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# Weak* fixed point property and asymptotic centre for the Fourier–Stieltjes algebra of a locally compact group

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**Abstract**

In this paper we show that the Fourier–Stieltjes algebra $B(G)$ of a non-compact locally compact group $G$ cannot have the weak* fixed point property for nonexpansive mappings. This answers two open problems posed at a conference in Marseille-Luminy in 1989. We also show that a locally compact group is compact exactly if the asymptotic centre of any non-empty weak* closed bounded convex subset $C$ in $B(G)$ with respect to a decreasing net of bounded subsets is a non-empty norm compact subset. In particular, when $G$ is compact, $B(G)$ has the weak* fixed point property for left reversible semigroups. This generalizes a classical result of T.C. Lim for the circle group. As a consequence of our main results we obtain that a number of properties, some of which were known to hold for compact groups, in fact characterize compact groups.


*Keywords:* Weak*-fixed point property; Fourier–Stieltjes algebra; Compact groups; Asymptotic centre; Left reversible semigroups; Fell group

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## 1. Introduction

Let $E$ be a Banach space and $K$ be a non-empty bounded closed convex subset of $E$. We say that $K$ has the *fixed point property* if every nonexpansive mapping $T : K \to K$ (i.e. $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$) has a fixed point. We say that $E$ has the *weak fixed point property* if every weakly compact convex subset of $E$ has the fixed point property. A dual Banach space $E$ is said to have the weak* fixed point property if each weak* compact convex subset of $E$ has the fixed point property. Since weakly compact sets (in a dual Banach space) are weak* compact, the weak* fixed point property implies the weak fixed point property. A well known result of Bruck [6] shows that a Banach space $E$ with the weak fixed point property, already has the weak fixed point property for commuting semigroups, i.e., if $K$ is a weakly compact convex subset of $E$ and $\mathcal{S} = \{T_s \mid s \in S\}$ is a representation of a commutative semigroup $S$ as nonexpansive self-mappings on $K$, then $K$ contains a common fixed point for $\mathcal{S}$. Consequently, it follows from [26] that, for a compact group $G$, the Fourier–Stieltjes algebra $B(G)$ (which in this case coincides with $A(G)$, the Fourier algebra of $G$), has the weak fixed point property for commuting semigroups. It also follows from [27, Theorem 4.2 and Proposition 5.1] that if $G$ is an [AU]-group (definition in Section 2), then $B(G)$ has the weak fixed point property for left reversible semigroups. Furthermore, if $G$ is separable and compact, then $B(G)$ has the weak* fixed point property for left reversible semigroups. Note that [AU]-groups include all compact groups. However, there are important non-compact locally compact groups such as the Fell group which are [AU]-groups (see [40]). In a recent paper, N. Randrianantoanina [37] proved that $G$ is an [AU]-group if and only if $B(G)$ has the weak fixed point property for left reversible semigroups.

The main purpose of this paper is to show that, given a locally compact group $G$, its Fourier–Stieltjes algebra $B(G)$ regarded as the dual of the group $C^*$-algebra $C^*(G)$ has the weak* fixed point property for nonexpansive mappings (or weak* normal structure) if and only if $G$ is compact. This answers two open problems (Problem 8 and Problem 9) posed in the 1989 conference "Fixed point theory and applications" held in Marseille-Luminy [23] (see also Open Problem 6.6 in [27]).

In [37], Randrianantoanina shows that for a Hilbert space $H$ (not necessarily separable), the space $\mathcal{T}(H)$ of trace class operators on $H$ has the weak* fixed point property for left reversible semigroups using non-commutative $L^1$-spaces. From this one can derive that for any compact group $G$, $B(G)$ has the weak* fixed point property for left reversible semigroups. We give a more elementary proof of this fact and also show that this property characterizes compact groups among all locally compact groups. This answers a problem raised in Remark 4.3 [37] and improves a result of Lau and Mah [27] for separable compact groups.

This paper is organized as follows: In Section 3 we show for the classical example of a non-compact [AU]-group, the Fell group $F$, that there is a non-empty weak*-compact convex subset $K$ of $B(F)$ and an isometry $T$ from $K$ into $K$ which is fixed point free. In Section 4 we prove one of our main results: If $G$ is non-compact, then $B(G)$ cannot have the weak* fixed point property. In Section 5, we show that if $G$ is compact, then the asymptotic centre of a non-empty weak* closed convex subset $C$ in $B(G)$ with respect to a decreasing net of bounded subsets is a non-empty norm compact subset of $C$. In particular, $B(G)$ has the weak* fixed point property for left reversible semigroups. Section 6 contains a general result on relations between the above mentioned properties on dual Banach spaces and states some open problems.

## 2. Some preliminaries

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. Let $L^1(G)$ denote the group algebra of $G$ with convolution product. The group $C^*$-algebra $C^*(G)$ is defined to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_* = \sup \|\pi_{(f)}\|,$$

where the supremum is taken over all nondegenerate $*$-representations $\pi$ of $L^1(G)$ as a $*$-algebra of bounded operators on a Hilbert space. Let $\mathcal{B}(L^2(G))$ be the set of all bounded operators on the Hilbert space $L^2(G)$ and $\rho$ be the left regular representation of $G$, i.e. for $f \in L^1(G)$ $\rho(f)$ is the bounded operator in $\mathcal{B}(L^2(G))$ defined by convolution with $f$. Denote by $C^*_\rho(G)$ the completion of $L^1(G)$ with respect to the norm $\|\rho(\cdot)\|$, and denote by $VN(G)$ the closure of $\{\rho(f) \mid f \in L^1(G)\}$ in the weak operator topology in $\mathcal{B}(L^2(G))$. In the case when $G$ is left amenable, in particular when $G$ is compact, then $C^*(G)$ is isometrically isomorphic to $C^*_\rho(G)$. Denote the set of continuous positive definite functions on $G$ by $P(G)$, and the set of continuous functions on $G$ with compact support by $C_{00}(G)$. Define the Fourier–Stieltjes algebra of $G$, denoted by $B(G)$, to be the linear span of $P(G)$. The Fourier algebra of $G$, denoted by $A(G)$, is defined to be the closed linear span of $P(G) \cap C_{00}(G)$. Clearly, $A(G) = B(G)$ when $G$ is compact. It is known that $C^*(G)^* = B(G)$, where the duality is given by $\langle f, \phi\rangle = \int f(t)\phi(t)\, d\lambda(t)$, $f \in L^1(G)$, $\phi \in B(G)$, and $A(G)^* = VN(G)$ (see [11] and [21] for details).

A locally compact $G$ is said to be an [AU]-group if the von Neumann algebra generated by any continuous unitary representation of $G$ is atomic, which is equivalent to saying that every continuous unitary representation of $G$ is completely reducible, i.e. is the Hilbert sum of irreducible representations. For details see Appendix A.

