Is the five-flow conjecture almost false?

Jesper Lykke Jacobsen[1,2] and Jesús Salas[3,4]

[1] *Laboratoire de Physique Théorique, École Normale Supérieure*
24 rue Lhomond, 75231 Paris, FRANCE
JESPER.JACOBSEN@ENS.FR

[2] *Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris, FRANCE*

[3] *Grupo de Modelización, Simulación Numérica y Matemática Industrial*
Universidad Carlos III de Madrid
Avda. de la Universidad, 30
28911 Leganés, SPAIN
JSALAS@MATH.UC3M.ES

[4] *Grupo de Teorías de Campos y Física Estadística*
Instituto Gregorio Millán, Universidad Carlos III de Madrid
Unidad Asociada al IEM-CSIC
Madrid, SPAIN

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Abstract

The number of nowhere zero $\mathbb{Z}_Q$ flows on a graph G can be shown to be a polynomial in Q, defining the flow polynomial $\Phi_G(Q)$. According to Tutte's five-flow conjecture, $\Phi_G(5) > 0$ for any bridgeless G. A conjecture by Welsh that $\Phi_G(Q)$ has no real roots for $Q \in (4, \infty)$ was recently disproved by Haggard, Pearce and Royle. These authors conjectured the absence of roots for $Q \in [5, \infty)$. We study the real roots of $\Phi_G(Q)$ for a family of non-planar cubic graphs known as generalised Petersen graphs $G(m, k)$. We show that the modified conjecture on real flow roots is also false, by exhibiting infinitely many real flow roots $Q > 5$ within the class $G(nk, k)$. In particular, we compute explicitly the flow polynomial of $G(119, 7)$, showing that it has real roots at $Q \approx 5.0000197675$ and $Q \approx 5.1653424423$. We moreover prove that the graph families $G(6n, 6)$ and $G(7n, 7)$ possess real flow roots that accumulate at $Q = 5$ as $n \to \infty$ (in the latter case from above and below); and that $Q_c(7) \approx 5.2352605291$ is an accumulation point of real zeros of the flow polynomials for $G(7n, 7)$ as $n \to \infty$.

Key Words: Nowhere zero flows; flow polynomial; flow roots; Tutte's five–flow conjecture; Petersen graph; transfer matrix.

1 Introduction

Given an arbitrary graph G and a set of Q colours, the number of proper vertex Q–colourings of G is given by the chromatic polynomial $\chi_G(Q)$, which is indeed a polynomial in Q [9, 50, 51]. The four-colour theorem states that every planar graph admits a 4-vertex-colouring (i.e., $\chi_G(4) > 0$ for any planar graph G) [1].

The fact that $\chi_G(Q)$ is a polynomial in Q, allows us to promote Q from its initial definition as a positive integer to a complex variable $Q \in \mathbb{C}$. This suggests an algebraic or even analytic approach to the colouring problem. There exist many studies of the location in $\mathbb{C}$ of the roots of $\chi_G(Q)$, henceforth called chromatic roots. These studies concern either specific graphs, or all planar graphs, or other infinite families of graphs.

Birkhoff and Lewis [10] have made the following conjecture: if G is planar, $\chi_G(Q) > 0$ for $Q \in [4, \infty)$. Obviously, the statement of this conjecture is stronger than the four-colour theorem, but unfortunately has not yet been turned into a theorem. The corresponding result for $Q \in [5, \infty)$ has however been proved by the same authors [10] (see also [32, 42, 52]).

Beraha and Kahane [3] have exhibited an infinite family of planar graphs for which $Q = 4$ can be proved to be an accumulation point of *complex* chromatic roots. In that sense, the four-colour theorem is "almost false". Improving on this, Royle [36] has proved, for a slightly different family, that $Q = 4$ is also an accumulation point of *real* chromatic roots (converging to $Q = 4$ from below). Finally, Sokal [40] has proved that for a specific family of planar graphs (generalised Θ-graphs) chromatic roots are dense in $\mathbb{C}$ (except perhaps in the disc $|Q - 1| < 1$).

A close cousin of the chromatic polynomial is the so-called *flow polynomial* $\Phi_G(Q)$. Let $G = (V, E)$ be an arbitrary (not necessarily planar) graph G with vertex set V and edge set E, and let Γ be an additive Abelian group. A Γ*–flow* on G is a map $\phi\colon E \to \Gamma$ that attributes a variable $\phi(e)$ to each edge $e \in E$, subject to the conservation of these variables at each vertex, with respect to an arbitrary chosen orientation of E. An elementary example of a flow is the current ϕ in an electrical network, in which case the conservation constraint is known as Kirchhoff's first law [29].

A *nowhere zero* Γ*–flow* is a Γ–flow ϕ such that $\phi(e) \neq 0$ for all $e \in E$ [28, 46, 55]. If Γ is a *finite* Abelian group of order Q, it can be shown that the number of nowhere zero Γ–flows depends only on Q (not on the *specific* structure of the group Γ), and it is in fact the restriction to $Q \in \mathbb{N}$ of a polynomial in Q called the flow polynomial $\Phi_G(Q)$ [44]. One can then again extend the definition to $Q \in \mathbb{C}$ and study the location of (real or complex) flow roots.

A *nowhere zero* Q*–flow* of G is a nowhere zero $\mathbb{Z}$–flow ϕ such that $|\phi(e)| \leq |Q - 1|$ for all $e \in E$. Tutte [44] showed that G has a nowhere zero Q–flow if and only if it has a nowhere zero $\mathbb{Z}_Q$–flow; but these two concepts are different! Tutte's result immediately implies the following interesting (but far from obvious) property of nowhere zero $\mathbb{Z}_Q$-flows [45]:

Proposition 1.1 *If $\Phi_G(Q) > 0$ for some $Q \in \mathbb{N}$, then $\Phi_G(Q') > 0$ for all integers $Q' \geq Q$.*

When G is planar, one has [43] the duality relation $\chi_{G^*}(Q) = Q\, \Phi_G(Q)$, where G^* denotes the dual graph. In this case, the properties of $\Phi_G(Q)$ thus follow from those of $\chi_{G^*}(Q)$. But for non-planar G, the flow polynomial $\Phi_G(Q)$ is a genuinely new object.

It is worth stressing that the Birkhoff–Lewis theorem [10] provides a uniform upper bound for the real zeros of the chromatic polynomial of *all* loopless planar graphs, namely $Q = 5$. However, such an upper bound (if it actually exists at all!) is not known for the real zeros of the flow polynomial of arbitrary bridgeless graphs. (Obviously, $\Phi_G(Q) = 0$ if G has a bridge, because of the "nowhere zero" condition.) The existence of such uniform upper bound and its value, if it does exist, are long-standing open problems in Combinatorics.

Consider now arbitrary (not necessarily planar) bridgeless graphs G. Because there exist graphs not admitting a nowhere zero 4–flow, the strongest possible results for integer and real flow roots are given, respectively, by the following two well–known conjectures:

Conjecture 1.2 (Tutte's five–flow conjecture [28, 45, 46]) *For any bridgeless graph G, $\Phi_G(5) > 0$.*

Remarks. 1. This conjecture implies that $\Phi_G(Q) > 0$ for all integers $Q \geq 5$ by Proposition 1.1.

2. The Petersen graph—which is a special case $G(5,2)$ of the generalised Petersen graphs $G(m,k)$ to be defined in Section 3 below—has the flow polynomial $\Phi_{G(5,2)}(Q) = (Q-1)(Q-2)(Q-3)(Q-4)(Q^2-5Q+10)$, which vanishes at $Q = 4$. So it does not admit a nowhere zero 4–flow.

Conjecture 1.3 (Welsh [49]) *For any bridgeless graph G, $\Phi_G(Q) > 0$ for $Q \in (4, \infty)$.*

It should be noted that the Welsh conjecture parallels that of Birkhoff and Lewis for the chromatic polynomial: the only difference is that the endpoint $Q = 4$ is included in the Birkhoff–Lewis conjecture for the chromatic polynomial, but not in the Welsh conjecture for the flow polynomial. Some results by Jackson on zero–free intervals for the flow polynomials of cubic graphs [20, 21] also suggest this close parallelism between $\chi_G(Q)$ for planar G and $\Phi_G(Q)$ for *arbitrary* G. (Note that both polynomials are evaluated at the *same* value of Q.)

A number of weaker results have been proved over the years, notably:

Theorem 1.4 (Seymour [38]) *For any bridgeless graph G, $\Phi_G(6) > 0$.*

Theorem 1.5 (Steinberg [41]) *For any bridgeless graph G that is embeddable in the projective plane, $\Phi_G(5) > 0$.*

An immediate corollary of Seymour's theorem (using Proposition 1.1) is that $\Phi_G(Q) > 0$ for all integers $Q \geq 6$. Thus $Q = 5$ is a uniform upper bound for *integer* flow roots. But the above results give no clue about the existence of a uniform upper bound for *real* flow roots.

The first step into proving (or disproving) Conjectures 1.2–1.3 consists in studying the flow roots of "small" graphs. By computing the flow roots of small graphs with high girth (up to 32 vertices and girth at least 7), Haggard, Pearce, and Royle [18] have very recently found an explicit counterexample to the Welsh conjecture: the flow polynomial of the generalised Petersen graph $G(16,6)$ has two real roots larger than $Q = 4$: $Q_1 \approx 4.0252205$, and $Q_2 \approx 4.2331455$. However, the same authors conjectured the following modification of Conjecture 1.3, in which 4 is replaced by 5, and the endpoint $Q = 5$ is now included in accordance with Tutte's five-flow conjecture:

Conjecture 1.6 (Haggard–Pearce–Royle [18]) *For any bridgeless graph G, $\Phi_G(Q) > 0$ for $Q \in [5, \infty)$.*

Remark. Note that Kochol [30] proved that the smallest counterexample to Tutte's five–flow conjecture should have girth at least 9. Note also that Jackson [20, Corollary 39] observed, as a special case of a more general matroidal result proved but not stated(!) by Oxley [32], that if G *and all its 3-edge-connected minors* have girth $\le g$, then $\Phi_G(Q) > 0$ for all real $Q > g$. So any graph with a large real flow root must either have high girth or have a 3-edge-connected minor with high girth.

Even though the naive parallelism between $\chi_G(Q)$ and $\Phi_G(Q)$ has been invalidated by the above-mentioned counterexample to Conjecture 1.3, a related line of reasoning would be that all these conjectures and theorems might be related by replacing Q for the chromatic polynomial with $Q+1$ for the flow polynomial. Thus, the four-colour theorem [1] "translates" into the Tutte five-flow conjecture [45, 46], and the Birkhoff–Lewis conjecture translates into Conjecture 1.6. (Note that the translation of Royle's result [36], showing the existence of a family of plane triangulations with real chromatic roots converging to 4 from below, is consistent with the fact that Conjecture 1.3 [49] is false.)

In this paper we study the flow polynomial on the infinite family of graphs known as the generalised Petersen graphs $G(m, k)$. Our main results are the following:

Theorem 1.7 *The value $Q = 5$ is an isolated accumulation point of real zeros of the flow polynomial $\Phi_G(Q)$ for the families of bridgeless graphs $G(6n, 6)$ and $G(7n, 7)$ with $n \ge 3$. Moreover:*

- *(a) There is a sequence of real zeros $\{Q_n\}$ of the flow polynomials $\Phi_{G(6n,6)}$ that converges to $Q = 5$ from below.*
- *(b) There is a sequence of real zeros $\{Q_n\}$ of the flow polynomials $\Phi_{G(7n,7)}$ that converges to $Q = 5$. The sub-sequence with odd (resp. even) n converges to $Q = 5$ from above (resp. below).*

Theorem 1.8

- *(a) The bridgeless graph $G(119, 7)$ has flow roots at $Q \approx 5.00002$ and $Q \approx 5.16534$ (where $\approx$ means "within 10^{-5}").*
- *(b) The value $Q_c(7) \approx 5.235261$ (where $\approx$ means "within 10^{-6}") is an accumulation point of real zeros of the flow polynomials $\Phi_{G(7n,7)}(Q)$. In particular, the sub-sequence for odd n of the real zeros $\{Q_n\}$ of the flow polynomials $\Phi_{G(7n,7)}$ converges to $Q_c(7)$ from below.*

Remark. The largest real flow root we have explicitly found is $Q_0 \approx 5.1653424423$ for $G(119, 7)$.

Thus, the Welsh conjecture and Conjecture 1.6 (the "translated Birkhoff–Lewis conjecture") are both false, and the Tutte five-flow conjecture is "almost false" in the same sense that the four-colour theorem is "almost false" [3, 36]. On the other hand, Theorem 1.7 includes the "translated version" of the existence theorem of Royle [36] for real chromatic roots.

In this work, we have considered the family of graphs $G(nk, k)$ with $k \leq 7$ and $n > 2$. For each k, we have located the set of accumulation points in the complex Q-plane of the roots of the flow polynomial $\Phi_{G(nk,k)}$, as $n \to \infty$. Most accumulation points belong to limiting curves $\mathcal{B}_k$; and in particular, we are interested in locating the points $Q_c(k)$, defined as the largest real value where the limiting curves $\mathcal{B}_k$ cross the real axis. (These points are likely to be accumulation points as $n \to \infty$ of real zeros, as in Theorem 1.8(b); but not always: see Section 6.3.) We have been able to obtain the values of $Q_c(k)$ for $k \leq 11$; and the numerical extrapolation of these values to $k \to \infty$ yields $\lim_{k\to\infty} Q_c(k) = Q_0 \approx 5.69$ [25]. We expect that this value is the largest real accumulation point that one can get from the family $G(nk, k)$.

Based on this—and on the failure of Conjectures 1.3 and 1.6—we venture the following weaker conjecture:

Conjecture 1.9 *For any bridgeless graph G, $\Phi_G(Q) > 0$ for $Q \in [6, \infty)$.*

The disproof of Conjecture 1.6 leaves basically three possibilities:

1. $Q \to Q + 1$ *translation is valid.* Then Tutte's 5-flow conjecture is true (because the 4-colour theorem is true) and Conjecture 1.9 is true (because the Birkhoff–Lewis theorem is true), but the Birkhoff–Lewis conjecture is false (because Conjecture 1.6 is false).

2. $Q \to Q + 2$ *translation is valid.* Then Tutte's 5-flow conjecture is false (because not every planar graph is 3–colourable), but Seymour's 6-flow theorem [38] corresponds to the 4-colour theorem. The existence of graphs with real flow roots in $(5, 6)$ corresponds to the existence of planar graphs with real chromatic roots in $(3, 4)$; and Royle's theorem on the existence of plane triangulations with real chromatic roots converging to 4 from below suggests that there should exist graphs with real flow roots converging to 6 from below. Finally, the Birkhoff–Lewis conjecture and Conjecture 1.9 would either be both true or both false.

3. *No translation holds: the two problems are less closely related than previously thought.* In this case, Conjecture 1.9 might well be false. Indeed, it might even be the case that there does not exist *any* finite upper bound for the real flow roots of general graphs; this would signal the strongest possible failure of the analogy between real chromatic roots of planar graphs and real flow roots of general graphs.

Note that we do in fact exhibit *infinitely* many flow roots in an interval $Q \in [4, Q_0]$ with $Q_0 > 5$. This means, loosely speaking, that if Tutte's 5-flow conjecture is true one should look for a purely combinatorial proof, i.e., one that considers only integer Q. This is exactly the same situation as for the 4– and 5–colour theorems; they hold true even though one can find families of graphs with real roots approaching $Q = 4$ from below [36], and other families with complex roots approaching densely to $Q = 4$ [3] and $Q = 5$ [40]. If Conjecture 1.9 turns out to be false, then it is very plausible that there exist *no upper bound* for real flow roots of arbitrary bridgeless graphs.

On a more technical level, we exhibit a method for computing exactly the flow polynomial on very large generalised Petersen graphs (which can readily be adapted to other similar graph families). This method relies on a transfer matrix construction similar to the one employed in our previous work [23,24] on the chromatic polynomial for graphs with periodic longitudinal boundary conditions.

The paper is organised as follows. In Section 2 we define the flow polynomial carefully and exhibit its relation to the Q-state Potts model. Building on this, we show in Section 3 how the flow polynomial for generalised Petersen graphs can be built by a transfer matrix construction. Our results, given in Section 4, are obtained by implementing this construction on a computer and pushing the computation to as large graphs as possible. Note that although obtained by computational means, the flow polynomials are exact and involve no approximation whatsoever. In Section 5 we introduce the Beraha–Kahane–Weiss theorem, which plays an important role in establishing our results. In Section 6, we describe our analytic findings about the real zeros of the flow polynomial for this family of graphs. To conclude, in Appendix A, we prove some technical lemmas included in the text that are essential in the proofs of the main results of this paper (Theorems 1.7 and 1.8). In Appendix B we study some additional structural properties of the transfer matrices. Finally, in Appendix C, we give the coefficients of the flow polynomial for the generalised Petersen graph $G(119,7)$.

2 Flow polynomial

Let $G=(V,E)$ be a connected graph and Γ be an Abelian group. Assign an arbitrary orientation to each edge $e\in E$. With respect to any fixed vertex $i\in V$, the edges E_i incident on i can then be characterised as either ingoing or outgoing: $E_i = E_i^{\rm in}\cup E_i^{\rm out}$.

