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WIENER'S THEOREM ON HYPERGROUPS

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Abstract. The following theorem on the circle group $\mathbb{T}$ is due to Norbert Wiener: If $f \in L^1(\mathbb{T})$ has non-negative Fourier coefficients and is square integrable on a neighbourhood of the identity, then $f \in L^2(\mathbb{T})$. This result has been extended to even exponents including $p = \infty$, but shown to fail for all other $p \in (1, \infty]$. All of this was extended further (appropriately formulated) well beyond locally compact abelian groups. In this paper we prove Wiener's theorem for even exponents for a large class of commutative hypergroups. In addition, we present examples of commutative hypergroups for which, in sharp contrast to the group case, Wiener's theorem holds for *all* exponents $p \in [1, \infty]$. For these hypergroups and the Bessel-Kingman hypergroup with parameter $\frac{1}{2}$ we characterise those locally integrable functions that are of positive type and square-integrable near the identity in terms of amalgam spaces.

1. Introduction

On the unit circle $\mathbb{T}$ consider the following statement: If an integrable function on $\mathbb{T}$ has non-negative Fourier coefficients and is $p-$integrable on some neighbourhood of the identity, then f is $p-$integrable on all of $\mathbb{T}$. For $p = 2$ this is a theorem of Norbert Wiener. It was then shown to hold for all even $p \in \mathbb{N}$ and $p = \infty$, but to fail for all other $p \in (1, \infty]$ [15, 13]. All of this was extended (appropriately formulated) successively to compact abelian [12], locally compact abelian [7] and finally IN-groups [11] (groups having at least one relatively compact neighbourhood of the identity invariant under inner automorphisms). Since, in the original formulation, Wiener's theorem does not extend to non-compact groups (it fails even for the real line), the results on non-compact groups G are

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formulated with $L^p(G)$ replaced by the amalgam space $(L^p, \ell^\infty)(G)$. (for compact groups this is no change, as $(L^p, \ell^\infty)(G) = L^p(G)$ in this case). Related information can be found in [11, p. 1].

In Section 2 of this paper we extend the positive result to a large class of commutative hypergroups, namely those where the product of bounded continuous positive definite functions is itself positive definite (see Corollary 2.15 below). In particular this applies to strong hypergroups.

In Section 3 we consider Bessel-Kingman hypergroups. These are strong hypergroups, so the results of Section 2 apply to them. For the motion hypergroup, *i.e.* the Bessel-Kingman hypergroup with $\alpha = \frac{1}{2}$, we show (Theorem 3.6) that for $p = 2$ there is a characterization like the one in [7] of positive definite functions that are square integrable near the identity. Since the proof (following [7]) makes use of results about Fourier transforms, duality and interpolation for amalgam spaces defined via certain tilings, we need to show that on this hypergroup the norms for these spaces are equivalent to amalgam norms defined using translations. For groups this equivalence is well known (see [1] or [6]), but for hypergroups this is not clear. We obtain some results on translation, convolution and the Fourier transform for amalgam spaces on the motion hypergroup; these are needed for the proof of Theorem 3.6. We also compare our amalgam norms with some other ones, including those in [3].

Finally in Section 4 we look at the countable non-discrete hypergroups considered in [5] and [14]. We prove the analogue of Theorem 3.6 and show that for these hypergroups, in sharp contrast to the group case, Wiener's theorem holds for *all* exponents $p \in [1, \infty]$; see Theorem 4.10 and Corollary 4.12 below.

2. Wiener's theorem for $p \in \mathbb{N}$ or $p = \infty$

Let K be a hypergroup with Haar measure ω_K. In the following any unexplained notation will be taken from [2]. Recall that, although the product of two elements, say x, y of K, might not be defined, the convolution of the unit point masses ε_x and ε_y is defined. When the integral of a function f on K against the measure $\varepsilon_x * \varepsilon_y$ is defined, that integral is denoted by $f(x * y)$. We recall the definition of positive definiteness on hypergroups ([2, Definition 4.1.1]).

Definition 2.1. A function f on K is called *positive definite* if it is measurable and locally bounded, and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} f\left(x_i * x_j^-\right) \geq 0$$

for all choices of $c_i \in \mathbb{C}$, $x_i \in K$ and $n \in \mathbb{N}$.

The set of continuous positive definite functions will be denoted by $P(K)$. Note that, unlike for groups, there are hypergroups where such functions are not necessarily bounded (see [2, p. 268] or Remark 2.10 below). The subset of bounded functions in $P(K)$ is denoted by $P_b(K)$.

When f, g and h are functions on K, the notation $f(g * h)$ will mean the pointwise product of the function f with the convolution $g * h$, rather than meaning the integral of f against a measure $g * h$ as in the notation $f(x * y)$ above. We sometimes also write $(g * h)f$ or $f \cdot (g * h)$ (and this extends to cases where g is a measure).

Definition 2.2. A locally integrable function f is said to be of *positive type* if

$$\int f \cdot (g * g^*) \, d\omega_K \geq 0$$

for every $g \in C_c(K)$, where $g^*(x) := \Delta(x^-) g^\sim(x), g^\sim := \overline{g^-}$ and $g^-(x) := g(x^-)$ for all $x \in K$.

For continuous f this amounts to saying that f is positive definite (see [2], Lemma 4.1.4; when K is not unimodular, the function $g^\sim$ in part (iii) of that lemma should be replaced by the function g^*). In particular, if K is discrete the notions "of positive type" and "positive definite" coincide.

Remark 2.3. If K is any non-discrete hypergroup, there exist lower semicontinuous functions of positive type in $L^1(K)$ that are unbounded near the identity and hence don't belong to $P(K)$. To see this, note that using the outer regularity of ω_K for the null set $\{e\}$ there is a decreasing sequence of symmetric neighbourhoods U_n with $\omega_K(U_n) \to 0$, and we may assume $\omega_K(U_n) < 1/n$. Let $f = \sum \lambda_n \mathbf{1}_{U_n} * \mathbf{1}_{U_n}$ where $\lambda_n = 1/(n\,\omega_K(U_n))$ and $\mathbf{1}_{U_n}$ is the indicator function of U_n. Now

$$\|f\|_1 = \sum \lambda_n \omega_K(U_n)^2 \leq \sum \frac{1}{n^2} < \infty.$$

Being the supremum of continuous functions, f is lower semicontinuous, and we have

$$f(e) = \sum \lambda_n \omega_K(U_n) = \sum \frac{1}{n} = \infty$$

so f is unbounded near e. Since $\mathbf{1}_{U_n} = \mathbf{1}_{U_n}{}^\sim$, f is of positive type.

On several occasions in this paper we use that if f is a function of positive type and h is is a real-valued continuous function with compact support, then $h * f * h^-$ is of positive type. This can be seen from the definition using [2], (1.4.23), that is

$$\int (f * h) g \, d\omega_K = \int f \cdot (g * h^-) \, d\omega_K \tag{2.1}$$

and its left-hand version

$$\int (h * f) g \, d\omega_K = \int f \cdot ((\Delta^- h^-) * g) \, d\omega_K \tag{2.2}$$

which has a similar proof. (Note that $\Delta^- h^- = h^*$ since h is real-valued.) In the special case when $f \in L^1(K)$ and the hypergroup K is commutative, we can also see this using the Fourier transform.

Remark 2.4. Let K be a commutative hypergroup. A function $f \in L^1(K)$ is of positive type if and only if $\hat{f} \geq 0$ on the support of the Plancherel measure π_K.

Proof. (a) Let $f \in L^1(K)$ be of positive type and let $\chi \in \operatorname{supp} \pi_K$. By [2, 4.1.22], there is net (f_ι) in $C_c(K)$ such that $f_\iota * f_\iota^\sim \to \chi$ uniformly on compact sets. We may assume that $f_\iota * f_\iota^\sim(e) = 1$ for all ι. For $\varepsilon > 0$ choose a compact $C \subset K$ such that $\int_{K \setminus C} |f| \, d\omega_K < \varepsilon$. Since $|\chi| \leq 1$ and $|f_\iota * f_\iota^\sim| \leq 1$ (for the second inequality, note that $f_\iota * f_\iota^\sim \in P_b(K)$ by [2, Lemma 4.1.5(b)], and the bound follows from [2, Lemma 4.1.3(g)]), we have that

$$\left| \hat{f}(\chi) - \int f \cdot (f_\iota * f_\iota^\sim) \, d\omega_K \right| \leq 2\varepsilon + \int_C |f| \, |\chi - (f_\iota * f_\iota^\sim)| \, d\omega_K < 3\varepsilon$$

for suitable ι. By the assumption on f we have $\int f \cdot (f_\iota * f_\iota^\sim) \, d\omega_K \geq 0$ (note that $f_\iota^\sim = f_\iota^*$ since K is unimodular), and hence $\hat{f}(\chi) \geq 0$.

(b) Suppose $\hat{f} \geq 0$ on $\operatorname{supp} \pi_K$ and let $g \in C_c(K)$. We have using (2.1) and Plancherel's theorem

$$\int f \cdot (g * g^\sim) \, d\omega_K = \int (f * \overline{g}) \, \overline{\overline{g}} \, d\omega_K = \int \hat{f} \hat{g} \overline{\hat{g}} \, d\pi_K = \int \hat{f} \left| \hat{g} \right|^2 d\pi_K \geq 0 \qquad \square$$

As in [2, p. 8], the set of all probability measures on K will be denoted by $M^1(K)$.

Lemma 2.5. *Let K be a commutative hypergroup. For every relatively compact neighbourhood U of the identity there is a constant $C_U > 0$ such that*

$$\int g \cdot (\mu * \mathbf{1}_U) \, d\omega_K \leq C_U \int g \mathbf{1}_U \, d\omega_K \tag{2.3}$$

for all choices of $\mu \in M^1(K)$ and all non-negative $g \in P_b(K)$.

Proof. By Theorem 4.1.13 of [2] we may write $g(x)$ as a coefficient of a cyclic representation D of the hypergroup K on a Hilbert space $\mathcal{H}$, that is there is a cyclic vector $\mathbf{u} \in \mathcal{H}$ such that

$$g(x) = \langle D(x)\mathbf{u}, \mathbf{u} \rangle_{\mathcal{H}}$$

for all $x \in K$.

Choose a relatively compact neighbourhood V of e such that

$$\bigcup \{ \operatorname{supp} \, (\varepsilon_{x^-} * \varepsilon_y) : x, y \in V \} \subset U$$

and $\omega_K(V) \leq 1$; these conditions guarantee that

$$h := \mathbf{1}_V^\sim * \mathbf{1}_V \leq \mathbf{1}_U. \tag{2.4}$$

Since $h \in C_c^+(K)$ with $h(e) > 0$ and U is relatively compact, there exist $x_1, x_2, ..., x_n \in K$ and $\lambda_1, \lambda_2, ..., \lambda_n > 0$ such that $\mathbf{1}_U \leq \sum_{i=1}^n \lambda_i \tau_{x_i} h$, where

$$\tau_{x_i} h(y) = h(x_i * y)$$

is the x_i–translate of h.