A Banach space $E$ is said to have the Radon–Nikodym property if each closed convex subset $D$ of $E$ is dentable i.e., for any $\varepsilon > 0$, there exists an $x$ in $D$ such that $x \notin \overline{\mathrm{co}}(D \backslash B_\varepsilon(x))$, where $B_\varepsilon(x) = \{y \in X \mid \|x - y\| < \varepsilon\}$ and $\overline{\mathrm{co}}K$ is the closed convex hull of a set $K \subseteq E$. It was shown in [28, Lemma 3.1] that if the predual $\mathfrak{M}_*$ of a von Neumann algebra $\mathfrak{M}$ has the Radon–Nikodym property, then $\mathfrak{M}_*$ has the weak fixed point property.

It follows from [40, Theorem 3.5] that the class of groups $G$ for which $B(G)$ has the Radon–Nikodym property is precisely the [AU]-groups (see also [14]).

Let $K$ be a bounded closed convex subset of a Banach space $E$. A point $x$ in $K$ is called a *diametral point* if

$$\sup\big\{\|x - y\|\colon\ y \in K\big\} = \mathrm{diam}(K),$$

where $\mathrm{diam}(K)$ denotes the diameter of $K$. The set $K$ is said to have *normal structure* if every nontrivial (i.e., contains at least two points) convex subset $H$ of $K$ contains a non-diametral point of $H$ (see [13,20]). A Banach space $E$ has *weak normal structure* if every nontrivial weakly compact convex subset of $E$ has normal structure. A dual Banach space $E$ has *weak*$^*$ *normal structure* if every nontrivial weak$^*$ compact convex subset of $E$ has normal structure [13, p. 44], [19,26,31].

A dual Banach space $E$ is said to have property $UKK^*$ (weak$^*$ *uniformly Kadec–Klee property)* if for any $\varepsilon > 0$ there is a $0 < \delta < 1$ such that whenever $A$ is a subset of the closed unit ball of $E$ containing a sequence $(x_n)$ with $\mathrm{sep}((x_n)) := \inf\{\|x_n - x_m\|\colon n \neq m\} > \varepsilon$, then there is an $x$ in the weak$^*$ closure of $A$ such that $\|x\| \leqslant \delta$. The property $UKK^*$ was introduced by van Dulst and Sims [9]. They proved that if $E$ has property $UKK^*$, then $E$ has weak* normal structure and hence has the weak* fixed point property.

Let $S$ be a semitopological semigroup, i.e. $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to as$ and $s \to sa$ from $S$ into $S$ are continuous. $S$ is called *left reversible* if $\overline{aS} \cap \overline{bS} \neq \emptyset$ for any $a, b \in S$, where, in general, $\overline{K}$ denotes the closure of the set $K$. Clearly abelian semigroups and groups are left reversible. Let $CB(S)$ be the $C^*$-algebra of bounded continuous complex-valued functions on $S$ and for $a \in S$, let $\ell_a$ be the left translation operator on $CB(S)$ defined by $(\ell_a f)(t) = f(at)$ for $f \in CB(S)$ and $t \in S$. Then $S$ is *left amenable* if there is an $m \in CB(S)^*$ such that $\|m\| = m(1) = 1$ and $m(\ell_a f) = m(f)$ for all $f \in CB(S)$ and $a \in S$. If the topology on $S$ is normal and $S$ is left amenable, then $S$ is left reversible. In particular, if $S$ is left amenable as a discrete semigroup, then $S$ is left reversible. Left reversible semigroups have played an important role in the study of common fixed point theorems and ergodic type theorems for semigroups of nonexpansive mappings (see [17,22,23,29,30, 32,33,36]).

Let $S$ be a semitopological semigroup, and $K$ be a topological space. An action of $S$ on $K$ is a map $\psi$ from $S \times K$ to $K$, denoted by $\psi(s, k) = sk$, $s \in S$, $k \in K$, such that $s_1 s_2(k) = s_1(s_2 k)$, for all $s_1, s_2 \in S$, and $k \in K$. The action is separately continuous if $\psi$ is continuous in each of the variables when the other is kept fixed. Lau showed in [20] that if $E$ is a Banach space and $\mathcal{S} = \{T_s \mid s \in S\}$ is a separately continuous representation of a left reversible semitopological semigroup $S$ as nonexpansive self-maps on a norm compact convex subset $K$ of $E$, then $K$ contains a common fixed point for $\mathcal{S}$. We say a Banach space $E$ has the *weak fixed point property for left reversible semigroups* if whenever $\mathcal{S}$ is a left reversible semitopological semigroup and $K$ is a non-empty weakly compact convex subset of $E$ for which the action of $S$ on $K$ (with the norm topology) is separately continuous and nonexpansive, then $K$ has a common fixed point for $S$. Similarly a dual Banach space $E$ has the *weak* fixed point property for left reversible semigroups* if whenever $S$ is a left reversible semitopological semigroup and $K$ is a non-empty weak* compact convex subset of $E$ for which the action of $S$ on $K$ is separately continuous and nonexpansive, then $K$ has a common fixed point for $S$. In general, a weakly compact convex set of a Banach space need not have the fixed point property for left reversible semigroups, not even for commutative semigroups. Indeed, Alspach [1] (see also [5, Theorem 4.2], [4,8]) showed there is a weakly compact convex subset $K$ in $L^1[0,1]$ and an isometry $T : K \to K$ without a fixed point. Hence if $S = (\mathbb{N}, +)$ and $\mathcal{S} = \{T^n \mid n \in \mathbb{N}\}$, then $K$ does not have a common fixed point for $\mathcal{S}$. However, Bruck showed in [6] that a Banach space $E$ having the weak fixed point property has the weak fixed point property for commutative semigroups, and Lim showed in [33] that a Banach space with weak normal structure has the weak fixed point property for left reversible semigroups. For dual Banach spaces, it is known (see [33,34]) that $\ell_1$ and all uniformly convex Banach spaces have the weak* fixed point property for left reversible semigroups.

## 3. The case of the Fell group $F = Q_p \rtimes C$

In this section, we shall demonstrate for our motivating example, the Fell group $F = Q_p \rtimes C$, that $B(F)$ fails the weak* fixed point property. Here $p$ is a fixed prime number, $C$ is the compact multiplicative subgroup of $Q_p \backslash \{0\}$ consisting of those elements of $Q_p$ with valuation 1, and $Q_p$ is the additive group of the $p$-adic number field. Multiplication in $F$ is given by $(x, k)(x', k') = (x\tau_k(x'), kk')$, $x, x' \in Q_p$, $k, k' \in C$, and $\tau_k(x') = kx'$.