A Γ–*flow* on G is a map $\phi\colon E\to\Gamma$ that attributes a variable $\phi(e)$ to each edge $e\in E$, subject to the constraint

$$\sum_{e\in E_i^{\rm in}} \phi(e) \;=\; \sum_{e\in E_i^{\rm out}} \phi(e) \tag{2.1}$$

for any $i\in V$. The edge orientation is actually immaterial in these definitions: if one wants to change the orientation of an edge e_0, it suffices to change simultaneously the sign of the flow along that edge, $\phi(e_0)\to-\phi(e_0)$.

A *nowhere zero* Γ*–flow* is a Γ–flow ϕ such that $\phi(e)\neq 0$ for all $e\in E$. If Γ is a *finite* Abelian group, we denote $\Phi_G(\Gamma)$ the number of nowhere zero Γ–flows on G. In particular, a $\mathbb{Z}_Q$*–flow* (resp. a *nowhere zero* $\mathbb{Z}_Q$*–flow*) on G is a map $\phi\colon E\to\{0,1,\ldots,Q-1\}$ (resp. $\phi\colon E\to\{1,2,\ldots,Q-1\}$) for which the constraint (2.1) is imposed modulo Q.

Let Γ be a finite Abelian group of order Q. Clearly, the total number of Γ-flows on G is $Q^{c(E)}$, where for any subset $E'\subseteq E$, $c(E')$ denotes the number of independent cycles (cyclomatic number) in the induced graph $G'=(V,E')$. To obtain the number of nowhere zero Γ-flows, we first subtract for each $e\in E$ the flows for which $\phi(e)=0$. Since flows with two zero–flow edges will be subtracted off twice, these must be put back in the sum, and proceeding by inclusion-exclusion we find [53]

$$\Phi_G(\Gamma) \;=\; \sum_{E'\subseteq E} (-1)^{|E|-|E'|} Q^{c(E')}\,. \tag{2.2}$$

By this result, $\Phi_G(\Gamma)$ depends only on Q and is indeed the restriction to positive integers of a polynomial in Q, namely (2.2). We call (2.2) the *flow polynomial* of G and henceforth write it as $\Phi_G(Q)$.

Meanwhile, recall the partition function of the Q-state Potts model [33]

$$Z_G(Q,v) \;=\; \sum_{\sigma} \prod_{(ij)\in E} e^{K\delta\left(\sigma(i),\sigma(j)\right)}\,, \tag{2.3}$$

where the map $\sigma\colon V \to \{0,1,\ldots,Q-1\}$ is called the spin, and K is the coupling constant. The Kronecker delta function $\delta(x,y)$ is defined by $\delta(x,y)=1$ if $x=y$, and $\delta(x,y)=0$ otherwise. We have introduced the convenient parameter $v=e^K-1$. By expanding the edge product and performing the sum over σ, one recovers the partition function in the Fortuin-Kasteleyn cluster representation [16]

$$Z_G(Q,v) \;=\; \sum_{E'\subseteq E} v^{|E'|} Q^{k(E')}\,, \tag{2.4}$$

where $k(E')$ is the number of connected components in $G'=(V,E')$.

Graph theorists will recognise in (2.4) [a reparametrisation of] the Tutte polynomial [45] and interpret σ in (2.3) as a vertex colouring. Proper vertex colourings, i.e., those for which adjacent vertices are coloured differently, are obtained for $K\to-\infty$, and therefore

$$\chi_G(Q) \;=\; Z_G(Q,-1) \tag{2.5}$$

is the chromatic polynomial.

Setting instead $v=-Q$ in (2.4), and using the topological identity

$$k(E') \;=\; |V|-|E'|+c(E')\,, \tag{2.6}$$

one establishes the connection with the flow polynomial

$$\Phi_G(Q) \;=\; (-1)^{|E|}Q^{-|V|}Z_G(Q,-Q)\,. \tag{2.7}$$

Note that $\Phi_G(Q)=0$ if G contains a bridge $e_0\in E$. Indeed, by the constraint (2.1) one would have $\phi(e_0)=0$, preventing the existence of a nowhere zero flow.

In the case where G is planar, let G^* denote the dual graph. Recall the fundamental duality relation [54] of the Potts model partition function

$$Z_G(Q,v) \;=\; K\, Z_{G^*}(Q,v^*)\,, \tag{2.8}$$

where v^* is the dual of v

$$v\,v^* \;=\; Q\,, \tag{2.9}$$

and the proportionality factor is

$$K \;=\; Q^{1-|V^*|}v^{|E|} \;=\; Q^{|V|-|E|-1}v^{|E|}\,. \tag{2.10}$$

Noticing that $v=-Q$ is dual to $v=-1$ by (2.9) furnishes a relation between the flow polynomial of G and the chromatic polynomial of G^*. Indeed, using (2.8) and (2.10) we have [43]

$$\chi_{G^*}(Q) \;=\; Q\;\Phi_G(Q)\,. \tag{2.11}$$

Alternatively, the relation (2.11) can be proved by noting that there exists an obvious bijection between the nowhere zero Γ-flows on G and the proper colourings of the faces of G, with the colour on one face being fixed. Indeed, let ϕ be a flow on G. Then, turning around a vertex, each time one moves from a face i to an adjacent face j, if the separating edge e is seen oriented to the right (resp. left), its flow variable $\phi(e)$ defines the colour difference $c_j - c_i = \phi(e)$ (resp. $c_j - c_i = -\phi(e)$). Starting from the face with fixed colour, these differences define the face colouring of the whole graph. The mapping from proper colourings to flows follows similarly.

It is useful to note that for any bridgeless 3–connected graph G, one can deduce from (2.4)/(2.7) that $\Phi_G(Q)$ is a polynomial in Q of degree $|E| - |V| + 1$ in Q, and that the first two coefficients of $\Phi_G(Q)$ are given by

$$\Phi_G(Q) \;=\; Q^{|E|-|V|+1} \;-\; |E|\, Q^{|E|-|V|} + \dots \tag{2.12}$$

The first term comes from the fact that there is a unique spanning graph (V, E') in (2.4)/(2.7) with $E' = E$. The second term is given by the contribution of the $|E|$ spanning subgraphs (V, E') with $E' = E \setminus e$ for each $e \in E$, and the observation that $k(E') = 1$ since G is connected and bridgeless. Notice that if we consider the spanning graph (V, E') with $E' \setminus \{e, e'\}$ for any two distinct edges $e, e' \in E$, the 3–connectedness of G guarantees that the next term in (2.12) will be of order $Q^{|E|-|V|-1}$, as in this case we also have $k(E') = 1$.

3 Transfer matrix for flow polynomials of generalised Petersen graphs

3.1 Generalised Petersen graphs

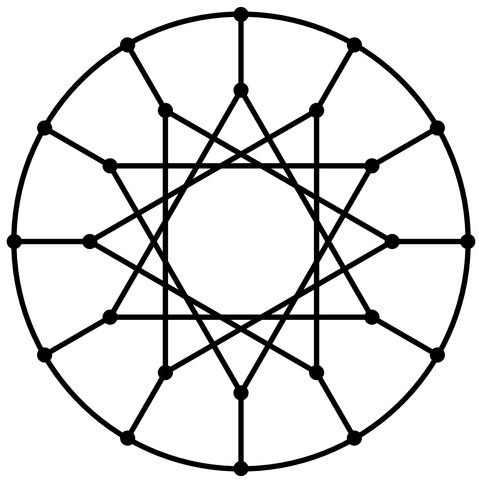
The goal of this paper is to evaluate the flow polynomial on a family of graphs $G(m, k)$ called *generalised Petersen graphs* and defined as follows: let m, k be positive integers such that $m > k$. Then $G(m, k)$ is a cubic graph with $2m$ vertices denoted i_p and j_p for $p = 1, 2, \dots, m$: i.e.,

$$V(G(m,k)) \;=\; \{i_1, \dots, i_m, j_1, \dots, j_m\}\,. \tag{3.1}$$

The edge set consists of $3m$ edges $(i_p j_p)$, $(i_p i_{p+1})$, $(j_p j_{p+k})$, for $p = 1, 2, \dots, m$, and with all indices considered modulo m: i.e.,

$$E(G(m,k)) \;=\; \{(i_p, j_p), (i_p i_{p+1}), (j_p j_{p+k}) \mid 1 \le p \le m\}\,. \tag{3.2}$$

Note that $G(m,k)$ is simple for $m \neq 2k$; but it has double edges when $m = 2k$. These graphs were introduced by Watkins [48]. As an example, $G(12,4)$ can be drawn as follows:



The graphs $G(m,k)$ are clearly bridgeless. They are non-planar for all pairs (m,k) except for the case $(3,2)$ and the two sub-families $(p,1)$ and $(2p,2)$ with $p \geq 1$. They have girth 8 for m and k sufficiently large. We have thus a two-parameter family of non-planar cubic graphs with high girth, and based on exhaustive studies of small graphs [18] we expect this family to produce large real flow roots. However, it is easy to see that $Q = 5$ is not a flow root:

Lemma 3.1 *For every generalised Petersen graph $G(m,k)$ with m,k positive integers such that $m > k$, $\Phi_{G(m,k)}(5) > 0$. In fact, every graph $G(m,k)$ other than the ordinary Petersen graph $G(5,2)$ has $\Phi_{G(m,k)}(4) > 0$.*

Proof. It is well known [11] that every generalised Petersen graph $G(m,k)$ (with the exception of the Petersen graph $G(5,2)$ itself), admits a Tait colouring: i.e., an edge 3–colouring such that at every vertex, the three incident edges take distinct colours.

It is worth noting that the definition of the generalised Petersen graph $G(m,k)$ in Refs. [11, 48] explicitly excludes the case $m = 2k$. However, it is easy to see that any $G(2k,k)$ has a Tait colouring: e.g., the edges (i_p, j_p) take colour 1, the edges (i_p, i_{p+1}) take alternatively colours 2 and 3 (as p goes from 1 to m), and for each p, one of the double edges (j_p, j_{p+k}) takes colour 2, and the other edge, colour 3.

The existence of such edge 3–colourings is equivalent, for cubic loopless graphs, to the existence of a nowhere zero 4–flow for the same graph [27, Proposition 2(b)]. Therefore, for all $G(m,k)$ except the Petersen graph $G(5,2)$, $\Phi_{G(m,k)}(4) > 0$, and furthermore, $\Phi_{G(m,k)}(5) > 0$ by Proposition 1.1. The case $G(5,2)$ is dealt with directly: from the exact expression for $\Phi_{G(5,2)}$ (see the second remark after Conjecture 1.2 above), we conclude that $\Phi_{G(5,2)}(5) = 240 > 0$. ∎

We shall however show that the five-flow conjecture is "almost false", in the sense of Theorem 1.7.

3.2 Potts model transfer matrix

We wish to evaluate $Z_{G(m,k)}(Q,v)$—of which the flow polynomial $\Phi_{G(m,k)}(Q)$ is a special case—by a transfer matrix construction.

Contrary to an often repeated but false statement, evaluating $Z_G(Q,v)$ by a transfer matrix construction is possible for *any* graph G, and does not require G to consist of a number of identical layers [2]. However, when G does have a layered structure—as is the case here—$Z_G(Q,v)$ can be computed by the repeated application of the *same* transfer matrix.

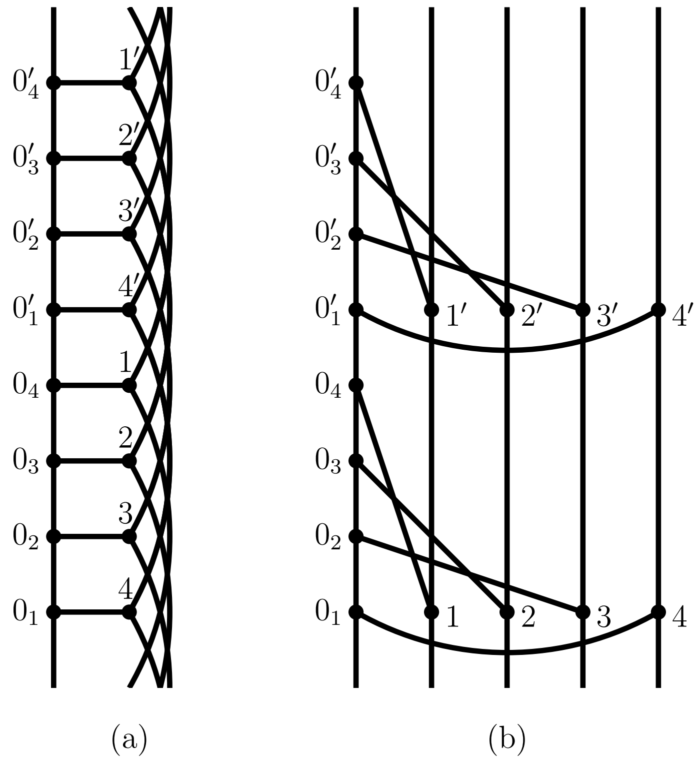


Figure 1: (a) Generalised Petersen graph $G(m,k)$, here with $k=4$. There are m layers of two vertices in the vertical direction, but some edges link layers at distance k. The boundary conditions are periodic in the vertical direction. (b) When $m=nk$, $G(nk,k)$ can be redrawn as shown. There are $n=m/k$ layers of width $k+1$ vertices, each comprising a total of $2k$ vertices. All edges now link vertices within the same layer, or in two adjacent layers.

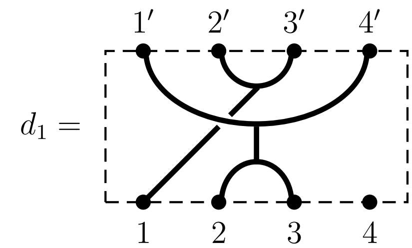
Let us suppose for simplicity that m is a multiple of k: i.e., $m=nk$. Then the generalised Petersen graph $G(nk,k)$ can be redrawn as in Figure 1. This turns $G(nk,k)$ into a graph of n identical layers of width $L=k+1$ vertices with periodic boundary conditions in the vertical direction. (We shall henceforth refer to this as periodic *longitudinal* boundary conditions, in accordance with the fact that the transfer matrix builds up the graph vertically.) We now claim that this implies that $Z_{G(nk,k)}(Q,v)$ can be written as a Markov trace

$$Z_{G(nk,k)}(Q,v) \;=\; \mathrm{Tr}(\mathsf{T}_L)^n \tag{3.3}$$

of the n-th power of a transfer matrix T_L to be defined shortly.

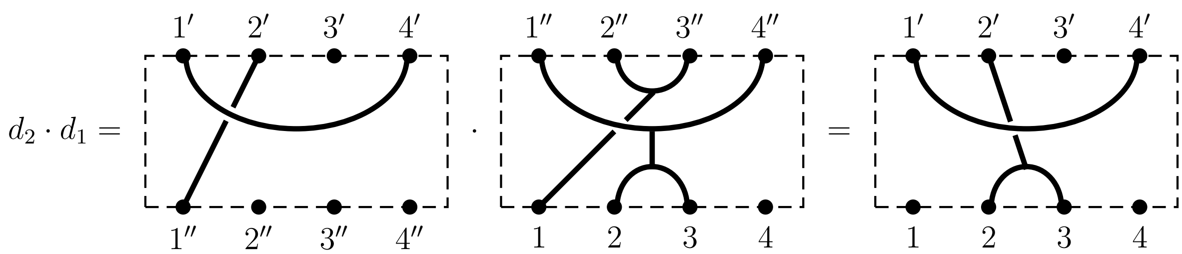
In general, for a layered graph of width L, T_L acts on basis states A_L which are set partitions of $2L$ points $\{1, 2, \ldots, L, 1', 2', \ldots, L'\}$. These basis states can be depicted as *partition diagrams*, which are hypergraphs on $2L$ vertices, drawn inside a rectangle with L vertices (labelled $1', 2', \ldots, L'$) on top and L vertices (labelled $1, 2, \ldots, L$) on bottom. Each hyperedge represents one block in the partition. A block that contains at least one vertex from both the top and bottom rows is called a *link*. A block containing precisely one vertex is called a *singleton*. The number of links in a diagram d is denoted $\ell(d)$.

The following example with $L = 4$



represents the partition $(1, 2', 3')(2, 3, 1', 4')(4)$. It has two links and one singleton.

The multiplication $d = d_2 \cdot d_1$ of two partition diagrams is defined by stacking the diagrams vertically. Specifically, the top row of d_2 becomes the top row of d, the bottom row of d_1 becomes the bottom row of d, and the top row of d_1 is identified with the bottom row of d_2. Any blocks not containing points in the top or bottom rows of d are removed in the process. This gives, for example:



This diagram multiplication turns A_L into an associative *partition monoid* [19, 31] with identity $I = (1, 1')(2, 2') \cdots (L, L')$. Observe that

$$\ell(d_2 \cdot d_1) \;\leq\; \min\left(\ell(d_2), \ell(d_1)\right) . \tag{3.4}$$

The idea is now that these diagrams will represent the edge subset appearing in the cluster representation (2.4) of the Potts model partition function. The factors of v can be dealt with locally, and the tricky part is to get a handle on the non-local factors Q. To this end, it is natural to associate an element of $\mathbb{C}$ with each diagram, which will play the role of the Boltzmann weight, i.e., the weight of a partially built configuration E' in (2.4). In the diagram multiplication $d = d_2 \cdot d_1$, let $\kappa(d_1, d_2)$ be the number of blocks which are removed because they contain no point in the top or bottom rows of d. The non-local part of the Boltzmann weight is then $Q^{\kappa(d_1, d_2)}$.

