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**TOWARD A FINITE-DIMENSIONAL FORMULATION OF  
QUANTUM FIELD THEORY <sup>†</sup>***Miguel Navarro\**

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**Abstract**

Rules of quantization and equations of motion for a finite-dimensional formulation of Quantum Field Theory are proposed which fulfill the following properties: a) both the rules of quantization and the equations of motion are covariant; b) the equations of evolution are second order in derivatives and first order in derivatives of the space-time co-ordinates; and c) these rules of quantization and equations of motion lead to the usual (canonical) rules of quantization and the (Schrödinger) equation of motion of Quantum Mechanics in the particular case of mechanical systems. We also comment briefly on further steps to fully develop a satisfactory quantum field theory and the difficulties which may be encountered when doing so.

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# 1 Introduction

At present the main goal of Theoretical Physics is to unify Quantum (Field) Theory and General Relativity. This task will probably require a previous reformulation of either of these theories or both of them.

The standard way of quantizing a field theory – and hence the actual form of standard Quantum Field Theory (QFT) – relies on the fact that Classical Field Theory (CFT) can be considered to be a generalization of Classical Mechanics (CM) in which the finite number of degrees of freedom of the latter is replaced with an infinite (continuum) number in the former. In this formulation the fields are considered to be functions  $\varphi^a(\mathbf{x})(t) \equiv \varphi_{\mathbf{x}}^a(t)$ ; that is, the spatial co-ordinates are regarded as labels (the discrete superindex  $a$  labels the different fields in the theory). This description is supported primarily by the fact that it is a direct generalization of Quantum Mechanics (QM), which, as a theory with a vast range of predictions, is a source of great confidence. The standard framework requires, nonetheless, the use of functionals in place of ordinary functions as well as infinite-dimensional differential calculus, which is plagued with ambiguities. These ambiguities are at the root of the renormalization problem.

This fact also leads to a problem of foundation: if Classical Field Theory, which is based on a small number of ordinary functions over the space-time, gives a description of the world, albeit rough and primitive, why must Quantum Field Theory be described with functionals – that is, functions with an infinite (continuum) number of arguments?

All this raises the question of whether a description of the quantum theory of fields in terms of ordinary functions is possible or not.

In fact, the kinematical description of a QFT of this type arises naturally from CFT provided that the latter, as a generalization of Classical Mechanics,

is interpreted in a way different from the one that leads to the standard QFT [10]. In this reading of CFT, all the co-ordinates of the space-time are considered to be on the same footing, no special role is played by time. The fields are not taken to be an infinite (continuum) set of functions of time but rather a discrete set of functions of all the space-time co-ordinates:  $\varphi^a = \varphi^a(x)$ , with  $x = (\mathbf{x}, t)$  and  $a$  a discrete label. Since there is a finite number of functions we shall refer to this approach as finite-dimensional QFT as opposed to the standard or infinite-dimensional QFT.

The first steps towards a covariant finite-dimensional formulation of field theory were given by Born [1], Weyl [2], de Donder [3] and Carathéodory [4] as early as the 1930s. Further attempts are due to Good [5] and Liotta [6], and more recent ones to Tapia [7] and Kanatchikov [8]. However, much of this effort has been focused on following routes to the quantum theory which closely mimick the one which, starting from Classical Mechanics, leads to the standard Quantum Mechanics. These routes pass, therefore, through developing a covariant canonical (Hamiltonian) formulation of the theory. The basic idea underlying this approach – which can be referred to as the *bottom to top* approach – is that, if a complete canonical formulation of the finite-dimensional description of Classical Field Theory were found, the finite-dimensional quantum theory would then naturally follow.

This procedure is legitimate, but it would perhaps be more profitable to try to construct directly, by whatever the means, a self-consistent finite-dimensional covariant QFT. After all, Quantum Mechanics should arise as only a limiting case of this QFT, and nothing guarantees that Poisson brackets, for instance, will play any role in the structure of the more general theory.

This opposite route, which we might term *top to bottom*, has, in fact,

been recently inaugurated with a proposal by Good of rules of quantization and equations of motion which give rise to a finite-dimensional QFT [9] . However, in the particular case of mechanical systems, Good's proposal does not lead to the standard Quantum Mechanics and, therefore, the resultant theory does not reproduce basic, long experimentally verified, predictions of standard QM [10]. The proposal, as a whole, should therefore be discarded.