Now the group $C^*$-algebra of the Fell group $C^*(F)$ is liminal (i.e. *CCR*), and its dual object, the set of (equivalence classes of) its irreducible representations, is $\widehat{C^*(F)} = \widehat{F} = \widehat{C} \dot{\cup} \{T^j \mid j \in \mathbb{Z}\}$ in the notation of Baggett (see [2, pp. 142–143]), where "$\dot{\cup}$" denotes the disjoint union of two sets, and the direct sum of the $\{T^j\}$ (with infinite multiplicity) equals the regular representation $R$ of $F$.

Defining $\hat{a}(\pi) = \pi(a)$ for $a \in C^*(F)$, $\pi \in \widehat{F}$, we may identify $C^*(F)$ $*$-isometrically with a $C^*$-algebra of operator-valued functions on $\widehat{F} = \widehat{C} \cup \{T^j \mid j \in \mathbb{Z}\}$. The topology of $\widehat{F}$ relativized either to $\widehat{C}$ or to $\{T^j \mid j \in \mathbb{Z}\}$ is discrete, and every $\gamma \in \widehat{C}$ is the limit of any sequence $(T^{j_n})$ where $j_n$ converges to minus infinity [2, p. 143]. Denoting the compact linear operators on a Hilbert space $H$ by $\mathcal{K}(H)$, we now show that $C^*(F)$ contains the $c_0$-sum $\bigoplus_0 \mathcal{K}(H_{T^j})$ isometrically: Let $E \subset \{T^j \mid j \in \mathbb{Z}\}$ be finite and $I = \bigcap_{j \notin E} \operatorname{Ker} T^j$. For $i \in E$, since $T^i$ is not in the closure of $\{T^j \mid j \notin E\}$, $\operatorname{Ker} T^i$ does not contain $I$. Hence $T^i(I) \neq 0$ and $T^i|_I$ is irreducible [7, 2.11.2]. Being a sub-$C^*$-algebra of $C^*(F)$, $I$ is liminal too. So, given $S_i \in \mathcal{K}(H_i)$ for $i \in E$, there is some $b \in I$ with $T^i(b) = S_i$ for all $i \in E$ [7, 4.2.4 and 4.2.5]. For $j \notin E$ we have $T^j(b) = 0$ by the definition of $I$. For $\gamma \in \widehat{C}$, $\gamma(b) = 0$ holds too, since $|\gamma(a)| \leqslant \underline{\lim}_n \|T^{j_n}(a)\|$ for any $a \in A$ and any sequence $j_n \to -\infty$. So, the algebraic direct sum of all $\mathcal{K}(H_{T^j})$, $j \in \mathbb{Z}$, is naturally contained in $C^*(F)$. Taking the norm closure, we obtain $J := \bigoplus_0 \mathcal{K}(H_{T^j}) \subset C^*(F)$.

Although we do not really need it, let us say a bit more about the structure of $C^*(F)$. Since $J$ is a closed ideal, by [7, 2.11.2] we have $\widehat{A/J} = \widehat{C}$ in a canonical way, which implies that $A/J$ is isometrically isomorphic to the commutative $C^*$-algebra $C_0(\widehat{C}) \cong C^*(C)$. We have $\lim_{j\to\infty} \|T^j(a)\| = 0$ for all $a \in C^*(F)$, but the corresponding statement for $j \to -\infty$ holds for $a \in J$ only.

By [2], the trivial representation is limit of every sequence $(T^{i_n})$ with $i_n \to -\infty$, so by Dixmier [7] the constant function $\varphi_0 = 1$ is weak* limit of states associated with the $T^{i_n}$. Since $F$ is separable, so is $C^*(F)$, which implies that on bounded sets of $B(F)$ the weak* topology is metrizable. It follows that $\varphi_0$ is the weak* limit of some sequence of states $\varphi_1, \varphi_2, \ldots$ with $\varphi_n$ associated with some $T^{j_n}$, $T^{j_n} \neq T^{j_m}$ for $n \neq m$, and $j_n \to -\infty$.

Let $K = \{\sum_0^\infty \alpha_j \varphi_j \mid \alpha_j \geqslant 0, \sum_0^\infty \alpha_j = 1\}$.

**Lemma 3.1.** *$K$ is a convex weak* compact subset of $B(F)$.*

**Proof.** The set $Q = \{\varphi_n,\ n = 0, 1, 2, \ldots\}$ is countable and weak* compact. Its weak* closed convex hull $\overline{\mathrm{co}}\,Q$, being norm bounded, is weak* compact and, by [38, Theorem 3.28] coincides with the set of all integrals $\int \varphi \, d\mu(\varphi)$ where $\mu$ is a Borel probability measure on $Q$. Since 1-point sets are Borel in $Q$, every such $\mu$ must be discrete. So $\overline{\mathrm{co}}\,Q = K$, in particular $K$ is weak* compact. □

Define $T : K \to K$ by $T(\sum_0^\infty \alpha_i \varphi_i) = \sum_1^\infty \alpha_{i-1} \varphi_i$. Then $T$ is well defined, because the coefficients $\alpha_i$ can be recovered from the sum $\varphi = \sum_0^\infty \alpha_i \varphi_i$. One can see this by using the fact that $\bigoplus_0 \mathcal{K}(H_{T^j}) \subset C^*(F)$: If $S_n \in \mathcal{K}(H_{T^{j_n}})$ is chosen such that $\varphi_n(S_n) = 1$ and $x_n \in \bigoplus_0 \mathcal{K}(H_{T^j})$ has $j_n$-coordinate $S_n$ and all other coordinates zero, then $\varphi(x_n) = \alpha_n \cdot \varphi_n(S_n) = \alpha_n$. This works for all $n \geqslant 1$, and $\alpha_0$ is recovered by $\alpha_0 = 1 - \sum_1^\infty \alpha_i$.

**Theorem 3.2.** (i) *$T$ is isometric (i.e. $\|T\varphi - T\psi\| = \|\varphi - \psi\|$ for $\varphi, \psi \in K$).*

(ii) *$T$ has no fixed point in $K$.*

**Proof.** (i) Let $\varphi, \psi \in K$, $\varphi = \sum_0^\infty \alpha_i \varphi_i$, $\psi = \sum_0^\infty \beta_i \varphi_i$. Then

$$\varphi - \psi = \sum_0^\infty (\alpha_i - \beta_i)\varphi_i.$$

We clearly have $\|\varphi - \psi\| \leqslant \sum_0^\infty |\alpha_i - \beta_i|$. To get the reverse inequality, let $\varepsilon > 0$ and let $a \in C^*(F)$ with $\|a\| \leqslant 1$ be such that $|(\alpha_0 - \beta_0)\varphi_0(a) - |\alpha_0 - \beta_0|| < \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n>n_0} |\alpha_i - \beta_i| < \varepsilon$. Adjust $a$ by adding a suitable element $b$ of $\bigoplus_1^{n_0} \mathcal{K}(H_{T^j})$ such that $\|a + b\| \leqslant 1$ (for this, it suffices to have $\|T^j(a + b)\| \leqslant 1$ for $j = 1, \ldots, n_0$) and $|\sum_1^{n_0} (\alpha_i - \beta_i)\varphi_i(a + b) - \sum_1^{n_0} |\alpha_i - \beta_i|| < \varepsilon$. Then we have

$$\left| \sum_0^\infty (\alpha_i - \beta_i)\varphi_i(a + b) - \sum_0^\infty |\alpha_i - \beta_i| \right| < 4\varepsilon.$$