These considerations motivate the definition of the *partition algebra* [19, 31] $\mathbb{C}A_L(Q)$ as the associative algebra over $\mathbb{C}$ with basis A_L and multiplication defined by

$$d_2 \, d_1 \;=\; Q^{\kappa(d_1, d_2)} \, (d_2 \cdot d_1) . \tag{3.5}$$

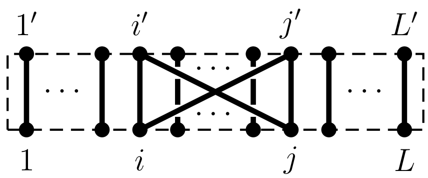
The partition algebra $\mathbb{C}A_L(Q)$ can be represented faithfully as an algebra of matrices in $\mathbb{C}^{A_L \times A_L}$ whose rows and columns are indexed by the partition monoid A_L: namely, the matrix $M(d)$ representing $d \in A_L$ has matrix elements

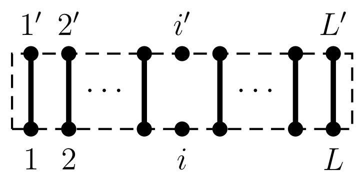
$$M(d)_{d''d'} \;=\; \begin{cases} Q^{\kappa(d',d)} & \text{if } d'' = d \cdot d' \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

This is indeed the point of view that we shall take when constructing the transfer matrix of the flow polynomial and manipulating it explicitly (see Section 4.2). The elements of A_L can then be interpreted as the basis states of this representation.

Remark. With no risk of confusion, we shall therefore use the notation A_L to refer both to the partition monoid and to the set of basis states. However, we shall adopt a notation that distinguishes an element $\mathsf{O}_L \in \mathbb{C}A_L(Q)$ in the partition algebra from its corresponding matrix representation $\mathcal{O}_L \in \mathbb{C}^{A_L \times A_L}$.

We now define a set of generators for the monoid A_L. These generators will be the elementary building blocks used to define the transfer matrix T_L. Apart from the identity I, the necessary generators are the *join operators* J_{ij} that amalgamate the blocks containing points i, j, i' and j', and the *detach operators* D_i that remove point i' from its block and turn it into a singleton. In the pictorial representation this gives rise to the diagrams

$\mathsf{J}_{ij} =$ 

$\mathsf{D}_i =$ 

As a consequence of the above definitions, the product $\mathsf{D}_i d$ within $\mathbb{C}A_L(Q)$ produces a factor Q if the point i' is a singleton in diagram d, and a factor 1 otherwise. In particular, we have

$$\mathsf{D}_i^2 \;=\; Q\, D_i\,. \tag{3.7}$$

From these building blocks we can now form the operators representing the addition of an edge to the graph that is being built up by the transfer matrix. These are

$$\mathsf{H}_{ij} \;=\; I + v\mathsf{J}_{ij}\,, \tag{3.8a}$$

$$\mathsf{V}_i \;=\; vI + \mathsf{D}_i\,. \tag{3.8b}$$

The letters H and V stand for horizontal and vertical, where a horizontal edge is understood to link vertices within the same layer of the graph (recall Figure 1), and a vertical edge links vertices from two adjacent layers.

Inspecting Figure 1, and labelling the points as in the figure, we can now finally define the transfer matrix (with $L = k+1$):

$$\mathsf{T}_L \;=\; \mathsf{H}_{01} \left(\prod_{i=k}^{2} \mathsf{V}_0 \mathsf{H}_{0i} \right) \left(\prod_{i=0}^{k} \mathsf{V}_i \right) \tag{3.9a}$$

$$\;=\; \mathsf{H}_{01}\mathsf{V}_0\mathsf{H}_{02}\mathsf{V}_0\mathsf{H}_{03}\cdots\mathsf{V}_0\mathsf{H}_{0k}\mathsf{V}_k\mathsf{V}_{k-1}\cdots\mathsf{V}_0 \tag{3.9b}$$

Note the order of indices in the products.

3.3 Markov trace and eigenvalue amplitudes

It remains to explain the meaning of the Markov trace Tr in (3.3). The Markov trace of any partition diagram $d \in A_L$ is by definition $Q^{\kappa(d)}$, where $\kappa(d)$ is the number of connected components in the diagram obtained from d by identifying the points i and i' for all $i = 1, 2, \ldots, L$. This identification corresponds to implementing the periodic longitudinal boundary conditions in Figure 1. The definition of the Markov trace extends to the partition algebra $\mathbb{C}A_L(Q)$ by linearity.

With this definition, (3.3), and (3.9), we are in principle equipped to compute the partition function $Z_{G(nk,k)}(Q, v)$ as a polynomial in Q and v. A practical problem for going to large k is however that the dimension of T_L, i.e., the number of basis states A_L, grows very fast with L:

$$\dim \, \mathsf{T}_L \;=\; |A_L| \;=\; B_{2L} \,, \tag{3.10}$$

where B_n are the Bell numbers with exponential generating function (egf)

$$\sum_{n=0}^{\infty} \frac{B_n z^n}{n!} \;=\; \exp(\mathrm{e}^z - 1) \,. \tag{3.11}$$

Considerable progress can nevertheless be made if one takes advantage of the structure of the partition algebra [19]. In practical terms this means that the number of points participating in the partitions can be halved from $2L$ to L. We now explain how this comes about.

Denote by $A_L^{(\ell)}$ the elements of the partition monoid A_L with exactly ℓ links, and define for $i = 0, 1, \ldots, L$ the set of elements with at most i links:

$$\mathcal{A}_{L,i} \;=\; \bigcup_{\ell=0}^{i} A_L^{(\ell)} \,. \tag{3.12}$$

Thanks to (3.4), the $\mathcal{A}_{L,i}$ are in fact ideals which, moreover, constitute a filtration of the monoid:

$$\mathcal{A}_{L,0} \;\subseteq\; \mathcal{A}_{L,1} \;\subseteq\; \cdots \;\subseteq\; \mathcal{A}_{L,L} \;=\; A_L \,. \tag{3.13}$$

This implies immediately that for any element O_L (and T_L in particular) in the partition algebra $\mathbb{C}A_L(Q)$, the corresponding matrix $\mathcal{O}_L \in \mathbb{C}^{A_L \times A_L}$ has a block-triangular structure with respect to ℓ. The eigenvalues of $\mathcal{O}_L$ can therefore be found by restricting to $A_L^{(\ell)}$ for $\ell = 0, 1, \ldots, L$. From the point of view of the matrix representation of $\mathbb{C}A_L(Q)$, this restriction amounts to replacing a matrix $\mathcal{O}_L$ by another matrix $\mathcal{O}'_L$ in which all the off-diagonal blocks have been set to zero, i.e., $\mathcal{O}'_L$ is block-diagonal with respect to ℓ. Although $\mathcal{O}'_L$ does not represent any element of $\mathbb{C}A_L(Q)$, it is still a well-defined matrix and we can study its eigenvalues, which are the same as those of $\mathcal{O}_L$.

In fact $\mathcal{O}'_L$ is block-diagonal with respect to a more refined partition of basis states. To see this, it suffices to observe that $\mathcal{O}'_L$ cannot change the blocks of the partition that contain *only* points from the bottom set $\{1, 2, \ldots, L\}$, since the multiplication has been defined by acting on the top points only; nor can it amalgamate two blocks into one, or "abandon" a link by failing to connect it to the top row. Therefore, $\mathcal{O}'_L$ is block-diagonal, with the blocks of $\mathcal{O}'_L$ being indexed by partitions of the bottom points $\{1, 2, \ldots, L\}$ together with a marking of ℓ of them as "links". Moreover, all the blocks corresponding to a given value of ℓ are *identical*, by virtue of the definition of the generators of A_L and the restrictions imposed when going from $\mathcal{O}_L$ to $\mathcal{O}'_L$.

As far as the determination of the eigenvalues goes, one can therefore restrict further the basis states of $A_L^{(\ell)}$ to partitions of the top points $\{1', 2', \ldots, L'\}$ only, with precisely ℓ blocks (which were the links in the full partition monoid) being marked $1, 2, \ldots, \ell$ (to indicate that they are connected, respectively, to the first, second,..., ℓ-th marked block on the bottom row). Note that the marked blocks carry *distinct* labels, since the action of $\mathcal{O}'_L$ can still exchange their order (relative to the now-forgotten fixed order of the links with respect to the bottom points $\{1, 2, \ldots, L\}$). One can then finally block-diagonalise $\mathcal{O}'_L$ by rearranging these restricted basis states into linear combinations that are irreducible representations λ of the symmetric group S_ℓ.

To summarise, all distinct eigenvalues of $\mathcal{O}_L$ can be found by studying the irreducible representations labelled by ℓ and λ. We have thus the following decomposition of the Markov trace

$$\mathrm{Tr}\ \mathcal{O}_L \ = \ \sum_{\ell=0}^{L} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda}\ \mathrm{tr}_{\ell,\lambda}\, \mathcal{O}_L \,, \tag{3.14}$$

where now $\mathrm{tr}_{\ell,\lambda}$ are ordinary matrix traces. The coefficients $\alpha_{\ell,\lambda}$ (which are polynomials in Q as we shall see below) are eigenvalue amplitudes, which can also be interpreted as the dimensions of the commutant of the partition algebra.

Consider now $\lambda \in S_\ell$ through its corresponding Young diagram, $Y(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where λ_i is the number of boxes in the i–th row. If there are less than ℓ rows in $Y(\lambda)$, the expression is of course padded with zeros. One then has the result [19, Proposition 3.24]

$$\alpha_{\ell,\lambda} \ = \ \frac{\dim \lambda}{\ell!} \prod_{i=0}^{\ell-1} (Q - i - \lambda_{\ell-i}) \,. \tag{3.15}$$

We recall that the dimension $\dim \lambda$ of the representation λ is given by the hook formula [37]

$$\dim \lambda \ = \ \frac{\ell!}{\prod\limits_{x \in Y(\lambda)} h_x} \,, \tag{3.16}$$

where h_x is the hook length of the box $x \in Y(\lambda)$, i.e., the number of boxes to its right, plus the number of boxes below it, plus the box itself. We shall sometimes need the total amplitude β_ℓ for a given number of marked blocks. This reads then

$$\beta_\ell \ = \ \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \dim \lambda \,. \tag{3.17}$$

Remarks. 1. A couple of other cases of decompositions of the Markov trace, analogous to (3.14)–(3.15), have previously been considered in the literature. Indeed, had the graph been planar and with periodic *transverse* boundary conditions (in addition to the periodic longitudinal boundary conditions that we assume throughout), the transfer matrix could only have changed the cyclic order of the links, and the relevant group would not have been S_ℓ, but rather the cyclic group C_ℓ. Its representation theory leads to very different expressions [35, Eqs. (1.2)/(1.3)] for the analogue of (3.14)–(3.15). A similar remark holds for planar graphs with free transverse boundary conditions, in which case the links cannot be permuted at all, and the group acting on the links is the trivial group E consisting of only the identity. The corresponding decomposition of the Markov trace can be found in [34, Eqs. (8)–(10)].

2. Obviously the graph does not need to be non-planar for (3.14)–(3.15) to be applicable. Rather, since $E \subseteq C_\ell \subseteq S_\ell$, the two planar cases discussed in the preceding remark can be treated in the general non-planar formalism. But when doing so, some of the representations $\lambda \in S_\ell$ will lead to zero eigenvalues and/or eigenvalues corresponding to different representations λ will coincide. Discarding the former representations, and summing up the amplitudes $\alpha_{\ell,\lambda}$ of the latter, then reproduces the results of [34, 35].

3.4 Flow polynomial transfer matrix

The transfer matrix that produces the flow polynomial $\Phi_{G(nk,k)}(Q)$ can be taken simply as T_L of Sections 3.2–3.3, i.e., by specialising (3.3) to $v = -Q$.

One can however reduce the dimension of the relevant partition algebra by remarking that for $v = -Q$, the vertical operator V_i in (3.8b) is a projector (up to a constant). Indeed, by (3.7) one finds $\mathsf{V}_i^2 = (vI + \mathsf{D}_i)^2 = v^2 I + (2v + Q)\mathsf{D}_\mathsf{i}$, which is a multiple of V_i if and only if $v = -Q$. The normalised projector $(-Q)^{-1}\mathsf{V}_i$ annihilates any partition diagram in which the point i' is a singleton. Concerning the reduced partitions of the points $\{1', 2', \ldots, L'\}$ with precisely ℓ marked blocks—as described in Section 3.3—the precise statement is: $(-Q)^{-1}\mathsf{V}_i$ annihilates any reduced diagram in which i' is an un-marked singleton.

Since V_i and V_j commute for any i, j, the following operator

$$\mathsf{P}_L \;=\; (-Q)^{-L} \prod_{i=0}^{L-1} \mathsf{V}_i \tag{3.18}$$

is also a projector. It annihilates any reduced diagram containing an un-marked singleton. We can therefore replace (3.3) by

$$Z_{G(nk,k)}(Q, -Q) \;=\; \mathrm{Tr}(\widetilde{\mathsf{T}}_L)^n \,, \tag{3.19}$$

where $\widetilde{\mathsf{T}}_L = \mathsf{P}_L \mathsf{T}_L$ with $v = -Q$, and consider the trace only over states without un-marked singletons. This implies that the flow polynomial [cf. (2.7)] can be finally written as

$$\Phi_{G(nk,k)}(Q) \;=\; (-1)^{nk}\, Q^{-2nk}\; \mathrm{Tr}(\widetilde{\mathsf{T}}_L)^n \,, \tag{3.20}$$

as the generalised Petersen graph $G(m, k)$ has $3m$ edges and $2m$ vertices. The prefactor $(-1)^{nk}\, Q^{-2nk}$ can be absorbed in the definition of the transfer matrix: if we define

$$\widehat{\mathsf{T}}_L \;=\; (-1)^k\, Q^{-2k}\, \widetilde{\mathsf{T}}_L \,, \tag{3.21}$$

then (3.20) becomes

$$\Phi_{G(nk,k)}(Q) \;=\; \mathrm{Tr}(\widehat{\mathsf{T}}_L)^n \,. \tag{3.22}$$

The decomposition (3.14)–(3.15) goes through as before, now only with the "no un-marked singleton" constraint imposed on the representations labelled by ℓ and λ.

Remarks. 1. All the entries in the matrix $\widetilde{\mathsf{T}}_L$ are polynomials in Q; but this property does not hold in general for the matrix elements of $\widehat{\mathsf{T}}_L$. Some of them may contain terms with inverse powers of Q.

Let us give an example for $k = 3$. When we apply the transfer matrix $\widetilde{\mathsf{T}}_4$ to the partition $(1', 2', 3', 4')(0, 1, 2, 4)$, we get several partitions with coefficients that are polynomial in Q. In particular, we obtain the partition $(1', 4')(2', 3')(1, 2, 3, 4)$ with the coefficient $(-Q)^5$. If we divide this polynomial by the prefactor $(-1)^k Q^{-2k}$ [cf. (3.21)], we obtain $1/Q$, which is *not* a polynomial in Q.

2. The structural properties of $\widetilde{\mathsf{T}}_L$ and $\widehat{\mathsf{T}}_L$ are obviously the same.

3.5 Dimensions of representations

Let us first consider the number of partitions $A_m^{(\ell)}$ of m points with ℓ marked and distinguishable blocks. It is given by

$$|A_m^{(\ell)}| \;=\; m!\,[z^m]\left((\mathrm{e}^z - 1)^\ell \exp\left(\mathrm{e}^z - 1\right)\right) \,, \tag{3.23}$$

as is easily seen by elementary manipulations of the egf of the Bell numbers (the case $\ell = 0$). Indeed, we are interested in the particular case $m = k+1$. Using (3.11), and the fact that the Stirling numbers of the second kind $\left\{ k \atop \ell \right\}$ (or Stirling subset numbers) [17] have the following egf [15]

$$\left\{ k \atop \ell \right\} \;=\; \frac{k!}{\ell!}\,[z^k]\,(\mathrm{e}^z - 1)^\ell \,, \tag{3.24}$$

we can derive the following closed form for $|A_m^{(\ell)}|$:

$$|A_m^{(\ell)}| \;=\; \ell! \sum_{p=0}^{m} \binom{m}{p} \left\{ p \atop \ell \right\} B_{m-p} \tag{3.25a}$$

$$\;=\; \ell! \sum_{s=0}^{m-\ell} \binom{\ell+s}{\ell} \left\{ m \atop \ell+s \right\} \tag{3.25b}$$

where we have gone from (3.25a) to (3.25b) by using the well-known expression of the Bell numbers in terms of the Stirling subset numbers [15]

$$B_n \;=\; \sum_{s=0}^{n} \left\{ n \atop s \right\}, \tag{3.26}$$

and using Eq. (6.28) of Ref. [17], valid for integers $p, n, m \geq 0$:

$$\left\{ n \atop p+m \right\} \binom{p+m}{p} \;=\; \sum_k \left\{ k \atop p \right\} \left\{ n-k \atop m \right\} \binom{n}{k} . \tag{3.27}$$

It is clear from (3.25) that $\ell! \mid |A_m^{(\ell)}|$, $|A_m^{(m)}| = m!$, and $|A_m^{(0)}| = B_m$.

