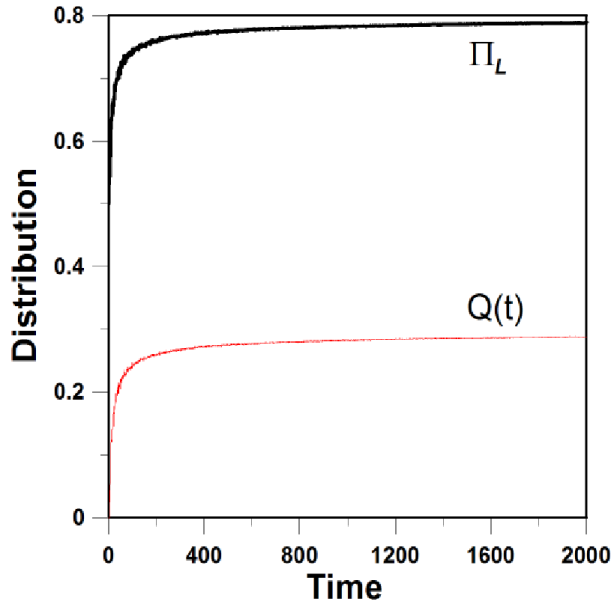
Figure 3: The left component of the GCD and the interference term as functions of the time step, for $p = 0.2$

has a corresponding map. For our numerical calculations, instead of working with the density operator, a statistical description can be obtained by combining many runs of the form Eq. (2) with the appropriate statistical weights. Such a description, of course, is equivalent to working with the density matrix, as long as a sufficiently large ensemble is considered. For our purposes, we have found that this procedure is more efficient than a whole calculation involving a large number of time steps and, consequently, of matrix positions, if one was to deal with the density matrix. We have checked that the number of runs in the ensemble is large enough so as to achieve convergence.

To implement the algorithm we proceed as follows. At each time step t, the usual QW map is applied to each position at the left of the origin, while at the right half-line the usual QW map, or the map obtained by the additional action of the Pauli matrix σ_z, are applied with probabilities $1 - p$ and p, respectively, p being the only parameter in the model. We take as the initial conditions a walker starting at the central position $|0\rangle$ with chirality $\frac{1}{\sqrt{2}}(1, i)^T$, $\gamma = \pi/4$, and we consider an ensemble of 100 dynamical trajectories with 2000 time steps. Finally, magnitudes are averaged over the whole ensemble.

Fig. 1 shows the position distribution of the QW given (for a single run) by $P_k(t) \equiv |a_k(t)|^2 + |b_k(t)|^2$, as a function of the position k at $t = 2000$. The plot, as any other

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quantity we will show, reflects the final averaging over the ensemble. This figure shows, for $p = 0.2$, a very different behavior to the left and to the right of the origin. In particular, the behavior of the system on the left half-line is close to one of the QW without decoherence, while on the right half-line the behavior is typical of a classical walker with a Gaussian distribution. It is worth noticing that most of the probability goes to the left, as shown in this figure. In other words, the region on the right, where decoherence is acting, tends to reflect the walker towards the left, where it can freely propagate. This effect is important in order to understand other results that we show below.

As it is well known, one of the most striking properties of the one-dimensional QW is its ability to spread over the line linearly in time, as characterized by the standard deviation $\sigma(t) \sim t$, while its classical analog spreads out as the square root of time $\sigma(t) \sim t^{1/2}$. In our case, the standard deviation of the system is presented in Fig. 2 for different values of p. We see that the standard deviation grows subballistically, as a power law $\sigma(t) \sim t^c$, where c takes a constant value. It is interesting to note that this behavior remains for the whole range of values of p. For values of p close to zero, the QW spreads almost ballistically, as expected. For intermediate values (between 0 and 1), most of the probability goes to the left, and this prevents decoherence from effectively reducing the QW to a diffusive behavior. Actually, this is only a simplified vision, as the actual process is more complicated: Indeed, the QW on the left is actually evolving under the form of waves propagating **both** to the left and to the right. These right-moving waves will penetrate on the right region and eventually be reflected to the left. It is this complicated interplay between right and left motions that leads to the subballistic behavior explained above. For values of p close to unity, the evolution on the right side is dictated by the operator $\sigma_z U(\gamma)$, which is obviously unitary. Therefore, the QW on both sides will recover the main properties of the standard QW.

In Fig. 3 we present the left (L) component of the GCD and the interference term $Q(t)$: Here it is seen that these quantities have definite limits. We have checked numerically that Eq.(20) is satisfied independently of both the initial condition and the values of p, for $p > 0$.

V. CONCLUSIONS

We have analyzed a model for spatially-dependent decoherence on the quantum walk, which we implement via the introduction of appropriate Kraus operators. In our simplified

model, the decoherence acts only on one side of the lattice. We introduced a particular model, defined as an additional operation acting on the coin, although we have explored other choices of the Kraus operators, as the ones defined in [30, 31] with similar results. In spite of the noise introduced by these operators, some characteristic properties of the QW, such as the long term linear behavior of the standard deviation, still survive. We have calculated the GCD, and showed that it has a given limit as a function of time. We conclude that the dynamical evolution of the GCD is not Markovian, and has an asymptotic value that depends on the initial conditions and on the probability of decoherence. This is an unexpected result, since decoherence tends to destroy the correlations arising from the unitary evolution, thus paving the way to a classical-like Markovian process. It is also in agreement with other results discussed here, showing that, in spite of an apparently extreme decoherence, acting on an semi-infinite lattice, some quantum features are preserved.