Let $\nu = \sum_{i=1}^{n} \lambda_i \varepsilon_{x_i}$. Then

$$\begin{aligned}\int g \cdot (\mu * \mathbf{1}_U)\, d\omega_K &\le \int g \left(\mu * \left(\sum_{i=1}^{n} \lambda_i \tau_{x_i} h\right)\right) d\omega_K \\ &= \left\langle D(\mu * \nu^- * h)\mathbf{u}, \mathbf{u}\right\rangle_{\mathcal{H}} \\ &= \left\langle D(\mu * \nu^- * \mathbf{1}_V)\mathbf{u}, D(\mathbf{1}_V)\mathbf{u}\right\rangle_{\mathcal{H}} \\ &= \left\langle D(\mu * \nu^-)D(\mathbf{1}_V)\mathbf{u}, D(\mathbf{1}_V)\mathbf{u}\right\rangle_{\mathcal{H}} \\ &\le \|D(\mu * \nu^-)\|_{B(\mathcal{H})} \|D(\mathbf{1}_V)\mathbf{u}\|_{\mathcal{H}}^2 \\ &\le \|\nu\| \int hg\, d\omega_K\end{aligned}$$

since $\|\mu\| = 1$, and since

$$\begin{aligned}\|D(\mathbf{1}_V)\mathbf{u}\|_{\mathcal{H}}^2 &= \langle D(\mathbf{1}_V)\mathbf{u}, D(\mathbf{1}_V)\mathbf{u}\rangle_{\mathcal{H}} = \langle D(\mathbf{1}_V)^* D(\mathbf{1}_V)\mathbf{u}, \mathbf{u}\rangle_{\mathcal{H}} \\ &= \langle D(\mathbf{1}_V^{\sim} * \mathbf{1}_V)\mathbf{u}, \mathbf{u}\rangle_{\mathcal{H}} = \langle D(h)\mathbf{u}, \mathbf{u}\rangle_{\mathcal{H}} = \int hg\, d\omega_K.\end{aligned}$$

So, letting $C_U = \|\nu\|$, we have that

$$\int g \cdot (\mu * \mathbf{1}_U)\, d\omega_K \le C_U \int g \mathbf{1}_U\, d\omega_K.$$

□

Corollary 2.6. *Let K be a commutative hypergroup such that $P_b(K) \cdot P_b(K) \subset P_b(K)$ and let $p \in \mathbb{N}$ be even. For every relatively compact neighbourhood U of the identity there is a constant $C_U > 0$ such that for all choices of $\mu \in M^1(K)$ and $f \in P_b(K)$*

$$\int |f|^p \cdot (\mu * \mathbf{1}_U)\, d\omega_K \le C_U \int |f|^p \mathbf{1}_U\, d\omega_K. \tag{2.5}$$

Proof. Let $p \in \mathbb{N}$ be even. Since $f \in P_b(K)$, the same is true for $\overline{f}$. It follows that

$$|f|^p = (\overline{f} f)^{p/2} \in P_b(K)$$

and it is also positive. Inserting $g = |f|^p$ in inequality (2.3) yields the inequality (2.5). □

Remark 2.7. We remind the reader that for strong hypergroups,

$$P_b(K) \cdot P_b(K) \subset P_b(K).$$

(Use Bochner's theorem to write two functions f and g in $P_b(K)$ as inverse transforms of two nonnegative measures μ, ν respectively on $K^\wedge$. Then fg is the inverse transform of $\mu * \nu$ and hence belongs to $P_b(K)$ as well.) In particular, Corollary 2.6 and much of what follows holds for all strong hypergroups.

We now extend inequality (2.5) to integrable functions f of positive type.

Corollary 2.8. *Let K be a commutative hypergroup such that $P_b(K) \cdot P_b(K) \subset P_b(K)$ and take $p \in \mathbb{N}$ to be even. For every relatively compact neighbourhood U of the identity there is a constant $C_U > 0$ such that for all choices of $\mu \in M^1(K)$*

and $f \in L^1(K)$ *of positive type (equivalently:* $f \in L^1(K)$ *with* $\hat{f} \geq 0$ *on* $\operatorname{supp} \pi_K$*) we have*

$$\int |f|^p \cdot (\mu * \mathbf{1}_U)\, d\omega_K \leq C_U \int |f|^p \, \mathbf{1}_U \, d\omega_K. \tag{2.6}$$

Proof. Let f be such a function with $\int |f|^p \, \mathbf{1}_U \, d\omega_K < \infty$ and write $f_\iota = k_\iota * f * k_\iota^-$ where $k_\iota \in C_c^+(K)$, $\int k_\iota \, d\omega_K = 1$ and $\operatorname{supp} k_\iota \downarrow \{e\}$. (If K is first countable, then this approximate identity can in fact be chosen to be a sequence.) Clearly f_ι is bounded, continuous and integrable. Since f_ι is of positive type (see the paragraph immediately preceding Remark 2.4), it is also in $P_b(K)$. Now the values of f_ι on U depend on the values of f on a slightly larger neighbourhood U', and we cannot rule out *a priori* the possibility that $\int |f|^p \, \mathbf{1}_{U'} \, d\omega_K = \infty$. For this technical reason we first use a compact neighbourhood W of e contained in the interior of U.

For sufficiently large ι the values of f_ι on W only depend on the values of f on U, and we have

$$\|(f - f_\iota)\, \mathbf{1}_W\|_p \leq \left\| f \mathbf{1}_U - k_\iota * (f \mathbf{1}_U) * k_\iota^- \right\|_p \to 0 \tag{2.7}$$

since $f \mathbf{1}_W = f \mathbf{1}_U \mathbf{1}_W$ and $f_\iota \mathbf{1}_W = [k_\iota * (f \mathbf{1}_U) * k_\iota^-]\, \mathbf{1}_W$ for sufficiently large ι. We also have

$$\|f_\iota - f\|_1 \to 0 \tag{2.8}$$

and we can extract a sequence (f_n) from (f_ι) satisfying both (2.7) and (2.8), and (if necessary, passing to a subsequence thereof) converging pointwise *a.e.* to f. Using Fatou's lemma we obtain

$$\begin{aligned}
\int |f|^p \cdot (\mu * \mathbf{1}_W)\, d\omega_K &\leq \liminf_n \int |f_n|^p \, \mu * \mathbf{1}_W \, d\omega_K \\
&\leq C_W \liminf_n \int |f_n|^p \, \mathbf{1}_W \, d\omega_K \\
&\leq C_W \int |f|^p \, \mathbf{1}_W \, d\omega_K
\end{aligned}$$

where, for the middle inequality, we have appealed to (2.5), and the last inequality follows from (2.7). Choose $x_1, x_2, \ldots, x_n \in K$ and $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ such that $\mathbf{1}_U \leq \sum_{i=1}^n \lambda_i \, \tau_{x_i} \mathbf{1}_W$. We then have

$$\begin{aligned}
\int |f|^p \cdot (\mu * \mathbf{1}_U)\, d\omega_K &\leq \sum_{i=1}^n \lambda_i \int |f|^p \cdot (\mu * \tau_{x_i} \mathbf{1}_W)\, d\omega_K \\
&= \sum_{i=1}^n \lambda_i \int |f|^p \cdot \left(\mu * \varepsilon_{x_i^-} * \mathbf{1}_W\right) d\omega_K \\
&\leq C_W \left(\sum_{i=1}^n \lambda_i\right) \int |f|^p \, \mathbf{1}_W \, d\omega_K \\
&\leq \left(\sum_{i=1}^n \lambda_i\right) C_W \int |f|^p \, \mathbf{1}_U \, d\omega_K
\end{aligned} \tag{2.9}$$

and this ends the proof of the corollary. □

To prepare for Remark 2.11, we insert the following definition.

Definition 2.9. For $p \in [1, \infty)$ we say that a measurable function f belongs to the amalgam space $(L^p, \ell^\infty)(K)$ if $\|f\|_{p,\infty,U} := \sup_x \left\| f \left(\tau_x \mathbf{1}_U\right)^{1/p} \right\|_p$ is finite for some relatively compact neighbourhood U of the identity.

In the discussion following Corollary 2.15 below, we show that replacing U by a different relatively compact neighbourhood of the identity yields an equivalent norm and hence the same space $(L^p, \ell^\infty)(K)$. Note that

$$L^1(K) \subset (L^1, l^\infty)(K) \subset L^1_{loc}(K).$$

Remark 2.10. In the group case, Corollary 2.8 extends to locally integrable functions f of positive type (see [11, 1.1 and Theorem 1.6]), but for hypergroups this is not always possible. Indeed the Naimark hypergroup ([2, p. 99], but note the misprint in line 5, the second occurrence of a^n should be deleted) is a counterexample. For this hypergroup on $\mathbb{R}^+$ with Haar measure $d\omega(x) = \sinh^2 x\, dx$ there are unbounded (positive definite) characters of the form $\chi_a(x) = \frac{\sinh(rx)}{r \sinh x}$ where $r > 1$ and $a = -r^2$. Then $\chi_a(x)$ behaves like $e^{(r-1)x}$ as $x \to \infty$. Writing $U := [0, 1]$, for $x > 1$ we have $0 \le \tau_x \mathbf{1}_U \le 1$, $\operatorname{supp}(\tau_x \mathbf{1}_U) \subset J_x := [x-1, x+1]$ and $\int \tau_x \mathbf{1}_U\, d\omega = \int \mathbf{1}_U\, d\omega =: c$, so that $\tau_x \mathbf{1}_U \ge \frac{c}{2\omega(J_x)}$ on a set with measure at least $\frac{c}{2}$. Therefore

$$\left\| \chi_a \left(\tau_x \mathbf{1}_U\right)^{1/p} \right\|_p \ge \|\chi_a \tau_x \mathbf{1}_U\|_p \ge \left(\min_{J_x} \chi_a\right) \frac{c}{2\omega(J_x)} \left(\frac{c}{2}\right)^{\frac{1}{p}}.$$

For a sufficiently small ($a < -9$ will do), the right-hand side of this inequality tends to ∞ as $x \to \infty$ (and hence $J_x \to \{\infty\}$), which shows that Corollary 2.8 does not hold on this hypergroup.

Remark 2.11. The proof of Corollary 2.8 works for any (locally integrable) function f of positive type for which the convolutions f_ι all belong to L^∞. Those convolutions are continuous, of positive type and (by assumption) bounded, hence positive definite. The L^1-convergence in (2.8) can then be replaced by local L^1-convergence, that is by convergence in $L^1(C)$ for every compact set C.

In particular, the proof works for all $f \in (L^1, \ell^\infty)(K)$ of positive type because the k_ι in our proof all belong to $C_c(K)$. So $f * k_\iota^- \in L^\infty$, as we show in a moment, and hence so does $f_\iota = k_\iota * f * k_\iota^-$, which shows that f_ι is bounded for each ι.

For any relatively compact neighbourhood $U \ni e$, and ι chosen suitably large so that $\operatorname{supp}(k_\iota) \subset U$, we have

$$\begin{aligned}\left|f * k_\iota^-\right|(x) &\leq \int \left|f(x*y)\, k_\iota^-(y^-)\right| d\omega(y)\\ &\leq \|k_\iota\|_\infty \int |f(x*y)|\, \mathbf{1}_U(y)\, d\omega(y)\\ &\leq \|k_\iota\|_\infty \int |f|(x*y)\, \mathbf{1}_U(y)\, d\omega(y)\\ &= \|k_\iota\|_\infty \int |f(y)|\, \mathbf{1}_U(x^- * y)\, d\omega(y)\\ &= \|k_\iota\|_\infty \left\| |f|\, \tau_{x^-} \mathbf{1}_U \right\|_1\\ &\leq \|k_\iota\|_\infty \|f\|_{1,\infty,U}\end{aligned}$$

where for the first equality we refer to [2], Theorem 1.3.21, and hence $f * k_\iota^-$ is bounded.

Theorem 2.12. *Let K be a commutative hypergroup such that $P_b(K) \cdot P_b(K) \subset P_b(K)$ and let $p \in \mathbb{N}$ be even. For every relatively compact neighbourhood U of the identity there is a constant $C_U > 0$ such that for all choices of $\mu \in M^1(K)$ and $f \in (L^1, \ell^\infty)(K)$ of positive type we have*

$$\|f \cdot (\mu * \mathbf{1}_U)\|_p \leq \left\| f \cdot (\mu * \mathbf{1}_U)^{1/p} \right\|_p \leq C_U^{1/p} \left\| f\, (\mathbf{1}_U)^{1/p} \right\|_p = C_U^{1/p} \|f\, \mathbf{1}_U\|_p. \quad (2.10)$$

In particular this holds for $f \in L^1(K)$ of positive type (equivalently: $f \in L^1(K)$ with $\hat{f} \geq 0$ on $\operatorname{supp} \pi_K$).

Proof. The first inequality in (2.10) holds for all finite exponents $p > 1$ since $0 \leq \mu * \mathbf{1}_U \leq 1$. The next inequality in (2.10) uses Corollary 2.8, the assumption that $p \in \mathbb{N}$ is even and Remark 2.11. □

Corollary 2.13. *Let K be a commutative hypergroup such that $P_b(K) \cdot P_b(K) \subset P_b(K)$. For $f \in (L^1, \ell^\infty)(K)$ of positive type we have*

$$\|f\|_\infty \leq \|f \mathbf{1}_U\|_\infty . \quad (2.11)$$

In particular, since $0 \leq \tau_x \mathbf{1}_U \leq 1$, we have

$$\|f \tau_x \mathbf{1}_U\|_\infty \leq \|f \mathbf{1}_{U_x}\|_\infty \leq \|f \mathbf{1}_U\|_\infty \quad (2.12)$$

where $U_x = \{y \,|\, \tau_x \mathbf{1}_U(y) > 0\}$.