Meanwhile, the sum of the eigenvalue amplitudes for a given ℓ and all possible Young diagrams λ is given by (3.15)–(3.17):

$$\beta_\ell \;=\; \ell! \sum_{i=0}^{\ell} \frac{(-1)^i}{i!} \binom{Q}{\ell - i} . \tag{3.28}$$

This is indeed a polynomial in Q; it can be rewritten in terms of the falling factorials $Q^{\underline{i}} = \prod_{j=1}^{i}(Q+1-j)$ [17] as:

$$\beta_\ell \;=\; \sum_{i=0}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} Q^{\underline{i}} \,. \tag{3.29}$$

The values we need in this paper are:

$$\begin{aligned}
\beta_0 &= 1 && (3.30a)\\
\beta_1 &= Q-1 && (3.30b)\\
\beta_2 &= Q^2-3Q+1 && (3.30c)\\
\beta_3 &= Q^3-6Q^2+8Q-1 && (3.30d)\\
\beta_4 &= Q^4-10Q^3+29Q^2-24Q+1 && (3.30e)\\
\beta_5 &= Q^5-15Q^4+75Q^3-145Q^2+89Q-1 && (3.30f)\\
\beta_6 &= Q^6-21Q^5+160Q^4-545Q^3+814Q^2-415Q+1 && (3.30g)\\
\beta_7 &= Q^7-28Q^6+301Q^5-1575Q^4+4179Q^3-5243Q^2+2372Q-1 && (3.30h)
\end{aligned}$$

Just as in the case of the planar partition algebra [see Ref. [26], in particular Eqs. (2.16) and (2.20)] the compatibility between the dimensions $|A_m^{(\ell)}|$ and the amplitudes β_ℓ can be expressed in the form of a sumrule:

$$\sum_{\ell=0}^{m} \frac{1}{\ell!}\, \beta_\ell \, |A_m^{(\ell)}| \;=\; Q^m \,. \tag{3.31}$$

This expresses that the number of degrees of freedom per vertex of the graph is indeed Q, as expected. We also have the sumrule

$$B_m^{(0)} \;=\; \sum_{\ell=0}^{m} \frac{1}{\ell!} \, |A_m^{(\ell)}| \;=\; \sum_{r=0}^{m} 2^r \left\{ {m \atop r} \right\} , \tag{3.32}$$

where the integers $B_m^{(0)}$ form the sequence `A001861` of [39]. Their egf is

$$\sum_{m=0}^{\infty} \frac{B_m^{(0)} z^m}{m!} \;=\; \exp\big(2(\mathrm{e}^z - 1)\big) \,, \tag{3.33}$$

as can be deduced from (3.23).

For fixed m and ℓ, we now introduce the "no un-marked singletons" constraint. We then obtain a smaller set of partitions $\widetilde{A}_m^{(\ell)}$. We are interested in the cardinality of the set of partitions $\widetilde{A}_m^{(\ell)}$ with $m = k+1$, denoted by $\widetilde{N}_k(\ell) = |\widetilde{A}_{k+1}^{(\ell)}|$.

The number of partitions of m points with ℓ marked and distinguishable blocks satisfying the "no un-marked singletons" constraint is

$$|\widetilde{A}_m^{(\ell)}| \;=\; m! \, [z^m] \, \Big((\mathrm{e}^z - 1)^\ell \exp\left(\mathrm{e}^z - 1 - z\right) \Big) \,. \tag{3.34}$$

The key point is that the number of partitions with no singletons is given by the egf $\exp(\mathrm{e}^z - 1 - z)$ [15, p. 111]. The numbers associated with this egf are given by

$$S_n \;=\; n!\,[z^n]\exp(\mathrm{e}^z - 1 - z) \;=\; (-1)^n \sum_{q=0}^{n} \binom{n}{q} (-1)^q\, B_q\,, \tag{3.35}$$

where the B_n are the Bell numbers (3.11). Then, a closed form for the numbers $|\widetilde{A}_m^{(\ell)}|$ reads:

$$|\widetilde{A}_m^{(\ell)}| \;=\; \ell! \sum_{p=0}^{m} \binom{m}{p} \left\{ {p \atop \ell} \right\} S_{m-p}\,. \tag{3.36}$$

This formula implies that $\ell! \mid |\widetilde{A}_m^{(\ell)}|$, and that $|\widetilde{A}_m^{(m)}| = m!$.

The sumrule corresponding to (3.31) now reads

$$\sum_{\ell=0}^{m} \frac{1}{\ell!}\, \beta_\ell\, |\widetilde{A}_m^{(\ell)}| \;=\; (Q-1)^m\,. \tag{3.37}$$

The fact that Q has been replaced by $Q-1$ is a manifestation of the "nowhere zero" constraint. We also have the sumrule corresponding to (3.32)

$$\widetilde{B}_m^{(0)} \;=\; \sum_{\ell=0}^{m} \frac{1}{\ell!}\, |\widetilde{A}_m^{(\ell)}| \;=\; \sum_{r=0}^{m} \binom{m}{r} B_r^{(0)} (-1)^{m-r}\,, \tag{3.38}$$

where the integers $B_m^{(0)}$ are given by (3.32). Their egf is

$$\sum_{m=0}^{\infty} \frac{\widetilde{B}_m^{(0)} z^m}{m!} \;=\; \exp\left(2(\mathrm{e}^z - 1) - z\right)\,. \tag{3.39}$$

Notice that the difference $|A_m^{(\ell)}| - |\widetilde{A}_m^{(\ell)}|$ gives the number of partitions of the set $\{1, 2, \ldots, m\}$ with ℓ marked points and with at least one un-marked singleton. These partitions do not contribute to the final result, as they are associated to null eigenvalues.

Remark. For simplicity, as $L = k+1$, we will consider hereafter the bottom-row (resp. top-row) points labelled as $\{0, 1, \ldots, k\}$ (resp. $\{0', 1', \ldots, k'\}$). The monoid A_L will contain the partitions of the set $\{0, 1, \ldots, k, 0', 1', \ldots, k'\}$.

4 Flow polynomial for the generalised Petersen graphs

4.1 General theory

Let us start with the simplest case $k = 1$. The graph $G(n,1)$ is isomorphic to a cyclic ladder of width 2. From the known Potts-model partition function [12, and references therein], one can easily derive

$$\Phi_{G(n,1)}(Q) \;=\; (Q^2 - 3Q + 1)(-1)^n + (Q-1)(Q-3)^n + (Q-2)^n\,, \tag{4.1}$$

where the eigenvalues $\mu = -1, Q-3, Q-2$ correspond to the sectors with $\ell = 2, 1, 0$ links, respectively.

Let us now focus on $k \geq 2$. The flow polynomial (3.22) for the generalised Petersen graph $G(nk, k)$ can be written using (3.14) as

$$\Phi_{G(nk,k)}(Q) \;=\; \sum_{\ell=0}^{k+1} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \, \mathrm{tr}_{\ell,\lambda} (\widehat{\mathsf{T}}_{k+1})^n \,, \tag{4.2}$$

where the amplitudes $\alpha_{\ell,\lambda}$ are given by (3.15). This formula is the most general one. In terms of the non-zero eigenvalues $\mu_{k,\ell,\lambda,s}$ of the transfer matrix $\widehat{\mathsf{T}}_{k+1}$, it reads:

$$\Phi_{G(nk,k)}(Q) \;=\; \sum_{\ell=0}^{k+1} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \sum_{s=1}^{\widetilde{N}_k(\ell,\lambda)} \mu^n_{k,\ell,\lambda,s} \,, \tag{4.3}$$

where $\widetilde{N}_k(\ell,\lambda)$ is given by

$$\widetilde{N}_k(\ell,\lambda) \;=\; \widetilde{N}_k(\ell) \, \frac{\dim \lambda}{\ell!} \tag{4.4}$$

(see (B.7) in the proof of Lemma B.2), and $\widetilde{N}_k(\ell) = |\widetilde{A}^{(\ell)}_{k+1}|$ [cf. (3.34)/(3.36)].

The flow polynomial $\Phi_{G(nk,k)}$ is obtained in (4.2) as a linear combination of ordinary matrix traces with definite coefficients $\alpha_{\ell,\lambda}$ given by (3.15). This is *all* that we need to compute $\Phi_{G(nk,k)}$ rigorously from the various diagonal blocks of $\widehat{\mathsf{T}}_L$. It is worth stressing that Eq. (4.3) holds true irrespective of whether some eigenvalues happen to be identical or not.

To simplify the notation, we will denote by $\widehat{\mathsf{T}}_{k+1,\ell}$ the diagonal block of the full transfer matrix $\widehat{\mathsf{T}}_{k+1}$ corresponding to partitions with exactly ℓ links. We will denote by $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ the diagonal block of $\widehat{\mathsf{T}}_{k+1,\ell}$ corresponding to the irreducible representation λ of the group S_ℓ. Similar notation will be used for the diagonal blocks of the transfer matrix $\widetilde{\mathsf{T}}_{k+1}$. Then, Eq. (4.2) can be rewritten as:

$$\Phi_{G(nk,k)}(Q) \;=\; \sum_{\ell=0}^{k+1} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \, \mathrm{tr} (\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n \,, \tag{4.5}$$

where tr is an ordinary trace. The dimension of the matrix $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ is given by (4.4).

We have symbolically computed *all* blocks $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ for $1 \leq k \leq 7$, $0 \leq \ell \leq k+1$, and all representations $\lambda \in S_\ell$. Therefore, we can compute the *exact* flow polynomial $\Phi_{G(nk,k)}$ for $1 \leq k \leq 7$ by using Eq. (4.5), as all the elements involved are exactly known.

The dimension of the blocks for $k = 6, 7$ is in some cases very large, and the computation of the traces in (4.5) is very memory– and CPU-consuming even for modest values of n. For instance, the block for $k = 7$, $\ell = 3$, and $\lambda = (2, 1)$ has dimension 14 364.

Fortunately, the blocks $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ have for any $k \geq 1$, any $1 \leq \ell \leq k+1$, and any $\lambda \in S_\ell$ an additional internal structure. This is given by the following lemma proved in Appendix A:

Lemma 4.1 *Fix $k \geq 1$. Then for any $0 \leq \ell \leq k+1$, and any irreducible representation $\lambda \in S_\ell$, the diagonal block $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ can be written as an upper-block-triangular matrix when the basis vectors are ordered appropriately:*

$$\widehat{\mathsf{T}}_{k+1,\ell,\lambda} \;=\; \left(\begin{array}{cc} \widehat{\mathsf{D}}_{k+1,\ell,\lambda} & \widehat{\mathsf{S}}_{k+1,\ell,\lambda} \\ 0 & \widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda} \end{array} \right) , \tag{4.6}$$

where $\widehat{\mathsf{D}}_{k+1,\ell,\lambda}$ is a diagonal matrix with all its diagonal elements equal to $\mu_{k,k+1} = (-1)^k$, whose rows and columns are indexed by partitions of $\{0,1,\ldots,k\}$ with no un-marked singletons, ℓ marked clusters, and vertex 0 is a marked singleton.

Remarks. 1. The eigenvalue $\mu_{k,k+1} = (-1)^k$ will be called "trivial" in the following. The non-trivial eigenvalues come from the blocks $\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda}$ [hence the superscript "(nt)"].

2. It follows from the description of $\widehat{\mathsf{D}}_{k+1,\ell,\lambda}$ that for $\ell = 0$ the block $\widehat{\mathsf{D}}_{k+1,0}$ has zero dimension (i.e., all eigenvalues coming from the sector $\ell = 0$ are non-trivial), and that for $\ell = k+1$ the non-trivial block $\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,k+1,\lambda}$ has zero dimension (i.e., all eigenvalues coming from the sector $\ell = k+1$ are trivial).

3. The traces $\operatorname{tr}(\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n$ are trivially written as:

$$\operatorname{tr}(\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n \;=\; (-1)^{kn} \dim \widehat{\mathsf{D}}_{k+1,\ell,\lambda} \;+\; \operatorname{tr}(\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda})^n \,. \tag{4.7}$$

In this way we significantly reduce the burden of the computation. For instance, the non-trivial block for $k = 7$, $\ell = 3$, and $\lambda = (2,1)$ has dimension 11 816, compared to 14 364 for the whole matrix $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$.

4. There is of course a similar upper-block-triangular decomposition for the matrix $\widetilde{\mathsf{T}}_{k+1,\ell,\lambda}$. The trivial eigenvalues are given in this case by $(-1)^k\,(-1)^k Q^{2k} = Q^{2k}$ [cf., (3.21)].

5. In Appendix B, we shall prove an explicit formula for $\dim \widehat{\mathsf{D}}_{k+1,\ell,\lambda}$ for arbitrary k,ℓ,λ (Lemma B.1). But we stress that this result plays no role in our proof of Theorems 1.7 and 1.8, since for $1 \leq k \leq 7$ we have determined these dimensions by explicit computation.

As described in Section 4.2 below, we have *exactly* computed *all* non-trivial blocks $\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda}$, as well as the dimensions $\dim \widehat{\mathsf{D}}_{k+1,\ell,\lambda}$ for $1 \leq k \leq 7$, $0 \leq \ell \leq k+1$, and $\lambda \in S_\ell$. Then, we have computed the flow polynomials $\Phi_{G(nk,k)}$ for $1 \leq k \leq 7$ and selected values of $n \geq 1$ by using (4.5)/(4.7). In particular, Eq. (4.5) and Lemma 4.1 are the *essential* elements in our method to compute the flow polynomials on large graphs $G(nk,k)$, establishing in particular Theorem 1.8(a). The practical implementation of this method is explained in detail in the next section.

Now we will focus on those results that we need to prove Theorems 1.7 and 1.8(b). Both theorems are based on the Beraha–Kahane–Weiss theorem (see Section 5), which concerns analytic functions of the form (5.1). In particular, we should show that the eigenvalues $\mu_{k+1,\ell,\lambda,s}$ [cf., (4.3)] coming from the transfer matrices $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ satisfy all the hypotheses of this theorem. First we need to prove that the amplitudes $\alpha_{\ell,\lambda}$ [cf., (3.15)] and the eigenvalues $\mu_{k+1,\ell,\lambda,s}$ are analytic functions of Q in some domain D of the complex Q-plane. Indeed, the amplitudes are polynomials in Q, hence analytic functions of Q in the whole complex plane. The eigenvalues $\mu_{k+1,\ell,\lambda,s}$ are algebraic functions of Q; they are thus analytic in the whole complex Q-plane, except at the branch cuts. So we can choose D to be any connected open set of $\mathbb{C}$ not containing a branch cut.

In addition, there is a "no-degenerate-dominance" condition requiring that there must not exist two eigenvalues μ_i and μ_j with $i \neq j$ [the labels i, j are just a shorthand for our indices $k+1, \ell, \lambda, s$] such that 1) $\mu_i \equiv e^{i\delta}\mu_j$ for some real constant δ, and 2) the region $D_i \subseteq D$ where these two eigenvalues dominate (i.e., $|\mu_i| = |\mu_j| \geq |\mu_k|$ for all k) has a nonempty interior. The first step to check that this condition holds is to find out if there are equal eigenvalues among the $\mu_{k+1,\ell,\lambda,s}$. If two or more eigenvalues $\mu_{i_1}, \ldots, \mu_{i_p}$ are *exactly* equal *for all* Q, then we can amalgamate them into a single term μ_i of the sum (5.1), and absorb the multiplicity into the corresponding amplitude α_i. So each family of *equal* eigenvalues can be considered to be a single eigenvalue for the purpose of checking the "no-degenerate-dominance" condition.

We have already proved that the trivial eigenvalue $\mu_{k,k+1} = (-1)^k$ appears in all blocks $\widehat{T}_{k+1,\ell,\lambda}$ with $\ell \geq 1$. Thus, we may include this eigenvalue only once in the sum (5.1) arising in the Beraha–Kahane–Weiss theorem, and absorb the multiplicity in the amplitude $\alpha_{k,k+1}$. We now consider the non-trivial blocks $\widehat{T}^{(\mathrm{nt})}_{k+1,\ell,\lambda}$. The exact symbolic computation of these blocks for $1 \leq k \leq 7$ reveals that there is an exact degeneracy of eigenvalues when $\ell = k$. This is the content of the following lemma proved in Appendix A:

Lemma 4.2 *Fix $k \geq 1$ and $\ell = k$. Then, for all irreducible representations $\lambda \in S_\ell$, there are k eigenvalues $\mu_{k,k,s}$ in the non-trivial diagonal block $\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,k,\lambda}$. Each of these eigenvalues $\mu_{k,k,s}$ has multiplicity* $\dim\lambda$.

Thus, we may include each eigenvalue $\mu_{k,k,s}$ only once in the sum (5.1), and absorb the multiplicity in the amplitude $\alpha_{k,k,s}$.

Finally, we have to check that, for each fixed value of $k \in \{1, 2, \ldots, 7\}$, the eigenvalues $\mu_{k,\ell,\lambda,s}$ (for $0 \leq \ell \leq k-1$), $\mu_{k,k,s}$ (for $\ell = k$) and $\mu_{k,k+1} = (-1)^k$ satisfy the "no-degenerate-dominance" condition. This can be achieved by numerically computing the values of *all* these eigenvalues at a generic value of Q, and finding that there are no two eigenvalues with the same absolute value. In our case, we choose $Q = \pi + i\sqrt{3}$. Therefore we conclude that:

Lemma 4.3 *Fix $k \in \{1, 2, \ldots, 7\}$. Then the non-trivial eigenvalues $\mu_{k,\ell,\lambda,s}$ (for $0 \leq \ell \leq k-1$) and $\mu_{k,k,s}$ (for $\ell = k$), and the trivial eigenvalue $\mu_{k,k+1} = (-1)^k$ satisfy:*

1. *For every Q except perhaps a finite set, the eigenvalues are all distinct.*

2. *The "no-degenerate-dominance" condition holds.*

Remark. We do not know how to prove the extension of this result to $k \geq 8$, since our proof for $1 \leq k \leq 7$ is by explicit computation. We nevertheless conjecture that Lemma 4.3 holds true for all $k \geq 1$.