Hence $\|\varphi - \psi\| = \sum_0^\infty |\alpha_i - \beta_i|$, since $\varepsilon > 0$ was arbitrary. Applying this to $T\varphi - T\psi$ we obtain $\|T\varphi - T\psi\| = \sum_1^\infty |\alpha_{i-1} - \beta_{i-1}| = \sum_0^\infty |\alpha_i - \beta_i| = \|\varphi - \psi\|$.

(ii) If $\varphi = \sum_0^\infty \alpha_i \varphi_i$ is fixed under $T$, from $\varphi = T\varphi$ we obtain $\alpha_0 = 0$, $\alpha_i = \alpha_{i-1}$ for $i \geqslant 1$, hence $\alpha_i = 0$ for all $i$. But $\varphi = 0$ is not a point in $K$. □

We thus have proved that the Fourier–Stieltjes algebra of the Fell group does not have the weak* fixed point property.

## 4. The general case

The Fell group example provides a guideline for the general case. However, since we do not know what the $C^*$-algebra of a general [AU]-group looks like, we shall use von Neumann algebra arguments.

**Lemma 4.1.** *Let $G$ be a locally compact group and $\sigma$ be the Hilbert sum of some irreducible representations of $G$. If $\pi$ is one of the summands of $\sigma$, and $P$ is the projection onto the space $H^\pi = \bigoplus_{\pi' \approx \pi} H_{\pi'}$, then $P$ is in the von Neumann algebra generated by $\sigma$.*

**Proof.** It suffices to show that $P$ commutes with every $T$ in the commutant of $\sigma$, which is equivalent to $TH^\pi \subset H^\pi$ for all such $T$. If $\xi \in H^\pi$ and $T\xi \notin H^\pi$, there is a summand $\pi_0 \not\approx \pi$ of $\sigma$ such that $H_{\pi_0}$-component of $T\xi$ is nonzero. This implies that for some $\pi' \approx \pi$ the $H_{\pi_0}$-component of $T\xi_{\pi'}$ is nonzero, where $\xi_{\pi'}$ denotes the $H_{\pi'}$-component of $\xi$. The operator $M_{\pi'}$ (resp. $M_{\pi_0}$) on $H_\sigma$ which is the identity on the $H_{\pi'}$ (resp. $H_{\pi_0}$) component and zero on all other components of $H_\sigma$ clearly is in the commutant of $\sigma$; hence so is the operator $S = M_{\pi_0} T M_{\pi'} \neq 0$. By construction we have $SH_{\pi'} \subset H_{\pi_0}$; so $S$ intertwines $\pi'$ and $\pi_0$. Since $\pi'$ and $\pi_0$ are irreducible, this implies $\pi' \approx \pi_0$, which is a contradiction. We conclude that $TH^\pi \subset H^\pi$ for all $T$ in the commutant of $\sigma$. □

Let $A$ be a $C^*$-algebra and let $f$ be a positive form on $A$, $f(x) = (\pi_f(x)\xi|\xi)$. Let us describe the unique positive normal extension of $f$ to the universal von Neumann algebra $\omega(A)''$. Define $\tilde{\xi} = (\xi_g)$ in the universal representation space $\bigoplus_{g \text{ pos.}} H_{\pi_g}$ by $\xi_g = \xi$ for $g = f$ and $\xi_g = 0$ for all other $g$. Then $\tilde{f}(y) = (y\tilde{\xi}|\tilde{\xi})$ for $y \in \omega(A)''$ defines a positive normal form on $\omega(A)''$. Its restriction to $A$ is $f$ (in the sense $\tilde{f}(\omega(x)) = f(x)$ for $x \in A$, where $\omega$ denotes the universal representation $\omega = \bigoplus \pi_g$); so by [7] $\tilde{f}$ must be the unique positive normal extension of $f$.

**Lemma 4.2.** *If $\pi$ is an irreducible $*$-representation of the $C^*$-algebra $A$ and $x \mapsto f(x) = (\pi(x)\xi|\xi)$ is a positive form associated with $\pi$, then the support of $f$ is $\leqslant P_\pi$, where $P_\pi$ is the projection onto the subspace $H^\pi = \bigoplus_{\pi' \approx \pi} H_{\pi'}$ of the universal representation space of $A$.*

**Proof.** Letting $\sigma = \omega$ in Lemma 4.1 we see that $P_\pi \in \omega(A)''$. With $\tilde{f}$ as above we have $\tilde{f}(1 - P_\pi) = ((1 - P_\pi)\xi|\xi) = (0|\xi) = 0$; so supp $f \leqslant P_\pi$ (see [7]). □

Probably the following lemma is known but we are not aware of a reference and so we give a short proof.

**Lemma 4.3.** *If $G$ is $\sigma$-compact, then for the weak* topology on the unit sphere $S$ of $B(G)$, every function $\varphi \in S$ has a countable base of neighbourhoods.*

**Proof.** By assumption there are compact sets $K_n \subset G$ for $n \in \mathbb{N}$ such that $G = \bigcup_1^\infty K_n$. By enlarging $K_n$ if necessary, we may assume that $\mathring{K}_n \supset K_{n-1}$. This implies that, for compact $K$, there is $n_0 \in \mathbb{N}$ such that $K \subset K_{n_0}$. Therefore, for $\varphi \in S$, the sets $U_n = \{\psi \in B(G) \mid |\psi - \varphi| < \frac{1}{n}$ on $K_n\}$ form a neighbourhood base of $\varphi$ for the topology of uniform convergence on compact sets, which by [15] coincides with the weak* topology on $S$. □

Let $H$ be an open subgroup of $G$. For $u \in B(H)$ let $u_0 : G \to G$ be the extension of $u$ by zero outside $H$.

The following lemma is well known for positive definite functions [16, 32.43], but to extend it to all of $B(G)$, e.g. by use of polar decomposition, appears more involved than a direct proof.

