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# Involutions on the algebra of physical observables from reality conditions

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Some aspects of the algebraic quantization program proposed by Ashtekar are revisited in this article. It is proven that, for systems with first-class constraints, the involution introduced on the algebra of quantum operators via reality conditions can never be projected unambiguously to the algebra of physical observables, i.e., of quantum observables modulo constraints. It is nevertheless shown that, under sufficiently general assumptions, one can still induce an involution on the algebra of physical observables from reality conditions, though the involution obtained depends on the choice of particular representatives for the equivalence classes of quantum observables. © 1996 American Institute of Physics. [S0022-2488(96)03801-8]

## I. INTRODUCTION

Recently, Ashtekar *et al.*<sup>1-3</sup> have elaborated a program for the nonperturbative quantization of dynamical systems with first-class constraints. This program is specially designed to deal with the problem of quantizing general relativity, and has already been carried out successfully in a number of lower dimensional gravitational models, including minisuperspaces<sup>4-6</sup> and 2+1 gravity.<sup>1,3,7</sup> The program proposed by Ashtekar is an extension, based on the algebraic approach to quantum mechanics,<sup>8</sup> of Dirac's canonical quantization method.<sup>9</sup> One of the main novelties with respect to Dirac's procedure is the introduction of a prescription to find the inner product in the space of quantum states. This allows one to adhere to the standard probabilistic interpretation of quantum mechanics when the quantization can be achieved.

Ashtekar's program consists of a series of steps that, after completion, should provide us with a consistent quantum theory. It can be applied, in principle, to any classical system whose phase space  $\Gamma$  is a real symplectic manifold.<sup>1</sup>

One must first choose a subspace  $S$  of the vector space of smooth complex functions on  $\Gamma$ . This subspace must contain the unit function and be closed both under complex conjugation and Poisson brackets.<sup>2</sup> In addition,  $S$  has to be complete, in the sense that any sufficiently regular complex function on phase space should be expressible as a sum of products of elements in  $S$  (or as a limit of this type of sums).<sup>2</sup>

Each element  $X$  in  $S$  is to be regarded as an elementary classical variable which is unambiguously associated with an abstract operator  $\hat{X}$ . One then constructs the free associative algebra  $\mathcal{F}$  generated by these elementary quantum operators. On this algebra, one imposes the commutation relations that follow from the classical Poisson brackets, namely, if  $X, Y \in S$ , one must demand that (at least up to terms proportional to  $\hbar^2$ )

$$[\hat{X}, \hat{Y}] - i\hbar\{\widehat{X}, \widehat{Y}\} = \hat{0}. \quad (1.1)$$

If there exist algebraic relations of the form

$$f_i(X_1, \dots, X_n) = 0 \quad (i = 1, \dots, m) \quad (1.2)$$

between the elements in  $S$  (e.g., when the dimension of  $S$  is greater than that of  $\Gamma$ ), such relations have also to be imposed on the corresponding quantum operators, with a suitable choice of factor ordering, if needed.<sup>2</sup> The imposition of these commutation and algebraic relations simply amounts to taking the quotient of the free algebra  $\mathcal{F}$  by the ideal  $\mathcal{I}_{\mathcal{F}}$  generated by the left-hand sides of Eq. (1.1) and the quantum counterpart of Eq. (1.2). The algebra of operators obtained in this way will be called  $\mathcal{A}$ .

At this point one should promote the complex conjugation relations in  $S$  to an involution on  $\mathcal{A}$ . We recall that an involution  $\star$  on the algebra  $\mathcal{A}$  is a map  $\star: \mathcal{A} \rightarrow \mathcal{A}$  that satisfies

$$(\hat{X}^\star)^\star = \hat{X}, \tag{1.3}$$

$$(\hat{X} + \lambda \hat{Y})^\star = \hat{X}^\star + \bar{\lambda} \hat{Y}^\star, \quad (\hat{X} \hat{Y})^\star = \hat{Y}^\star \hat{X}^\star, \tag{1.4}$$

for all  $\hat{X}, \hat{Y} \in \mathcal{A}$  and complex numbers  $\lambda$ . Here,  $\bar{\lambda}$  is the complex conjugate to  $\lambda$ . To introduce the desired involution on  $\mathcal{A}$ , one can proceed in the following manner. For every  $X, Y \in S$  such that  $Y$  is the complex conjugate to  $X$ , define  $\hat{X}^\star = \hat{Y}$ , and use properties (1.4) to extend this definition to all the operators in the free algebra  $\mathcal{F}$ . It is straightforward to check that one then gets an involution on  $\mathcal{A}$  provided that the ideal  $\mathcal{I}_{\mathcal{F}}$  of operators which vanish modulo commutation and algebraic relations is invariant under the  $\star$ -operation, i.e., that  $\mathcal{I}_{\mathcal{F}}$  is a  $\star$ -ideal of  $\mathcal{F}$ . We will assume hereafter that this is in fact the case, and denote the resulting  $\star$ -algebra by  $\mathcal{A}^{(\star)}$ . The  $\star$ -relations in  $\mathcal{A}^{(\star)}$  are usually called reality conditions,<sup>1</sup> for they capture the complex conjugation relations between elementary classical variables.

The next step in the quantization consists in finding a faithful representation for the abstract algebra  $\mathcal{A}$  by linear operators acting on a complex vector space  $V$ . If the classical system possesses first-class constraints  $\{C_i\}$ , these constraints must now be explicitly represented by operators  $\{\hat{C}_i\}$ . In general, a choice of factor ordering, and of regularization in infinite dimensional systems,<sup>2,3</sup> are needed at this point in order to get a consistent algebra of quantum constraints,<sup>9</sup> that is, to guarantee that

$$[\hat{C}_i, \hat{C}_j] = \hat{f}_{ij}^k \hat{C}_k, \tag{1.5}$$

where  $\hat{f}_{ij}^k \in \mathcal{A}$  and we use the convention that pairs of contracted indices are summed over.