Lemmas 4.2 and 4.3 are essential for proving that our eigenvalues satisfy the hypotheses of the Beraha–Kahane–Weiss theorem. This theorem is the starting point for proving Theorems 1.7 and 1.8(b).

4.2 Practical procedure

We have written a PERL script to compute the *symbolic* transfer matrix T_{k+1} using ideas similar to those already explained in [23, 24]. For $1 \le k \le 4$, we have checked our programs using MATHEMATICA. Further checks were performed with code written in C that allows us to numerically compute the leading eigenvalue for given values of k, ℓ, and $\lambda = (\ell) =$ the completely symmetric irreducible representation of S_ℓ.

The first step is to obtain the relevant diagonal blocks $\widetilde{\mathsf{T}}_{k+1,\ell,\lambda}$ of the transfer matrix $\widetilde{\mathsf{T}}_{k+1}$. We first fix the value of ℓ ($0 \le \ell \le k+1$) and a bottom-row configuration compatible with the chosen value of ℓ and the "no un-marked singletons" condition.

Remark. Our choice for the bottom-row partition is the simplest one. We take the partition $\{\{0, 1, \ldots, k\}\}$ for $\ell = 0$, the partition $\{\{\overline{0}, \overline{1}, \ldots, \overline{k}\}\}$ for $\ell = 1$, the partition $\{\{\overline{0}, \overline{1}, \ldots, \overline{k-1}\}, \{\overline{k}\}\}$ for $\ell = 2, \ldots$ and the partition $\{\{\overline{0}\}, \{\overline{1}\}, \ldots, \{\overline{k}\}\}$ for $\ell = k+1$. The overline over a site means that this site (and the block it belongs to) is connected to a block of the top-row partition by a link.

We then determine the basis of the relevant partition space of dimensionality $\widetilde{N}_k(\ell)$ [cf. (3.36)]. Indeed, the result does not depend on the chosen bottom-row partition.

Remark. This statement is true if we explicitly mark ℓ blocks of the bottom-row partition, and leave un marked the rest of the blocks (if any). If we instead chose not to do this marking, we would arrive at a basis of dimensionality $p\widetilde{N}_k(\ell)$ for some integer $p \ge 2$. This would happen e.g., for $k = 3$ and $\ell = 1$, if we chose the bottom-row partition $\{\{0, 1\}, \{2, 3\}\}$ without explicitly saying which block is marked. Therefore, for a given top-row connectivity, e.g., $\{\{0', 1', 2'\}, \{\overline{3'}\}\}$, there would correspond *two* distinct partitions of the full set $\{0, \ldots, 3, 0', \ldots, 3'\}$: namely, $(0', 1', 2')(3', 0, 1)(2, 3)$ and $(0', 1', 2')(0, 1)(3', 2, 3)$. Each eigenvalue of the transfer matrix would then be repeated $p = 2$ times. This extra factor comes obviously from the two ways we can mark one of the two blocks in $\{\{0, 1\}, \{2, 3\}\}$. We stress that with the choice of the preceding remark this problem can never occur.

We now choose an irreducible representation λ of the symmetric group S_ℓ of dimensionality $\dim \lambda$. To obtain the relevant diagonal block $\widetilde{\mathsf{T}}_{k+1,\ell,\lambda}$ corresponding to λ, we simply take as our basis vectors those linear combinations of the "standard" basis vectors with the appropriate properties under S_ℓ. Finally, in order to "extract" the trivial eigenvalues Q^{2k} for $\ell \ge 1$, we exploit the structure of $\widetilde{\mathsf{T}}_{k+1,\ell,\lambda}$ given by Lemma 4.1. We thus obtain the nontrivial block $\widetilde{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda}$ and the dimension $\dim \widetilde{\mathsf{D}}_{k+1,\ell,\lambda}$. We then compute the powers $(\widetilde{T}^{(nt)}_{k+1,\ell,\lambda})^n$ and their traces, for the desired values of n.

For small values of k, this computation can be performed symbolically for any not-too-large value of n, using a symbolic algebraic manipulator program, such as MATHEMATICA. However, for larger values of k (say, $k = 6, 7$) this is not feasible, as we have blocks of dimension as large as 11 816 (for $k = 7$, $\ell = 3$, and $\lambda = (2, 1)$), and the symbolic computation of the powers of such large blocks is extremely time– and memory–consuming, beyond our current computer capabilities.

A key issue in the subsequent analysis is the dependence of the traces $\mathrm{tr}(\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda})^n$ on Q, n, and k. The needed information is given by the following lemma (the proof can be found in Appendix A):

Lemma 4.4 *Let $k \geq 1$, $0 \leq \ell \leq k+1$, and λ be an irreducible representation of S_ℓ. Then, for each $n \geq 0$, $\mathrm{tr}(\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda})^n$ [cf. (4.6)] is a polynomial in Q of degree at most $n[k+\min(1-\ell,0)]$.*

Our computation then made use of the following tricks:

- By Lemma 4.4, we know that the traces $\mathrm{tr}(\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda})^n$ are polynomials in Q of degree at most $d = n[k + \min(1 - \ell, 0)]$. Therefore, it suffices to compute the *evaluation* of each trace at $d + 1$ integer values of $Q \neq 0$, and then reconstruct the corresponding polynomial using Lagrange's interpolation method. In order to check the result, we always compute at least $d + 2$ values of the trace. Please note that we compute the evaluation of the trace $\mathrm{tr}(\widehat{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda})^n$ by first computing the evaluation of the trace $\mathrm{tr}(\widetilde{\mathsf{T}}^{(\mathrm{nt})}_{k+1,\ell,\lambda})^n$, and then multiplying the result by the factor $(-1)^{nk}Q^{-2nk}$ [cf. (3.21)]. This is why we take $Q \neq 0$.

- *Not* all the entries of the transfer matrix $\widetilde{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda}$ are integers for integer values of Q; rather they are rational numbers. As we want to perform the trace computation with (infinite-precision) integer arithmetic, for each value of Q we multiplied the matrix by the minimum (positive) integer value such that all entries are integers. After the computation is done, we reconstructed the true solution by dividing by the appropriate factor.

- The integers involved in the actual calculations are very large. Therefore, we compute the value of the trace for a given value of $Q \neq 0$ using modular arithmetic for a given set of prime numbers $p \leq 65\,521 \lesssim 2^{16}$ (we need up to 65 different primes). We then reconstruct the value using the Chinese remainder theorem using infinite-precision arithmetic in MATHEMATICA. We always use at least one more prime than needed, in order to check the result. To accelerate the computation of the trace using modular arithmetic, we use a program written in C.

- For $1 \leq k \leq 6$, we are able to compute the traces for many different values of n. However, for $k = 7$, the computation is so demanding, that we have focused on powers of the type $n = 2^q + 1$ with integer $q \geq 1$. The reason why we consider odd powers for $k = 7$ will become clear in Section 6.

Once the traces are computed, we can form the flow polynomial using (4.5). Notice that for $1 \leq k \leq 7$, everything in this formula is exactly known. The zeros of the flow polynomials are then obtained using the program MPSOLVE [7,8]. This software has the advantage that if one requests the zeros with 50–digit precision (as in our case), the results are guaranteed to have at least such precision.

Remark. To give a clear idea of what has been achieved, consider the case $k = 7$, and more specifically the computation of Φ_G for the graph $G = G(17k, k)$, which is the largest computation undertaken in this work. Applying naively (3.9) within the diagrammatic basis would imply computing the 17th power of T_{k+1}, which according to (3.10) is a matrix of dimension $B_{2(k+1)} = 10\,480\,142\,147$ whose entries are polynomials in Q of degree at most $2k = 14$. The decomposition of T_{k+1} and use of the "no un-marked singleton" constraint has reduced the computation to the sum over 31 blocks, the largest of which has dimension 11 816. Even with these tricks, the computation took around six months calendar time, using 50–80 processors, corresponding to some 30 years of CPU time.

k	Maple	HPR Tutte [18]	BJ Tutte [2]
1	$1 \le n \le 15$	$1 \le n \le 50$	$1 \le n \le 50$
2	$1 \le n \le 7$	$1 \le n \le 25$	$1 \le n \le 25$
3	$1 \le n \le 5$	$1 \le n \le 11$	$1 \le n \le 12$
4	$1 \le n \le 4$	$1 \le n \le 8$	$1 \le n \le 7$
5	$1 \le n \le 3$	$1 \le n \le 6$	$1 \le n \le 5$
6	$1 \le n \le 2$	$1 \le n \le 5$	$1 \le n \le 4$
7	$1 \le n \le 2$	$1 \le n \le 4$	$1 \le n \le 4$

Table 1: Tests performed on our transfer-matrix computations of the flow polynomial for the generalised Petersen graphs $G(kn, k)$. For each value of k in the interval $1 \le k \le 7$, we show the values of n for which we have computed $\Phi_{G(kn,k)}$ using a) Maple (second column), b) the Tutte code developed by Haggard, Pierce, and Royle [18] (third column), and c) the Tutte code developed by Bedini and Jacobsen [2] (fourth column). In all cases, the agreement between these computations and our transfer-matrix results is perfect.

4.3 Additional checks

Because our results are derived using software, we have performed some tests in order to ensure that the results are correct. First of all, for the smallest members of each family $G(nk, k)$, we have computed the flow polynomial using three different software programs: Maple, the program Tutte developed by Haggard, Pierce and Royle [18], and the program Tutte developed by Bedini and Jacobsen [2]. The pairs (k, n) for which the checks have been performed are shown in Table 1. In all cases, the agreement with our transfer-matrix computations is perfect.

For the cubic graphs $G(m, k)$ that we are considering, we may improve on (2.12) by adding a few more terms:

$$\begin{aligned} \Phi_{G(m,k)}(Q) \;&=\; Q^{|E|-|V|+1} - |E| Q^{|E|-|V|} + \left(\frac{|E|(|E|-1)}{2} - |V| \right) Q^{|E|-|V|-1} \\ &- \;\left(\frac{|E|(|E|-1)(|E|-2)}{6} - |V|(|E|-2) \right) Q^{|E|-|V|-2} + \ldots . \end{aligned} \tag{4.8}$$

In this expression, the coefficient of $Q^{|E|-|V|-1}$ arises from two contributions in which $E \setminus E'$ is respectively two edges, and three edges all incident on the same vertex. The contributions to the coefficient of $Q^{|E|-|V|-2}$ are slightly more complicated to characterise. Inserting $|V| = 2m$ and $|E| = 3m$ we obtain

$$\Phi_{G(m,k)}(Q) \;=\; Q^{m+1} \left(1 - \frac{3m}{Q} + \frac{m}{2} \frac{9m-7}{Q^2} - \frac{m}{2} \frac{(3m-2)(3m-5)}{Q^3} + \ldots \right) . \tag{4.9}$$

We have checked that for all the graphs $G(nk, k)$ we have considered, the flow polynomials obtained from the procedure outlined above indeed satisfy (4.9).

There are some theorems that give us some information about the location of the real zeros of the flow polynomial. The first theorem applies to a general bridgeless graph, while the second one applies only to cubic graphs (i.e., it is valid for $G(nk, k)$ with $n > 2$):

Theorem 4.5 (Wakelin [47]; see also Refs. [14, 22]) *Let G be a bridgeless graph with $|V|$ vertices, $|E|$ edges, b blocks, and no isolated vertices. Then:*

- *$\Phi_G(Q)$ is non-zero with sign $(-1)^{|E|-|V|+1}$ for $Q \in (-\infty, 1)$.*
- *$\Phi_G(Q)$ has a zero of multiplicity b at $Q = 1$.*
- *$\Phi_G(Q)$ is non-zero with sign $(-1)^{|E|-|V|+b+1}$ for $Q \in (1, \frac{32}{27}]$.*

Theorem 4.6 (Jackson [20, 21]) *Let G be a 3–connected cubic graph with $|V|$ vertices and $|E|$ edges. Then:*

- *$\Phi_G(Q)$ is non-zero with sign $(-1)^{|E|-|V|}$ for $Q \in (1, 2)$.*
- *$\Phi_G(Q)$ has a zero of multiplicity 1 at $Q = 2$.*
- *$\Phi_G(Q)$ is non-zero with sign $(-1)^{|E|-|V|+1}$ for $Q \in (2, \delta)$, where $\delta \approx 2.546$ is the flow root of the cube in the interval $(2, 3)$ [i.e., the zero in this interval of $Q^3-9Q^2+29Q-32$].*

The generalised Petersen graphs $G(nk, k)$ satisfy $|V| = 2nk$, $|E| = 3nk$, and $b = 1$. Therefore, the above theorems imply that, for any $n > 2$, $\Phi_G(Q)$ has only two simple real roots in the interval $(-\infty, \delta)$, namely $Q = 1$ and $Q = 2$. The sign for $Q \in (-\infty, 1) \cup (2, \delta)$ is that of $(-1)^{nk+1}$, and it has the opposite sign for $Q \in (1, 2)$. For large enough $Q > 0$, the sign of $\Phi_G(Q)$ is always positive. Therefore, for even k and any $n \geq 3$, or for odd k and even $n \geq 4$, this implies the existence of a real zero in $[\delta, \infty)$.

Remark. The lower bound is sharp, as the cube is isomorphic to the generalised Petersen graph $G(4, 3)$, which has a zero at $Q = \delta$.

We have explicitly checked that all the computed flow polynomials $\Phi_{G(nk,k)}$ have only two simple roots in the whole interval $(-\infty, \delta)$, namely $Q = 1, 2$. Furthermore, for all even (resp. odd) k, and all $n \geq 3$ (resp. all even $n \geq 4$), the polynomial $\Phi_{G(nk,k)}$ has at least one root in $[\delta, \infty)$.

If G is a cubic graph, one can easily see whether $Q = 3$ is a flow root or not [13, Proposition 6.4.2]:

Theorem 4.7 *A bridgeless cubic graph G has a nowhere zero 3–flow if and only if it is bipartite.*

As the graphs $G(nk, k)$ are bipartite if and only if k is odd and n is even, Theorem 4.7 implies that $\Phi_{G(nk,k)}(Q)$ has at least one factor $Q - 3$ whenever $G(nk, k)$ is not bipartite: namely, when k is even or n is odd (or both). When $G(nk, k)$ is bipartite, then $\Phi_{G(nk,k)}(3) > 0$. We have checked these facts in all the flow polynomials we have explicitly computed in this work.

5 The Beraha–Kahane–Weiss theorem

A central role in the subsequent analysis is played by a theorem on analytic functions due to Beraha, Kahane and Weiss (BKW) [3–6] and generalised slightly by Sokal [40]. The situation is as follows: let D be a domain (connected open set) in the complex plane, and let $\alpha_1, \ldots, \alpha_M, \mu_1, \ldots, \mu_M$ ($M \geq 2$) be analytic functions on D, none of which is identically zero. For each integer $n \geq 0$, define

$$f_n(z) \;=\; \sum_{k=1}^{M} \alpha_k(z)\, \mu_k(z)^n \;. \tag{5.1}$$

We are interested in the zero sets

$$\mathcal{Z}(f_n) \;=\; \{z \in D\colon\; f_n(z) = 0\} \tag{5.2}$$

and in particular in their limit sets as $n \to \infty$:

$$\liminf \mathcal{Z}(f_n) \;=\; \{z \in D\colon\; \text{every neighbourhood } U \ni z \text{ has a nonempty intersection with all but finitely many of the sets } \mathcal{Z}(f_n)\} \tag{5.3}$$

$$\limsup \mathcal{Z}(f_n) \;=\; \{z \in D\colon\; \text{every neighbourhood } U \ni z \text{ has a nonempty intersection with infinitely many of the sets } \mathcal{Z}(f_n)\} \tag{5.4}$$

Let us call an index k *dominant at* z if $|\mu_k(z)| \geq |\mu_l(z)|$ for all l ($1 \leq l \leq M$); and let us write

$$D_k \;=\; \{z \in D\colon\; k \text{ is dominant at } z\} \;. \tag{5.5}$$

Then the limiting zero sets can be completely characterised as follows:

Theorem 5.1 (Beraha–Kahane–Weiss [3–6, 40]) *Let D be a domain in $\mathbb{C}$, and let α_1, …, α_M, $\mu_1, \ldots, \mu_M$ ($M \geq 2$) be analytic functions on D, none of which is identically zero. Let us further assume a "no-degenerate-dominance" condition: there do not exist indices $k \neq k'$ such that $\mu_k \equiv \omega \mu_{k'}$ for some constant ω with $|\omega| = 1$ and such that D_k ($= D_{k'}$) has nonempty interior. For each integer $n \geq 0$, define f_n by*

$$f_n(z) \;=\; \sum_{k=1}^{M} \alpha_k(z)\, \mu_k(z)^n \;.$$

Then $\liminf \mathcal{Z}(f_n) = \limsup \mathcal{Z}(f_n)$, *and a point z lies in this set if and only if either:*

(a) There is a unique dominant index k at z, and $\alpha_k(z) = 0$; or

(b) There are two or more dominant indices at z.

Note that case (a) consists of isolated points in D, while case (b) consists of curves (plus possibly isolated points where all the μ_k vanish simultaneously). Henceforth we shall denote by $\mathcal{B}$ the locus of points satisfying condition (b).