Proof. The second quantity in (2.10) is the L^p norm of f relative to the measure $(\mu * \mathbf{1}_U)\, d\omega$. Since the total mass of this measure is finite, letting $p \to \infty$ in (2.10) gives the essential supremum of $|f|$ on the set where $\mu * \mathbf{1}_U > 0$. Apply this with $\mu = \varepsilon_x$ for various points x in K, and use the fact that U_x is a neighbourhood of x^-, to obtain $\|f\|_\infty \leq \|f \mathbf{1}_U\|_\infty$. □

Remark 2.14. Note that taking $\mu = \varepsilon_x$ in Theorem 2.12 gives that for all even $p \in \mathbb{N}$

$$\|f\, \tau_x \mathbf{1}_U\|_p \leq \left\| f\, (\tau_x \mathbf{1}_U)^{1/p} \right\|_p \leq C_U^{1/p} \|f\, \mathbf{1}_U\|_p . \quad (2.13)$$

It is useful to recall at this stage that for fixed p, the quantities $\|f\,\tau_x\mathbf{1}_U\|_p$ and $\left\|f\,(\tau_x\mathbf{1}_U)^{1/p}\right\|_p$ agree on groups but not necessarily on hypergroups (see the end of Remark 3.4 below).

We restate (2.12) and (2.13) using Definition 2.9.

Corollary 2.15. *(Wiener's theorem for functions in $(L^1,\ell^\infty)(K)$) Let K be a commutative hypergroup such that $P_b(K)\cdot P_b(K)\subset P_b(K)$ and take $p\in\mathbb{N}$ even or $p=\infty$. If $f\in(L^1,\ell^\infty)(K)$ is of positive type, and satisfies $f\,\mathbf{1}_U\in L^p(K)$ for some relatively compact neighbourhood U of e, then*

$$f\in(L^p,\ell^\infty)(K)\quad\text{and}\quad\|f\|_{p,\infty,U}\leq C_U^{1/p}\,\|f\,\mathbf{1}_U\|_p\,.$$

In particular this holds for $f\in L^1(K)$ satisfying the same conditions.

Note that, by the equivalence proved next, if K is compact, then $(L^p,\ell^\infty)=L^p$ and $\|\cdot\|_{p,\infty,U}$ equals (up to equivalence) the L^p norm on K (take $\|\cdot\|_{p,\infty,K}$ and use $\tau_x\mathbf{1}_K=\mathbf{1}_K$).

We now compare $\|f\|_{p,\infty,U}$ for different choices of U (even on non-commutative hypergroups). Let U and V be relatively compact neighbourhoods of e, and denote the corresponding amalgam spaces by $(L^p,\ell^\infty)_U$ and $(L^p,\ell^\infty)_V$ respectively. There are $\lambda_i>0$ and $x_i\in K$ such that $1_U\leq\sum_{i=1}^n\lambda_i\tau_{x_i}\mathbf{1}_V$. Let $f\in(L^p,\ell^\infty)_V$ and $x\in K$. When $1\leq p<\infty$ we have

$$\begin{aligned}\left\|f\,(\tau_x 1_U)^{1/p}\right\|_p^p &= \int|f|^p\,\tau_x 1_U\,d\omega_K\leq\int|f|^p\,\tau_x\left(\sum_{i=1}^n\lambda_i\tau_{x_i}\mathbf{1}_V\right)\,d\omega_K\\ &=\sum_{i=1}^n\lambda_i\left\|f\,(\tau_x\tau_{x_i}\mathbf{1}_V)^{1/p}\right\|_p^p\leq\left(\sum_{i=1}^n\lambda_i\right)\|f\|_{p,\infty,V}^p\end{aligned}$$

by Lemma 2.16 below (set $\mu=\varepsilon_{x^-}*\varepsilon_{x_i^-}$). Hence

$$f\in(L^p,\ell^\infty)_U\quad\text{and}\quad\|f\|_{p,\infty,U}\leq C\|f\|_{p,\infty,V}$$

with $C=\left(\sum_{i=1}^n\lambda_i\right)^{1/p}$, so that the amalgam space $(L^p,\ell^\infty)(K)$ does not depend on the chosen neighbourhood.

Note that, since necessarily $\sum\lambda_i\geq 1$, this sum can serve as a constant for all finite p. So we have constants of equivalence which only depend on U and V, but not on p.

If $p=\infty$ and (as before) we denote by U_x the set where $\tau_x\mathbf{1}_U>0$, then $\|f\|_{\infty,\infty,U}=\sup_x\|f\mathbf{1}_{U_x}\|_\infty$. It follows that $\|f\|_{\infty,\infty,U}=\|f\|_\infty$ since U_x is a neighbourhood of x. So in this case, if we use V instead of U, we obtain not only an equivalent norm but in fact the very same norm.

Lemma 2.16. *Let $p\in[1,\infty]$. For $f\in(L^p,\ell^\infty)_V$ and μ a probability measure with compact support we have $f\,(\mu*\mathbf{1}_V)^{1/p}\in L^p$ and $\|f\,(\mu*\mathbf{1}_V)^{1/p}\|_p\leq\|f\|_{p,\infty,V}$.*

Proof. By [10, Proposition 13.64] and the remarks following it, the set S of all convex linear combinations of Dirac measures is weakly dense in $M^1(K)$. So there is a net (μ_ι) in S with $\mu_\iota\to\mu$ weakly. In the present case we may assume

$\operatorname{supp}\mu_\iota \subset \operatorname{supp}\mu$ (in the proof of [10, 13.64], if $A_j \cap \operatorname{supp}\mu \neq \emptyset$, choose x_j in this set and not just in A_j). By [2, Theorem 1.6.18(b)] we obtain $\|\mu_\iota * g - \mu * g\|_1 \to 0$ for all $g \in L^1(K)$. From the net $(\mu_\iota * \mathbf{1}_V)$ we may extract a sequence $(\mu_n * \mathbf{1}_V)$ converging in $\|\cdot\|_1$ and (if necessary, passing to a subsequence thereof) also pointwise *a.e.* to $\mu * \mathbf{1}_V$. Hence

$$(\mu_n * \mathbf{1}_V)^{1/p} \to (\mu * \mathbf{1}_V)^{1/p} \ a.e.$$

All these functions have absolute value ≤ 1 (see [2, 1.4.6]) and have support in the compact set $\operatorname{supp}(\mu) * \operatorname{supp}(\mathbf{1}_V)$ (see [2, 1.2.12])), hence are dominated by $h = \mathbf{1}_{\operatorname{supp}(\mu)*\operatorname{supp}(\mathbf{1}_V)}$. There are $\beta_k > 0$ and $y_k \in K$ such that $h \leq \sum_{k=1}^{l} \beta_k \left(\tau_{y_k}\mathbf{1}_V\right)^{1/p}$, so

$$\|f\,h\|_p \leq \sum_{k=1}^{l} \beta_k \|f\,(\tau_{y_k}\mathbf{1}_V)^{1/p}\|_p < \infty.$$

By dominated convergence we obtain $\|f(\mu_n * \mathbf{1}_V)^{1/p} - f(\mu * \mathbf{1}_V)^{1/p}\|_p \to 0$. Now, since μ_n is a convex combination $\sum_{j=1}^{m} \gamma_j \varepsilon_{x_j}$, we have

$$\left\|f\,(\mu_n * \mathbf{1}_V)^{1/p}\right\|_p^p = \left\|f\left(\sum_{j=1}^{m} \gamma_j \tau_{x_j^-}\mathbf{1}_V\right)^{1/p}\right\|_p^p = \int |f|^p \sum_{j=1}^{m} \gamma_j \tau_{x_j^-}\mathbf{1}_V \, d\omega_K$$

$$= \sum_{j=1}^{m} \gamma_j \left\|f\,\left(\tau_{x_j^-}\mathbf{1}_V\right)^{\frac{1}{p}}\right\|_p^p \leq \sum_{j=1}^{m} \gamma_j \|f\|_{p,\infty,V}^p = \|f\|_{p,\infty,V}^p.$$

Hence $\|f\,(\mu * \mathbf{1}_V)^{1/p}\|_p \leq \|f\|_{p,\infty,V}$ as asserted. □

Remark 2.17. All of the results obtained so far hold for a large class of commutative hypergroups, in particular for strong hypergroups, and hence also for those examples to be considered below. Furthermore, much of this section extends to some non-commutative hypergroups. A version of Lemma 2.5 holds without the assumption that K is commutative. Instead, we assume that there is a relatively compact neighbourhood V of the identity with the property that $\mathbf{1}_V$ is central in the convolution algebra $L^1(K)$ and hence in the measure algebra on K. The conclusion of the lemma then holds for neighbourhoods U of e that include the support of the product $\mathbf{1}_{\tilde{V}} * \mathbf{1}_V$. The centrality assumption implies that K is unimodular. In particular, $(\mathbf{1}_V)^* = \mathbf{1}_{\tilde{V}}$ (as in the commutative case). Therefore the proof of the lemma remains almost the same (replace the sentence concerning the supports of the $\varepsilon_{x^-} * \varepsilon_y$ up to and including inequality (2.4) by "Let $h = \mathbf{1}_{\tilde{V}} * \mathbf{1}_V$."). With the same modified hypothesis, Corollary 2.6 holds with no change in its proof. For Corollary 2.8 we also require that the support of $\mathbf{1}_{\tilde{V}} * \mathbf{1}_V$ be contained in the interior of U, rather than just in U. In the proof of Corollary 2.8 take W equal to this support. Then for such U, Theorem 2.12 and hence Remark 2.14 as well as Corollary 2.15 for even p also hold. For $p = \infty$, Corollary 2.13 and hence the corresponding part of Corollary 2.15 hold on general hypergroups (without any centrality assumption):

Let $f \in (L^1, \ell^\infty)(K)$ be of positive type. If U is a relatively compact neighbourhood of e and $f_\iota = k_\iota * f * k_\iota^-$ where the k_ι are as in the proof of Corollary

2.8, take ι large enough so that supp $(k_\iota^* * k_\iota) \subset U$. Then (see Remark 2.11) f_ι is continuous, positive definite and bounded, so by [2, Lemma 4.1.3(g)] for the first equality and (2.2) for the third equality below, we have

$$\begin{aligned} \|f_\iota\|_\infty &= k_\iota * f * k_\iota^- (e) \\ &= \int (k_\iota * f)\, k_\iota \, d\omega_K \\ &= \int f \cdot (k_\iota^* * k_\iota)\, d\omega_K \\ &\le \|f \mathbf{1}_U\|_\infty \|k_\iota^* * k_\iota\|_1 \\ &\le \|f \mathbf{1}_U\|_\infty . \end{aligned}$$

Since $f_\iota \to f$ locally in L^1-norm (that is, $\|(f_\iota - f)\,\mathbf{1}_C\|_1 \to 0$ for every compact $C \subset K$), we obtain $\|f\|_\infty \le \|f \mathbf{1}_U\|_\infty$.

3. Hypergroups on $\mathbb{R}_+$

In this section we consider some hypergroups on $\mathbb{R}_+$ to which all of Section 2 applies. For one of them we show that the version of Wiener's theorem presented in [7] for locally compact abelian groups also holds (Theorem 3.6 below), as indeed do other positive results about translation, convolution and Fourier transforms, which we need for the proof of the theorem.

3.1. **Bessel-Kingman hypergroups.** For these hypergroups the reader is referred to [2, , Section 3.5.61], but we give here some basic properties. Let $\alpha > -\frac{1}{2}$. For $x, y \in \mathbb{R}_+$ consider the convolution

$$\varepsilon_x *_\alpha \varepsilon_0 = \varepsilon_x = \varepsilon_0 *_\alpha \varepsilon_x$$

and for $x, y > 0$,

$$\varepsilon_x *_\alpha \varepsilon_y (f) = \int_{|x-y|}^{x+y} K_\alpha (x, y, z)\, f(z)\, z^{2\alpha+1}\, dz,\ f \in C_0(\mathbb{R}_+)$$

where

$$K_\alpha (x, y, z) := \left(\frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\alpha + \frac{1}{2}\right) 2^{2\alpha-1}} \right) \frac{\left[\left(z^2 - (x-y)^2\right) \left((x+y)^2 - z^2\right) \right]^{\alpha - \frac{1}{2}}}{(xyz)^{2\alpha}}.$$

Then $(\mathbb{R}_+, *_\alpha)$ is a commutative hypergroup with the identity involution and Haar measure $\omega_\alpha(dz) = z^{2\alpha+1} dz$. Its characters are given by $\varphi_\lambda(x) := j_\alpha(\lambda x)$, $x \in \mathbb{R}_+$ for each $\lambda \ge 0$ where j_α denotes the modified Bessel function of order α given by

$$j_\alpha(x) := \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{2^{2k} k! \Gamma(\alpha+k+1)} x^{2k},\ x \in \mathbb{R}.$$

Note that $\varphi_0 \equiv 1$.