In this context, the natural next step toward a finite-dimensional QFT should be to find an alternative proposal which, while preserving the basic features of Good's framework, give rise to the standard QM in the case of mechanical systems, hence avoiding the experimental failure of Good's rules. The task therefore is to find rules of quantization and equations of motion such that:

1. Both rules of quantization and equations of motion must be explicitly covariant; i.e., space and time co-ordinates are treated on the same footing.
2. Within the limits of mechanical systems these rules of quantization and equations of motion must reduce themselves to the familiar canonical rules of quantization and Schrödinger equation of evolution of ordinary Quantum Mechanics.
3. The equations of evolution must be second order in derivatives and first order in derivatives of the space-time co-ordinates.

This task in fact constitutes the main goal of the present letter: to show, using an example to be presented below, that proposals which fulfill all the requirements above do exist.

Although fully developing a quantum theory is far beyond the scope of the present letter, some basic guidelines to carry on the present analysis are

briefly commented in Section 3.

## 2 An improved proposal for equations of motion and rules of quantization

Let us consider the Schrödinger equation of ordinary Quantum Mechanics:

$$i\frac{d}{dt}\Psi = \widehat{H}\Psi \quad (1)$$

To generalize this equation to field theories, which are defined over a four-dimensional space-time manifold, we need a generalized Hamiltonian to be placed on its r.h.s., and a generalized “time derivative” operator to be placed on its l.h.s. A generalization for the Hamiltonian is well known [9, 10]: the covariant Hamiltonian  $\mathcal{H}$  which is obtained, from a Lagrangian  $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu\phi^a)$ , by means of the generalized covariant Legendre transform<sup>†</sup>:

$$\mathcal{H} = \pi_a^\mu \partial_\mu \phi^a - \mathcal{L} . \quad (2)$$

The covariant momenta  $\pi_a^\mu$  are defined by:

$$\pi_a^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \quad (3)$$

If we now write the Lagrangian in the following covariant Hamiltonian form

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<sup>†</sup>If attention is paid to other features of the Hamiltonian in Mechanics, its conservation properties for instance, the energy momentum tensor  $\Theta^\mu{}_\nu = \phi^a{}_{,\nu} \pi_a^\mu - \delta_\nu^\mu \mathcal{L}$  may appear to be a more natural generalization in field theory. However, for the purposes of the present letter, the covariant Hamiltonian  $\mathcal{H}$  is equally good and allows us to keep in line with Good’s proposal. For the sake of being specific, we shall limit our present discussion to this case only.

$$\mathcal{L} = \pi_a^\mu \partial_\mu \phi^a - \mathcal{H}(\phi^a, \pi_a^\mu) \quad (4)$$

its Lagrange equations of motion will also have a covariant Hamiltonian form:

$$\partial_\mu \phi^a = \frac{\partial \mathcal{H}}{\partial \pi_a^\mu} \quad (5)$$

$$\partial_\mu \pi_a^\mu = -\frac{\partial \mathcal{H}}{\partial \phi^a} \quad (6)$$

With these ingredients, Good postulated the following quantum equation of motion [9] (see also [10]):

### Good's quantum equation of motion

$$-\frac{\partial^2}{\partial x^\nu \partial x_\nu} \Psi(\varphi^a, x) = \widehat{\mathcal{H}} \Psi(\varphi^a, x) \quad (7)$$

This equation of motion was supplemented with the following rules of quantization:

### Good's quantization rules

$$\begin{aligned} \varphi^a &\longrightarrow \widehat{\varphi}^a = \varphi^a \\ \pi_a^\mu &\longrightarrow \widehat{\pi}_a^\mu = -\frac{\partial^2}{\partial \varphi^a \partial x_\mu} \end{aligned} \quad (8)$$

This proposal, along with many attractive features, involves a number of undesired properties which prevent it from being a good starting point for a finite-dimensional formulation of QFT:

- a) The equation of motion is higher order in derivatives and (at least) second order in (space-)time derivatives.
- b) The proposal does not, in either the quantization rules or the evolution equation, reproduce Quantum Mechanics in the particular case of a mechanical system.

Either of these drawbacks is serious enough to rule out this proposal as a good candidate for a finite-dimensional QFT. Moreover, it was shown in ref. [10] that this theory leads to (measurable) predictions which do not agree with standard Quantum Mechanics. Hence, this proposal should be discarded.