**Lemma 4.4.** *If $H$ is an open subgroup of $G$, the map $Q : u \mapsto u_0$ is an isometric isomorphism from $B(H)$ into $B(G)$.*

**Proof.** (i) We first show that $C^*(H)$ is contained in $C^*(G)$ (with the same norm). For $f \in \mathcal{L}^1(H)$ clearly $\|f\|_{C^*(G)} = \sup_\pi \|\pi|_H(f)\| \leqslant \|f\|_{C^*(H)}$. On the other hand, if $\sigma$ is a continuous unitary representation of $H$, since $H$ is open, $\sigma$ is a subrepresentation of the induced representation $\mathrm{ind}_H^G(\sigma)$. To see this, use the isometric embedding $\xi \mapsto g_\xi$ of $H_\sigma$ into $H_{\mathrm{ind}_H^G(\sigma)}$ defined by $g_\xi(h) = \sigma(h^{-1})\xi$ for $h \in H$, and $g_\xi = 0$ on $G \backslash H$, and observe that $\mathrm{ind}_H^G(\sigma)(k)g_\xi = g_{\sigma(k)\xi}$ for $k \in H$ (for definitions see [12, pp. 152, 153]). So for $f \in \mathcal{L}^1(H)$ we obtain $\|\mathrm{ind}_H^G(\sigma)(f)\| \geqslant \|\sigma(f)\|$, which shows that the inequality above is really an equality, and we obtain $C^*(H) \subset C^*(G)$.

(ii) Since by (i) $C^*(H)$ is a closed subspace of $C^*(G)$, we have $B(H) = C^*(H)^* \cong B(G)/C^*(H)^\perp$ isometrically, where $C^*(H)^\perp$ denotes the annihilator of $C^*(H)$ in $B(G)$. So, given $f \in B(H)$ and $\varepsilon > 0$, there is some $F \in B(G)$ with $F|_H = f$ and $\|F\|_{B(G)} < \|f\|_{B(H)} + \varepsilon$. Multiplying $F$ with the continuous positive definite function $1_H$ (characteristic function of $H$), we obtain $f_0 = F1_H$ and $\|f_0\|_{B(G)} \leqslant (\|f\|_{B(H)} + \varepsilon)$. Since $\varepsilon > 0$ was arbitrary, it follows that $\|f_0\|_{B(G)} = \|f\|_{B(H)}$. □

**Theorem 4.5.** *Let $G$ be a locally compact group. If $B(G)$ has the weak* fixed point property, then $G$ is compact.*

**Proof.** If $B(G)$ has the weak* fixed point property, then a fortiori $B(G)$ has the weak fixed point property, so by [37, Theorem 4.1] $G$ is an [AU] group, and hence unimodular [40, Theorem 4.4]. It therefore suffices to prove the assertion for [AU] groups.

(i) Suppose $G$ is a $\sigma$-compact non-compact [AU] group. The proof of Theorem 3.4 in [35] works for $B_\rho(G)$, too and shows that the set $I$ of isolated points in $\widehat{G}_\rho$ is dense in $\widehat{G}_\rho$. Since [35, Theorem 3.1(ii)] applies to these points (they have a weak* strongly exposed positive definite coefficient in $P_\rho(G)$), they are also closed in $\widehat{G}_\rho$. By [41, Theorem 7.6] however there must be some $\pi_0 \in \widehat{G}_\rho$ which is not of this sort, so has no weak* strongly exposed positive definite coefficient in $P_\rho(G)$. Let $\varphi_0$ be a (pure) state associated with this $\pi_0$. We know that $\varphi_0$ is weak* limit of (pure) states associated with representations $\pi \in I$. By Lemma 4.3, there is a sequence of such states $\varphi_n$ say with $\varphi_n \to \varphi_0$ in the weak* topology. Since points in $I$ are closed, $\pi_0$ is not in the closure of any finite subset of $I$, so that we may assume that the representations $\pi_n$ belonging to $\varphi_n$ are inequivalent for different $n$. In particular, by Lemma 4.2 the supports of the $\varphi_n$ are pairwise orthogonal. As in the example of the Fell group, let $K = \{\sum_0^\infty \alpha_i\varphi_i \mid \alpha_i \geqslant 0,\ \sum_0^\infty \alpha_i = 1\}$. Using the arguments from the Fell group example in Section 3, one sees that $K$ is weak* compact convex, and the map $(\alpha_i) \mapsto \sum_0^\infty \alpha_i\varphi_i$ is one-to-one because of the orthogonality of the supports of the $\varphi_i$.

Define the map $T : K \to K$ as before. For $\varphi = \sum \alpha_i\varphi_i$ and $\psi = \sum \beta_i\varphi_i$ in $K$ we have $\varphi - \psi = \sum_{\alpha_i \geqslant \beta_i}(\alpha_i - \beta_i)\varphi_i - \sum_{\alpha_i < \beta_i}(\beta_i - \alpha_i)\varphi_i$. Since the two sums have orthogonal supports, we obtain

$$\|\varphi - \psi\| = \left\| \sum_{\alpha_i \geqslant \beta_i} \cdots \right\| + \left\| \sum_{\alpha_i < \beta_i} \cdots \right\| = \sum_0^\infty |\alpha_i - \beta_i|.$$

Hence $\|T\varphi - T\psi\| = \sum_0^\infty |\alpha_i - \beta_i| = \|\varphi - \psi\|$, so $T$ is isometric. Also the argument of the example shows that $T$ has no fixed point in $K$. In particular, $B(G)$ does not satisfy the weak* fixed point property.

(ii) Suppose $G$ is an [AU] group that is not $\sigma$-compact. We first observe that since $G$ is non-compact, $G$ must contain an open $\sigma$-compact subgroup $H$ which is not compact. To see this, take a compact neighbourhood $U$ of $e$. Then there is an infinite sequence $\{x_n\}$ such that $\{Ux_n\}$ is pairwise disjoint. Let $V$ be the union of all $Ux_n$ and $H$ be the union of powers of $\{V \cup V^{-1}\}$. Then $H$ is $\sigma$-compact open subgroup of $G$ which is non-compact since the infinite sequence $\{x_n\}$ has no convergent subnet.

Let $B^0(H)$ denote the subalgebra of $B(G)$ consisting of functions vanishing outside $H$. It follows from Lemma 4.4 and (ii) of its proof that the map $Q : u \to u^0$ (extension by zero outside $H$) is an isometric isomorphism from $B(H)$ onto $B^0(H)$ and the weak* topology on $B(H) \cong B^0(H)$ is the relative topology of the weak* topology on $B(G)$. Also, by Hahn–Banach theorem the weak topology on $B(H) \cong B^0(H)$ is the relative topology of the weak topology on $B(G)$.

We infer that $H$ has the weak* fixed point property, so by [37, Theorem 4.1], $H$ is an [AU] group, too, and since $H$ is $\sigma$-compact non-compact, (i) applies. We thus obtain a weak* compact convex set $K$ in $B(H) \cong B^0(H)$ (which is also weak* compact in $B(G)$) and a distance preserving map $T : K \to K$ which has no fixed point in $K$. In particular, $B(G)$ fails to have the weak* fixed point property. □

**Remark 4.6.** The reader may have noticed that in the case of [AU] groups we proved a slightly stronger statement than Theorem 4.5, namely: If $G$ is a non-compact [AU]-group, there is a non-void weak* compact convex set $K$ and an *isometric* map $T : K \to K$ which has no fixed point.

