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M and F-Theory Instantons, $N=1$ Supersymmetry and Fractional Topological Charge

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Abstract

We analize instanton generated superpotentials for three dimensional $N=2$ supersymmetric gauge theories obtained by compactifying on S^1 $N=1$ four dimensional theories. For $SU(2)$ with $N_f=1$, we find that the vacua in the decompactification limit is given by the singular points of the Coulomb branch of the $N=2$ four dimensional theory (we also consider the massive case). The decompactification limit of the superpotential for pure gauge theories without chiral matter is interpreted in terms of 't Hooft's fractional instanton amplitudes.

1 Introduction.

Some of the deepest dynamical problems in gauge theories, such as the confinement problem, appear as tractable issues once we pass from four to three dimensions [1]. Recently, the study of $N=2$ supersymmetry in three dimensions has began to shed some light on the more difficult dynamics of four dimensional $N=1$ theories [2]. Moreover, the interplay between $N=2$ in three dimensions and $N=4$ in four dimensions is the one existing between M and F-theory compactifications on Calabi-Yau fourfolds.

Using elliptically fibered Calabi-Yau fourfolds, the Coulomb branch of an ADE $N=2$ gauge theory in three dimensions can be defined by means of the resolution of the corresponding ADE singularity [3]. Instantons defined in terms of aritmetic genus one divisors [4] of the resolved fourfold provide an R-dependent superpotential [3], with R determined by the class of the elliptic fiber [4]. The $R \to 0$ limit reproduces the known results in $N=2$ three dimensional theories [5], while the $R \to \infty$ limit defines a superpotential in four dimensions, with $N=1$ supersymmetry, compatible with the $\mathrm{tr}(-1)^F$ computation [6], and the cluster derivation of gaugino condensates [7, 8]. The main interest of this four dimensional limit is that it can not be trivially described directly in four dimensions or, equivalently, in the F-theory compactification on the fourfold, where the absence of Coulomb branch forbids the resolution of singularities. Thus, the $R \to \infty$ limit provides some information on the strong infrared dynamics taking place in the confinement regime of the uncompactified theory.

In this letter we study the case of $SU(2)$ with 1 flavor. For 1 flavor, we find an R-dependent superpotential on the Coulomb branch of $N=2$ in three dimensions, which in the $R \to \infty$ limit provides a set of minima in agreement with the result arising from soft breaking ($N=2$ to $N=1$) the exact solution [9] for the Coulomb branch of four dimensional $N=2$ with $N_f=1$. The other issue we consider is the direct interpretation of the $R \to \infty$ limit of the pure $N=1$ four dimensional superpotential in terms of 't Hooft [10] fractional instanton amplitudes [11]. In fact, for $SU(N_C)$ gauge theories, the topology of the residual gauge transformations compatible with the set of twisted boundary conditions representing the non-vanishing $\mathbf{Z}_{N_C}$-magnetic flux [12], boundary conditions compatible with the confinement phase, reproduce in the four dimensions context the Dynkin structure [3] of the singularity resolution used in the definition of the Coulomb branch of three dimensional $N=2$.

2 M-theory Instantons.

We will consider M-theory compactifications on a Calabi-Yau fourfold X, which lead to three dimensional $N=2$ supersymmetry. If X admits an elliptic fibration,

$$E \longrightarrow X \xrightarrow{\Pi} B, \tag{2.1}$$

we can define F-theory compactifications on X which lead to four dimensional $N=1$ supersymmetry. If we assume that on a codimension one locus $C \subset B$ the elliptic fiber is degenerate, of ADE type in Kodaira's classification, then we obtain an $N=1$ ADE-gauge theory in four dimensions. This $N=1$ four dimensional theory results from compactifying on C the 7-brane worldvolume. The bare coupling constant is then given by [3]

$$\frac{1}{g_4^2} = V_C. \tag{2.2}$$

Through further compactification on S^1, we recover the $N=2$ three dimensional theory defined by the M-theory compactification on X.

Denoting by ϵ the class of the elliptic fiber, E, we can relate, by a chain of dualities, M-theory compactified on X with type II_B compactifications on $B \times S^1$, where the radius of S^1 scales like $\frac{1}{\epsilon}$. The decompactification limit $\epsilon \to 0$ ($R \to \infty$) corresponds to the F-theory compactification, while the $\epsilon \to \infty$ ($R \to 0$) limit does correspond to M-theory compactification.