The kernel  $V_p \subset V$  of the constraints  $\{\hat{C}_i\}$  supplies the vector subspace of quantum states. One must then determine the subalgebra  $\mathcal{A}_p \subset \mathcal{A}$  of operators which leave  $V_p$  invariant. These operators commute weakly with the quantum constraints,

$$\hat{A} \in \mathcal{A}_p \Leftrightarrow [\hat{A}, \hat{C}_i] = \hat{h}_i^j \hat{C}_j \quad (\hat{h}_i^j \in \mathcal{A}). \tag{1.6}$$

Let us define now

$$\mathcal{I}_C \equiv \{\hat{X}^i \hat{C}_i; \hat{X}^i \in \mathcal{A}\}. \tag{1.7}$$

Using Eqs. (1.5) and (1.6) one can show that  $\mathcal{I}_C \subset \mathcal{A}_p$  and that,  $\forall \hat{I} \in \mathcal{I}_C$  and  $\forall \hat{A} \in \mathcal{A}_p$ , both  $\hat{A} \hat{I}$  and  $\hat{I} \hat{A}$  belong to  $\mathcal{I}_C$ , so that  $\mathcal{I}_C$  is an ideal of  $\mathcal{A}_p$ . On the other hand, if  $\hat{A} \in \mathcal{A}_p$ , all the operators of the form  $\hat{B} = \hat{A} + \hat{I}$ , with  $\hat{I} \in \mathcal{I}_C$ , have exactly the same action on quantum states, for  $V_p$  is annihilated by the quantum constraints. In order to obtain the algebra  $\mathcal{A}'_p$  of operators with a well-defined action on  $V_p$ , one should therefore take the quotient of  $\mathcal{A}_p$  by the ideal  $\mathcal{I}_C$ :<sup>2</sup>

$$\mathcal{A}'_p \equiv \mathcal{A}_p / \mathcal{I}_C. \tag{1.8}$$

The operators in  $\mathcal{A}'_p$  are the quantum physical observables of the system.<sup>10</sup>

The quantization program presented so far leaves a certain freedom in the following steps: (a) the selection of the subspace  $S$  of elementary classical variables, (b) the construction of the linear

representation for the algebra  $\mathcal{A}$  of quantum operators, and (c) the choice of factor ordering in the quantum constraints  $\{\hat{C}_j\}$ . The final result of the quantization process will depend on these inputs.<sup>2</sup> In particular, Ashtekar and Tate<sup>2</sup> assumed at this stage that, with a judicious choice of such inputs and at least for a large variety of physical systems, the involution defined on  $\mathcal{A}^{(\star)}$  would induce an involution on  $\mathcal{A}'_p$ .

It is worth remarking in this sense that the  $\star$ -relations will project unambiguously to the algebra of physical observables only if two conditions are fulfilled. On the one hand,  $\mathcal{A}'_p \subset \mathcal{A}$  must be invariant under the  $\star$ -operation:  $\forall \hat{A} \in \mathcal{A}'_p, \hat{A}^\star \in \mathcal{A}'_p$ . On the other hand, it is necessary that  $\mathcal{I}_C \subset \mathcal{A}'_p$  be a  $\star$ -ideal of  $\mathcal{A}'_p$ :  $\forall \hat{I} \in \mathcal{I}_C, \hat{I}^\star \in \mathcal{I}_C$ . When this is the case, the  $\star$ -operation provides a uniquely defined map between equivalence classes in  $\mathcal{A}'_p$  which satisfies the properties (1.3) and (1.4) of an involution. The involution induced on  $\mathcal{A}'_p$  will be denoted again by  $\star$ , and the resulting  $\star$ -algebra of physical observables by  $\mathcal{A}'_p^{(\star)}$ .

The idea suggested by Ashtekar<sup>1-3</sup> is to employ the involution on  $\mathcal{A}'_p^{(\star)}$  to select the inner product  $\langle \cdot, \cdot \rangle$  on  $V_p$  and, therefore, the Hilbert space  $\mathcal{H}$  of physical states (normalizable quantum states). More specifically, he proposed to determine the inner product on  $V_p$  by demanding that the  $\star$ -relations between physical observables are realized as adjoint relations on the Hilbert space  $\mathcal{H}$ , i.e.,

$$\langle \Psi, \hat{A}'\Phi \rangle = \langle \hat{B}'\Psi, \Phi \rangle \quad \forall \Phi, \Psi \in \mathcal{H}, \quad \forall \hat{A}', \hat{B}' = (\hat{A}')^\star \in \mathcal{A}'_p^{(\star)}. \quad (1.9)$$

Rendall showed<sup>11</sup> that this condition is such a severe restriction on the inner product that, if an admissible inner product exists, it is unique (up to a positive global factor) under very general assumptions.

This completes the quantization program put forward by Ashtekar. If this program can be carried out for a given classical system, one would arrive at a mathematically consistent quantum theory in which real physical observables would be represented by self-adjoint operators acting on a Hilbert space of physical states.

The purpose of this work is to demonstrate however that there exists an impediment to achieving one of the steps of the above quantization method. We will prove in Sec. II that the  $\star$ -relations in  $\mathcal{A}^{(\star)}$  never project unambiguously to the algebra of physical observables. This problem can be nonetheless overcome by slightly modifying Ashtekar's program, as we will show in Sec. III. The price to be paid is to allow a new freedom in the quantization process. A particular procedure to introduce an involution on  $\mathcal{A}'_p$  from reality conditions should then be adopted. The subtleties that arise in defining such an involution are illustrated in Sec. IV by considering some simple physical systems. We finally discuss the physical implications of our results and conclude in Sec. V.

## II. AMBIGUITIES IN THE REALITY CONDITIONS ON PHYSICAL OBSERVABLES

We want to prove that reality conditions (i.e., the  $\star$ -relations between quantum operators) never project unambiguously to the algebra of physical observables when there exist first-class constraints on the system. We will assume that the faithful, linear representation constructed for the algebra  $\mathcal{A}$  of quantum operators is irreducible. Otherwise, one should decompose it in irreducible components, and apply the proof to follow to each component separately.