We shall often refer to the functions μ_k as "eigenvalues", and to the α_k as "amplitudes", because that is exactly how they arise in the transfer matrix formalism.

In Ref. [6, p. 55], Beraha, Kahane, and Weiss give (without proof) the following corollary, concerning the convergence of *real* roots of f_n to *real isolated* limiting points, based on their proof of Theorem 5.1:

Corollary 5.2 *Assume the hypotheses of Theorem 5.1. Let z_0 be a real isolated limiting point, and suppose that the functions f_n, the dominant eigenvalue $\mu_\star$ and its coefficient $\alpha_\star$ are all real in an interval $(z_0 - \epsilon, z_0 + \epsilon)$ for some $\epsilon > 0$ [with of course $\alpha_\star(z_0) = 0$] and suppose further than $\alpha'_\star(z_0) \neq 0$. Then z is the limit of a real sequence $\{z_n\}$, defined for all sufficiently large n, for which $f_n(z_n) = 0$.*

The proof is simple: for any sufficiently small $\epsilon > 0$, $\alpha_\star(z_0 \pm \epsilon)$ are non-zero of opposite sign, and $\mu_\star(z_0 \pm \epsilon)$ is still dominant. Then, for all sufficiently large n (depending on ϵ), $f_n(z_0 \pm \epsilon)$ are real of opposite sign. Therefore there exists a root in-between.

In the next section we need some general results on the existence of a real sequence of zeros $\{z_n\}$ such that it converges to a real *non-isolated* limiting point. These results can be summarised in the following

Lemma 5.3 *Assume the hypotheses of Theorem 5.1. Let us suppose that z_0 is a real non-isolated limiting point, such that exactly two dominant eigenvalues μ_1 and μ_2 become equimodular at z_0. Let us further suppose that:*

(a) The two dominant eigenvalues are analytic functions in a neighbourhood of z_0.

(b) The eigenvalue μ_1 (resp. μ_2) is dominant (resp. subdominant) in the interval $[z_0-\epsilon, z_0)$, and is subdominant (resp. dominant) in the interval $(z_0, z_0 + \epsilon]$, for some $\epsilon > 0$.

(c) The corresponding amplitudes α_1 and α_2 do not vanish at z_0.

(d) The eigenvalues μ_1, μ_2 and the amplitudes α_1, α_2 are real in a real neighbourhood of z_0.

Then:

1. *If $\alpha_1(z_0)\alpha_2(z_0) > 0$ and $\mu_1(z_0) = -\mu_2(z_0)$, then for odd n, there is a sequence of real zeros $\{z_n\}$ converging to z_0.*

2. *If $\alpha_1(z)\alpha_2(z) < 0$ and $\mu_1(z_0) = -\mu_2(z_0)$, then for even n, there is a sequence of real zeros $\{z_n\}$ converging to z_0.*

3. *If $\alpha_1(z_0)\alpha_2(z_0) < 0$ and $\mu_1(z_0) = \mu_2(z_0)$, then for all n, there is a sequence of real zeros $\{z_n\}$ converging to z_0.*

Remark. When we say that "there is a sequence of real zeros $\{z_n\}$ converging to z_0", we mean that for all $\epsilon > 0$ there exists $n_0 = n_0(\epsilon) < \infty$ such that for *all* $n \geq n_0(\epsilon)$ [or all odd or all even n, as the case may be] there is a zero z_n of f_n satisfying $|z_n - z_0| < \epsilon$.

Proof of Lemma 5.3. The function f_n defined by (5.1) can be written as

$$f_n(z) \;=\; \alpha_1(z)\mu_1(z)^n + \alpha_2(z)\mu_2(z)^n + \sum_{j=3}^{N} \alpha_j(z)\mu_j(z)^n \,, \tag{5.6}$$

where the $N-2$ other eigenvalues are subdominant in a neighbourhood of z_0. Then, for every sufficiently small $\epsilon > 0$ we can find real numbers $0 < r_1, r_2 < 1$ such that

$$\left| \frac{\mu_i(z_0 - \epsilon)}{\mu_1(z_0 - \epsilon)} \right| \;\leq\; r_1 \,, \tag{5.7}$$

for all $i \geq 2$, and

$$\left| \frac{\mu_i(z_0 + \epsilon)}{\mu_2(z_0 + \epsilon)} \right| \;\leq\; r_2 \,, \tag{5.8}$$

for all $i = 1$ or $i \geq 3$. We choose ϵ small enough so that the signs of the dominant eigenvalues and amplitudes are the same as at z_0. Then, we have that

$$f_n(z_0 - \epsilon) \;=\; \mu_1^n \left[\alpha_1 + \sum_{j=2}^{N} \alpha_j \left(\frac{\mu_j}{\mu_1} \right)^n \right] \tag{5.9a}$$

$$f_n(z_0 + \epsilon) \;=\; \mu_2^n \left[\alpha_2 + \alpha_1 \left(\frac{\mu_1}{\mu_2} \right)^n + \sum_{j=3}^{N} \alpha_j \left(\frac{\mu_j}{\mu_2} \right)^n \right] \tag{5.9b}$$

where in (5.9a) [resp. (5.9b)] all quantities are evaluated at $z_0 - \epsilon$ [resp. $z_0 + \epsilon$]. Then, for large enough n, the quantities $f_n(z_0 \pm \epsilon)$ have the opposite sign in the following cases:

1. $\alpha_1 \alpha_2 > 0$ and $(\mu_1/\mu_2)^n < 0$, which occurs if $\mu_1 \mu_2 < 0$ and n is odd.

2. $\alpha_1 \alpha_2 < 0$ and $(\mu_1/\mu_2)^n > 0$, which occurs if $\mu_1 \mu_2 > 0$ (then n can have either parity) or if $\mu_1 \mu_2 < 0$ and n is even.

In any of these cases, the continuous function f_n attains values of distinct signs at the endpoints of the interval $[z_0 - \epsilon, z_0 + \epsilon]$, therefore there should be a zero at some point inside this interval. ∎

Remarks. 1. In the fourth case $\alpha_1 \alpha_2 > 0$ and $\mu_1 = \mu_2$, the zeros converging to z_0 are *non-real.*

2. If the derivative of the ratio μ_1/μ_2 is nonvanishing at z_0, then condition (b) [or the same condition with μ_1 and μ_2 interchanged] necessarily holds. However, the converse is false: it is possible for condition (b) to hold even if μ_1/μ_2 has a vanishing derivative at z_0.

As remarked in Ref. [6, p. 55], the convergence rate for isolated and non-isolated limiting points is rather different: exponentially fast for the former $|z_0 - z_n| \leq A\, r^n$, and $|z_0 - z_n| \leq A\, n^{-1}$ for the latter, as $n \to \infty$.

6 Real zeros of the flow polynomials $\Phi_{G(nk,k)}$

In this section we will discuss the real zeros of the flow polynomials $\Phi_{G(nk,k)}$ for $1 \leq k \leq 7$. In particular, we will focus on the real zeros around $Q = 5$, and on the existence of real zeros $Q > 5$.

6.1 $k \leq 5$

For $1 \leq k \leq 3$, we compute the flow polynomials of $G(nk, k)$ and their roots for all n in the range $1 \leq n \leq 30$. All the flow roots we have found are smaller than $Q = 4$. We conjecture that this holds for larger n as well. For $k = 4, 5$ we find flow polynomials with real roots greater than $Q = 4$:

- $G(28, 4)$ has two real roots greater than $Q = 4$: $Q_1 \approx 4.0002086861$ and $Q_2 \approx 4.3876416603$. As n grows, the maximal real root of $\Phi_{G(4n,4)}$ tends to the value $Q_c(4) \approx 4.5697435537$. The convergence to $Q_0 = Q_c(4)$ is due to case (3) of Lemma 5.3: at this point the dominant eigenvalues come from the sectors with $\ell = 1$ and $\ell = 3$ links, and $\alpha_{1,(1)}(Q_0)\alpha_{3,(3)}(Q_0) < 0$, and $\mu_{5,1,(1)}(Q_0) = \mu_{5,3,(3)}(Q_0) \approx 11.9477$.

- $G(30, 5)$ has two real roots greater than $Q = 4$: $Q_1 \approx 4.0000786673$ and $Q_2 \approx 4.4867394006$. As n grows, the maximal real root of $\Phi_{G(5n,5)}$ tends to the value $Q_c(5) \approx 4.9029018077$. The convergence to $Q_0 = Q_c(5)$ is due to case (2) of Lemma 5.3: at this point the dominant eigenvalues come from the sectors with $\ell = 0$ and $\ell = 3$ links, and $\alpha_0(Q_0)\alpha_{3,(3)}(Q_0) < 0$, and $\mu_{5,0}(Q_0) = -\mu_{5,3,(3)}(Q_0) \approx -453.306$. Therefore, there are real zeros close to $Q_c(5)$ only for even values of n.

Both families thus provide counter-examples to Welsh's conjecture (Conjecture 1.3).

Remark. The values of $Q_c(k)$ mentioned above are obtained by determining the eigenvalue crossing that corresponds to case (b) of Theorem 5.1.

6.2 $k = 6, 7$: The isolated limiting point $Q = 5$

The family $G(6n, 6)$ provides a very strong counter-example to the Welsh conjecture (Conjecture 1.3), as it displays a real zero converging to $Q = 5$ from below; for instance, $G(144, 6)$ has a real zero at $Q \approx 4.9987003379$.

This empirical observation will be made rigorous by applying Corollary 5.2 to the family $G(6n, 6)$ at $Q = 5$. In this case, there is a unique (and therefore, real) leading eigenvalue, which corresponds to $\ell = 3$ and $\lambda = (3)$: $\mu_{3,(3),\star}(5) \approx 177.122$. The corresponding amplitude $\alpha_{3,(3)}$ is given by (3.15),

$$\alpha_{3,(3)}(Q) \;=\; \frac{1}{8} Q(Q-1)(Q-5)\,, \tag{6.1}$$

and is a polynomial in Q with a single zero at $Q = 5$. Therefore, the BKW theorem implies that $Q = 5$ is an isolated limiting point for this family. The fact that the limiting point $Q = 5$ is isolated implies that there is an interval of radius ϵ around $Q = 5$ where there are no other limiting points. Therefore, in this interval the eigenvalue $\mu_{3,(3),\star}$ is still dominant, and hence real. Therefore, Corollary 5.2 implies that there is a sequence of *real* zeros $\{Q_n\}$ converging to $Q = 5$. We can go a little bit further and show that there exists a sequence of real zeros converging to $Q = 5$ *from below*:

Corollary 6.1 *The point $Q = 5$ is an isolated limiting point for the family $G(6n, 6)$. There is a sequence of real zeros $\{Q_n\}$ of the flow polynomial $\Phi_{G(6n,6)}$ that converges to $Q = 5$ from below.*

PROOF. Let us consider the point $Q = 5-\epsilon$ with ϵ small enough so that: 1) the leading eigenvalue $\mu_{3,(3),\star}(Q) > 0$; 2) all other N sub-leading eigenvalues μ_j satisfy $|\mu_j(Q)/\mu_{3,(3),\star}(Q)| \le r < 1$; 3) the sub-leading amplitudes $|\alpha_j| \le M$ are bounded; and 4) $\alpha_{3,(3)}(Q) < 0$. We can always choose ϵ so that these conditions are fulfilled, as the eigenvalues and amplitudes are analytic functions of Q in a neighbourhood of $Q = 5$. In addition, we know that $\Phi_{G(6n,6)}(5) > 0$, due to Lemma 3.1. Therefore, we only need to show that $\Phi_{G(6n,6)}(Q) < 0$ for n large enough. This is easy as,

$$\Phi_{G(6n,6)}(Q) \;=\; \mu_{3,(3),\star}(Q)^n \left[\alpha_{3,(3)}(Q) + \sum_{j=1}^{N} \alpha_j(Q) \left(\frac{\mu_j(Q)}{\mu_{3,(3),\star}(Q)}\right)^n\right] . \tag{6.2}$$

We can always choose N_0 such that $NMr^{N_0} < |\alpha_{3,(3)}(5-\epsilon)|$. Then, for all $n \ge N_0$, the sign of $\Phi_{G(6n,6)}(Q)$ is that of $\alpha_{3,(3)}(Q)$ (i.e., negative), so that there should be a point Q_0 in $(5-\epsilon, 5)$ such that $\Phi_{G(6n,6)}(Q_0) = 0$. ∎

The same situation applies to the family $G(7n, 7)$ at $Q = 5$. In this case, the unique (and real) leading eigenvalue is $\mu_{3,(3),\star}(5) \approx -621.779$. Therefore, $Q = 5$ is an isolated limiting point for this family, and there is a *real* sequence of zeros $\{Q_n\}$ converging to $Q = 5$. Because $\mu_{3,(3),\star}(5) < 0$, the above arguments imply that the convergence of the sequence $\{Q_n\}$ to $Q = 5$ is a bit more complicated:

Corollary 6.2 *The point $Q = 5$ is an isolated limiting point for the family $G(7n, 7)$ with $n \ge 3$. There is a sequence of real zeros $\{Q_n\}$ of the flow polynomial $\Phi_{G(7n,7)}$ converging to $Q = 5$. The sub-sequence with odd n (resp. even n) converges to $Q = 5$ from above (resp. below).*

PROOF. If we consider even n, then the proof is as before; therefore the sequence $\{Q_{2p}\}_{p\in\mathbb{N}}$ converges to $Q = 5$ from below. On the contrary, for odd n, then the sign of the leading term $\alpha_{3,(3)}\mu_{3,(3),\star}(Q)^n$ is positive for $Q < 5$, and negative for $Q > 5$. Therefore, a trivial modification of the above arguments leads to the convergence of the sub-sequence $\{Q_{2p+1}\}_{p\in\mathbb{N}}$ to $Q = 5$ from above. ∎

Corollaries 6.1 and 6.2 imply Theorem 1.7. Moreover, Corollary 6.2 implies the existence of real flow roots arbitrarily close to $Q = 5$ on *both* sides (above and below). Thus, the Haggard–Pearce–Royle conjecture (Conjecture 1.6) is false, and a counterexample is given by $G(7n, 7)$ for all sufficiently large odd n.

6.3 $k = 7$: Real flow polynomial zeros larger than $Q = 5$

There are also real non-isolated limiting points for the families $G(nk, k)$ with $1 \le k \le 7$. These non-isolated limiting points correspond to "crossings" between two dominant eigenvalues, each of them coming from a block of the transfer matrix $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ with different values of ℓ. Therefore, both dominant eigenvalues are real and analytic in some real interval around the non–isolated limiting point $Q_c(k)$. In some cases, there is a corresponding sequence of real zeros $\{Q_n\}$ converging to that limiting point.

The family $G(6n,6)$ would be in principle a good candidate for having real roots larger than $Q=5$: a direct calculation shows that $Q_c(6) \approx 5.1079785012$ is a non-isolated limiting point for this family. However, the actual computation of all the members of this family up to $G(144,6)$ does not reveal any zero $Q>5$. The explanation is simple: at $Q=Q_c(6)$ both dominant eigenvalues are equal $\lambda_{1,(1),\star}(Q_c(6)) = \lambda_{3,(3),\star}(Q_c(6)) \approx 169.757$, and the corresponding amplitudes are both positive. Therefore, the hypotheses of Lemma 5.3 are not satisfied, and we cannot find *real* zeros converging to the non-isolated limiting point $Q_c(6)$. Rather, the zeros converging to $Q_c(6)$ should be non-real.

However, we have found that the family $G(7n,7)$ with *odd* n does have members with the desired property. Even though $G(63,7)$ does not have any zeros larger than $Q=5$, the next member that we have computed, $G(119,7)$, has two such zeros: $Q_1 \approx 5.0000197675$, and $Q_2 \approx 5.1653424423$. The flow polynomial for $G(119,7)$ has degree 120 and can be written as

$$\Phi_{G(119,7)}(Q) \;=\; (Q-1)(Q-2)(Q-3)P_{117}(Q)\,, \tag{6.3}$$

where $P_{117}(Q) = Q^{117} - 351Q^{116} + 61191Q^{115} - 7064107Q^{114} + \ldots$ is a polynomial in Q of degree 117. The coefficients of this polynomial are given in Appendix C. We can formalise the existence of such real roots greater than $Q=5$ in the following way:

Proposition 6.3 *Let $\Phi_G(Q)$ be the flow polynomial of the generalised Petersen graph $G = G(119,7)$. Then it has a real zero in the interval $(5+10^{-5}, 5+2\times 10^{-5})$, and another real zero in the interval $(516534\times 10^{-5}, 516535\times 10^{-5})$.*

Proof. We can evaluate the polynomial $\Phi_G(Q)$ at the two end-points of the interval $(5+10^{-5}, 5+2\times 10^{-5})$ using exact rational arithmetic and find results of distinct sign: $\Phi_G(5+10^{-5}) \approx +2.21791\times 10^{42}$, and $\Phi_G(5+2\times 10^{-5}) \approx -5.27937\times 10^{40}$. Therefore, the intermediate value theorem ensures the existence of a zero of Φ_G in the open interval $(5+10^{-5}, 5+2\times 10^{-5})$.

The same procedure can be carried out for the second interval: $\Phi_G(516534\times 10^{-5}) \approx -1.46592\times 10^{42}$, and $\Phi_G(516535\times 10^{-5}) \approx +4.53729\times 10^{42}$. ∎

Remark. As a curiosity, $\Phi_{G(119,7)}(5) = 4488918995790513676672232799446257724715600$, and $\Phi_{G(119,7)}(4) = 1133172760943853528$.