It is well known that $(\mathbb{R}_+, *_\alpha) \cong (\mathbb{R}_+, *_\alpha)^\wedge$, where the hypergroup isomorphism is given by $\lambda \longmapsto \varphi_\lambda$ (so that $(\mathbb{R}_+, *_\alpha)$ is not only strong but even Pontryagin); see [16, Example 7.2]. Wiener's theorem as in Corollary 2.15 therefore holds for these Bessel-Kingman hypergroups.

For $\alpha = \frac{1}{2}$ (the motion hypergroup) the convolution is given by

$$\varepsilon_x *_{\frac{1}{2}} \varepsilon_y (f) = \frac{1}{2xy} \int_{|x-y|}^{x+y} f(z)\, z\, dz \tag{3.1}$$

in which case the characters are just

$$\varphi_\lambda (x) = j_{\frac{1}{2}} (\lambda x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{3}{2}\right)}{2^{2k} k! \Gamma\left(k+\frac{3}{2}\right)} (\lambda x)^{2k} = \frac{\sin \lambda x}{\lambda x}, \ \lambda \geq 0.$$

The term 'motion hypergroup' is justified by the fact that $\left(\mathbb{R}_+, *_{\frac{1}{2}}\right)$ is isomorphic to the double coset space $M(3) /\!/ SO(3)$.

For $f \in L^1 (\mathbb{R}_+, *_\alpha), \alpha > -\frac{1}{2}$, its Fourier transform is defined by

$$\hat{f} (\varphi_\lambda) := \int_{\mathbb{R}_+} f \varphi_\lambda \, d\omega_\alpha$$

and the convolution of two functions f, g is given by

$$f *_\alpha g (x) := \int_{\mathbb{R}_+} f (x *_\alpha y)\, g(y)\, \omega_\alpha (dy).$$

Recall that

$$(f *_\alpha g)^\wedge = \hat{f}\hat{g}.$$

When $\alpha = \frac{1}{2}$ we have

$$\hat{f} (\varphi_\lambda) = \begin{cases} \dfrac{1}{\lambda} \displaystyle\int_{\mathbb{R}_+} f(x) (\sin \lambda x)\, x\, dx, & \lambda \neq 0, \\ \int_{\mathbb{R}_+} f(x)\, x^2\, dx, & \lambda = 0. \end{cases}$$

and, in particular,

$$\left(\mathbf{1}_{[0,\varepsilon)}\right)^\wedge (\varphi_\lambda) = \begin{cases} \dfrac{1}{\lambda^3} (\sin \lambda\varepsilon - \lambda\varepsilon \cos \lambda\varepsilon), & \lambda \neq 0, \\ \dfrac{\varepsilon^3}{3}, & \lambda = 0. \end{cases} \tag{3.2}$$

3.2. **The amalgam spaces $(L^p, \ell^q)(\mathbb{R}_+, *_\alpha)$ for $1 \leq p, q \leq \infty$.** In preparation for Theorem 3.6 in Section 3.4, we need to develop some properties of certain *discrete* amalgam spaces. We define them so that the norms $\|\cdot\|_{p,\infty}$ used in this section are equivalent to the corresponding *continuous* norms $\|\cdot\|_{p,\infty,U}$ used in Section 2, and we prove this equivalence in Section 3.3. At the end of the current subsection, we consider other families of discrete amalgam norms, in particular those introduced in [3], and show that they are mostly not equivalent to the norms that we use.

For each $n \in \mathbb{N}$ write $I_n := [n-1, n)$ and for $1 \leq p, q \leq \infty$ define

$$\|f\|_{p,q} := \left(\sum_{n=1}^{\infty} \omega_\alpha (I_n) \left(\frac{1}{\omega_\alpha (I_n)} \int_{n-1}^{n} |f|^p \, d\omega_\alpha \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{3.3}$$

with the usual convention when one or both of p, q is ∞, that is

$$\|f\|_{\infty,q} = \left(\sum_{n=1}^{\infty} \omega_\alpha(I_n) \sup_{x\in I_n} |f(x)|^q\right)^{\frac{1}{q}}$$

$$\|f\|_{p,\infty} = \sup_n \left(\frac{1}{\omega_\alpha(I_n)} \int_{n-1}^{n} |f|^p \, d\omega_\alpha\right)^{\frac{1}{p}}$$

$$\text{and} \quad \|f\|_{\infty,\infty} = \sup_n \sup_{x\in I_n} |f(x)| = \|f\|_\infty .$$

The (p, q) –amalgam space is defined as the subspace of all measurable functions f given by

$$(L^p, \ell^q)(\mathbb{R}_+, *_\alpha) = \left\{f : \|f\|_{p,q} < \infty\right\}.$$

We have the following result.

Proposition 3.1. *Let f be a measurable function. Then*

$$\|f\|_{p_1,q} \leq \|f\|_{p_2,q} \quad \textit{for } p_1 \leq p_2$$

$$\textit{and} \quad \|f\|_{p,q_1} \leq C\|f\|_{p,q_2} \quad \textit{for } q_1 \geq q_2,$$

where C is a constant. In particular, for $p_1 \leq p_2$ and $q_1 \geq q_2$

$$(L^{p_2}, \ell^{q_2})(\mathbb{R}_+, *_\alpha) \subset (L^{p_1}, \ell^{q_1})(\mathbb{R}_+, *_\alpha),$$

$$\textit{so that} \quad (L^p, \ell^q)(\mathbb{R}_+, *_\alpha) \subset L^p(\mathbb{R}_+, *_\alpha) \cap L^q(\mathbb{R}_+, *_\alpha) \quad \textit{for } p \geq q$$

$$\textit{and} \quad L^p(\mathbb{R}_+, *_\alpha) \cup L^q(\mathbb{R}_+, *_\alpha) \subset (L^p, \ell^q)(\mathbb{R}_+, *_\alpha) \quad \textit{for } p \leq q.$$

Proof. This is straightforward using Hölder's inequality together with the property that $\omega_\alpha(I_n) \geq C > 0$ for all n. □

Note that $(L^\infty, \ell^1)(\mathbb{R}_+, *_\alpha)$ is the smallest amalgam space and $(L^1, \ell^\infty)(\mathbb{R}_+, *_\alpha)$ is the largest.

Remark 3.2. We now use indicator functions on subintervals of I_n to show that for $p \neq q$ our amalgam norms are not equivalent to the discrete amalgam norms in [3], which are computed on sets with measures uniformly bounded away from 0 and ∞. There is no division or multiplication by measures of tiles in the computation of those norms. In the present case we obtain norms equivalent to those in [3] by splitting $\mathbb{R}_+$ into disjoint intervals of Haar measure 1; at least $\omega_\alpha(I_n) - 2$ of these subintervals are included in I_n. Let f be the indicator function of one such subinterval. Then the norm of f in our space $(L^p, \ell^q)(\mathbb{R}_+, *_\alpha)$ is $\omega_\alpha(I_n)^{1/q-1/p}$, while its norm in the corresponding space in [3] is 1. Since $\omega_\alpha(I_n) \to \infty$ as $n \to \infty$, these norms are not equivalent unless $p = q$.

Both families of discrete amalgams on Bessel-Kingman hypergroups are constructed in such a way as to have norms equal to the usual L^p norm, and hence to each other when $p = q$. In the examples above, the functions f are not positive definite, and we do not know whether there are corresponding examples involving positive definite functions. Finally, most other choices give amalgam norms that are not equivalent to ours, for example the partition choice having the I_n without normalization, and the continuous amalgam norm as in Definition 2.9

but without the $1/p$ power. The only cases where our discrete amalgam norm is equivalent to the one without weights are those where $p = q$, and the only case where the two kinds of continuous amalgam norms are equivalent is that where $p = 1$ (see the end of Remark 3.4).

3.3. **Equivalence of the discrete amalgam norm $\|\cdot\|_{p,\infty}$ with the continuous amalgam norm defined by translations in the case $\alpha = \frac{1}{2}$.** For the following subsections of Section 3 we only consider the Bessel-Kingman hypergroup $\left(\mathbb{R}_+, *_{\frac{1}{2}}\right)$ (and to simplify the notation we write ω in place of $\omega_{\frac{1}{2}}$). Values of $\alpha > \frac{1}{2}$ are treated in [9].

Proposition 3.3. *For $p \in [1, \infty)$,*

$$\|f\|_{p,\infty} \leq C \sup_{y\in[0,\infty)} \left(\int |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega\right)^{1/p}.$$

Proof. We have using (3.1)

$$\begin{aligned}
\tau_y \mathbf{1}_{[0,1]}(x) &= \mathbf{1}_{[0,1]}\left(y *_{\frac{1}{2}} x\right) \\
&= \varepsilon_y *_{\frac{1}{2}} \varepsilon_x \left(\mathbf{1}_{[0,1]}\right) \\
&= \frac{1}{2xy} \int_{[|x-y|, x+y]\cap[0,1]} t\, dt \\
&= \begin{cases} 1, & x + y \leq 1, \\ \frac{1}{4xy}\left(1 - (x-y)^2\right), & x + y > 1 \text{ and } |x - y| < 1, \\ 0, & |x - y| \geq 1. \end{cases}
\end{aligned} \tag{3.4}$$

For $y = n + \frac{1}{2}, n \in \mathbb{N}$, we obtain

$$\tau_{n+\frac{1}{2}} \mathbf{1}_{[0,1]}(x) = \begin{cases} \frac{1-\left(n+\frac{1}{2}-x\right)^2}{4x\left(n+\frac{1}{2}\right)}, & \left|n + \frac{1}{2} - x\right| < 1, \\ 0, & \left|n + \frac{1}{2} - x\right| \geq 1. \end{cases}$$

On the interval I_{n+1} this is larger than

$$\frac{3/4}{4\left(n+\frac{1}{2}\right)(n+1)} \geq \frac{3/16}{2\omega\left(I_{n+1}\right)}$$

which holds for all $n \in \mathbb{N}$. On I_1 we have the trivial estimate $\tau_0 \mathbf{1}_{[0,1]} \geq 1 = \frac{1}{3}\frac{1}{\omega(I_1)}$, and putting these together gives

$$\sup_{y\in[0,\infty)} \left(\int |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega\right)^{1/p} \geq \frac{3}{32} \sup_{n\in\mathbb{N}} \left(\int_{I_n} \frac{1}{\omega\left(I_n\right)} |f|^p \, d\omega\right)^{1/p}. \qquad \square$$

Remark 3.4. In Proposition 3.3 we compared the norm $\|f\|_{p,\infty}$ with the continuous amalgam norm $\|f\|_{p,\infty,[0,1]} = \sup_{y\in[0,\infty)} \left\| f\left(\tau_y \mathbf{1}_{[0,1]}\right)^{\frac{1}{p}}\right\|_p$ for $p \in [1,\infty)$. We consider the same comparison with $p=\infty$. Letting

$$A(y) = \left\{t \in [0,\infty) : \tau_y \mathbf{1}_{[0,1]}(t) > 0\right\}$$

we have $\|f\|_{\infty,\infty,[0,1]} = \sup_{y\in[0,\infty)} \left\| f \mathbf{1}_{A(y)}\right\|_\infty$. Clearly $A(y)$ is an open neighbourhood of y and hence

$$\sup_{y\in[0,\infty)} \left\| f \mathbf{1}_{A(y)}\right\|_\infty = \|f\|_\infty = \|f\|_{\infty,\infty}.$$

This means that for $p=\infty$ we have $C=1$ and in fact equality in Proposition 3.3.

We warn the reader that for every $p\in(1,\infty]$ the seemingly similar (and, in the group case, identical) norm $\sup_{y\in[0,\infty)} \left\| f\tau_y \mathbf{1}_{[0,1]}\right\|_p$ is smaller and not equivalent to $\sup_{y\in[0,\infty)} \left\| f\left(\tau_y \mathbf{1}_{[0,1]}\right)^{\frac{1}{p}}\right\|_p$. In fact, for this smaller norm, Proposition 3.3 fails for all choices of the constant C. The reason for this is that the sup-norm of $\tau_y \mathbf{1}_{[0,1]}$ tends to zero as $y\to\infty$.