We have seen that, in order to obtain a uniquely defined involution on physical observables from reality conditions, it is necessary that both  $\mathcal{A}'_p$  and  $\mathcal{I}_C$  be invariant under the  $\star$ -operation. In particular, we should have

$$\forall \hat{I} \in \mathcal{I}_C, \quad \hat{I}^0 \equiv \hat{I}^\star \in \mathcal{I}_C. \quad (2.1)$$

Select now in  $\mathcal{T}_C$  one of the quantum constraints, e.g.,  $\hat{C}_1$ , and consider also all the operators of the form  $\hat{I}_1 = \hat{Z}\hat{C}_1 \in \mathcal{T}_C$ , with  $\hat{Z} \in \mathcal{A}$ . Employing condition (2.1), and recalling definition (1.7), we obtain

$$\hat{C}_1^* = \hat{Y}_1^j \hat{C}_j, \quad (2.2)$$

$$(\hat{I}_1)^* = \hat{C}_1^* \hat{Z}^* = \hat{Y}_1^j \hat{C}_j \hat{Z}^* \equiv \hat{I}_1^0 = \hat{X}_1^k \hat{C}_k, \quad (2.3)$$

for some  $\hat{Y}_1^j, \hat{X}_1^k \in \mathcal{A}$ .

On the other hand, the image  $\hat{Z}^*$  of all the operators  $\hat{Z} \in \mathcal{A}$  is again the whole algebra  $\mathcal{A}$ , because the  $\star$ -operation is an involution. Relation (2.3) therefore implies that,  $\forall \hat{Z} \in \mathcal{A}$ , there exist  $\hat{X}_1^k \in \mathcal{A}$  such that

$$\hat{Y}_1^j \hat{C}_j \hat{Z} = \hat{X}_1^k \hat{C}_k. \quad (2.4)$$

This identity between operators must hold on any element of  $V$ , the vector space on which  $\mathcal{A}$  has been represented. Choosing then  $\Phi \in V_p \subset V$  with  $\Phi$  different from zero, it follows from Eq. (2.4) that,  $\forall \hat{Z} \in \mathcal{A}$ ,

$$\hat{Y}_1^j \hat{C}_j (\hat{Z}\Phi) = \hat{X}_1^k \hat{C}_k \Phi = 0, \quad (2.5)$$

for the physical state  $\Phi$  is annihilated by all quantum constraints. Besides, since the representation constructed is irreducible and  $\Phi \neq 0$ , the range of  $\hat{Z}\Phi$  ( $\forall \hat{Z} \in \mathcal{A}$ ) must be the whole vector space  $V$ . So, the above equation states that  $V$  is the kernel of the operator  $\hat{Y}_1^j \hat{C}_j$ . Being the representation for  $\mathcal{A}$  faithful, we then must have

$$\hat{Y}_1^j \hat{C}_j = \hat{0}. \quad (2.6)$$

But this is clearly inconsistent with the fact that the  $\star$ -operation is an involution, because, using Eqs. (2.2) and (2.6), we get that  $\hat{C}_1 = (\hat{C}_1^*)^* = \hat{0}$ . In this way, we conclude that, when there exist first-class constraints,  $\mathcal{T}_C$  can never be invariant under the  $\star$ -operation and, therefore, reality conditions do not project unambiguously to the algebra of physical observables.

Moreover, it is generally the case that the  $\star$ -image of  $\mathcal{T}_C$  is not even contained in  $\mathcal{A}_p$ , so that  $\mathcal{A}_p$  is not a  $\star$ -subalgebra of  $\mathcal{A}$ . This can be proved, for instance, under the assumptions that (a) Eq. (2.2) holds for a certain quantum constraint  $\hat{C}_1$  and (b) there exists at least one operator  $\hat{W} \in \mathcal{A}$  whose commutator with  $\hat{C}_1$  does not belong to  $\mathcal{A}_p$ . One can then check that, while  $\hat{W}^* \hat{Y}_1^j \hat{C}_j$  belongs to  $\mathcal{T}_C$ , its  $\star$ -conjugate, given by  $\hat{C}_1 \hat{W}$ , is not included in  $\mathcal{A}_p$ . Hence, even though one could find a representative  $\hat{A}$  for a given physical observable  $\hat{A}'$  such that  $\hat{A}^* \in \mathcal{A}_p$ , the  $\star$ -conjugates of all other operators in the equivalence class  $\hat{A}'$  (i.e., the operators  $\hat{A} + \hat{I}$ , with  $\hat{I} \in \mathcal{T}_C$ ) will in general not belong to the algebra  $\mathcal{A}_p$ .

For the sake of an example, let us consider a classical system whose phase space admits a set of global coordinates of the form  $s \equiv \{t, H, x, p\}$ , with  $t, H, x, p \in \mathbb{R}$ , and  $H$  and  $p$  the momenta canonically conjugate to  $t$  and  $x$ , respectively. Suppose, in addition, that there exists only one first-class constraint on the system, given by  $H=0$ . This extremely simple example describes, for instance, a Kantowski–Sachs model with positive cosmological constant.<sup>5</sup>