As it is well known [2], $N=2$ supersymmetric three dimensional pure gauge theories posses a classical Coulomb branch of dimension equal to the rank r of the gauge group. This Coulomb branch is parametrized , if we define the theory by compactifying the $N=1$ theory in four dimensions, in terms of the Wilson line in the internal direction. At the classical level we have a set of $r+1$ singular points corresponding to Wilson lines in the center of the group. If fundamental fermions are absent, then these point represent classical restoration of non abelian symmetry (see some comments on this issue in the last section). In three dimensions, instantons are characterized by their monopole magnetic charge. They generate non perturbatively a superpotential [5] of the type

$$W = \sum_{i=1}^{r} \exp(-\Phi_i), \tag{2.3}$$

with Φ_i the r complex scalars used to parametrize the Coulomb branch. The existence of this superpotential is mainly due to the fact that instantons in three dimensions have only two fermionic zero modes, as superconformal invariance is absent.

The M-theory origin of the superpotential (2.3) has recently been presented in references [4, 3]. In fact, instantons in M-theory are defined [4] wrapping the euclidean 5-brane on a 6-cycle D, contained in X, satisfying

$$\chi(\mathcal{O}_{\mathcal{D}}) = 1 - h^{3,0} - h^{2,0} + h^{1,0} = 1. \tag{2.4}$$

Condition (2.4) is equivalent to the existence, in uncompactified spacetime, of two fermionic zero modes, which implies the generation of a superpotential.

When X admits an elliptic fibration, vertical instantons, defined by a divisor D satisfying (2.4), and such that $\Pi(D)$ is codimension one in B, survive in the uncompactified F-theory limit defined as $\epsilon \to 0$ [4]. In fact, through a chain of dualities the 5-brane

instanton can be interpreted in the type II_B language as a 3-brane wrapping the four cycle $\Pi(D)$.

Let us now consider a divisor D with $\Pi(D) = C \subset B$, with C the codimension one locus in B where the elliptic fiber develops an ADE singularity. We will also assume that $h^{1,0}(C) = h^{2,0}(C) = 0$, which from the point of view of F-theory implies the absence of adjoint matter in the four dimensional theory. Using Hirzebuch-Riemann-Roch theorem, we get for this divisor

$$\chi(\mathcal{O}_D) = C_2(G), \tag{2.5}$$

with $C_2(G)$ the dual Coxeter number. Clearly, this divisor D does not contribute to the superpotential. In fact, to go to the Coulomb branch in three dimensions is equivalent to performing the resolution of the singular elliptic fiber at C [3]. By this procedure, we get a set of $r+1$ irreducible components E_i satisfying $E_i \cdot E_i = -2$, with intersection matrix defined by the corresponding affine Dynkin diagram. Each of these components can be used in order to define a divisor D_i, obtained by fibering E_i on C, with

$$\chi(\mathcal{O}_{D_i}) = 1. \tag{2.6}$$

Moreover, as pointed out in reference [3], these divisors are constrained by the standard relations between the roots of a Lie algebra. Denoting α_i the roots of the Lie algebra, we have, for a group of rank r,

$$\sum_{i=1}^{r} a_i \alpha_i = \tilde{\alpha}, \tag{2.7}$$

with $\tilde{\alpha}$ the biggest root, which defines the extra point in the affine Dynkin diagram. Moreover (see table),

$$\sum_{i=1}^{r} a_i = C_2(G) - 1. \tag{2.8}$$

$\mathbf{A_{n-1}}$	$\tilde{\alpha} = \sum_{i=1}^{n-1} \alpha_i$	$C_2(G) = n$
$\mathbf{D_n}$	$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$C_2(G) = 2n-2$
$\mathbf{E_6}$	$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$C_2(G) = 12$
$\mathbf{E_7}$	$\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	$C_2(G) = 18$
$\mathbf{E_8}$	$\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	$C_2(G) = 30$

A set of r irreducible components E_i can be related to the roots α_i, while the extra one, that we will denote E_0, can be associated to $-\tilde{\alpha}$ [1], which defines the extra point of the affine Dynkin diagram. Thus, we get the relation

$$\sum_{i=1}^{r} a_i E_i + E_0 = E, \tag{2.9}$$

[1]Notice that the extra component does not contribute to the Picard group of the fourfold.