Proposition 3.5. *For* $p \in [1,\infty)$,

$$\|f\|_{p,\infty} \geq C \sup_{y\in[0,\infty)} \left(\int |f|^p \tau_y \mathbf{1}_{[0,1]}\, d\omega\right)^{1/p}.$$

Proof. (i) For $y\in[0,1)$ the expression in (3.4) takes the simpler form

$$\tau_y \mathbf{1}_{[0,1]}(x) = \begin{cases} 1, & x \leq 1-y, \\ \frac{1}{4xy}\left(1-(x-y)^2\right) \leq 1, & 1-y<x<1+y, \\ 0, & x \geq 1+y. \end{cases}$$

Since $\tau_y \mathbf{1}_{[0,1]} \leq \mathbf{1}_{[0,2)}$ this gives

$$\begin{aligned} \int |f|^p \tau_y \mathbf{1}_{[0,1]}\, d\omega &\leq \int |f|^p \mathbf{1}_{[0,1)}\, d\omega + \int |f|^p \mathbf{1}_{[1,2)}\, d\omega \\ &\leq \frac{1}{\omega(I_1)} \int |f|^p \mathbf{1}_{[0,1)}\, d\omega + \frac{3}{\omega(I_2)} \int |f|^p \mathbf{1}_{[1,2)}\, d\omega \\ &= \int_{I_1} \frac{1}{\omega(I_1)} |f|^p\, d\omega + 3\int_{I_2} \frac{1}{\omega(I_2)} |f|^p\, d\omega \end{aligned}$$

the second inequality holding since $\omega(I_1) = \frac{1}{3} < 1$ and $\omega(I_2) = \frac{7}{3} < 3$. Hence

$$\begin{aligned} \left(\int |f|^p \tau_y \mathbf{1}_{[0,1]}\, d\omega\right)^{1/p} &\leq \left(\int_{I_1} \frac{1}{\omega(I_1)} |f|^p\, d\omega\right)^{1/p} + 3^{1/p}\left(\int_{I_2} \frac{1}{\omega(I_2)} |f|^p\, d\omega\right)^{1/p} \\ &\leq \left(1+3^{1/p}\right) \|f\|_{p,\infty} \end{aligned}$$

(ii) For $y \in [1,2)$ we have $\tau_y \mathbf{1}_{[0,1]} \leq \mathbf{1}_{[0,3)}$ which leads to

$$\begin{aligned}
&\int |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega \\
&\quad \leq \int_{I_1} |f|^p \, d\omega + \int_{I_2} |f|^p \, d\omega + \int_{I_3} |f|^p \, d\omega \\
&\quad \leq \int_{I_1} \frac{1}{\omega(I_1)} |f|^p \, d\omega + 3 \int_{I_2} \frac{1}{\omega(I_2)} |f|^p \, d\omega + 7 \int_{I_3} \frac{1}{\omega(I_3)} |f|^p \, d\omega
\end{aligned}$$

since $\omega(I_3) < 7$, and hence

$$\begin{aligned}
&\left(\int |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega \right)^{1/p} \\
&\quad \leq \left(\int_{I_1} \frac{1}{\omega(I_1)} |f|^p \, d\omega \right)^{1/p} + 3^{1/p} \left(\int_{I_2} \frac{1}{\omega(I_2)} |f|^p \, d\omega \right)^{1/p} \\
&\quad + 7^{1/p} \left(\int_{I_3} \frac{1}{\omega(I_3)} |f|^p \, d\omega \right)^{1/p} \\
&\quad \leq \left(1 + 3^{1/p} + 7^{1/p}\right) \|f\|_{p,\infty} \, .
\end{aligned}$$

(iii) For $y \geq 2$ we have

$$\tau_y \mathbf{1}_{[0,1]}(x) = \begin{cases} \frac{1}{4xy}\left(1 - (x-y)^2\right), & y-1 < x < y+1, \\ 0, & \text{otherwise.} \end{cases}$$

If $y \in I_k$, then $k \geq 3$ and $(y-1, y+1)$ intersects at most I_{k-1}, I_k, I_{k+1}. For $x \in (y-1, y+1)$ we have

$$4xy > 4(y-1)y > 4(k-2)(k-1).$$

Now $k \geq 3$ implies $4(k-2) \geq k$ and $3(k-1) \geq k+2$ so that

$$4xy \geq \frac{1}{3} k(k+2) \geq \frac{1}{3}\left(k^2 + k + \frac{1}{3}\right) = \frac{1}{3}\omega(I_{k+1}) \geq \frac{1}{3}\omega(I_k) \geq \frac{1}{3}\omega(I_{k-1})$$

Thus we obtain for $j = k-1, k, k+1$

$$\begin{aligned}
\int_{I_j} |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega &= \int_{I_j} |f|^p \frac{1}{4xy}\left(1 - (x-y)^2\right) \mathbf{1}_{(y-1,y+1)} \, d\omega \\
&\leq 3 \int_{I_j} \frac{1}{\omega(I_j)} |f|^p \, d\omega
\end{aligned}$$

and

$$\left(\int |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega \right)^{1/p} \leq 3^{1/p} \sum_{j=k-1}^{k+1} \left(\int_{I_j} \frac{1}{\omega(I_j)} |f|^p \, d\omega \right)^{1/p} \leq 3^{1+1/p} \|f\|_{p,\infty} \, .$$

(iv) Taking C to be the maximum of the constants in (i)-(iii) we have

$$\left(\int |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega\right)^{1/p} \leq C \, \|f\|_{p,\infty}$$

for all $y \in [0, \infty)$ and hence

$$\sup_{y \in [0,\infty)} \left(\int |f|^p \, \tau_y \mathbf{1}_{[0,1]} \, d\omega\right)^{1/p} \leq C \, \|f\|_{p,\infty} \, . \qquad \square$$

3.4. **Functions that are square integrable on a neighbourhood of the identity.** For $p = 2$ we have the following characterisation along the lines of [7], Theorem 3.1.

Theorem 3.6. *For $f \in L^1\left(\mathbb{R}_+, *_{\frac{1}{2}}\right)$ with $\hat{f} \geq 0$ the following are equivalent:*

(1) *f is square integrable in a neighbourhood of the identity;*
(2) $\hat{f} \in (L^1, \ell^2)\left(\mathbb{R}_+, *_{\frac{1}{2}}\right)$;
(3) $f \in (L^2, \ell^\infty)\left(\mathbb{R}_+, *_{\frac{1}{2}}\right)$.

Proof. The proof of Theorem 3.1 in [7] applies, but we need to check that the results used there are still valid in our setting. This requires the equivalence of the continuous and the discrete amalgam norms, which we showed in Propositions 3.3 and 3.5, together with uniform boundedness of translation along with the Hausdorff-Young theorem for these amalgam spaces. We prove the latter two properties in the next three sections. $\square$

3.5. **Translation in** $(L^\infty, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$**.** In this section we show that translation is uniformly bounded on the amalgam space $(L^\infty, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$. Denote the Haar measure $\omega(I_n)$ of the interval I_n by ω_n. It is easily checked that $\omega_n = n^2 - n + \frac{1}{3}$. Given a locally integrable function f on $\mathbb{R}_+$ let $P_n f := f \mathbf{1}_{I_n}$ and consider

$$\tau_y f(x) = f\left(x *_{\frac{1}{2}} y\right) = \frac{1}{2xy} \int_{[|x-y|, x+y]} f(t) t \, dt.$$

Note that $|\tau_y f| \leq \tau_y(|f|)$ pointwise, and that $\tau_y(|f|) \leq \tau_y g$ if $|f| \leq g$ almost everywhere. We want to show uniform boundedness of the translation operators τ_y on $(L^\infty, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$.

Consider an index n and a positive number y, and write $f_n := \mathbf{1}_{I_n}$. It will be enough to show that

$$\|\tau_y f_n\|_{(L^\infty, \ell^1)} \leq C \|f_n\|_{(L^\infty, \ell^1)}$$

for a number C that is independent of y and n. Indeed, letting $c_n = \|P_n f\|_\infty$ and $g = \sum_n c_n f_n$, we then have that $|f| \leq g$ pointwise, and thus $\|\tau_y f\|_{(L^\infty, \ell^1)} \leq$

$\|\tau_y g\|_{(L^\infty,\ell^1)}$. But also $\tau_y g \le \sum_n c_n \tau_y(f_n)$ pointwise so that

$$\|\tau_y f\|_{(L^\infty,\ell^1)} \le \|\tau_y g\|_{(L^\infty,\ell^1)} \le \sum_n c_n \|\tau_y f_n\|_{(L^\infty,\ell^1)}$$
$$\le \sum_n c_n C \|f_n\|_{(L^\infty,\ell^1)} = C\|f\|_{(L^\infty,\ell^1)}.$$

Fix y and n, and call a non-negative integer k *exceptional* if $k=1$ or if there is some number x in the interval I_k such that $|x-y|$ or $x+y$ lies in I_n. Denote the set of exceptional indices by E, and let G be the set of *generic* indices forming the complement of E in $\mathbb{Z}_+$.

If k is generic, then the intersection of the interval $[|x-y|, x+y]$ with I_n is either empty for all x in I_k, or this intersection is all of I_n for all such x. Then $\tau_y f_n$ either vanishes on the whole interval I_k or it coincides on I_k with

$$\frac{1}{2xy}\int_{n-1}^{n} t\,dt. \tag{3.5}$$

Since $k \ge 2$, the expression above does not change by more than a factor of 2 as x runs through the interval I_k.

So for each generic index k there is a non-negative constant d_k with $d_k \le \tau_y f_n(x) \le 2d_k$ for all x in I_k. Then

$$\omega_k \|P_k(\tau_y f_n)\|_\infty \le \omega_k 2d_k \le 2\|P_k(\tau_y f_n)\|_1.$$

Note too that $\omega_n\|f_n\|_\infty = \|f_n\|_1$ since f_n is constant ($=1$) on its support I_n. Therefore,

$$\sum_{k\in G} \omega_k \|P_k(\tau_y f_n)\|_\infty \le \sum_{k\in G} 2\|P_k(\tau_y f_n)\|_1 \le \sum_{k\in\mathbb{Z}_+} 2\|P_k(\tau_y f_n)\|_1$$
$$= 2\|\tau_y(f_n)\|_1 \le 2\|f_n\|_1 = 2\omega_n\|f_n\|_\infty = 2\|f_n\|_{(L^\infty,\ell^1)},$$

the last inequality holding since the norm of any translation on $L^1(\mathbb{R}_+, *_{\frac{1}{2}})$ is 1.

One way for k to be exceptional is to have $x+y$ belong to I_n for some x in I_k, that is, the sets $y+I_k$ and I_n have non-empty intersection; equivalently, the set $I_n - y$ overlaps I_k. There are at most two such values of k, and none when $y > n$. Any other exceptional indices k come from cases where $I_n + y$ or $y - I_n$ overlap I_k, or $k=1$. It follows easily that there are at most seven exceptional indices, and in fact there are at most five of them.