As elementary classical variables, we can choose the complex vector space spanned by  $s$  and the unity. The  $\star$ -operation on the corresponding algebra  $\mathcal{A}$  of quantum operators is defined by

$$\hat{t}^* = \hat{t}, \quad \hat{H}^* = \hat{H}, \quad (2.7)$$

$$\hat{x}^* = \hat{x}, \quad \hat{p}^* = \hat{p}, \quad \hat{1}^* = \hat{1}, \quad (2.8)$$

and the properties (1.4) of an involution. The only quantum constraint is  $\hat{H}=0$ . On the other hand, it is not difficult to prove that the equivalence classes of the operators  $\hat{1}$ ,  $\hat{x}$ , and  $\hat{p}$  form a complete set of physical observables. We point out, nevertheless, that the algebra  $\mathcal{A}_p$  is not generated by  $\{\hat{1}, \hat{x}, \hat{p}, \hat{H}\}$ , but contains also operators like  $\hat{i}\hat{H}$  and  $(\hat{i})^2\hat{H}$  which are elements in  $\mathcal{T}_C$ . From Eq. (2.8) it follows that each equivalence class of observables possesses at least a representative whose  $\star$ -conjugate belongs to  $\mathcal{A}_p$ . In general, however, the  $\star$ -image of different representatives do not coincide modulo the constraint  $\hat{H}=0$ , because the  $\star$ -operation does not leave the ideal  $\mathcal{T}_C$  invariant. Namely, from Eqs. (2.7), (2.8) and the commutator  $[\hat{i}, \hat{H}] = i\hbar\hat{1}$ , we get

$$(\hat{i}\hat{H})^\star = \hat{i}\hat{H} - i\hbar\hat{1}, \quad ((\hat{i})^2\hat{H})^\star = (\hat{i})^2\hat{H} - 2i\hbar\hat{i}, \tag{2.9}$$

so that the  $\star$ -conjugate to  $\hat{i}\hat{H}$  belongs to  $\mathcal{A}_p$ , but not to  $\mathcal{T}_C$ , whereas the  $\star$ -conjugate to  $(\hat{i})^2\hat{H}$  is not even in  $\mathcal{A}_p$ .

### III. INVOLUTIONS ON PHYSICAL OBSERVABLES

We have seen that the  $\star$ -relations in  $\mathcal{A}^{(\star)}$  do not project unambiguously to  $\mathcal{A}'_p$ , because the  $\star$ -operation never maps all the representatives of a class of physical observables into another equivalence class. In order to define the  $\star$ -conjugate to a physical observable, one is therefore forced to choose first a particular representative for it. We now want to discuss under which circumstances it is possible to introduce an involution on  $\mathcal{A}'_p$  by this procedure.

To construct an involution  $\star$  on  $\mathcal{A}'_p$ , it actually suffices to define the  $\star$ -operation on an (over-) complete set of physical observables, and demand that this operation verifies conditions (1.4). Suppose then that  $\{\hat{U}'_a\}$  is a complete set in  $\mathcal{A}'_p$ , that is, that  $\mathcal{A}'_p$  can be obtained from the free associative algebra  $\mathcal{B}'$  generated by  $\{\hat{U}'_a\}$  by imposing the commutation relations between the observables  $\hat{U}'_a$ , as well as any algebraic relation that could exist between them. Assume also that one can find representatives  $\{\hat{U}_a\}$  of the observables  $\{\hat{U}'_a\}$  such that their  $\star$ -conjugates  $\{\hat{U}^\star_a\}$  belong to  $\mathcal{A}_p$ . One might then hope that the  $\star$ -operation on  $\mathcal{A}'_p$  could be defined by

$$(\hat{U}'_a)^\star = (\hat{U}^\star_a)', \tag{3.1}$$

where  $(\hat{U}^\star_a)'$  denotes the equivalence class of  $\hat{U}^\star_a$ . However, we will prove that the assumptions introduced above do not guarantee that Eq. (3.1) leads to a well-defined involution on the algebra of physical observables.

The proof makes use of the fact that, being  $\{\hat{U}'_a\}$  complete in  $\mathcal{A}'_p$ , any operator in the algebra  $\mathcal{A}'_p$  should be expressible, modulo an element in the ideal  $\mathcal{T}_C$  (1.7), as (possibly a limit of) a sum of products of the representatives  $\{\hat{U}_a\}$ . In particular, since every  $\hat{U}^\star_a \in \mathcal{A}_p$ , one gets

$$\hat{U}^\star_a = \sum_n \lambda_a^{b_1 \dots b_n} \hat{U}_{b_1} \dots \hat{U}_{b_n} + \hat{X}_a^i \hat{C}_i, \tag{3.2}$$

with  $\hat{X}_a^i \in \mathcal{A}$  and the  $\lambda_a^{b_1 \dots b_n}$ 's some complex numbers. Hence, from Eq. (3.1),

$$(\hat{U}'_a)^\star = \sum_n \lambda_a^{b_1 \dots b_n} \hat{U}'_{b_1} \dots \hat{U}'_{b_n}. \tag{3.3}$$

This  $\star$ -operation will be an involution on  $\mathcal{A}'_p$  only if  $((\hat{U}'_a)^\star)^\star = \hat{U}'_a$  for all  $\hat{U}'_a$ . This, together with Eqs. (1.4), (3.1), and (3.3), implies

$$\hat{U}'_a = \sum_n \bar{\lambda}_a^{b_1 \dots b_n} (\hat{U}^\star_{b_n})' \dots (\hat{U}^\star_{b_1})'. \tag{3.4}$$

On the other hand, we have from Eq. (3.2)

$$\hat{U}_a = \sum_n \bar{\lambda}_a^{b_1 \cdots b_n} \hat{U}_{b_n}^* \cdots \hat{U}_{b_1}^* + \hat{C}_i^* \hat{X}_a^{i*}, \tag{3.5}$$

since the  $\star$ -operation is an involution on  $\mathcal{A}^{(\star)}$ . Consistency of Eq. (3.4) with (3.5) requires then

$$\hat{C}_i^* \hat{X}_a^{i*} = \hat{Y}_a^i \hat{C}_i, \tag{3.6}$$

for some operators  $\hat{Y}_a^i \in \mathcal{A}$ . This condition will not be satisfied by generic operators  $\hat{X}_a^i \hat{C}_i \in \mathcal{T}_C$ , because the ideal  $\mathcal{T}_C$  is not invariant under the  $\star$ -operation when there exist first-class constraints on the system. Therefore, the  $\star$ -relations (3.3) will not supply in general an involution on  $\mathcal{A}'_p$ . To obtain that involution, it is necessary that both conditions (3.2) and (3.6) are satisfied by the representatives of our complete set of physical observables.