with $E \cdot E = 0$, the class of the elliptic fiber. The contributions of these divisors to the superpotential are given by

$$W = \sum_{i=0}^{r} \exp[-V(C) \cdot V(E_i)]. \tag{2.10}$$

Using (2.9), and for $V(E) = \epsilon (\sim \frac{1}{R})$, the superpotential (2.10) becomes [3]

$$W = \sum_{i=0}^{r} \exp[-V(C) \cdot V(E_i)] + \gamma \exp[\sum_{i=1}^{r} a_i \ V(C) \cdot V(E_i)], \tag{2.11}$$

with

$$\gamma = exp\left(-\frac{1}{g_3^2 \cdot R}\right). \tag{2.12}$$

The main interest of (2.11) is that it provides us with an R-dependent superpotential, with well behaved limits both in the $N=2$ three dimensional case and the $N=1$ case in four dimensions. In fact, the $R \to 0$ limit leads to the $N=2$ superpotential (2.3) generated by instantons, and in the $R \to \infty$ limit it gives a set of $C_2(G)$ minima, in perfect agreement with the Witten index computation [6].

The superpotential (2.11) generalizes the one obtained in reference [2], for the particular case of $SU(2)$, to arbitrary gauge groups. The derivation in [2] starts with the solution to the $N=4$ theory in three dimensions, which is given by the Atiyah-Hitchin [13] manifold

$$y^2 = x^2 v + \gamma x, \tag{2.13}$$

where $v = x - u$, and γ is the dynamically generated scale, in adequate units. This theory corresponds to the $R \to 0$ limit of the $N=2$ supersymmetric pure gauge theory on $\mathbf{R}^3 \times S^1$. If we now softly break $N=2$ to $N=1$ by the addition of a superpotential ϵu, the R dependent superpotential for the $N=1$ theory becomes

$$W = \gamma^2 \lambda(\tilde{y}^2 - \tilde{x}^2 v - \tilde{x}) + \epsilon(\gamma \tilde{x} - v), \tag{2.14}$$

with $\gamma \tilde{x} = x$, $\gamma \tilde{y} = y$. Now, an effective superpotential for $\tilde{x}$ can be written by solving $\frac{\partial W}{\partial \lambda} = \frac{\partial W}{\partial \tilde{y}} = \frac{\partial W}{\partial v} = 0$,

$$W = \epsilon \left(\gamma \tilde{x} + \frac{1}{\tilde{x}}\right), \tag{2.15}$$

which coincides with (2.11) when the identification [2] $\frac{\epsilon}{\tilde{x}} = exp(-V(C) \cdot V(E_1))$ is made[2]

[2]If we take the $\epsilon \to \infty$ limit, we can define a blow up through the double limit $\lim_{\epsilon \to \infty, \tilde{x} \to \infty} \frac{\epsilon}{\tilde{x}} = exp(-V(C) \cdot V(E_1))$. By interpretting the blow up parameters $V(E_1)$ in terms of x, we observe that their definition implies $\tilde{x} \to \infty$, and $\gamma \to 0$.

3 The QCD case.

Let us now add flavors transforming in the fundamental representation. For simplicity, we will consider the case of $SU(2)$. The R-dependent superpotential can be derived using the Atiyah-Hitchin manifold for massless $N_f = 1$ $N=4$ supersymmetric $SU(2)$ theory,

$$y^2 = x^2 v + \gamma_1. \tag{3.1}$$

In this case, the superpotential we obtain, following the same steps as in [2], is

$$W = \epsilon \left(\gamma_1 \tilde{x} + \frac{1}{\tilde{x}^2} \right), \tag{3.2}$$

which for $R \to 0$, gives us the superpotential

$$W = \frac{\epsilon}{\tilde{x}^2} \tag{3.3}$$

The minima, in the $R \to \infty$ four dimensional limit, are given by the three roots of unity, in agreement with the $\mathbf{Z}_3$ symmetric set of singular points of the Seiberg-Witten solution for $N = 2$ four dimensional $SU(2)$ gauge theory, with $N_f = 1$ [9]. In fact, the minima of (3.2) are at the points $\frac{1}{\tilde{x}} = \frac{\Lambda e^{2\pi i n/3}}{(2)^{1/3}}$, and therefore we get

$$\frac{\epsilon}{\tilde{x}} \simeq e^{2\pi i n/3} \Lambda \epsilon, \qquad n = 0, 1, 2, \tag{3.4}$$

where $\gamma_1 \equiv \Lambda^3$, with Λ the scale of the $N = 2$ theory. In the $\epsilon \to \infty$ limit, the three ground states of the Coulomb branch approach each other [9].