If $G$ is unimodular and not an [AU] group, from [24] we obtain a still stronger statement: like above, but with "weak" in place of "weak*". We do not know whether this stronger statement also holds for non-unimodular groups.

**Theorem 4.7.** *Let G be a locally compact group. The following are equivalent*:

(a) *$G$ is compact.*
(b) *$B(G)$ has the UKK* property.*
(c) *$B(G)$ has weak* normal structure.*
(d) *$B(G)$ has the weak* fixed point property for nonexpansive mappings.*

**Proof.** (a) $\Longrightarrow$ (b) was proved in [9]

(b) $\Longrightarrow$ (c) was proved in [26].

(c) $\Longrightarrow$ (d) was proved in [34].

(d) $\Longrightarrow$ (a) follows from Theorem 4.5. □

## 5. Asymptotic centre

Let $C$ be a non-empty subset of a Banach space $X$ and $\{D_\alpha \mid \alpha \in \Lambda\}$ be a decreasing net of bounded non-empty subsets of $X$. For each $x \in C$, and $\alpha \in \Lambda$, let

$$r_\alpha(x) = \sup\{\|x - y\| \mid y \in D_\alpha\},$$
$$r(x) = \lim_\alpha r_\alpha(x) = \inf_\alpha r_\alpha(x),$$
$$r = \inf\{r(x) \mid x \in C\}.$$

The set (possibly empty)

$$\mathcal{AC}(\{D_\alpha \mid \alpha \in \Lambda\}) = \{x \in C \mid r(x) = r\}$$

is called the *asymptotic centre* of $\{D_\alpha \mid \alpha \in \Lambda\}$ with respect to $C$ and $r$ is the *asymptotic radius* of $\{D_\alpha \mid \alpha \in \Lambda\}$ with respect to $C$.

The notion of asymptotic centre is due to M. Edelstein [10]. See also [33].

In this section, we shall show that the asymptotic centre of a non-empty weak* closed convex set $C$ in $B(G)$ of a compact group $G$ with respect to a decreasing net of bounded subsets of $B(G)$ is a non-empty norm compact subset of $C$. This was first proved by T.C. Lim when $G$ is the circle group [34], and for separable compact groups by Lau and Mah [27]. We begin with the following lemma. It was proved in [27] for sequences. Note that in consideration of *nets*, we have to avoid the diagonalization process employed in [34] and [27]. Since this is not entirely obvious we give some details of the proof.

**Lemma 5.1.** *Let $G$ be a compact group, and let $\{D_\alpha \mid \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of $B(G)$, and $(\varphi_\mu)_{\mu\in M}$ be a weak* convergent bounded net with weak* limit $\varphi$. Then*

$$r(\varphi) + \limsup_\mu \|\varphi_\mu - \varphi\| = \limsup_\mu r(\varphi_\mu), \tag{5.1}$$

*i.e.*:

$$\begin{aligned}\lim_\alpha \sup\bigl\{\|\varphi - \psi\| \bigm| \psi \in D_\alpha\bigr\} + \limsup_\mu \|\varphi_\mu - \varphi\| \\ = \limsup_\mu \lim_\alpha \sup\bigl\{\|\varphi_\mu - \psi\| \bigm| \psi \in D_\alpha\bigr\}.\end{aligned} \tag{5.2}$$

**Proof.** As in [27] it can be seen that the left hand side of (5.2) majorates the right hand side.

To prove the reverse inequality we may assume without loss of generality that $\varphi = 0$, so we shall prove

$$\lim_\alpha \sup\bigl\{\|\psi\| \bigm| \psi \in D_\alpha\bigr\} + \overline{\lim_\mu}\|\varphi_\mu\| \leqslant \overline{\lim_\mu} \lim_\alpha \sup\bigl\{\|\varphi_\mu - \psi\| \bigm| \psi \in D_\alpha\bigr\}. \tag{5.3}$$

For $\alpha \in \Lambda$ and $n \in \mathbb{N}$ choose $\psi_{\alpha,n} \in D_\alpha$ such that $\|\psi_{\alpha,n}\| > \sup\{\|\psi\| \mid \psi \in D_\alpha\} - \frac{1}{n}$. We order $\Lambda \times N$ by the setting $(\alpha, n) \geqslant (\alpha', n')$ if $\alpha \geqslant \alpha'$ and $n \geqslant n'$. Then $\overline{\lim}_{(\alpha,n)}\|\psi_{\alpha,n}\| = \lim_\alpha \sup\{\|\psi\| \mid \psi \in D_\alpha\}$. Since $\overline{\lim}_{(\alpha,n)}\|\varphi_\mu - \psi_{\alpha,n}\| \leqslant \lim_\alpha \sup\{\|\varphi_\mu - \psi\| \mid \psi \in D_\alpha\}$, it now suffices to prove the following inequality:

$$\overline{\lim_{(\alpha,n)}}\|\psi_{\alpha,n}\| + \overline{\lim_\mu}\|\varphi_\mu\| \leqslant \overline{\lim_\mu}\,\overline{\lim_{(\alpha,n)}}\|\varphi_\mu - \psi_{\alpha,n}\|. \tag{5.4}$$

If we pass to a subnet of $(\psi_{\alpha,n})$, $(\psi_\beta)$ say (abuse of notation), such that $\overline{\lim}_{(\alpha,n)}\|\psi_{\alpha,n}\| = \lim_\beta \|\psi_\beta\|$, this may decrease the right side of (5.4) and thus make the inequality harder to prove. The same applies, if we then pass to a subnet $(\varphi_{\mu'})$ of $(\varphi_\mu)$ such that $\overline{\lim}_\mu\|\varphi_\mu\| = \lim_{\mu'}\|\varphi_{\mu'}\|$ and if after this we pass to a subnet $(\varphi_{\mu''})$ of $(\varphi_{\mu'})$ such that

$$\overline{\lim_{\mu'}}\,\overline{\lim_\beta} = \|\varphi_{\mu'} - \psi_\beta\| = \lim_{\mu''}\overline{\lim_\beta}\|\varphi_{\mu''} - \psi_\beta\|.$$