We will study now the case in which these requirements hold for our particular choice of representatives. Our previous discussion shows that the  $\star$ -operation defined by Eqs. (3.3) and (1.4) is then an involution on  $\mathcal{B}'$ , the free associative algebra generated by  $\{\hat{U}'_a\}$ . Recalling that the algebra  $\mathcal{A}'_p$  of physical observables can be obtained from  $\mathcal{B}'$  by imposing on its generators the commutation relations and any existing algebraic relations, we conclude that the  $\star$ -operation introduced on  $\mathcal{B}'$  straightforwardly supplies an involution on  $\mathcal{A}'_p$  provided that such an operation is compatible with the relations imposed on the generators  $\{\hat{U}'_a\}$ . In other words, the ideal of  $\mathcal{B}'$  generated by those relations should be invariant under the  $\star$ -operation. When this requisite is fulfilled, one gets an involution on  $\mathcal{A}'_p$  which captures the reality conditions on quantum operators.

Notice that the involution at which one arrives depends, nevertheless, on two choices: the complete set of physical observables and the representatives for them. In general, distinct choices may lead to different involutions on the algebra of physical observables. We will comment on this point further in Sec. V.

A situation which is often encountered in physical applications<sup>4,5</sup> is that one can find a complete set in  $\mathcal{A}'_p$  admitting representatives  $\{\hat{U}_a\}$  such that the complex vector space spanned by them is closed under reality conditions, i.e.,

$$\hat{U}_a^* = \lambda_a^b \hat{U}_b. \tag{3.7}$$

In this case, assumption (3.2) holds with  $\hat{X}_a^i \hat{C}_i = \hat{0}$ , so that Eq. (3.6) is trivially satisfied. It is then at least possible to obtain an involution on the free algebra  $\mathcal{B}'$  by replacing the operators  $\hat{U}_a$  in Eq. (3.7) with their corresponding equivalence classes of physical observables.

#### IV. EXAMPLES

Let us illustrate our discussion by dealing with some examples. Consider, for instance, the physical system that was analyzed at the end of Sec. II. A complete set of physical observables for this system is  $\mathcal{O}' \equiv \{\hat{1}', \hat{x}', \hat{p}'\}$ , where  $\hat{1}'$ ,  $\hat{x}'$ , and  $\hat{p}'$  are the equivalence classes of the operators  $\hat{1}$ ,  $\hat{x}$ , and  $\hat{p}$ , respectively. We can select these operators as the representatives of  $\mathcal{O}'$ . The associated reality conditions, which are given by Eq. (2.8), have the form (3.7). So, hypotheses (3.2) and (3.6) apply. We can therefore try to induce an involution on  $\mathcal{A}'_p$  by the procedure explained in Sec. III. Since there exist no algebraic relations in  $\mathcal{O}'$ , the only consistency requirement that must be satisfied in order to get the desired involution is that reality conditions (2.8) are compatible with the commutators of the physical observables in  $\mathcal{O}'$ . There is just one commutator different from zero:  $[\hat{x}', \hat{p}'] = i\hbar \hat{1}'$ . On the other hand, we obtain from Eqs. (2.8) and (3.1)

$$(\hat{x}')^* = \hat{x}', \tag{4.1}$$

$$(\hat{p}')^* = \hat{p}', \quad (\hat{1}')^* = \hat{1}'. \quad (4.2)$$

Taking then the  $\star$ -conjugate to  $[\hat{x}', \hat{p}']$ , we get

$$([\hat{x}', \hat{p}']^* = [(\hat{p}')^*, (\hat{x}')^*] = [\hat{p}', \hat{x}'] = -i\hbar \hat{1}', \quad (4.3)$$

which is precisely  $(i\hbar \hat{1}')^*$ . All other commutators between  $(\hat{1}')^*$ ,  $(\hat{x}')^*$ , and  $(\hat{p}')^*$  vanish identically. Hence, the  $\star$ -operation constructed is compatible with the structure of  $\mathcal{A}'_p$ , and provides an involution on this algebra.

Let us consider now other choices of representatives of  $\mathcal{O}'$ . Adopt, e.g., the choice  $\{\hat{1}, \hat{x} + \hat{t}(\hat{H})^2, \hat{p}\}$ . It follows from Eqs. (2.7) and (2.8) that

$$(\hat{x} + \hat{t}(\hat{H})^2)^* = \hat{x} + \hat{t}(\hat{H})^2 - 2i\hbar \hat{H}, \quad \hat{p}^* = \hat{p}, \quad \hat{1}^* = \hat{1}. \quad (4.4)$$

These reality conditions are of the type (3.2), with  $\hat{X}_a^i \hat{C}_i = -2i\hbar \hat{H}$  for  $\hat{U}_a = \hat{x} + \hat{t}(\hat{H})^2$ , vanishing otherwise. In particular, assumption (3.6) is verified. Therefore, one can introduce a  $\star$ -operation on  $\mathcal{A}'_p$  by applying Eq. (3.1) to the present case. In this way, one recovers the  $\star$ -relations (4.1) and (4.2), and thus the same involution on the algebra of physical observables that was obtained above.

Choose now the operators  $\hat{1}$ ,  $\hat{x} + \hat{t}\hat{H}$ , and  $\hat{p}$  as representatives of  $\mathcal{O}'$ . The reality conditions are then given by

$$(\hat{x} + \hat{t}\hat{H})^* = \hat{x} + \hat{t}\hat{H} - i\hbar \hat{1}, \quad \hat{p}^* = \hat{p}, \quad \hat{1}^* = \hat{1}. \quad (4.5)$$

These reality conditions are of the form (3.7), and induce on  $\mathcal{A}'_p$  the  $\star$ -operation defined through Eq. (4.2) and

$$(\hat{x}')^* = \hat{x}' - i\hbar \hat{1}'. \quad (4.6)$$

Since Eqs. (4.2) and (4.6) imply again relation (4.3), and  $(\hat{1}')^*$  commutes with  $(\hat{x}')^*$  and  $(\hat{p}')^*$ , the introduced  $\star$ -operation is compatible with the commutators of the physical observables, and is therefore an involution on  $\mathcal{A}'_p$ . However, this involution differs from that obtained in Eqs. (4.1) and (4.2). This proves that the involution induced on  $\mathcal{A}'_p$  from reality conditions depends on the particular selection of representatives made for the complete set of physical observables under consideration.