Using the mass deformed Atiyah-Hitchin manifold studied by Dancer [14],

$$y^2 = x^2 v - i 2 m \gamma_1 x + \gamma_1^2, \tag{3.5}$$

we can derive the superpotential for the massive case,

$$W = \epsilon \left(\gamma_1 \tilde{x} - \frac{2im}{\tilde{x}} + \frac{1}{\tilde{x}^2} \right), \tag{3.6}$$

which in the three dimensional $R \to 0$ limit leads to the superpotential

$$W = \epsilon \left(-\frac{2im}{\tilde{x}} + \frac{1}{\tilde{x}^2} \right). \tag{3.7}$$

Now, we could consider the type of non perturbative effects able to generate, in the $N = 2$ three dimensional theory with finite ϵ, the superpotentials (3.3) and (3.7). In the three dimensional theory, we should use the Callias version of Atiyah-Hitchin index theorem [15]. In the notation of [15], the index is given by

$$\text{index} = [j(j+1) - \{m\}(\{m\}+1)] \cdot n, \tag{3.8}$$

for instanton number equal n. The $2n$ gluino zero modes correspond to $j=1$ and $\{m\}=0$ in (3.8). For fermions in the fundamental representation, $j=\frac{1}{2}$, and we get index $=1$, for $\{m\}=-\frac{1}{2}$, and index $=0$, for $\{m\}=\frac{1}{2}$[3]. In order to reproduce the mass term in (3.6), we should consider $\{m\}=\frac{1}{2}$. The superpotential $\frac{\epsilon}{\tilde{x}^2}$ is however more difficult to interpret. According to the power of the denominator, we might think of some sort of 2-instanton effect. Taking into account the powers of ϵ, and interpretting the instanton effect as associated to $\frac{\epsilon}{\tilde{x}}$, we get an instanton effect of the type $\frac{\epsilon}{\tilde{x}}$, and other of the type $\frac{\epsilon}{\tilde{x}}\cdot\left(\frac{1}{\epsilon}\right)$. Thinking of this second instanton effect in similar terms as the massive contribution $\frac{\epsilon}{\tilde{x}}(2im)$, we will get a net vertex $\lambda\lambda\lambda\lambda\psi\psi$, which can generate the superpotential $\epsilon\frac{1}{\tilde{x}^2}$, if we pair up λ and ψ zero modes, and lift them. This is only an heuristic way to interpret the quadratic term in (3.2) for finite values of ϵ. This effect is supressed in the $\epsilon\to\infty$ limit [16]. For finite ϵ that any singularity of the four dimensional Coulomb branch leads to a confinement ground state characterized by a vacuum expectation value for the monopole fields. It would be interesting superpotentials of the type (3.2), for finite ϵ, with monopole superpotentials.

For $N=1$ four dimensional $SU(2)$ gauge theory, with one massless flavor, we get in the instanton background six zero modes, four of them corresponding to superesymmetric and superconformal transformations acting on the instanton configuration, and the other two to the flavor fermionic zero modes. Using the technique of constrained instantons, it was shown in [17] that a superpotential can be generated for $N_f=1$ by lifting four of the zero modes which can be paired. To derive this superpotential in the F-theory approach, we should consider the D-instanton defined fibering on C the degenerate elliptic fiber of A_1 type. Again, taking into account the existence of an $SU(2)$ gauge connection on C, with topological number $1=N_f$, we get for this D instanton aritmetic genus equal one [18]. Notice that this four dimensional instanton has been directly defined in the F-theory context, where no resolution of the singularity is allowed.