It therefore suffices to prove the inequality

$$\lim_\beta \|\psi_\beta\| + \lim_\mu \|\varphi_\mu\| \leqslant \lim_\mu \overline{\lim_\beta}\|\varphi_\mu - \psi_\beta\| \tag{5.5}$$

where, for convenience, we wrote $\mu$ for $\mu''$ again. Let $q = \lim_\mu \|\varphi_\mu\|$ and $r = \lim_\mu \overline{\lim}_\beta\|\varphi_\mu - \psi_\beta\|$. If (5.5) is false, there is some $p > 0$ with

$$\lim_\beta \|\psi_\beta\| = r - q + p. \tag{5.6}$$

To obtain a contradiction it is sufficient to show that for each $\varepsilon > 0$ we can find two sequences $\beta_1 \leqslant \beta_2 \leqslant \cdots$ and finite subsets $\sigma_1 \subset \sigma_2 \subset \cdots$ of $I$ such that for $\beta \geqslant \beta_k$

$$\sum_{i\in\sigma_k\setminus\sigma_{k-1}} \bigl\|\psi_\beta(i)\bigr\| > (p - \varepsilon)/2, \quad \text{with } \sigma_0 = \emptyset.$$

This contradicts the boundedness of $(\psi_\beta)$ because for $\beta \geqslant \beta_k$,

$$\|\psi_\beta\| > \sum_{i \in \sigma_k} \|\psi_\beta(i)\| \geqslant k(p-\varepsilon)/2.$$

The construction of the two sequences is as in the proof of Lemma 3.1 of [27] pages 362–364. □

**Definition 5.2.** Let $E$ be a dual Banach space. We say that $E$ has the *lim-sup property* for decreasing nets of bounded subsets if (5.2) holds for any decreasing net $\{D_\alpha \mid \alpha \in \Lambda\}$ of bounded subsets of $E$, and any weak* convergent bounded net $(\varphi_\mu)$ with weak* limit $\varphi$. We say that $E$ has the *asymptotic centre property* for decreasing nets of bounded subsets if for any non-empty weak* closed convex subset $C$ in $E$ and any decreasing net $\{D_\alpha \mid \alpha \in \Lambda\}$ of bounded non-empty subsets of $C$, the asymptotic centre of $\{D_\alpha \mid \alpha \in \Lambda\}$ with respect to $C$ is a non-empty norm compact convex subset of $C$.

The "lim-sup property" for sequences was introduced by T.C. Lim in [34]. It was called "Lim's condition" in [25]. The name was formally introduced in [27] in honour of T.C. Lim.

**Theorem 5.3.** *Let $G$ be a locally compact group. The following are equivalent*:

(a) *$G$ is compact.*
(b) *$B(G)$ has the lim-sup property.*
(c) *$B(G)$ has the asymptotic centre property.*
(d) *$B(G)$ has the weak* fixed point property for left reversible semigroups.*
(e) *$B(G)$ has the weak* fixed point property for nonexpansive mappings.*
(f) $\|\varphi\| + \limsup_\mu \|\varphi_\mu - \varphi\| = \limsup_\mu \|\varphi_\mu\|$ *for any bounded net $(\varphi_\mu)$ in $B(G)$ which converges to $\varphi \in B(G)$ in the weak* topology.*
(g) *For any net $(\varphi_\mu)$ in $B(G)$ and any $\varphi \in B(G)$ we have that $\|\varphi_\mu - \varphi\| \to 0$ if and only if $\varphi_\mu \to \varphi$ in the weak* topology and $\|\varphi_\mu\| \to \|\varphi\|$.*
(h) *On the unit sphere of $B(G)$ the weak* and the norm topology coincide.*

**Proof.** (a) $\Longrightarrow$ (b) follows from Lemma 5.1.

(b) $\Longrightarrow$ (c): Let $C$ be a weak* closed convex non-empty subset of $B(G)$ and $\{D_\alpha \mid \alpha \in \Lambda\}$ be a decreasing net of bounded non-empty subset of $C$. Let $r(\varphi)$ be as defined at the beginning of this section.

First, we show that the convex function $\varphi \mapsto r(\varphi)$ is weak* lower semi-continuous. To this end, it suffices to prove that the level set $K_s := \{\varphi \in C \mid r(\varphi) \leqslant s\}$ is weak* closed for each $s$. We may assume that $s \geqslant 0$. Let $(\varphi_\mu)$ be a net in $K_s$ which converges to $\varphi$ in the weak* topology. By (b)

$$r(\varphi) = \limsup_\mu r(\varphi_\mu) - \limsup_\mu \|\varphi_\mu - \varphi\| \leqslant s. \tag{5.7}$$

Hence $\varphi \in K_s$, and $K_s$ is weak* closed.

Now denote the asymptotic centre by $K$ and the asymptotic radius by $r$. Let $s > r$, then $r = \inf\{r(x) \mid x \in C \cap K_s\}$ and $K = \{x \in C \cap K_s \mid r(x) = r\}$. The set $K_s$ is norm bounded and weak* closed, so it is weak* compact. Thus the weak* lower semi-continuous convex functional must attain its minimum on the set $C \cap K_s$. Hence $K \neq \emptyset$.

Next, we prove that $K$ is norm compact. Clearly $K = \bigcap_{s>r} K_s$ is weak* compact. If $(\varphi_\mu)$ is a net in $K$, there is a subnet $(\varphi_{\mu'})$ converging weak* to some $\varphi \in K$. Since $r(\varphi_\mu) = r(\varphi) = r$, we have $\overline{\lim}\|\varphi_{\mu'} - \varphi\| = 0$ by (b). Thus the net $(\varphi_{\mu'})$ is norm convergent to $\varphi$, hence $K$ is norm compact.

(c) $\Longrightarrow$ (d): Let $S$ be a left reversible semitopological semigroup, and $C$ a weak* compact convex non-empty subset of $B(G)$ for which the action of $S$ on $(C, \|\cdot\|)$ is separately continuous and nonexpansive. Let $S$ be directed by $a \geqslant b$ if $aS \subseteq \overline{bS}$. For a fixed $u \in C$, let $W_s = \overline{sS(u)}$ for all $s \in S$. Then $\{W_s \mid s \in S\}$ is a decreasing net of subsets of $C$. Let $K$ be the asymptotic centre of $\{W_s \mid s \in S\}$ with respect to $C$. Then $K$ is a non-empty norm compact convex subset of $C$. Moreover, it is $S$-invariant. For, let $x \in K$, $s \in S$, and $\varepsilon > 0$ be arbitrary. Since $x \in K$, there exists $t \in S$ such that $tS(u) \subset W_t \subset B[x, r+\varepsilon]$, where $r$ is the asymptotic radius and $B[x, r]$ denotes the closed ball of radius $r$ centred at $x$. Since $s$ is nonexpansive, we have $stS(u) \subset B[s(x), r+\varepsilon]$, so that $W_{st} \subset B[s(x), r+\varepsilon]$. Thus, $s(x) \in K$. It now follows from Corollary 1 in [17] that $K$, and hence $C$, contains a common fixed point for $S$.