Fortunately, there are other proposals for quantization rules and evolution equations which are similar to Good's but behave much better.

To motivate our proposal, let us consider the ordinary harmonic oscillator and the Dirac field. The respective Lagrangians can be written:

$$\mathcal{L}_{HO} = a^*(i\dot{a} - a) \tag{9}$$

$$\mathcal{L}_D = \bar{\varphi}(i\rlap{\not{\partial}}\varphi - \varphi) \tag{10}$$

where  $a$  ( $a^*$ ) is the annihilation (creation) operator,  $\rlap{\not{\partial}} \equiv \gamma^\mu \partial_\mu$ , with  $\gamma_\mu$  the Dirac's matrices, and  $\bar{\varphi} = \varphi^\dagger \gamma^0$ .

Eqs. (9) and (10) tell us that the Dirac field is a higher-dimensional generalization of the ordinary harmonic oscillator. The generalization is accomplished by replacing the time derivative  $d/dt$  with the operator  $\rlap{\not{\partial}} = \gamma^\mu \partial_\mu$ .

Mimicking that generalization, we can postulate the following equation of motion for our finite-dimensional QFT, which is intended to generalize ordinary QM<sup>‡</sup>

### Dirac-like quantum equation of motion

$$i\Gamma^\mu \partial_\mu \Psi = \widehat{\mathcal{H}}\Psi \tag{11}$$

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<sup>‡</sup>While the present letter was being refereed, Kanatchikov, independently and following different reasoning from ours, also arrived to this equation, and to other conclusions which are similar to ours [12].

Here  $\Gamma_\mu$  are quantities which play a role similar to Dirac's matrices in the relativistic theory of the electron. However, we shall see later that further development of the theory may require the quantities  $\Gamma_\mu$  not to be the Dirac matrices. For the moment, and for the sake of specificity, we may think of these as if they were the Dirac matrices, the Kemmer matrices, or similar ones.

The next step is to construct the operator  $\widehat{\mathcal{H}}$ ; that is, we need quantization rules. Fortunately, once the quantities  $\Gamma_\mu$  are at our disposal, it is straightforward to propose quantization rules as well. These are:

### Dirac-like quantization rules

$$\begin{aligned}\varphi^a &\longrightarrow \widehat{\varphi}^a = \varphi^a \\ \pi_a^\mu &\longrightarrow \widehat{\pi}_a^\mu = -i\Gamma^\mu \frac{\partial}{\partial \varphi^a}\end{aligned}\tag{12}$$

The rules of quantization (12) and the evolution equation (11) fulfill the three properties listed in the Introduction. In particular, unlike in Good's proposal, ordinary Quantum Mechanics is contained in this new proposal. Therefore, the vast amount of experimental predictions of ordinary QM is entirely and automatically incorporated into our proposal. Hence, to rule out the new proposal we would have to look for a test which implied a genuine field system.

Our proposal, as far as the quantum equation of motion is concerned, almost revives Good's proposal. In fact, if we identify the quantities  $\Gamma_\mu$  with the Dirac matrices and "square" the equation of motion (11), we obtain an equation similar to Good's, differing only in that the operator in its r.h.s. is not the covariant Hamiltonian  $\widehat{\mathcal{H}}$ , but rather its square. On the contrary, the quantization rules are sharply different and give rise to Hamiltonian operators which also are strongly different.



### 3 Discussion and perspectives

Our proposals for rules of quantization (12) and equations of motion (11) fulfil the three properties in the Introduction. In this way, we improve Good's proposal in fundamental respects and actually solve the most serious objections which have been raised against it [10].

A detailed analysis of the quantum theories that our proposal – and related ones – would lead to is a complex task which is beyond the scope of the present letter. Let us, however, briefly comment on it.

The next step should be to find a proposal with a natural, well-behaved, positive-definite scalar product  $\langle | \rangle$ . In particular, and in analogy with QM, it seems natural to require that the scalar product of two wave functions  $\Psi, \Phi$  should be space-time independent:

$$\partial_\mu \langle \Psi | \Phi \rangle = 0 \quad (13)$$

This can be achieved if the equations of motion can be brought to the form

$$i\partial_\mu \Psi = \widehat{\mathcal{H}}_\mu \Psi \quad (14)$$

with  $\mathcal{H}_\mu$  self-adjoint operators.