Note, nonetheless, that the two involutions induced on  $\mathcal{A}'_p$  in the example discussed above are equivalent in the sense that the two resulting  $\star$ -algebras  $\mathcal{A}'_p^{(\star)}$  are isomorphic, as one can easily check by identifying the physical observable  $\hat{x}'$  in Eq. (4.1) with  $\hat{x}' - i(\hbar/2)\hat{1}'$  in (4.6). For infinite dimensional systems, however, one should expect that the involutions constructed by choosing different complete sets in  $\mathcal{A}'_p$  and appropriate representatives for them would lead in general to inequivalent  $\star$ -algebras of physical observables.

Suppose, on the other hand, that we can represent the  $\star$ -relations on  $\mathcal{A}'_p$  as adjoint relations on a Hilbert space of physical states, as suggested by Ashtekar. From the involution provided by Eqs. (4.1) and (4.2), we would then arrive at a quantum theory in which the observable  $\hat{x}'$  would be self-adjoint. The involution defined through Eqs. (4.2) and (4.6) would lead instead to a quantum theory in which  $\hat{x}'$  would not be represented by a self-adjoint operator, so that it should not correspond to a real physical observable of the system. In order to resolve this ambiguity one can insist, for instance, on that the real classical variable  $x$  should be represented by the quantum observable  $\hat{x}'$ . One would thus expect that the spectrum of  $\hat{x}'$  should be real to guarantee that this observable has always real expectation values. Hence,  $\hat{x}'$  should be self-adjoint. By itself, this condition supports the use of involution (4.1), (4.2) in the quantization, rather than other possible  $\star$ -relations on  $\mathcal{A}'_p$  which, like relation (4.6), are inconsistent with the identification of  $\hat{x}'$  as the quantum physical observable corresponding to  $x$ .



To close this section, we will present an example in which the involution induced on  $\mathcal{B}'$  via reality conditions is not compatible with the structure of the algebra of physical observables. Let us consider a physical system with a first-class constraint of the form  $H=0$ , where  $H \in \mathbb{R}$  is the momentum canonically conjugate to a certain variable  $t \in \mathbb{R}$ . We will assume that the reduced phase space of the system is the cotangent bundle over the unit circle  $S^1$ . As elementary variables, we can choose the complex vector space spanned by  $\{1, t, H, c_\theta \equiv \cos \theta, s_\theta \equiv \sin \theta, p_\theta\}$ . Here,  $\theta \in S^1$ , and  $p_\theta \in \mathbb{R}$  is the momentum conjugate to  $\theta$ . The reality conditions on the corresponding algebra  $\mathcal{A}^{(\star)}$  of quantum operators are given by Eq. (2.7) and

$$\hat{c}_\theta^\star = \hat{c}_\theta, \quad \hat{s}_\theta^\star = \hat{s}_\theta, \quad \hat{p}_\theta^\star = \hat{p}_\theta, \quad \hat{1}^\star = \hat{1}. \quad (4.7)$$

Besides, since  $\cos^2 \theta + \sin^2 \theta = 1$ , we will impose the algebraic relation

$$(\hat{c}_\theta)^2 + (\hat{s}_\theta)^2 = \hat{1}. \quad (4.8)$$

A complete set of physical observables is  $\mathcal{O}' \equiv \{\hat{1}', \hat{c}'_\theta, \hat{s}'_\theta, \hat{p}'_\theta\}$ , the prime denoting equivalence classes. The only nonvanishing commutators in  $\mathcal{O}'$  are

$$[\hat{c}'_\theta, \hat{p}'_\theta] = -i\hbar \hat{s}'_\theta, \quad [\hat{s}'_\theta, \hat{p}'_\theta] = i\hbar \hat{c}'_\theta. \quad (4.9)$$

In addition, relation (4.8) implies that the physical observables in  $\mathcal{O}'$  must satisfy

$$(\hat{c}'_\theta)^2 + (\hat{s}'_\theta)^2 = \hat{1}'. \quad (4.10)$$

If one chooses  $\hat{1}, \hat{c}_\theta, \hat{s}_\theta$ , and  $\hat{p}_\theta$  as the representatives of  $\mathcal{O}'$ , the procedure explained in Sec. III allows one to obtain a  $\star$ -operation on  $\mathcal{B}'$  (the free associative algebra generated by  $\mathcal{O}'$ ) which is compatible with the commutators (4.9) and the algebraic relation (4.10), and hence provides an involution on  $\mathcal{A}'_p$ . Let us select instead the representatives  $\mathcal{O} \equiv \{\hat{1}, (\hat{c}_\theta + i\hat{H}), \hat{s}_\theta, \hat{p}_\theta\}$ . From Eqs. (2.7) and (4.7) (and the commutator of  $\hat{t}$  and  $\hat{H}$ ), we get

$$\hat{1}^\star = \hat{1}, \quad (\hat{c}_\theta + i\hat{H})^\star = (\hat{c}_\theta + i\hat{H}) - i\hbar \hat{1}, \quad \hat{s}_\theta^\star = \hat{s}_\theta, \quad \hat{p}_\theta^\star = \hat{p}_\theta. \quad (4.11)$$

These reality conditions are of the type (3.7). Thus, we can apply the results of Sec. III to arrive at an involution on  $\mathcal{B}'$  which is defined through the  $\star$ -relations (4.11), but imposed on equivalence classes in  $\mathcal{O}'$ . However, such a  $\star$ -operation is incompatible with the algebraic relation (4.10), because

$$((\hat{c}'_\theta)^2 + (\hat{s}'_\theta)^2 - \hat{1})^\star = (\hat{c}'_\theta - i\hbar \hat{1}')^2 + (\hat{s}'_\theta)^2 - \hat{1}' \neq 0. \quad (4.12)$$

So, the involution introduced on  $\mathcal{B}'$  does not supply a well-defined involution on the algebra  $\mathcal{A}'_p$  of physical observables. This example shows that the freedom in choosing representatives of the complete set of physical observables is in general restricted by the consistency of the algebraic structures with the  $\star$ -operation constructed on  $\mathcal{A}'_p$ .