Acknowledgments

[1] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993).

[2] D. A. Meyer, J. Stat. Phys. **85**, 551 (1996), quant-ph/9604003.

[3] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, in *STOC* (2001), pp. 37–49.

[4] A. Ambainis, International Journal of Quantum Information **1**, 507 (2003).

[5] J. Kempe, Contemporary Physics **44**, 307 (2003).

[6] V. Kendon, Math. Struct. in Comp. Sci **17(6)**, 1169 (2006), quant-ph/0606016.

[7] V. Kendon, Phil. Trans. R. Soc .A **364**, 3407 (2006).

[8] N. Shenvi, J. Kempe, and K. B. Whaley, Phys. Rev. A **67**, 052307 (2003).

[9] A. Ambainis, SIAM Journal on Computing **37**, 210 (2007).

[10] A. M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, and D. A. Spielman, STOC Proc. pp. 59–68 (2003), quant-ph/0209131.

[11] A. M. Childs and J. Goldstone, Phys. Rev. A **70**, 022314 (2004).

[12] A. Tulsi, Phys. Rev. A **78**, 012310 (2008).

[13] A. M. Childs, Phys. Rev. Lett. **102**, 180501 (2009).

[14] N. B. Lovett, S. Cooper, M. Everitt, M. Trevers, and V. Kendon, Phys. Rev. A **81**, 042330 (2010).

[15] P. L. Knight, E. Roldán, and J. E. Sipe, Optics Communications **227**, 147 (2003), ISSN 0030-4018.

[16] B. C. Travaglione and G. J. Milburn, Phys. Rev. A **65**, 032310 (2002).

[17] W. Dür, R. Raussendorf, V. M. Kendon, and H.-J. Briegel, Phys. Rev. A **66**, 052319 (2002).

[18] J. Du, H. Li, X. Xu, M. Shi, J. Wu, X. Zhou, and R. Han, Phys. Rev. A **67**, 042316 (2003).

[19] C. A. Ryan, M. Laforest, J. C. Boileau, and R. Laflamme, Phys. Rev. A **72**, 062317 (2005).

[20] P. Xue, B. C. Sanders, A. Blais, and K. Lalumière, Phys. Rev. A **78**, 042334 (2008).

[21] D. Witthaut, Phys. Rev. A **82**, 033602 (2010).

[22] A. Schreiber, K. N. Cassemiro, V. Potoček, A. Gábris, P. J. Mosley, E. Andersson, I. Jex, and C. Silberhorn, Phys. Rev. Lett. **104**, 050502 (2010).

[23] P. Xue, B. C. Sanders, and D. Leibfried, Phys. Rev. Lett. **103,**, 183602 (2009).

[24] Z. Zhao, J. Du, H. Li, T. Yang, Z.-B. Chen, and J.-W. Pan (2007).

[25] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2007).

[26] U. Weiss, *Quantum Dissipative Systems* (World Scientific Publishing Company, Singapore, 2008).

[27] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. A **67**, 032304 (2003).

[28] C. C. Lopez and J. P. Paz, Phys. Rev. A **68,**, 052305 (2003), quant-ph/0308104.

[29] V. Kendon and B. Tregenna, Phys. Rev. A **67**, 042315 (2003).

[30] M. Annabestani, S. J. Akhtarshenas, and M. R. Abolhassani, Phys. Rev. A **81,**, 032321 (2009).

[31] M. Annabestani, S. J. Akhtarshenas, and M. R. Abolhassani (2010), 1004.4352.

[32] M. A. Broome, A. Fedrizzi, B. P. Lanyon, I. Kassal, A. Aspuru-Guzik, and A. G. White, Phys. Rev. Lett. **104**, 153602 (2010).

[33] A. Romanelli, Phys. Rev. A **81**, 062349 (2010).

[34] A. Romanelli and G. Hernandez, Physica A: Statistical Mechanics and its Applications **390**, 1209 (2011).

[35] A. Ahlbrecht, H. Vogts, A. H. Werner, and R. F. Werner, Journal of Mathematical Physics

52, 042201 (pages 36) (2011).

[36] A. Schreiber, K. N. Cassemiro, V. Potoček, A. Gábris, I. Jex, and C. Silberhorn, Phys. Rev. Lett. **106**, 180403 (2011).

[37] A. Romanelli, Phys. Rev. A **85**, 012319 (2012), URL http://link.aps.org/doi/10.1103/PhysRevA.85.012319.

[38] M. Castagnino and S. Fortin, International Journal of Theoretical Physics **50**, 2259 (2011).

[39] A. Romanelli, A. C. S. Schifino, G. Abal, R. Siri, and R. Donangelo, Physics Letters A **313**, 325 (2003).

[40] A. Romanelli, A. C. S. Schifino, R. Siri, G. Abal, A. Auyuanet, and R. Donangelo, Physica A: Statistical Mechanics and its Applications **338**, 395 (2004).