Consider now that in our proposal we take  $\Gamma_\mu$  to be the Dirac matrices  $\gamma_\mu$ . The natural “scalar product” is then:

$$\langle \Psi | \Phi \rangle = \int d\varphi \bar{\Psi} \Phi, \quad \bar{\Psi} = \Psi^\dagger \gamma^0 \quad (15)$$

However, neither is the norm  $||\Psi||^2 = \langle \Psi | \Psi \rangle$  positive-definite nor is the product (15) preserved by the space-time evolution.

The first problem could be solved in a manner similar to the way in which the non-positivity of the Hamiltonian is solved when second quantizing the

Dirac field – by requiring that the wave function  $\Psi$  be not a real field but rather a Grassmannian one. However, this solution raises the question of whether or not the resultant scalar product is independent on the representation of the gamma matrices.

That the scalar product is space-time dependent can be seen by considering wave functions  $\Psi$  for which  $\widehat{\mathcal{H}}\Psi = \mu\Psi$ , with  $\mu \in \mathfrak{R}$ . Then eq. (11) reduces to the Dirac equation, which have solutions in which the scalar product (15) is not space-time independent. This problem can be attributed to the fact that, for the case under consideration, the operators  $\widehat{\mathcal{H}}_\mu$  that appear on the r.h.s. of eq. (14) involve space-time derivatives. In fact, if we multiply eq. (11) by  $\gamma_\mu$  and use the equality

$$\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\Sigma_{\mu\nu} \quad (16)$$

we get

$$i\partial_\mu\Psi = \left(i\Sigma_{\mu\nu}\partial^\nu + \gamma_\mu\widehat{\mathcal{H}}\right)\Psi \equiv \widehat{\mathcal{H}}_\mu\Psi \quad (17)$$

Therefore, a satisfactory development of our proposal would require quantities  $\Gamma_\mu$  which are not the Dirac matrices. We should remark here that no reason has actually been put forward to identify the quantities  $\Gamma_\mu$  with the Dirac matrices. By now these quantities remain (almost) completely arbitrary; the only requirements are that the equations in which they are involved should be covariant under changes of reference frame. This can be achieved with Dirac's matrices, but also with Kemmer's and others. The hope is that further development of our proposal will put stronger restrictions on those quantities and eventually determine them completely. This would ultimately justify our introduction of them, which may appear to be a rather arbitrary ingredient of our proposal.

A particularly interesting line of development would be to consider, rather than the covariant Hamiltonian  $\mathcal{H}$ , the energy-momentum tensor  $\Theta_{\mu\nu}$ . In this regard the developments in ref. [7], where the classical dynamics of fields is developed in terms of that quantity and generalized Poisson brackets, may be especially valuable.

Finally, and for the sake of comparison, let us briefly show what the situation is in the standard QFT.

In the pure Heisenberg picture of (standard) QFT, the momentum operators  $P_\mu$  are such that

$$e^{iP_\mu a^\mu} \widehat{O}_H(x) e^{-iP_\mu a^\mu} = \widehat{O}_H(x + a) \quad (18)$$

for any operator  $\widehat{O}_H(x)$ , in particular the fields  $\widehat{\varphi}(x)$ .

By analogy with the Schrödinger picture of QM, let us remove *all* the space-time dependence from the operators and translate it into the wave functional by defining:

$$\begin{aligned} \widehat{O}_S &= e^{-iP_\mu x^\mu} \widehat{O}_H(x) e^{iP_\mu x^\mu} \\ \Psi(x)_S &= e^{iP_\mu x^\mu} \Psi_H \end{aligned} \quad (19)$$

The wave functionals  $\Psi(x)_S$  now obeys generalized Schrödinger equations (14):

$$i\partial_\mu \Psi(x)_S = P_\mu \Psi(x)_S \quad (20)$$

In this way, we have recovered a picture of the standard QFT (which can be called *pure Schrödinger* picture of QFT), which incorporates much of the spirit of the finite-dimensional QFT, although not the basic requirement – that the wave functions must be ordinary functions. On the contrary, the

functional  $\Psi$  in eqs. (19,20) are defined, not over the space of fields  $\varphi^a$ , but over the *phase space* of the theory which, in general, is infinite-dimensional [11].

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