(d) $\Longrightarrow$ (e): This is obvious by taking the semigroup generated by a single nonexpansive map.

(e) $\Longrightarrow$ (a): By Theorem 4.5.

(b) $\Longrightarrow$ (f): Take $D_\alpha = \{0\}$, for all $\alpha$.

(f) $\Longrightarrow$ (g): Clearly norm convergence of $\varphi_\mu$ to $\varphi$ implies convergence of the respective norms and weak* convergence. The converse is immediate from (f).

(g) $\Longrightarrow$ (h) is obvious.

(h) $\Longrightarrow$ (a): This is in [3, Theorem 3.9]. □

## 6. Some remarks and open problems

Our proof of Theorem 5.3 (a) $\Longrightarrow$ (d) $\Longrightarrow$ (e) is inspired by the case where $G = \mathbb{T}$, the circle group, by Lim [34]. In this case $B(\mathbb{T}) \cong \ell^1(\mathbb{Z})$. It generalizes the argument given from Theorem 4.6 in [28] for $B(G)$ of a separable [IN]-group. The argument given in Theorem 5.3 yields the following general result:

**Theorem 6.1.** *Let $E$ be a dual Banach space. Then the following implications hold*:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \quad \textit{and} \quad (a) \Longrightarrow (d) \Longrightarrow (e) \Longrightarrow (f)$$

*where*

(a) *$E$ has the lim-sup property.*
(b) *$E$ has the asymptotic centre property.*
(c) *$E$ has the weak* fixed point property for left reversible semigroups.*
(d) $\|\varphi\| + \limsup_\mu \|\varphi_\mu - \varphi\| = \limsup_\mu \|\varphi_\mu\|$ *for any bounded net $(\varphi_\mu)$ in $E$ converging weak* to $\varphi \in E$.*
(e) *For any net $(\varphi_\mu)$ in $E$ and any $\varphi \in E$ we have that $\|\varphi_\mu - \varphi\| \to 0$ if and only if $\varphi_\mu \to \varphi$ in the weak* topology and $\|\varphi_\mu\| \to \|\varphi\|$.*
(f) *The weak* topology and the norm topology coincide on the unit sphere $S$ of $E$.*

**Open problem 1.** Let $G$ be a locally compact group. Let $B_\rho(G)$ denote the reduced Fourier–Stieltjes algebra of $B(G)$, i.e. $B_\rho(G)$ is the weak* closure of $C_{00}(G) \cap B(G)$. Then $B_\rho(G) = C_\rho(G)^*$. Does the weak* fixed point property on $B_\rho(G)$ imply $G$ is compact? This is true when $G$ is amenable by Theorem 5.3, since $B(G) = B_\rho(G)$ in this case.

**Open problem 2.** Let $G$ be a locally compact group. Does the asymptotic centre property on $B_\rho(G)$ imply that $G$ is compact?

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## Appendix A

In [40, p. 184] Keith Taylor remarked that a (continuous unitary) representation $\pi$ of a locally compact group $G$ is completely decomposable (i.e. $\pi$ is the direct sum of irreducible representations) if and only if the von Neumann algebra $M$ generated by $\pi$ is atomic (i.e. every nonzero (selfadjoint) projection in $M$ dominates a minimal nonzero projection in $M$). The equivalence can be seen using [39, p. 301] together with some additional arguments, but this is tedious. Keith Taylor kindly supplied an elementary argument for the equivalence. With slight modifications on our side it runs as follows:

(a) Let $H$ be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. Let $\pi$ be a representation of $G$ on $H$, let $M_\pi$ be the von Neumann algebra generated by $\pi(G)$, i.e. the double commutant $M_\pi = \{\pi(x) \mid x \in G\}''$, and let $Proj(M'_\pi)$ denote the set of projections in $M'_\pi$, the commutant of $M_\pi$. If $K$ is a closed subspace of $H$, let $P_K$ be the orthogonal projection of $H$ onto $K$. Then $K$ is $\pi$-invariant if and only if $P_K \in Proj(M'_\pi)$. Furthermore, $K$ is an irreducible $\pi$-invariant subspace of $H$ if and only if $P_K$ is a minimal nonzero element of the lattice $Proj(M'_\pi)$. Thus, $\pi$ is completely reducible if and only if the identity in $Proj(M'_\pi)$ is the orthogonal sum of minimal elements of $Proj(M'_\pi)$.

(b) If $M$ is a von Neumann algebra with identity $I$ acting on a Hilbert space $H$ and $P$ is a projection in $M'$, then $M$ is $PM + (I - P)M$ and the latter is a direct sum. $P$ is a minimal nonzero projection (among projections in $M'$) if and only if $PM'P$ is one dimensional and this holds if and only if $PM = B(PH)$ (see 5.5.6 of [18]). Thus, $I$ being written as an orthogonal sum of minimal projections in $M'$ is the same as writing $H$ as an orthogonal sum of subspaces such that $M$ restricted to any of those subspaces consists of all the bounded operators on that subspace.

(c) Let $\mathcal{G}$ be a set of pairwise orthogonal minimal nonzero projections in $M'$ with $\sum_{P \in \mathcal{G}} P = I$, and define an equivalence $\overset{M}{\sim}$ by $P \overset{M}{\sim} Q$ if the (irreducible) representations of $M$ defined by restricting to $PH$ and $QH$ are equivalent. If $j$ is an equivalence class of $\overset{M}{\sim}$, the projection $E_j = \sum_{P \in j} P$ is in $M$ (proof like for Lemma 4.1) and clearly in $M'$ too. So $E_j$ is a central projection of $M$ and

$$ME_j = I_{H_j} \otimes B(P_j H), \tag{A.1}$$

where $P_j$ is some representative of $j$ and $H_j$ is the orthogonal complement of $P_j H$ in $E_j H$. If $R$ denotes the set of equivalence classes of $\mathcal{G}$, we have $E_i E_j = 0$ for $i \neq j$, $i, j \in R$. In addition

$\sum_{j\in R} E_j = I$, and $M$ is the von Neumann algebra direct sum $\sum_{j\in R} ME_j$, where each $ME_j$ looks like in (A.1).

(d) Now, if $P$ is a nonzero projection in $M$, there is some $i \in R$ with $PE_i \neq 0$. By (A.1) $PE_i$ must be of the form $I_{H_i} \otimes P'$, where $P'$ is a nonzero projection in $B(P_i H)$. If $Q$ is a minimal nonzero projection in $B(P_i H)$ dominated by $P'$, then $I_{H_i} \otimes Q$ is a minimal nonzero projection in the summand $ME_i$ of $M$ and it is dominated by $P$.

Conclusion: Parts (b), (c) and (d) show that if a von Neumann algebra is atomic then so is its commutant, and atomicity is equivalent to the identity being an orthogonal sum of minimal projections. This together with (a) proves the above stated equivalence.

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