In fact, the family $G(7n,7)$ has, for every large enough odd n, two real zeros larger than $Q=5$: one converging to $Q=5$ from above (as proven in Corollary 6.2), and the other one converging to the limiting point $Q_c(7) \approx 5.2352605291$ from below. The complete statement of the second part is given by the following result:

Proposition 6.4 *The point $Q_c(7) \in (5235260\times 10^{-6}, 5235261\times 10^{-6})$ is a non-isolated limiting point for the family $G(7n,7)$. There is a sequence of real zeros $\{Q_n\}$ of the flow polynomial $\Phi_{G(7n,7)}$ converging to $Q_c(7)$ from below for odd n.*

Proof. In the interval $[5235260\times 10^{-6}, 5235261\times 10^{-6}]$, we only find two dominant eigenvalues $\lambda_{0,\star}$ and $\lambda_{3,(3),\star}$, each of them coming from a different ℓ sector. Therefore, both eigenvalues and their sum $\lambda_{0,\star} + \lambda_{3,(3),\star}$ are analytic functions of Q in this interval. If we evaluate $\lambda_{0,\star} + \lambda_{3,(3),\star}$ at the two end-points of this interval using Mathematica, we find results of distinct sign: $\lambda_{0,\star}(5235260 \times 10^{-6}) + \lambda_{3,(3),\star}(5235260 \times 10^{-6}) \approx 0.000917$, and $\lambda_{0,\star}(5235261 \times 10^{-6}) + \lambda_{3,(3),\star}(5235261 \times 10^{-6}) \approx -0.000817$. Therefore, there exists an intermediate value $Q_c(7)$ in the open interval $(5235260 \times 10^{-6}, 5235261 \times 10^{-6})$ such that $\lambda_{0,\star}(Q_c(7)) + \lambda_{3,(3),\star}(Q_c(7)) = 0$. Exactly at $Q = Q_c(7)$, these two eigenvalues have opposite signs $\lambda_{0,\star}(Q_c(7)) = -\lambda_{3,(3),\star}(Q_c(7)) \approx -565.833$. The corresponding amplitudes are both positive for any $Q > 5$. Therefore, according to case (1) of Lemma 5.3 we find real zeros converging to $Q_c(7)$ only for large enough odd values of n.

The fact that the convergence is from below comes from the fact that $\alpha_{0,\star}(Q_c(7)) - \alpha_{3,(3),\star}(Q_c(7)) > 0$, so the sign of the flow polynomial at $Q_c(7)$ is positive $\Phi_{G(7n,7)}(Q_c(7)) \approx [\alpha_{0,\star}(Q_c(7)) - \alpha_{3,(3),\star}(Q_c(7))]\lambda_{0,\star}(Q_c(7))^n > 0$. However, its sign at $Q_c(7) - \epsilon$ (for small enough values of $\epsilon > 0$) $\Phi_{G(7n,7)}(Q_c(7) - \epsilon) \approx \alpha_{3,(3),\star}(Q_c(7))\lambda_{3,(3),\star}(Q_c(7))^n < 0$ for odd n. Therefore, there should be a root in-between. ∎

Propositions 6.3 and 6.4 imply Theorem 1.8.

A Proofs of Lemmas 4.1, 4.2, and 4.4

In this appendix we will provide the proofs of the lemmas that are essential for proving the main Theorems 1.7 and 1.8.

It is useful to rewrite the transfer matrix T_L [cf., (3.9)] as the product of two operators $\mathsf{T}_L = \mathsf{H}\,\mathsf{V}$, given by

$$\mathsf{H} \;=\; \prod_{i=k}^{1} \mathsf{H}_{0i}\mathsf{V}_0 \;=\; \mathsf{H}_{01}\mathsf{V}_0\mathsf{H}_{02}\mathsf{V}_0\mathsf{H}_{03}\cdots\mathsf{V}_0\mathsf{H}_{0k}\mathsf{V}_0 \tag{A.1a}$$

$$\mathsf{V} \;=\; \prod_{i=1}^{k} \mathsf{V}_i \;=\; \mathsf{V}_1\mathsf{V}_2\cdots\mathsf{V}_{k-1}\mathsf{V}_k \tag{A.1b}$$

where $L = k+1$, the operators V_i and H_{0i} are defined in (3.8) with $v = -Q$, and we have used the property $[\mathsf{V}_i, \mathsf{V}_j] = [\mathsf{D}_i, \mathsf{D}_j] = 0$ for all i, j.

In this appendix, we will work with a basis consisting on partitions of the top-row $\{0, 1, \ldots, k\}$, where we have omitted the primes for simplicity. An overline over a site means that this site (and the block it belongs to) is connected to a block of the bottom-row partition by a link.

Proof of Lemma 4.1.

Let us consider first the transfer matrix $\widetilde{\mathsf{T}}_L = \mathsf{P}_L\mathsf{T}_L$ with $\mathsf{T}_L = \mathsf{H}\,\mathsf{V}$ given by (A.1), and the projector P_L defined in (3.18). Let us fix ℓ $(0 \le \ell \le k+1)$ and a representation $\lambda \in S_\ell$ of dimensionality $\dim\lambda$. The set $\widetilde{A}_L^{(\ell)}$ can be split into two disjoint sets:

- $\mathcal{C}_1^\ell$ is the set of all partitions of $\{0, 1, \ldots, k\}$ with no un-marked singletons, ℓ marked clusters, and vertex 0 is a marked singleton.

- $\mathcal{C}_2^\ell$ is the set of all partitions of $\{0, 1, \ldots, k\}$ with no un-marked singletons, ℓ marked clusters, and vertex 0 is not a singleton. In this case, the vertex 0 can be either marked or un-marked.

It is clear that $|\widetilde{A}_L^{(\ell)}| = |\mathcal{C}_1^\ell| + |\mathcal{C}_2^\ell|$. Notice that we do not need to consider the case of vertex 0 being an un-marked singleton, since by definition partitions of this type do not belong to $\widetilde{A}_L^{(\ell)}$.

In order to find out the structure of the transfer matrix $\widetilde{\mathsf{T}}_L$, we do not need to specify the representation λ; therefore, we will omit this index (as well as the index k) to make the notation clearer.

Let us first consider a partition $\mathcal{P}_1 \in \mathcal{C}_1^\ell$ represented by the vector $\boldsymbol{e}_{\mathcal{P}_1}$ in the space of partitions. Then,

$$\mathsf{V}\, \boldsymbol{e}_{\mathcal{P}_1} \;=\; (-Q)^k\, \boldsymbol{e}_{\mathcal{P}_1} + \sum_{\mathcal{P}\in\widetilde{C}_1^\ell} A_{\mathcal{P}}\, \boldsymbol{e}_{\mathcal{P}}\,, \tag{A.2}$$

where the sum is over the set of all partitions $\widetilde{C}_1^\ell$ of $\{0,1,\ldots,k\}$ with ℓ marked blocks, with vertex 0 a marked singleton, and such that there is at least one un-marked singleton. This is because V only contains the identity operator and detach operators D_i with $1 \le i \le k$.

If we consider a partition $\mathcal{P}_2 \in \mathcal{C}_2^\ell$, then

$$\mathsf{V}\, \boldsymbol{e}_{\mathcal{P}_2} \;=\; (-Q)^k\, \boldsymbol{e}_{\mathcal{P}_2} + \sum_{\mathcal{P}\in\widetilde{C}_1^\ell} A_{\mathcal{P}}\, \boldsymbol{e}_{\mathcal{P}} + \sum_{\mathcal{P}\in\widetilde{C}_2^\ell} B_{\mathcal{P}}\, \boldsymbol{e}_{\mathcal{P}} \;+ \sum_{\mathcal{P}\in C_3^\ell} D_{\mathcal{P}}\, \boldsymbol{e}_{\mathcal{P}}\,, \tag{A.3}$$

where

- $\widetilde{C}_2^\ell$ is the set of all partitions of $\{0,1,\ldots,k\}$ with ℓ marked blocks, with vertex 0 not a singleton, and such that there is at least one un-marked singleton.
- C_3^ℓ is the set of all partitions with 0 an un-marked singleton. But the corresponding terms in (A.3) will all be annihilated by the application of the first operator V_0 in H (A.1a). We can therefore disregard those terms in what follows.

We now notice that the operator $\mathsf{H}_{0i}\mathsf{V}_0$ can be written as:

$$\mathsf{H}_{0i}\mathsf{V}_0 \;=\; -Q\, I + \mathsf{D}_0 + Q^2\, \mathsf{J}_{0i} - Q\, \mathsf{J}_{0i}\, \mathsf{D}_0 \tag{A.4}$$

where J_{0i} and D_0 are the join and detach operators, respectively. It is not hard to see that for all partitions $\mathcal{P} \in \widetilde{A}_L^{(\ell)}$:

$$\mathsf{H}_{0i}\mathsf{V}_0\boldsymbol{e}_{\mathcal{P}} \;=\; -Q\boldsymbol{e}_{\mathcal{P}} + \sum_{\mathcal{P}'\in\mathcal{C}_2^\ell} A_{\mathcal{P}'}\, \boldsymbol{e}_{\mathcal{P}'}\,, \tag{A.5}$$

where if $\mathcal{P} \in \mathcal{C}_2^\ell$, $\mathcal{P}'$ might coincide with $\mathcal{P}$. For partitions with at least one un-marked singleton $\mathcal{P} \in \widetilde{\mathcal{C}}_1^\ell \cup \widetilde{\mathcal{C}}_2^\ell$, we have

$$\mathsf{H}_{0i}\mathsf{V}_0\boldsymbol{e}_{\mathcal{P}} \;=\; -Q\boldsymbol{e}_{\mathcal{P}} + \sum_{\mathcal{P}'\in\mathcal{C}_2^\ell} A_{\mathcal{P}'}\, \boldsymbol{e}_{\mathcal{P}'} + \sum_{\mathcal{P}'\in\widetilde{\mathcal{C}}_2^\ell} B_{\mathcal{P}'}\, \boldsymbol{e}_{\mathcal{P}'}\,. \tag{A.6}$$

Notice that when we apply the full operator H, the partitions belonging to $\widetilde{\mathcal{C}}_1^\ell \cup \widetilde{\mathcal{C}}_2^\ell$ are eliminated by the application of the operator P_L (3.18).

We now put together the above observations, and conclude that if $\mathcal{P} \in \mathcal{C}_1^\ell$, then

$$\widetilde{\mathsf{T}}_L\boldsymbol{e}_{\mathcal{P}} \;=\; (-Q)^{2k}\boldsymbol{e}_{\mathcal{P}} + \sum_{\mathcal{P}'\in\mathcal{C}_2^\ell} A_{\mathcal{P}'}\, \boldsymbol{e}_{\mathcal{P}'}\,. \tag{A.7}$$

On the other hand, if $\mathcal{P} \in \mathcal{C}_2^\ell$, then

$$\widetilde{\mathsf{T}}_L \boldsymbol{e}_{\mathcal{P}} \;=\; \sum_{\mathcal{P}' \in \mathcal{C}_2^\ell} A_{\mathcal{P}'}\, \boldsymbol{e}_{\mathcal{P}'}\,. \tag{A.8}$$

This means that, if we order the partition states appropriately, the transfer matrix $\widetilde{\mathsf{T}}_L$ has a block triangular form (4.6). This result holds for all representations $\lambda \in S_\ell$.

The block corresponding to the partitions in $\mathcal{C}_1^\ell$ is diagonal with all eigenvalues equal to $(-Q)^{2k} = Q^{2k}$. In terms of the matrix $\widehat{T}_L$ (3.21), this common eigenvalue takes the value $\mu_{k,k+1} = Q^{-2k}\,(-1)^k Q^{2k} = (-1)^k$. $\blacksquare$

Remarks. 1. The case $\ell = 0$ is special, as $|\mathcal{C}_1^0| = 0$; therefore, the diagonal block does not exist.

2. The case $\ell = k+1$ is also special, as $|\mathcal{C}_2^{k+1}| = 0$; therefore, the block $\widehat{T}_{k+1,k+1,\lambda}$ is a diagonal matrix with all its elements equal to the trivial eigenvalue $\mu_{k,k+1}$.

Proof of Lemma 4.2.

Let us first consider the following top-row partition $\{\{\overline{0},\overline{1}\},\{\overline{2}\},\ldots,\{\overline{k}\}\}$ with $\ell = k$ links. This partition does not lead to the trivial eigenvalue $\mu_{k,k+1} = (-1)^k$, as it belongs to the class $\mathcal{C}_2^k$ defined in the proof of Lemma 4.1 above. Then the action of the transfer matrix T_{k+1} [cf., (A.1)] on this partition generates exactly k partitions: $\{\{\overline{0},\overline{1}\},\{\overline{2}\},\ldots,\{\overline{k}\}\}$, $\{\{\overline{1}\},\{\overline{0},\overline{2}\},\ldots,\{\overline{k}\}\}$, …, $\{\{\overline{1}\},\{\overline{2}\},\ldots,\{\overline{0},\overline{k}\}\}$. It is important to note that the ordering of the k links is preserved in the whole process. (Loosely speaking, there is "no room" to switch two links if we have $k+1$ sites and k links.) This also means that the action of the transfer matrix T_{k+1} on any of these partitions is independent of the actual ordering of the links. Hence, for each of the $k!$ possible orderings of the k links, the transfer matrix will be the same (modulo some reordering of the partitions), and will have dimension k. Therefore, all irreducible representations of the symmetric group S_k will give the same k non-trivial eigenvalues, and the multiplicity of each of these eigenvalues in $\widehat{\mathsf{T}}_{k+1,k,\lambda}$ is just the dimension of the representation λ. $\blacksquare$

Proof of Lemma 4.4.

Let us consider the full matrices $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ and $\widetilde{\mathsf{T}}_{k+1,\ell,\lambda}$; we will deal with their non-trivial blocks $\widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda}$ and $\widetilde{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda}$ at the end.

All the matrix elements of $\widetilde{\mathsf{T}}_{k+1,\ell,\lambda}$ are polynomials in Q. Therefore, $\operatorname{tr}(\widetilde{\mathsf{T}}_{k+1,\ell,\lambda})^n$ will be a polynomial in Q. However, not all the matrix elements of $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ are polynomials in Q; so we cannot conclude that $\operatorname{tr}(\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n$ is a polynomial in Q.

We need a different argument. Let us consider any partition $\mathcal{P}$ belonging to $\widetilde{\mathcal{A}}^{(\ell)}_{k+1}$. Then, its contribution to $\operatorname{tr}(\widetilde{\mathsf{T}}_{k+1,\ell,\lambda})^n$ will be the sum of the contributions of all the diagrams that start with the partition $\mathcal{P}$ and end with the same partition after n steps. By inspection of the form of the transfer matrix T_{k+1} (3.9) and its components V_i and H_{0i} (3.8), it is clear that there is a contribution proportional to $(-Q)^{2kn}$ due to the application of all the identity operators in T_{k+1} (for each layer, there are $3k$ of them, and only $2k$ take the factor $-Q$). Indeed, this is the minimum number of $-Q$ factors one can possibly obtain for a diagram of this type. Therefore, the minimum power of Q that appears in $\mathrm{tr}(\widetilde{\mathsf{T}}_{k+1,\ell,\lambda})^n$ is Q^{2kn}, which is exactly the inverse of the n-th power of the prefactor Q^{-2k} in the definition of $\widehat{\mathsf{T}}_{k+1}$ (3.20)/(3.21). In conclusion, $\mathrm{tr}(\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n$ is indeed a polynomial in Q.

Once the polynomial character of $\mathrm{tr}(\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n$ is established, we have to take care of the degree of this polynomial. The lemma is proved if we are able to show that all the entries in $\widetilde{\mathsf{T}}_{k+1,\ell,\lambda}$ have powers of Q of degree at most $3k + \min(1-\ell, 0)$. This would imply, that $\mathrm{tr}(\widetilde{\mathsf{T}}_{k+1,\ell,\lambda})^n$ is a polynomial in Q of degree at most $n[3k+\min(1-\ell,0)]$, and that $\mathrm{tr}(\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n$ is a polynomial in Q of degree at most $n[k + \min(1-\ell, 0)]$, as claimed.

Let us first consider the case $\ell = 0$. Let us start with an arbitrary partition of the top row $\mathcal{P} \in \widetilde{\mathcal{A}}^{(0)}_{k+1}$. When we apply $\widetilde{\mathsf{T}}_{k+1}$ to that partition, the maximum number of $-Q$ factors that can appear is $(-Q)^{3k}$: we apply the part $-Q\mathbf{1}$ for each vertical operator V_i, and the join operator $-Q\mathsf{J}_{0i}$ for each horizontal operator H_{0i}. We obtain for all $\mathcal{P} \in \widetilde{\mathcal{A}}^{(0)}_{k+1}$ the same final partition: $\{\{0, 1, \ldots, k\}\}$. Therefore, $\mathrm{tr}(\widetilde{\mathsf{T}}_{k+1,0,\lambda})^n$ is a polynomial in Q of degree at most $3kn$.