It remains to estimate $\omega_k\|P_k(\tau_y f_n)\|_\infty$ for each exceptional index k. When $k \le 3n$ use the estimate

$$\tau_y f_n(x) \le \frac{1}{2xy}\int_{[|x-y|,x+y]} t\,dt = \frac{1}{4xy}\{(x+y)^2 - |x-y|^2\} = 1$$

to see that

$$\omega_k\|P_k(\tau_y f_n)\|_\infty \le \omega_k \le \omega_{3n} \le 19\omega_n = 19\|f_n\|_{(L^\infty,\ell^1)}.$$

When k is exceptional and $k > 3n$, one of the sets $y \pm I_n$ must overlap I_k. The smallest value that y could take would then satisfy $y + n = k-1$, making

$y + \frac{1}{3}k > k - 1$ and $y > \frac{2}{3}k - 1 > \frac{1}{3}k$ since $k > 3$. In particular, $y > \frac{1}{3}x$ for all x in I_k in these cases. For this k and such x use the upper bound

$$\tau_y f_n(x) \leq \frac{1}{2xy} \int_{n-1}^{n} t\, dt = \frac{1}{4xy}\{n^2 - (n-1)^2\} < \frac{2n}{x^2} \leq \frac{2n}{(k-1)^2}$$

where the first inequality follows from (3.5), to see that

$$\omega_k \|P_k(\tau_y f_n)\|_\infty \leq \frac{k^2(2n)}{(k-1)^2} \leq 8n \leq 24\omega_n \leq 24\|f_n\|_{(L^\infty,\ell^1)}.$$

3.6. **Translation and convolution on $(L^p, \ell^q)(\mathbb{R}_+, *_{\frac{1}{2}})$.** In this section we deduce that translation is uniformly bounded on $(L^p, \ell^q)(\mathbb{R}_+, *_{\frac{1}{2}})$ and note that Young's inequality for convolution also holds for the amalgam spaces on $(\mathbb{R}_+, *_{\frac{1}{2}})$. The uniform boundedness of translation on $(L^\infty, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$ implies by duality that it also holds on $(L^1, \ell^\infty)(\mathbb{R}_+, *_{\frac{1}{2}})$. To confirm this, first note that matters reduce to the case of a non-negative function, g say, in $(L^1, \ell^\infty)(\mathbb{R}_+, *_{\frac{1}{2}})$, and that $\tau_y g$ is then also non-negative. This translate belongs to $(L^1, \ell^\infty)(\mathbb{R}_+, *_{\frac{1}{2}})$ if and only if

$$\int_{\mathbb{R}_+} (\tau_y g(x))\, f(x)\, d\omega(x) < \infty$$

for all non-negative functions f in the unit ball of $(L^\infty, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$. In this case, the norm of $\tau_y g$ in $(L^1, \ell^\infty)(\mathbb{R}_+, *_{\frac{1}{2}})$ is equal to the supremum of these integrals over all such functions f. By [2, Theorem 1.3.21], and the fact that $y^- = y$, these integrals are equal to

$$\int_{\mathbb{R}_+} g(z)\, (\tau_y f(z))\, d\omega(z) \leq \|g\|_{(L^1,\ell^\infty)} \|\tau_y f\|\|_{(L^\infty,\ell^1)} \leq C\|g\|_{(L^1,\ell^\infty)}.$$

We thus have uniform boundedness of translation on $(L^p, \ell^q)(\mathbb{R}_+, *_{\frac{1}{2}})$ when the reciprocal indices $(1/p, 1/q)$ sit at any of the four corners of the unit square in the first quadrant. As in [8], complex interpolation then yields uniform boundedness of translation whenever $(1/p, 1/q)$ lies in this unit square, that is whenever $1 \leq p, q \leq \infty$. This also follows in a more elementary way from Hölder's inequality.

As in the case of locally compact abelian groups, Young's inequality for convolution of L^p-functions extends to these amalgams. The general statement is that if reciprocal indices in the unit square satisfy the condition

$$\left(\frac{1}{p}, \frac{1}{q}\right) = \left(\frac{1}{p_1}, \frac{1}{q_1}\right) + \left(\frac{1}{p_2}, \frac{1}{q_2}\right) - (1, 1)$$

and if functions f_1 and f_2 belong to the respective amalgams $(L^{p_1}, \ell^{q_1})(\mathbb{R}_+, *_{\frac{1}{2}})$ and $(L^{p_2}, \ell^{q_2})(\mathbb{R}_+, *_{\frac{1}{2}})$, then the convolution of f_1 and f_2 is defined and belongs to $(L^p, \ell^q)(\mathbb{R}_+, *_{\frac{1}{2}})$. Moreover, we have

$$\|f_1 *_{\frac{1}{2}} f_2\|_{(L^p,\ell^q)} \leq C\|f_1\|_{(L^{p_1},\ell^{q_1})} \|f_2\|_{(L^{p_2},\ell^{q_2})}.$$

In fact, the inclusions between amalgams then imply that these statements still hold, usually with a different constant C, provided that $1/p \leq 1/p_1 + 1/p_2 - 1$ and

$1/q \geq 1/q_1 + 1/q_2 - 1$. Another way to state this is that $(1/p, 1/q)$ can be any point in the unit square lying northwest of the point $(1/p_1 + 1/p_2 - 1, 1/q_1 + 1/q_2 - 1)$, which is also required to lie in the unit square. Again the general case follows from a few extreme cases by complex interpolation or by repeated use of Hölder's inequality.

3.7. **Fourier transforms on $(L^p, \ell^q)(\mathbb{R}_+, *_{\frac{1}{2}})$.** Our goal in this section is to prove that if $f \in (L^p, \ell^q)(\mathbb{R}_+, *_{\frac{1}{2}})$ with $1 \leq p, q \leq 2$, then $\hat{f} \in (L^{q'}, \ell^{p'})(\mathbb{R}_+, *_{\frac{1}{2}})$. The cases where $p = q$ are already known (see [4]) with the same proof as for locally compact abelian groups, but if $p \neq q$, then this property of the Fourier transform requires some work. These cases will follow by complex interpolation from those where $p = q$ and the special ones where $(p, q) = (2, 1)$ or $(1, 2)$. (The latter is the one that arises in the proof of Theorem 3.6.) We show below that the two special cases are equivalent by duality, and we prove the first case using some easily-checked properties of transforms of the indicator functions $\mathbf{1}_{I_n}$.

From (3.2) we find that the Fourier transform of $\mathbf{1}_{I_1}$ belongs to $(L^\infty, \ell^q)(\mathbb{R}_+, *_{\frac{1}{2}})$ for all $q > \frac{3}{2}$, but does *not* belong to $(L^p, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$ for any value of p. Let $g_1 = 3\,\mathbf{1}_{I_1} *_{\frac{1}{2}} \mathbf{1}_{[0,2)}$ and $g_n = 3\,\mathbf{1}_{I_1} *_{\frac{1}{2}} \mathbf{1}_{[n-2,n+1)}$ when $n > 1$. We can check that $g_n(x) = 1$ for all x in I_n. When $n > 1$, Hölder's inequality gives

$$\begin{aligned}
\|\hat{g}_n\|_{(L^2,\ell^1)} &= 3\left\|\widehat{\mathbf{1}_{I_1}}\left(\widehat{\mathbf{1}_{I_{n-1}}} + \widehat{\mathbf{1}_{I_n}} + \widehat{\mathbf{1}_{I_{n+1}}}\right)\right\|_{(L^2,\ell^1)} \\
&\leq 3\left\|\widehat{\mathbf{1}_{I_1}}\right\|_{(L^\infty,\ell^2)}\left\|\widehat{\mathbf{1}_{I_{n-1}}} + \widehat{\mathbf{1}_{I_n}} + \widehat{\mathbf{1}_{I_{n+1}}}\right\|_{(L^2,\ell^2)} \\
&= C\left\|\widehat{\mathbf{1}_{I_{n-1}}} + \widehat{\mathbf{1}_{I_n}} + \widehat{\mathbf{1}_{I_{n+1}}}\right\|_2 \\
&= C\left\|\mathbf{1}_{I_{n-1}} + \mathbf{1}_{I_n} + \mathbf{1}_{I_{n+1}}\right\|_2 \\
&= C(\omega_{n-1} + \omega_n + \omega_{n+1})^{1/2} \\
&\leq C'\sqrt{\omega_n}.
\end{aligned}$$

By formula (3.3), if $f \in L^2(\mathbb{R}_+, *_{\frac{1}{2}})$ and f vanishes outside I_n, then $\|f\|_{(L^2,\ell^1)} = \sqrt{\omega_n}\|f\|_2$. Moreover, in this case $f = fg_n$ and it follows by Young's inequality for convolution of amalgams that

$$\begin{aligned}
\left\|\hat{f}\right\|_{(L^\infty,\ell^2)} &= \left\|\widehat{fg_n}\right\|_{(L^\infty,\ell^2)} = \left\|\hat{f} *_{\frac{1}{2}} \hat{g}_n\right\|_{(L^\infty,\ell^2)} \leq \left\|\hat{f}\right\|_{(L^2,\ell^2)}\left\|\hat{g}_n\right\|_{(L^2,\ell^1)} \\
&\leq \left\|\hat{f}\right\|_2 C\sqrt{\omega_n} = C\sqrt{\omega_n}\|f\|_2 = C\|f\|_{(L^2,\ell^1)}.
\end{aligned}$$

For a general function f in $(L^2, \ell^1)(\mathbb{R}_+,)$, applying the inequalities above to $P_n f := f\mathbf{1}_{I_n}$ yields that $\left\|\widehat{P_n f}\right\|_{(L^\infty,\ell^2)} \leq C\sqrt{\omega_n}\,\|P_n f\|_2$. Since for $(p, q) = (2, 1)$, formula (3.3) takes the special form $\|f\|_{(L^2,\ell^1)} = \sum_{n=1}^\infty \sqrt{\omega_\alpha(I_n)}\,\|P_n f\|_2$, it follows that $\left\|\hat{f}\right\|_{(L^\infty,\ell^2)} \leq C\,\|f\|_{(L^2,\ell^1)}$.

Suppose next that $g \in L^1(\mathbb{R}_+, *_{\frac{1}{2}})$. Then $\hat{g}$ belongs to $(L^2, \ell^\infty)(\mathbb{R}_+, *_{\frac{1}{2}})$ if and only if $\hat{g} f \in L^1(\mathbb{R}_+, *_{\frac{1}{2}})$ for all functions f in the unit ball of $(L^2, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$. In this case, $\|\hat{g}\|_{(L^2,\ell^1)}$ is equal to the supremum over all such functions f of the numbers $|\int \hat{g}(t) f(t)\omega(t)\, dt|$. But each of these integrals is equal to $\int g(x)\hat{f}(x)\omega(x)\, dx$ and so has absolute value less than or equal to

$$\|g\|_{(L^1,\ell^2)} \left\|\hat{f}\right\|_{(L^\infty,\ell^2)} \leq \|g\|_{(L^1,\ell^2)} C \|f\|_{(L^2,\ell^1)} = C\|g\|_{(L^1,\ell^2)}.$$

In other words, the Fourier transform is a bounded operator from $L^1(\mathbb{R}_+, *_{\frac{1}{2}})$ to $(L^2, \ell^\infty)(\mathbb{R}_+, *_{\frac{1}{2}})$ when $L^1(\mathbb{R}_+, *_{\frac{1}{2}})$ is viewed as a dense subspace of $(L^1, \ell^2)(\mathbb{R}_+, *_{\frac{1}{2}})$ with the norm $\|\cdot\|_{(L^1,\ell^2)}$. Extend this operator to all of $(L^1, \ell^2)(\mathbb{R}_+, *_{\frac{1}{2}})$.

This includes the usual extension of the Fourier transform operator from the intersection of the spaces $L^1(\mathbb{R}_+, *_{\frac{1}{2}})$ and $L^2(\mathbb{R}_+, *_{\frac{1}{2}})$ to an isometry from the space $L^2(\mathbb{R}_+, *_{\frac{1}{2}})$ to a dual copy of $L^2(\mathbb{R}_+, *_{\frac{1}{2}})$. It also includes the transform originally defined as a mapping of $L^1(\mathbb{R}_+, *_{\frac{1}{2}})$ to $L^\infty(\mathbb{R}_+, *_{\frac{1}{2}})$ and shown above to map the smaller space $(L^2, \ell^1)(\mathbb{R}_+, *_{\frac{1}{2}})$ to $(L^\infty, \ell^2)(\mathbb{R}_+, *_{\frac{1}{2}})$. So, the Hausdorff-Young theorem holds for amalgams in the four extreme cases where the indices (p, q) are $(1, 1)$, $(2, 2)$, $(2, 1)$ and $(1, 2)$, and the other cases then follow by complex interpolation.

4. Some countable non-discrete hypergroups

The positive conclusion in Wiener's theorem also holds for non-even exponents in the interval $[1, \infty)$ on some countable compact hypergroups H_a considered in [5] and [14], and on the countable locally compact hypergroup H below. Here a is a parameter in the interval $(0, 1/2]$. We let $a = 1/2$ and leave the other cases for the reader.