4 Fractional Instantons.

After the previous discussion on the superpotential a natural question arises, namely how to interpret the $R\to\infty$ limit of the superpotential (2.11) directly in terms of topologically non trivial configurations in four dimensions. The more natural guess would be to think on some kind of "fractional instanton", created by strong infrared dynamics: ordinary instantons in four dimensions posses too many zero modes to generate a superpotential. In fact, in the four dimensional F-theory context, where we can not perform any resolution of the singularity, the divisor D, with $\Pi(D)=C$, is of aritmetic genus $C_2(G)$. The fractional instanton should be thought as the M-theory 5-brane, formally wrapping D $\frac{1}{C_2(G)}$ times [4]. A different approach is the use of 't Hooft fractional instantons [10] to generate the

[3] $\{m\}$ is the largest eigenvalue of $\phi^a T^a$, with ϕ (the Higgs field in the adjoint) smaller than the fermion mass.

$R \to \infty$ limit of the superpotential (2.11). In the context of twisted boundary conditions on four dimensional spacetime [12], a contribution in pure $N=1$ supersymmetric Yang-Mills to the superpotential can be expected from a "toron" (topological number $\frac{1}{N_C}$) amplitude [11]. Following the notation of reference [12], this amplitude in the infinite volume limit, will be given by

$$< e, m=1|\lambda\lambda|e, m=1 >=< m=1|\lambda\lambda\ \Omega(k=1)|m=1 > e^{2\pi i e/N_C} = \Lambda^3 e^{2\pi i e/N_C} \quad (4.1)$$

where the electric flux e can take values $e = 0, \ldots, N_C - 1$, and where the state $|m=1>$ corresponds, in the temporal gauge $A^0 = 0$, to a fractional magnetic flux configuration, of magnetic flux $\frac{1}{N_C}$ in the third direction. The gauge transformation $\Omega(k=1)$ is part of the residual gauge symmetry, compatible with the twisted boundary conditions corresponding to the existence of the magnetic flux $|m=1>$. The phase factor in (4.1) comes from the definition of invariant states with respect to this residual symmetry:

$$|\overline{e}, \overline{m} > \equiv \frac{1}{N_C^3} \sum_k e^{2\pi i \overline{k}\overline{e}/N_C} \Omega(k) |\overline{m} > . \quad (4.2)$$

The fractional Pontryagin number corresponding to the tunnelling amplitude in (4.1) is given by

$$P = \frac{1}{N_C}. \quad (4.3)$$

Now, it is easy to observe that

$$\Omega(k=1)^{N_C} = T, \quad (4.4)$$

where T is a periodic gauge transformation on S^3, with $\Pi_3 = 1$. Taking into account that the non abelian instanton is defined by fibering the elliptic fiber E on C, relation (4.4) becomes the analog of the exponentiated version of Dynkin relation (2.9) for $SU(N_C)$. Defining the instanton action as $\exp -\frac{1}{g_4^2}$, the sum of contributions of type (4.1) to the superpotential produces, by means of (4.4), the desired result (2.11). Moreover, the minima for $SU(N_C)$ of the superpotential (2.11) reproduce exactly the phases in (4.1), which come from the definition of electric flux through (4.2). Notice that the minima of (2.11) are parametrized by the Wilson loop in the internal direction, which is precisely the meaning of e in (4.1). The fact that $< \lambda\lambda >$ is given in four dimensions in terms of e, comes in the toron computation directly from the the definition (4.2) of invariant states. Thus, we conclude that fractional instantons effectively appear in the uncompactified four dimensional limit. Notice that the Wilson loop in the internal direction, which parametrizes the Coulomb branch, can take values, if flavors are absent, in the center $\mathbf{Z}_{N_C}$ of the color group even when gauge invariance is restored. The discrete set of points in the moduli corresponding to Wilson lines in the center, becomes the ground states in the uncompactified limit. At these points, we have vortices instead of monopoles and torons

as their twisted version, that survive in the four dimensional limit. The torons appearing as the relevant topological configurations at these points. The crucial dynamics of torons that provide the right counting of zero modes consists in avoiding the superconformal zero mode modes of the instanton, without breaking supersymmetry; furthermore in the infinite volume limit they have vanishinjg field strength. In summary, we believe that fractional instanton effects in four dimensional $N=1$ are the right ingredient to understand the superpotentials derived through M-theory techniques, in the uncompactified limit.

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