The case $\ell = 1$ is similar. If we start with an arbitrary partition of the top row $\mathcal{P} \in \widetilde{\mathcal{A}}^{(1)}_{k+1}$ with 0 marked, and we apply the same operators as above, we end up with the partition $\{\{\overline{0}, \overline{1}, \ldots, \overline{k}\}\}$. The coefficient corresponding to this partition is a polynomial in Q of degree at most $3k$. Therefore, $\mathrm{tr}(\widetilde{\mathsf{T}}_{k+1,1,\lambda})^n$ is a polynomial in Q of degree at most $3kn$.

The case $2 \le \ell \le k+1$ is similar. Let us start with the following simple partition in $\widetilde{\mathcal{A}}^{(\ell)}_{k+1}$: $\{\{\overline{0}, \overline{\ell}, \ldots, \overline{k}\}, \{\overline{1}\}, \ldots, \{\overline{\ell-1}\}\}$. The argument is similar to the previous cases, except that we cannot join the site 0 to any of the other $\ell - 1$ blocks; since if we did, then the number of links would be smaller than ℓ. Therefore, we get a diagonal entry which is a polynomial in Q of degree at most $3k - (\ell - 1)$. Therefore, $\mathrm{tr}(\widetilde{\mathsf{T}}_{k+1,\ell,\lambda})^n$ is a polynomial in Q of degree at most $n[3k - (\ell - 1)]$.

Finally, Eq. (4.7) implies that $\mathrm{tr}(\widehat{\mathsf{T}}_{k+1,\ell,\lambda})^n$ and $\mathrm{tr}(\widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda})^n$ differ by a term of order Q^0. Therefore, $\mathrm{tr}(\widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda})^n$ is a polynomial in Q of degree at most $n[k + \min(1-\ell, 0)]$. ■

B More structural properties of the transfer matrix

Lemma 4.1 proved that the relevant diagonal blocks $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ have an upper-block-triangular structure (4.6). One interesting question is to know the dimension of the diagonal block $\widehat{\mathsf{D}}_{k+1,\ell,\lambda}$. This is the content of the following lemma:

Lemma B.1 *Fix $k \ge 1$, and $\ell = 0, \ldots, k+1$. Then, the dimension of the trivial diagonal block $\widehat{\mathsf{D}}_{k+1,\ell,\lambda}$ can be written as*

$$\dim \widehat{\mathsf{D}}_{k+1,\ell,\lambda} \;=\; \widetilde{N}_{k,1}(\ell) \dim \lambda \,, \tag{B.1}$$

where $\widetilde{N}_{k,1}(\ell)$ is given by

$$\widetilde{N}_{k,1}(\ell) \;=\; \begin{cases} 0 & \text{if } \ell = 0 \\[2mm] \dfrac{1}{(\ell-1)!}\, |\widetilde{A}^{(\ell-1)}_k| \;=\; \displaystyle\sum_{p=0}^{k} \binom{k}{p} \left\{ {p \atop \ell-1} \right\} S_{k-p} & \text{if } 1 \le \ell \le k+1 \end{cases} \tag{B.2}$$

and S_n is given in (3.35).

PROOF. As we have seen in the first remark after the proof of Lemma 4.1, $|\mathcal{C}_1^0| = 0$, so $\dim \widehat{\mathsf{D}}_{k+1,0,\lambda} = 0$. Thus, let us assume that $1 \leq \ell \leq k+1$. The dimension of the diagonal block $\widehat{\mathsf{D}}_{k+1,\ell,\lambda}$ should be equal to the number of partitions of the set $\{1, 2, \ldots, k\}$ with no un-marked singletons, $\ell - 1$ marked blocks, and corresponding to the representation $\lambda \in S_\ell$ of dimensionality $\dim \lambda$. But now, as we are considering linear combinations of the partitions with the right symmetries under S_ℓ, these blocks should be considered indistinguishable. Therefore,

$$\dim \widehat{\mathsf{D}}_{k+1,\ell,\lambda} \;=\; \frac{1}{(\ell-1)!}\, |\widetilde{A}_k^{(\ell-1)}| \, \dim \lambda \, . \tag{B.3}$$

The total number of trivial eigenvalues in $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ divided by $\dim \lambda$ is

$$\widetilde{N}_{k,1}(\ell) \;=\; \frac{1}{(\ell-1)!}\, |\widetilde{A}_k^{(\ell-1)}| \, . \tag{B.4}$$

Using (3.36) and after some algebra, it is not difficult to find the expression (B.2). ∎

Remark. Indeed, the dimensions of the matrices $\widehat{\mathsf{D}}_{k+1,\ell,\lambda}$ exactly found by a computer-assisted proof in Section 4.1 do coincide with the analytic formula above.

The next step is to compute the dimensionality of the non-trivial block $\widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda}$.

Lemma B.2 *Fix $k \geq 1$, $\ell \in \{0, 1, \ldots, k+1\}$, and the representation $\lambda \in S_\ell$. Then the dimension of the non-trivial diagonal block $\widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda}$*

$$\dim \widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda} \;=\; \widetilde{N}_{k,0}(\ell,\lambda) \, , \tag{B.5}$$

is given by

$$\widetilde{N}_{k,0}(\ell,\lambda) \;=\; \begin{cases} S_{k+1} + \displaystyle\sum_{p=0}^{k} \binom{k+1}{p+1} S_{k-p} & \text{if } \ell = 0 \\ \dim \lambda \displaystyle\sum_{p=1}^{k} \binom{k}{p} \left\{ {p \atop \ell} \right\} [S_{k-p} + S_{k+1-p}] & \text{if } 1 \leq \ell \leq k-1 \\ k \dim \lambda & \text{if } \ell = k \\ 0 & \text{if } \ell = k+1 \end{cases} \tag{B.6}$$

and S_n is given by (3.35).

PROOF. Let us first fix ℓ ($0 \leq \ell \leq k+1$) and the irreducible representation $\lambda \in S_\ell$ of dimensionality $\dim \lambda$. The dimension of the corresponding diagonal block $\widehat{\mathsf{T}}_{k+1,\ell,\lambda}$ is

$$\dim \widehat{\mathsf{T}}_{k+1,\ell,\lambda} \;=\; \frac{\widetilde{N}_k(\ell)}{\ell!} \, \dim \lambda \, . \tag{B.7}$$

The dimension of the trivial block $\mathsf{D}_{k+1,\ell,\lambda}$ is $\widetilde{N}_{k,1}(\ell)\dim\lambda$ by Lemma B.1. Therefore, the number of non-trivial eigenvalues in this block is given by

$$\dim \widehat{\mathsf{T}}^{(nt)}_{k+1,\ell,\lambda} \;=\; \widetilde{N}_{k,0}(\ell,\lambda) \;=\; \left[\frac{\widetilde{N}_k(\ell)}{\ell!} \;-\; \widetilde{N}_{k,1}(\ell)\right] \dim\lambda\,. \tag{B.8}$$

We can provide a closed form for $\widetilde{N}_{k,0}(\ell,\lambda)$ for $0\le \ell\le k-1$ by combining Lemma B.1, the definition $\widetilde{N}_k(\ell) = |\widetilde{A}^{(\ell)}_{k+1}|$ (3.36), and (B.8). After some algebra we find Eq. (B.6) for $\ell \le k-1$.

The case $\ell = k$ is derived directly from Lemma 4.2. There are k distinct non-trivial eigenvalues for every irreducible representation λ of S_k, each of them with multiplicity $\dim\lambda$. Therefore, $\widetilde{N}_{k,0}(k,\lambda) = k\,\dim\lambda$.

Finally, for $\ell = k+1$, the number of non-trivial eigenvalues is zero, as all eigenvalues are trivial in this sector (see the second remark after the proof of Lemma 4.1). ∎

Lemmas 4.1 and 4.2 imply that the flow polynomial (4.3) for the generalised Petersen graph $G(nk,k)$ is given by the following "complete" decomposition for any $k\ge 1$:

$$\Phi_{G(nk,k)}(Q) \;=\; \sum_{\ell=0}^{k-1}\sum_{\lambda\in S_\ell}\alpha_{\ell,\lambda}\sum_{s=1}^{\widetilde{N}_{k,0}(\ell,\lambda)}\mu^n_{k,\ell,\lambda,s} + \beta_k\sum_{s=1}^{k}\mu^n_{k,k,s} + \gamma_{k+1}(-1)^{nk} \tag{B.9}$$

where β_ℓ is given in (3.17), and γ_{k+1} is given by

$$\gamma_{k+1} \;=\; \beta_{k+1} + \sum_{\ell=1}^{k}\sum_{\lambda\in S_\ell}\alpha_{\ell,\lambda}\widetilde{N}_{k,1}(\ell)\dim\lambda \;=\; \beta_{k+1} + \sum_{\ell=1}^{k}\beta_\ell\widetilde{N}_{k,1}(\ell)\,. \tag{B.10}$$

Remark. Lemma 4.3 implies that all eigenvalues appearing in (B.9) are distinct for $1\le k\le 7$. We conjecture that all eigenvalues in (B.9) are also distinct for each $k\ge 8$.

We can compute the coefficients γ_{k+1} by using (B.10), the expressions (3.30) for the amplitudes β_ℓ, and the values (B.2) of $\widetilde{N}_{k,1}(\ell)$. The results for $1\le k\le 7$ are:

$$\gamma_2 \;=\; Q^2 - 3Q^2 + 1 \tag{B.11a}$$
$$\gamma_3 \;=\; Q^3 - 5Q^2 + 6Q - 1 \tag{B.11b}$$
$$\gamma_4 \;=\; Q^4 - 7Q^3 + 15Q^2 - 11Q + 1 \tag{B.11c}$$
$$\gamma_5 \;=\; Q^5 - 9Q^4 + 28Q^3 - 38Q^2 + 20Q - 1 \tag{B.11d}$$
$$\gamma_6 \;=\; Q^6 - 11Q^5 + 45Q^4 - 90Q^3 + 90Q^2 - 27Q + 1 \tag{B.11e}$$
$$\gamma_7 \;=\; Q^7 - 13Q^6 + 66Q^5 - 175Q^4 + 260Q^3 - 207Q^2 + 70Q - 1 \tag{B.11f}$$
$$\gamma_8 \;=\; Q^8 - 15Q^7 + 91Q^6 - 301Q^5 + 595Q^4 - 707Q^3 + 469Q^2 - 135Q + 1 \tag{B.11g}$$

For instance, for $k=3$, we have that $\gamma_4 = \beta_1 + 4\beta_2 + 3\beta_3 + \beta_4$.

For $1\le k\le 7$, the total number of *distinct* eigenvalues is given by

$$\widetilde{D}_k \;=\; 1 + k + \sum_{\ell=0}^{k-1}\sum_{\lambda\in S_\ell}\widetilde{N}_{k,0}(\ell,\lambda)\,, \tag{B.12}$$

where the $\widetilde{N}_{k,0}(\ell,\lambda)$ are given in (B.6). The values are $\widetilde{D}_k = 3, 7, 36, 229, 1658, 12803, 105934$ for $k = 1,\ldots,7$.

C The polynomial $\Phi_{G(119,7)}$

i	a_i
0	240453758183717079931230416441214627161100181583221695778758847017660
1	3778010581676303383947166862404894626185168386864045862610150115812052
2	30335569899732630785396756910315613411372095140384361997526113182374163
3	165825101972051263346220423069919998872239305113596582973635632055394977
4	693579540994844285783536577772823633624035569377766733899721814879025261
5	2365298420910039361031441041999881902182643546761840393901756834507719194
6	6843881667041711337123509683076504016112680686224789087846006938716709936
7	17263457862032800683223779137831846913199242114984324936441639648268053289
8	38713433713214705146001095156944656719927970414458008688062316276391488388
9	78323139898934533680532026518046626525971702819240021820160355852376101900
10	144596924706446634397013884950643553134413962554815223156161387400333508225
11	245805490880525223783583677086377232647690995050227537846281841251646914824
12	387578029394911704914155641300482722338098395075782035706404804765273096365
13	570251442350241862197997781738305884386349966454124613787648726411773264858
14	786830079084979854704322218255681568053263515605219068314256695784834361687
15	1022415665287852311682199189937274383849816815595817077282491287814210089327
16	1255617405561053292049115611630960519206807161888631329237623843787650902598
17	1461844034312439449575494091690068905133069052914279852509149780627087441657
18	1617740821249798765950531657277432254427341252461560806920840933023592630423
19	1705621288663026987807981694789591864355698512752538498113371227662898599836
20	1716721077841948079359918580031477807370750494656542267873960267539031216693
21	1652471749735377739204603329938965183162646296909313884512941374410264086521
22	1523600861689858982867164652432211354402872413669859092087649541160917763276
23	1347478648579683679991904590350043152972817068987716895096810710111624258414
24	1144543577841773436389066666439373326178625474359360165078175860151437888418
25	934744903425435172192816740334988586706047594412696933301815485640045053088
26	734761463017109267283189805699271455927373168195072916098687225728186938239
27	556406139434285183296690956661070857540502747488343675526234531927677714022
28	406248919977322871211475078592598791327840251152453917692944363153498710100
29	286204685555683178005609409552066982429730669340777755008844846916076522994
30	194690835588052864277724799113473665492499996408415866330514336755095511519

Table 2: Coefficients of the flow polynomial $\Phi_{G(119,7)}(Q) = (Q-1)(Q-2)(Q-3)\sum_{i=0}^{117}(-1)^{i+1}a_iQ^i$.

i	a_i
31	127959409173717651616971568749651845945079328780268936837379415435944126802
32	81303058361075417592304554906219204545070829443708169095378279034181616797
33	49966395799032436266189002396534646533235474102922584729978792549768709570
34	29716297381875173070101426039892101696811371497930415707024934873184994971
35	17109926011423464771467167427507814898291755276949983853151230858292134005
36	9541424411645170221820026129682497673840857753028326218052776323947917867
37	5155250259203698010398273841766642027281260863127250587163264787844562005
38	2699625968148386503782707617994694138490854643501950268752229332983496484
39	1370590172276847791788823798202187745155060493411059843973697377813429115
40	674811604481186691941181395279228274700479647745836519066846509110686871
41	322283263195622661124593258235203925484545860053525952388302407768566612
42	149338540704756141327132733140880585247255726988170348306522536601036428
43	67154254327764896250779574488797365684816293909986077788680077649219059
44	29310387712861712336349988437753297027267572645934195793910184667409525
45	12418975519744799441679142496250486367091770751393334192303593690937216
46	5108897246470415157235242805025854028977310258147102091244333174463070
47	2040790191809533248688898356534425242558953177861065156649563146370679
48	791669367678994305215119572575588439395174310193863932545018547493949
49	298263611514781343272224652335166519468558609433860386772388696144855
50	109143202315633324987330964029233262735802381917881292972386146290583
51	38793120529207946517836936596295050010421179961661603601012761033976
52	13393333822523527127943242411311403493467668611668190366289160119798
53	4491649304261905729281333948467503543032844169847443690259485449787
54	1463208911995358578670442816667443146618570037914577188051167519252
55	463002576567921398643877003491825254798116352410207026582354227639
56	142306049793794840801481088729855032253716088115496415256458286993
57	42482066138212698853729014913405235508231962094892775535623301954
58	12316937820630034457717661311813898450293797908401593794293724758
59	3468010035454592592294532401090077198723295654493559716526595085
60	948193746090263874793189178585310649056926723273402296835255341
61	251712395999861219430022121008687006528308619159639108633408792
62	64870522436154805702181162937007326080100049608985554185883024
63	16227914273399692643538894326321813038077624088578343763024693
64	3939856175099340812778271579647788724054817562072496203271622
65	928159554704604760538186427442530041159446537967362956747430
66	212130925228863323822024721431112601866613699451131860235710
67	47025221622368034908085340305586428332317539464533841648123
68	10108841512034189983877166351043431883332647922883010856327
69	2106711947665769238334508701464763203793865636828903442294
70	425522761043753110078251752857561183613194192306420512412
71	83277105914548332097899604836384635641176843712157856926
72	15786114232928385188569755800951974220820338221822797673
73	2897497349085917262770771233534175446735608061805601330
74	514765848404059046836837761417162573532059958487097353
75	88483984104925516580422896030293322509203586666446608

Table 2: (Continued.)

i	a_i
76	14709751385762293879136933265239581784434168823726105
77	2363940443321701747333111550458096052434867769928216
78	367070636910713149669510355126240632283426228583990
79	55045491030169621973045797552387375721405805002210
80	7967363593383576742789482483060195429602666353792
81	1112432655029526956212696750602542449173138786358
82	149736221917777447936668185649515326795606657139
83	19417037687233804941366229781880098811952696949
84	2423988165427613282917222688678473423362471885
85	291095664115059080227254369870423630080267269
86	33600127040413356951575245326064861734434090
87	3724448845910258224439985334986604271814630
88	396083067053161605214005007534380917557302
89	40370994515077502586895936288134103918883
90	3939422436008602219258246065351204112198
91	367585380265857178359590490307505526698
92	32755670407906919539877604731444999292
93	2783619278733903044429223967432069173
94	225251413853620678004411181186579261
95	17327620037637857872926704478452708
96	1264839938089389154908070831758552
97	87436328017906227077510426235281
98	5711550730808768130994711119064
99	351695047292333601570905687819
100	20358915651379087237820736779
101	1104604810216428455300363966
102	55981869638360027493471673
103	2640008664516116400095478
104	115338644735668374521021
105	4644751096461226418781
106	171402937526338140238
107	5756292666737489361
108	174485239351077202
109	4726362454421551
110	113000004047458
111	2347463061581
112	41510490693
113	607499109
114	7064107
115	61191
116	351
117	1

Table 2: (Continued.)

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