4.1. Compact countable commutative hypergroups.

Example 4.1. The one-point compactification $\mathbb{Z}_+ \cup \{\infty\}$ of the non-negative integers is a compact commutative hypergroup $\left(H_{\frac{1}{2}}, *\right)$ with convolution given by

$$\varepsilon_m * \varepsilon_n = \begin{cases} \sum_{k=1}^{\infty} \frac{1}{2^k} \varepsilon_{k+n}, & m = n \in \mathbb{Z}_+, \\ \varepsilon_\infty, & m = n = \infty, \\ \varepsilon_{\min\{m,n\}}, & m \neq n \in \mathbb{Z}_+ \cup \{\infty\}, \end{cases} \tag{4.1}$$

so that ε_∞ is the identity element. The Haar measure ω is given by $\omega(n) = \frac{1}{2^{n+1}}$ for $n < \infty$ and $\omega(\infty) = 0$. The characters χ_n are given by

$$\chi_n(m) = \begin{cases} 0, & m \leq n-2, \\ -1, & m = n-1, \\ 1, & m \geq n \end{cases}$$

where $n \in \mathbb{Z}_+$, and the Plancherel measure π is just

$$\pi(\chi_n) = \frac{1}{\|\chi_n\|_2^2} = \begin{cases} 2^{n-1}, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases} \tag{4.2}$$

We observe that the set of continuous positive definite functions is given by

$$P\left(H_{\frac{1}{2}}\right) = \left\{ f : f = \sum_{i=0}^{\infty} \alpha_i \chi_i : \alpha_i \geq 0, \sum_{i=0}^{\infty} \alpha_i < \infty \right\} \tag{4.3}$$

(indeed, in [5], equation (4.3) is the definition of $P\left(H_{\frac{1}{2}}\right)$). It is a consequence of Bochner's theorem ([2, Theorems 4.1.15 and 4.1.16]) that (4.3) holds if and only if $f \in P_b\left(H_{\frac{1}{2}}\right)$, and this space coincides with $P\left(H_{\frac{1}{2}}\right)$ because $H_{\frac{1}{2}}$ is compact.

If f is as in (4.3) then

$$f(n) = \left(\sum_{i=0}^{n} \alpha_i \right) - \alpha_{n+1} \tag{4.4}$$

for $n \in \mathbb{Z}_+$ and (because of continuity)

$$f(\infty) = \sum_{i=0}^{\infty} \alpha_i. \tag{4.5}$$

Remark 4.2. For $f \in P\left(H_{\frac{1}{2}}\right)$ we have $\|f\|_\infty = f(\infty)$, as seen from (4.4) and (4.5) (or from [2, Lemma 4.1.3(g)]).

4.2. **Operations on** $P\left(H_{\frac{1}{2}}\right)$**.** By (4.3) the function f is the inverse Fourier transform of

$$i \longmapsto \alpha_i / \pi(\chi_i)$$

and the latter function (on $\widehat{H_{\frac{1}{2}}}$) belongs to $L^1(\pi)$. The set of inverse transforms of functions in $L^1(\pi)$ is called the Fourier algebra of $H_{\frac{1}{2}}$, and is denoted by $A\left(H_{\frac{1}{2}}\right)$. It is shown in [5] that Lipschitz functions operate on $A\left(H_{\frac{1}{2}}\right)$; in particular, if $f \in A\left(H_{\frac{1}{2}}\right)$ and $1 \leq p < \infty$, then $|f|^p \in A\left(H_{\frac{1}{2}}\right)$ as well. We prove the corresponding statement for $P\left(H_{\frac{1}{2}}\right)$ and apply it in Section 4.7.

Proposition 4.3. *Let $1 \leq p < \infty$. Suppose that $f : H_{\frac{1}{2}} \to \mathbb{C}$ is p-integrable in a neighbourhood U of the identity e. If f is of positive type then so is $|f|^p$. In particular, if $f \in P\left(H_{\frac{1}{2}}\right)$ then $|f|^p \in P\left(H_{\frac{1}{2}}\right)$.*

Proof. The p-integrability of f near e implies global p-integrability, because the complement of U is finite. Since the Plancherel measure has full support, Remark 2.4 then reduces matters to checking that the Fourier coefficients of $|f|^p$ are non-negative if those of f are.

When $p = 1$, let $r(n) = f(n)\,\omega(n)$ for each n; then $r \in \ell^1$ since f is integrable. We claim that $\hat{f} \geq 0$ if and only if r is real-valued and

$$|r(n)| \leq r(n+1) + r(n+2) + \cdots \quad \text{for all } n \tag{4.6}$$

If these inequalities hold for f, then they also hold when all negative values $r(m)$ are replaced by $|r(m)|$, that is when f is replaced by $|f|$. So the case of the proposition where $p = 1$ follows from our claim.

The conditions above on r are equivalent to requiring for all n that

$$r(n) + r(n+1) + r(n+2) + \cdots \geq 0 \tag{4.7}$$

$$\text{and} \quad -r(n) + r(n+1) + r(n+2) + \cdots \geq 0. \tag{4.8}$$

Indeed, subtracting the two inequalities for the same value of n shows that $r(n)$ is real, and then inequality (4.6) follows since $|r(n)| = \max\{r(n), -r(n)\}$. The converse is obvious.

Condition (4.8) is equivalent to requiring that $\hat{f}(n+1) \geq 0$, while the 0^{th} case of condition (4.7) is equivalent to requiring that $\hat{f}(0) \geq 0$. If condition (4.8) holds for all n, and condition (4.7) holds for some value of n, then adding the corresponding case of condition (4.8) shows that condition (4.7) also holds for the next value of n. So the two conditions hold of all values of n if and only if f is of positive type.

To deal with exponents p in the interval $(1, \infty)$, consider the n-th instance of condition (4.6) with f replaced by $|f|^p$, that is

$$|f(n)|^p \omega(n) \leq |f(n+1)|^p \omega(n+1) + |f(n+2)|^p \omega(n+2) + \cdots .$$

Let $\omega_n(n+k) = \omega(n+k)/\omega(n)$ when $k = 1, 2 \cdots$. The inequality above is equivalent to requiring that

$$|f(n)| \leq \left[\sum_{k=1}^{\infty} |f(n+k)|^p \omega_n(n+k)\right]^{1/p} . \tag{4.9}$$

The expression on the right above is the L^p norm of the restriction of f to the set $\{n+1, n+2, \cdots\}$ with respect to the measure ω_n, which has total mass 1. By Hölder's inequality, that L^p norm majorizes the corresponding L^1 norm. So it is enough the prove inequality (4.9) when $p = 1$, and that was done in the first part of the proof. □

4.3. **A locally compact example.** We now analyse a non-compact example presented in [14]. For $N > 0$ the set U_N defined by

$$U_N := \{N, N+1, N+2, \cdots, \infty\} \tag{4.10}$$

is a proper subhypergroup of $H_{\frac{1}{2}}$ and is isomorphic to $H_{\frac{1}{2}}$, but with a scaled Haar measure. Define similar hypergroups U_N when $N \leq 0$ ($U_0 = H_{\frac{1}{2}}$), and let H be the union of these nested compact hypergroups. Then H is a locally compact commutative hypergroup with convolution given by

$$\varepsilon_m * \varepsilon_n = \begin{cases} \sum_{k=1}^{\infty} \frac{1}{2^k} \varepsilon_{k+n}, & m = n \in \mathbb{Z}, \\ \varepsilon_\infty, & m = n = \infty, \\ \varepsilon_{\min\{m,n\}}, & m \neq n \in \mathbb{Z} \cup \{\infty\}, \end{cases} \tag{4.11}$$

so that ε_∞ is the identity element, but H is not compact.

The functions χ_n in Example 4.1, with n now allowed to be any integer, comprise all the characters on H except for the character $\chi_{-\infty} \equiv 1$, which has Plancherel measure 0. The first case of formula (4.2) for the Plancherel measure of χ_n extends to all indices $n \leq 0$ (in particular we now have $\pi(\chi_0) = \frac{1}{2}$).

Note that H is Pontryagin since (up to the different parametrization of $H^\wedge$) it is self-dual via the mapping $n \to \chi_{-n}$. In fact it is straightforward to see that

$$\chi_m \chi_n = \begin{cases} \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{n-k}, & m = n \in \mathbb{Z}, \\ \chi_{-\infty}, & m = n = -\infty, \\ \chi_{\max\{m,n\}}, & m \neq n \in \mathbb{Z} \cup \{-\infty\}. \end{cases}$$

Remark 4.4. By [2, Corollary 2.4.20(ii)], $H_{\frac{1}{2}}$ is also Pontryagin. In particular, H and $H_{\frac{1}{2}}$ are strong hypergroups (that is, their canonical duals are also hypergroups). Now use Remark 2.7 to obtain

$$P_b(H_{\frac{1}{2}}) \cdot P_b(H_{\frac{1}{2}}) \subset P_b(H_{\frac{1}{2}}) \quad \text{and} \quad P_b(H) \cdot P_b(H) \subset P_b(H),$$

so that all the results of Section 2 apply to both $H_{\frac{1}{2}}$ and H. In particular the conclusion of Wiener's theorem holds on H, and again on $H_{\frac{1}{2}}$, for all even $p \geq 1$. In Section 4.7 we will show that the same conclusion holds on both $H_{\frac{1}{2}}$ and H for all $p \in [1, \infty]$.

4.4. **Localizing properties of functions.** Functions on H are positive definite if and only if their restrictions to each subhypergroup U_N are positive definite. The same is true for continuity of functions on H. If $g \in C_c(K)$ then the convolution $g^* * g$ vanishes outside U_N for some integer N. It follows that a (locally integrable) function is of positive type on H if and only if the restriction of that function to each U_N is of positive type. Lemma 4.6 below provides a converse to this.

It is again clear that every ℓ^1 sum of characters (including $\chi_{-\infty}$) with non-negative coefficients is continuous, bounded and positive definite. Conversely, given a function f in $P(H)$, denote its restriction to the subhypergroup U_N by $f|_{U_N}$. Then $f|_{U_N}$ is bounded as U_N is compact, and by [2, Lemma 4.1.3g],

$$\|f|_{U_N}\|_\infty = f|_{U_N}(\infty) = f(\infty)$$

for all $N \in \mathbb{Z}$. It follows that f is bounded on H, so then by Bochner's theorem again there exist non-negative $\alpha_{-\infty}$ and α_j, $j \in \mathbb{Z}$ with $\sum_j \alpha_j < \infty$ such that $f = \alpha_{-\infty}\chi_{-\infty} + \sum_j \alpha_j \chi_j$, and hence $\|f\|_\infty = f(\infty)$ and $P(H) = P_b(H)$.

The following proposition is a corollary of Proposition 4.3, using localization and the lines after (4.10), and will prove useful in Section 4.7.

Proposition 4.5. *Let $1 \le p < \infty$. Suppose that $f : H \to \mathbb{C}$ is p-integrable in a neighbourhood of the identity. If f is of positive type then so is $|f|^p$. In particular, if $f \in P(H)$ then $|f|^p \in P(H)$.*

Lemma 4.6. *Extend a function of positive type on the hypergroup U_N to all of H by making it vanish outside U_N. That extension is of positive type on H. In particular, the extension by zero of a function in $P(U_N)$ is in $P(H)$.*

Proof. Denote the original function by f_N and its extension by f. Since f_N is locally integrable and U_N is compact, $f_N \in L^1(U_N)$ and $f \in L^1(H)$.

To apply Remark 2.4, let χ be a character on H. Then its restriction $\chi|_{U_N}$ to U_N is a character on U_N, and $\hat{f}(\chi) = \overset{\wedge}{f_N}(\chi|_{U_N})$. Since every character on U_N has positive Plancherel measure, $\overset{\wedge}{f_N}(\chi|_{U_N}) \ge 0$, and $\hat{f}(\chi)$ is nonnegative too. □

4.5. **Discrete amalgam norms.** We used the amalgam norm

$$\|f\|_{p,\infty} = \sup_n \left(\frac{1}{\omega_\alpha(I_n)} \int_{n-1}^{n} |f|^p \, d\omega_\alpha \right)^{\frac{1}{p}} \tag{4.12}$$

to state Theorem 3.6 for Bessel-Kingman hypergroups. Consider the corresponding norm on H. Given the division by the mass $\omega_\alpha(I_n)$ here, the integral above should run over the interval I_n. In H that coincides with the set $\{n-1\}$, with the curious outcome that

$$\|f\|_{p,\infty} = \sup_n |f(n-1)| = \sup_n |f(n)| = \|f\|_\infty \tag{4.13}$$

no matter what p is.

When $p < \infty$, there are compactly supported functions in $L^p(H)$ that tend to ∞ at ∞. Any such function f has the property that

$$\sup_n \left(\int |f|^p \, \tau_n \mathbf{1}_U \, d\omega \right)^{\frac{1}{p}} < \infty \tag{4.14}$$

for each compact neighbourhood U of ∞ even though $\|f\|_{p,\infty} = \infty$. So the norm $\|\cdot\|_{p,\infty}$ is not equivalent to the one given in (4.14). We show below that the modified norm

$$\|f\|^*_{p,\infty} = \max\left\{ \left\|f \mathbf{1}_{H\setminus U_0}\right\|_{p,\infty}, \left\|f\mathbf{1}_{U_0}\right\|_p \right\}, \tag{4.15}$$

where U_0 can be replaced by any compact neighbourhood of ∞, is equivalent to the norm in (4.14).