## V. CONCLUSIONS AND FURTHER COMMENTS

We have shown that, in systems with first-class constraints, the involution defined on the algebra  $\mathcal{A}^{(\star)}$  of quantum operators does not ever project unambiguously to the algebra  $\mathcal{A}'_p$  of physical observables. The reason for this is that the  $\star$ -conjugates of all the representatives of any class of observables never belong to the same equivalence class in  $\mathcal{A}'_p$ , and, in general, not even to  $\mathcal{A}'_p$ .

We have also proved that, under sufficiently general circumstances, it is nevertheless possible to obtain a well-defined involution on  $\mathcal{A}'_p$  via reality conditions by making a particular choice of

representatives for the equivalence classes of physical observables. The procedure to arrive at this involution is the following. One must first find a complete set of physical observables  $\{\hat{U}'_a\}$  in  $\mathcal{A}'_p$ , and select representatives  $\{\hat{U}_a\}$  of them such that their  $\star$ -conjugates  $\{\hat{U}_a^*\}$  satisfy requirements (3.2) and (3.6), namely, such that every  $\hat{U}_a^*$  belongs to the free associative algebra generated by  $\{\hat{U}_a\}$  up to an operator which, as well as its  $\star$ -conjugate, vanish modulo quantum constraints. One can then introduce an involution  $\star$  in the free associative algebra  $\mathcal{B}'$  by defining  $(\hat{U}'_a)^*$  as the equivalence class of the observable  $\hat{U}_a^*$  [see Eqs. (3.2),(3.3)]. This involution on  $\mathcal{B}'$  straightforwardly supplies an involution on  $\mathcal{A}'_p$ , provided that the constructed  $\star$ -operation is compatible with the commutation and algebraic relations which exist between the physical observables in the complete set  $\{\hat{U}'_a\}$ .

The involution obtained in this way on  $\mathcal{A}'_p$  depends on the selection of a complete set of physical observables and of specific representatives for them. While these choices are severely restricted by the consistency conditions explained above, there is in general some freedom left, so that, by adopting different choices, one might in principle arrive at nonequivalent involutions on the algebra of physical observables.

This extra freedom in the quantization method suggested by Ashtekar, rather than being a supplementary complication, might actually become an additional help when attempting to complete the quantization. This is due to the fact that, given an involution  $\star$  on the algebra  $\mathcal{A}'_p$  and a certain representation for  $\mathcal{A}'_p$  on a vector space  $V_p$  of quantum states, there is *a priori* no guarantee that there exists an inner product on  $V_p$  with respect to which the  $\star$ -relations on physical observables are realized as Hermitian adjoint relations in the resulting Hilbert space. Thus if such an inner product does not exist for a particular involution on  $\mathcal{A}'_p$ , one can always try to induce a different involution on this algebra via reality conditions, and see whether it is possible to then find an inner product with the desired properties.

We notice, on the other hand, that the introduction of an involution on  $\mathcal{A}'_p$  amounts essentially to determine the  $\star$ -conjugate to a complete set of physical observables. When one expects that a set of this kind, or at least some of its elements, correspond classically to real observables of the system, it is reasonable to assume that they should be represented by self-adjoint operators. The involution defined on  $\mathcal{A}'_p$  should therefore ensure that these operators coincide with their  $\star$ -conjugates. These requirements clearly restrict the physically admissible involutions on  $\mathcal{A}'_p$ . Moreover, in the case that this type of physical arguments would apply to a complete set in  $\mathcal{A}'_p$ , one would fully specify the involution on this algebra. In this way, one can use physical intuition as a guideline to resolve (either partially or totally) the ambiguity encountered when inducing an involution on the algebra of physical observables from reality conditions.

Finally, an alternative strategy to remove such an ambiguity could consist in adopting a specific procedure to induce the involution  $\star$  on  $\mathcal{A}'_p$ . A procedure of this type might be, e.g., the following:<sup>12</sup> Let us denote by  $\mathcal{A}_s \subset \mathcal{A}_p$  the subalgebra formed by all the strong quantum observables of the theory (that is, the operators which commute exactly with all the quantum constraints  $\{\hat{C}_i\}$ ), and define  $\mathcal{I}_s \equiv \mathcal{I}_C \cap \mathcal{A}_s$ . It is immediate to check that  $\mathcal{I}_s$  is an ideal of  $\mathcal{A}_s$ . Suppose then that, in the system under consideration, the involution  $\star$  defined on  $\mathcal{A}^{(\star)}$  and the representation constructed for the algebra  $\mathcal{A}$  and for the constraints  $\{C_i\}$  are such that: (a) The complex vector space spanned by the quantum constraints  $\{\hat{C}_i\}$  is closed under reality conditions, i.e.,  $\hat{C}_i^* = \lambda_i^j \hat{C}_j$ , where the  $\lambda_i^j$ 's are complex numbers. (b) The algebra  $\mathcal{A}'_s \equiv \mathcal{A}_s / \mathcal{I}_s$  is isomorphic to  $\mathcal{A}'_p$ . (c) The ideal  $\mathcal{I}_s$  is invariant under the  $\star$ -operation. Notice that hypothesis (c) is in principle compatible with the fact that  $\mathcal{I}_C$  is not a  $\star$ -ideal of  $\mathcal{A}_p$ . Requirement (b), on the other hand, guarantees that each physical observable in  $\mathcal{A}'_p$  possesses (at least) one representative which is a strong observable.