Different choices of U in (4.14) give norms that are equivalent to each other, by the argument just after Corollary 2.15. Similar reasoning applies to (4.15), and it suffices to prove the equivalence between the latter and the norm in (4.14) when $U = U_0$. Split the calculation of the supremum in (4.14) into two cases corresponding to different instances of (4.11). For $n < 0$ we have $\tau_n \mathbf{1}_{U_0} = 2^{n+1}\mathbf{1}_{\{n\}}$, so that

$$\left(\int |f|^p \, \tau_n \mathbf{1}_{U_0} \, d\omega\right)^{\frac{1}{p}} = |f(n)| \, .$$

For $n \geq 0$ we obtain $\tau_n \mathbf{1}_{U_0} = \mathbf{1}_{U_0}$, and this gives

$$\left(\int |f|^p \, \tau_n \mathbf{1}_{U_0} \, d\omega\right)^{\frac{1}{p}} = \|f \mathbf{1}_{U_0}\|_p \, .$$

By formula (4.13), the norms in (4.14) and (4.15) coincide when $U = U_0$.

When $1 \leq q < \infty$, let

$$\|f\|^*_{p,q} = \left\{ \left\| f \mathbf{1}_{H \setminus U_0} \right\|^q_{p,q} + \|f \mathbf{1}_{U_0}\|^q_p \right\}^{1/q} \tag{4.16}$$

where

$$\left\| f \mathbf{1}_{H \setminus U_0} \right\|_{p,q} \equiv \left\{ \sum_{n<0} \omega(\{n\}) |f(n)|^q \right\}^{1/q}$$

actually doesn't depend on p. Whenever $1 \leq p, q \leq \infty$, denote the space of functions f on H for which $\|f\|^*_{p,q} < \infty$ by $(L^p, \ell^q)(H)$.

On H, the structure of these spaces is simpler than it is on the real line or on the Bessel-Kingman hypergroups. A function belongs to $(L^p, \ell^q)(H)$ if and only if its restriction to the set U_0 belongs to L^p and its restriction to the complement of U_0 belongs to L^q.

Since $\omega(U_0) = 1$, the restriction to U_0 then belongs to L^r for all $r \leq p$. Since each point in the complement of U_0 has mass at least 1, the restriction to the complement then belongs to L^r for all $r \geq q$. Extend those restrictions by 0 to see that $(L^p, \ell^q)(H)$ contains the same functions as $L^p(H) + L^q(H)$ when $p \leq q$, and the same functions as $L^p(H) \cap L^q(H)$ when $p \geq q$.

4.6. **Fourier transforms.** The norms $\| \cdot \|^*_{p,q}$ have good properties relative to Fourier transforms (see below). Define $\| \cdot \|_{p,q}$ on $\hat{H}$ as for H just by replacing ω by π. Let

$$U_0^\perp \equiv \left\{ n \in \hat{H} : n \leq 0 \right\}$$

and use $U_0^\perp$ and its complement in $\hat{H}$ to define $\| \cdot \|^*_{p,q}$ as in equations (4.15) and (4.16). We have the following counterpart of Theorem 3.6.

Theorem 4.7. *The following statements are equivalent for a (locally integrable) function f of positive type on the hypergroup H:*

(1) *f is square integrable in a neighbourhood of the identity;*

(2) *f is the (inverse) transform of a function in the space $(L^1, \ell^2)(\hat{H})$;*

(3) *$f \in (L^2, \ell^\infty)(H)$.*

Proof. Again this follows if the Fourier transform extends from $L^1(H) \cap L^2(H)$ to have appropriate mapping properties between suitable amalgam spaces, that is,

$$\text{if} \quad \|f\|_{p,q}^* < \infty, \quad \text{where} \quad 1 \le p, q \le 2, \quad \text{then} \quad \left\|\hat{f}\right\|_{q',p'}^* < \infty. \qquad (4.17)$$

By the observations at the end of Section 4.5, this is equivalent to checking, when $1 \le p, q \le 2$, that if $f \in L^p(H) + L^q(H)$ then $\hat{f} \in L^{q'}\left(\hat{H}\right) + L^{p'}\left(\hat{H}\right)$, and the same for $L^p(H) \cap L^q(H)$ and $L^{q'}\left(\hat{H}\right) \cap L^{p'}\left(\hat{H}\right)$. Both parts follow immediately from the Hausdorff-Young theorem [4] for hypergroups. □

Remark 4.8. In fact, $\left\|\hat{f}\right\|_{q',p'}^* \le \|f\|_{p,q}^*$ in all these cases. Complex interpolation again reduces matters to proving this in the extreme cases where (p, q) is one of $(1,1), (2,2), (1,2)$ and $(2,1)$. The first two cases are true because

$$\left\|\hat{f}\right\|_{\infty} \le \|f\|_1 \quad \text{and} \quad \left\|\hat{f}\right\|_2 = \|f\|_2.$$

The corresponding estimates in the other two extreme cases follow from each other by duality as in Section 3.7.

We elect to confirm the case where $(p, q) = (2, 1)$ and $(q', p') = (\infty, 2)$. Split f as $f_1 + f_2$, where $f_2 = f\mathbf{1}_{U_0}$ and f_1 vanishes on U_0. Since $\|f\|_{2,1}^* = \|f_1\|_1 + \|f_2\|_2$, it suffices to show that $\left\|\overset{\wedge}{f_1}\right\|_{\infty,2}^* \le \|f_1\|_1$ and $\left\|\overset{\wedge}{f_2}\right\|_{\infty,2}^* \le \|f_2\|_2$.

Note that $\overset{\wedge}{f_1}(n) = 0$ for all $n > 0$, since the support of f_1 is disjoint from that of χ_n when $n > 0$. So $\left\|\overset{\wedge}{f_1}\right\|_{\infty,2}^*$ simplifies to become $\left\|\overset{\wedge}{f_1}\mathbf{1}_{U_0^\perp}\right\|_\infty$, and

$$\left\|\overset{\wedge}{f_1}\right\|_{\infty,2}^* \le \left\|\overset{\wedge}{f_1}\right\|_\infty \le \|f_1\|_1 \quad \text{as required.}$$

Note also that the characters χ_n with $n \le 0$ are all equal to 1 on the set U_0, making $\overset{\wedge}{f_2}$ constant on the set $U_0^\perp$. Then $\left\|\overset{\wedge}{f_2}\mathbf{1}_{U_0^\perp}\right\|_\infty = \left\|\overset{\wedge}{f_2}\mathbf{1}_{U_0^\perp}\right\|_2$ since $\pi\left(U_0^\perp\right) = 1$. So

$$\begin{aligned}
\left\|\overset{\wedge}{f_2}\right\|_{\infty,2}^* &= \left\{\left(\left\|\overset{\wedge}{f_2}\mathbf{1}_{U_0^\perp}\right\|_\infty\right)^2 + \sum_{n>0} \pi(\{n\}) \left|\overset{\wedge}{f_2}(n)\right|^2\right\}^{\frac{1}{2}} \\
&= \left\{\left(\left\|\overset{\wedge}{f_2}\mathbf{1}_{U_0^\perp}\right\|_2\right)^2 + \sum_{n>0} \pi(\{n\}) \left|\overset{\wedge}{f_2}(n)\right|^2\right\}^{\frac{1}{2}} \\
&= \left\|\overset{\wedge}{f_2}\right\|_2 = \|f_2\|_2 \quad \text{as required.}
\end{aligned}$$

4.7. **Wiener's theorem for all exponents.** We will show that versions of Wiener's theorem hold on H for all exponents in the interval $[1,\infty]$, but we first note that Lemma 2.5 can be sharpened in the case of this hypergroup:

Remark 4.9. For $U = U_N$ we may choose the neighbourhood V in the proof of Lemma 2.5 to be U_N as well. Instead of inequality (2.4) we obtain

$$h := \mathbf{1}_V^{\sim} * \mathbf{1}_V = \omega(U_N)\mathbf{1}_{U_N}.$$

The next step in that proof then works with the singleton $x_1 = \{e\}$, the parameter $\lambda_1 = 1/\omega(U_N)$ and the measure $\nu = \lambda_1\varepsilon_e$. The long chain of equalities and inequalities there ends with the quantity $\|\nu\| \int hg\, d\omega_K$. For the special choice of h above, this is

$$\|\nu\| \left\{\omega(U_N)\int_{U_N} g\, d\omega_K\right\}$$

which gives the conclusion of Lemma 2.5 with

$$C_{U_N} = \|\nu\|\omega(U_N) = 1.$$

It follows that Corollary 2.6 holds with $C_U = 1$ when $U = U_N$. Since the proof of that corollary only requires that $|f|^p \in P_b(K)$, Proposition 4.5 yields the conclusion of the corollary for all exponents p in the interval $[1,\infty)$, again with $C_U = 1$ if $U = U_N$ for some N. The proof of Corollary 2.8 shows, for such exponents p, that if inequality (2.5) holds for all functions f in $P_b(K)$, then the inequality holds with the same constant C_U for all integrable functions f that are of positive type.

Theorem 4.10. *Let $p \in [1,\infty]$ and f be a function of positive type on H. Then*

$$\|f\|_{p,\infty}^* = \|f\|_{p,\infty,U_0} = \|f\mathbf{1}_{U_0}\|_p\,. \tag{4.18}$$

For a general relatively compact neighbourhood U of the identity there are constants C_U and C_U' (independent of p) such that

$$\|f\|_{p,\infty,U} \le C_U\, \|f\mathbf{1}_U\|_p \quad \text{and} \quad \|f\|_{p,\infty}^* \le C_U'\, \|f\mathbf{1}_U\|_p \tag{4.19}$$

for all (locally integrable) functions f of positive type.

Corollary 4.11. *Let $p \in [1,\infty]$. For every relatively compact neighbourhood U of the identity in H and every compact subset V of H there is a constant $C_{U,V}$ (independent of p) such that*

$$\|f\mathbf{1}_V\|_p \le C_{U,V}\, \|f\mathbf{1}_U\|_p \tag{4.20}$$

for all (locally integrable) functions f of positive type.

Corollary 4.12. *Let $p \in [1,\infty]$. For every neighbourhood U of the identity in the compact hypergroup $H_{\frac{1}{2}}$ there is a constant C_U (independent of p) such that*

$$\|f\|_p \le C_U\, \|f\mathbf{1}_U\|_p \tag{4.21}$$

for all functions f of positive type.

Proofs. As in Corollary 2.13, the cases where $p = \infty$ follow from those where $p < \infty$. In the latter cases, there is nothing to prove unless $\|f\mathbf{1}_U\|_p < \infty$. Restricting f to various subhypergroups U_N and extending those restrictions by 0 then reduces matters to cases where f has compact support and is therefore p-integrable, hence integrable.

The first equality in (4.18) was shown, when $1 \le p < \infty$, in the lines following (4.15). For the second equality, it is clear from the definition of $\|f\|_{p,\infty,U_0}$ that it is no smaller than $\|f\mathbf{1}_{U_0}\|_p$. The opposite inequality $\|f\|_{p,\infty,U_0} \le \|f\mathbf{1}_{U_0}\|_p$ holds because of the discussion after Remark 4.9. The same discussion yields the first inequality in line (4.19). The second inequality then follows by the equivalence of the norms $\|\cdot\|_{p,\infty,U}$ and $\|\cdot\|^*_{p,\infty}$. This completes the proof of Theorem 4.10.

For Corollary 4.11, use the chain of inequalities

$$\|f\mathbf{1}_V\|_p \le \|f\|_{p,\infty,V} \le C'_{U,V}\|f\|_{p,\infty,U} \le C'_{U,V}C_U\,\|f\mathbf{1}_U\|_p\,,$$

where the first step uses the definition of $\|\cdot\|_{p,\infty,V}$, the second step uses the equivalence of that norm with $\|\cdot\|_{p,\infty,U}$ and the last step uses the first inequality in (4.19). Corollary 4.12 follows because extending f by 0 gives a function of positive type on H. □

Remark 4.13. The first inequality in (4.19) provides an upper bound for $\|f\|_{p,\infty,U}$ in terms of $\|f\mathbf{1}_U\|_p$. When $p < \infty$, there is no such general bound for $\|f\|_p$. Indeed, since $\sum_{n=-\infty}^{\infty}\omega(n) = \infty$, the constant function $\mathbf{1}$ trivially belongs to the set $P(H)$ but to none of the spaces $L^p(H)$ with $0 < p < \infty$.

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