Using condition (a), it is possible to prove that the  $\star$ -operation leaves  $\mathcal{A}_s$  invariant. Assumption (c) ensures then that the  $\star$ -relations project unambiguously to  $\mathcal{A}'_s$ . One hence obtains a well-defined involution on  $\mathcal{A}'_s$  which, given condition (b), supplies a unique involution on  $\mathcal{A}'_p$  through the existing isomorphism between these two algebras. So, provided that hypotheses (a)–

(c) are satisfied, the above strategy allows one to induce an unambiguous involution on  $\mathcal{A}'_p$  from reality conditions.

In fact, this strategy is partially motivated by the refined version of Ashtekar's program which has been recently proposed.<sup>13</sup> In this new version of the program, it is supposed that all first-class constraints can be represented by self-adjoint operators on an auxiliary Hilbert space on which one has constructed a  $\star$ -representation of  $\mathcal{A}^{(\star)}$ . Hypothesis (a) must hence hold. Besides, rather than considering the whole algebra  $\mathcal{A}_p$ , one restricts one's attention to a certain  $\star$ -subalgebra of strong observables  $\mathcal{B}^{(\star)}_{\text{phys}} \subset \mathcal{A}_p$ .<sup>13</sup> After completion of the quantization, one obtains a physical Hilbert space  $\mathcal{H}_{\text{phys}}$  which carries a  $\star$ -representation of  $\mathcal{B}^{(\star)}_{\text{phys}}$ . Let us point out that this representation will not be faithful in general, for there may exist a subalgebra  $\mathcal{I}_{\mathcal{B}} \subset \mathcal{B}^{(\star)}_{\text{phys}}$  that annihilates the whole space of physical states. It seems reasonable to assume that  $\mathcal{I}_{\mathcal{B}} = \mathcal{I}_{\mathcal{C}} \cap \mathcal{B}^{(\star)}_{\text{phys}}$ , i.e., that the operators in  $\mathcal{B}^{(\star)}_{\text{phys}}$  with zero action on physical states are those which vanish modulo constraints.<sup>14</sup> On the other hand, it is possible to show<sup>14</sup> that Ashtekar's refined program can be consistently implemented only if  $\mathcal{I}_{\mathcal{B}}$  is a  $\star$ -ideal of  $\mathcal{B}^{(\star)}_{\text{phys}}$ , requirement which is the analog of hypothesis (c) above.

According to our discussion,  $\mathcal{H}_{\text{phys}}$  should finally provide a faithful representation of  $\mathcal{B}'_{\text{phys}} \equiv \mathcal{B}^{(\star)}_{\text{phys}} / \mathcal{I}_{\mathcal{B}}$ . In order that all relevant physical information can be extracted from the quantum theory obtained, it is then necessary that  $\mathcal{B}'_{\text{phys}}$  be isomorphic to  $\mathcal{A}'_p$  (or at least to a sufficiently large subalgebra of it). This last condition parallels hypothesis (b). In this sense, the strategy presented here to induce an involution on  $\mathcal{A}'_p$  actually is implicitly incorporated in Ashtekar's refined program, with the only generalization that one can consider a  $\star$ -subalgebra of strong observables  $\mathcal{B}^{(\star)}_{\text{phys}}$  as the substitute for  $\mathcal{A}_s$ .

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<sup>1</sup>A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity*, edited by L. Z. Fang and R. Ruffini (World Scientific, Singapore, 1991).

<sup>2</sup>A. Ashtekar and R. S. Tate, *J. Math. Phys.* **35**, 6434 (1994).

<sup>3</sup>A. Ashtekar, in *Gravitation and Quantizations*, in Les Houches Summer School Proceedings Vol. LVII, edited by B. Julia and J. Zinn-Justin (North-Holland, Amsterdam, 1993).

<sup>4</sup>A. Ashtekar, R. S. Tate, and C. Uggla, *Int. J. Mod. Phys. D* **2**, 15 (1993); N. Manojlović and G. A. Mena Marugán, *Phys. Rev. D* **48**, 3704 (1993); *Int. J. Mod. Phys. D* (to appear, 1995), gr-qc/9503056; G. A. Mena Marugán, *Phys. Rev. D* **50**, 3923 (1994); *Class. Quantum Gravit.* **11**, 2205 (1994); gr-qc/9504006.

<sup>5</sup>G. A. Mena Marugán, *Class. Quantum Gravit.* **11**, 589 (1994).

<sup>6</sup>H. Kastrop and T. Thiemann, *Nucl. Phys. B* **399**, 211 (1993); **436**, 681 (1995).

<sup>7</sup>A. Ashtekar, V. Husain, C. Rovelli, J. Samuel, and L. Smolin, *Class. Quantum Gravit.* **6**, L185 (1989); N. Manojlović and A. Miković, *Nucl. Phys. B* **385**, 571 (1992).

<sup>8</sup>A. Ashtekar and R. Geroch, *Rep. Prog. Phys.* **37**, 1211 (1974); A. Ashtekar, *Commun. Math. Phys.* **71**, 59 (1980).

<sup>9</sup>P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monograph Series No. 2 (Yeshiva University, New York, 1964).

<sup>10</sup>In the following, we will use primed and unprimed operators to refer, respectively, to observables defined modulo constraints and elements in  $\mathcal{A}$  (including those in the subalgebra  $\mathcal{A}_p$ ).

<sup>11</sup>A. Rendall, *Class. Quantum Gravit.* **10**, 2261 (1993); Erratum: gr-qc/9403001.

<sup>12</sup>This procedure implements a suggestion by A. Ashtekar (private communication).

<sup>13</sup>A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann, gr-qc/9504018.

<sup>14</sup>G. A. Mena Marugán, in *Proceedings of the Sintra Workshop on Modern Methods in Classical and Quantum Gravity*, Acta Cosmologica (to appear, 1996).