View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Digital.CSIC

IMAFF-FM-97/04, NIKHEF-97-048
HUTP-97/A055

Transmutations between Singular and Subsingular Vectors of the N=2 Superconformal Algebras

Matthias Dörrzapf[a] **and Beatriz Gato-Rivera**[b,c] *

[a] *Lyman Laboratory, Harvard University, Cambridge, 02138 MA, USA*

[b] *Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, Madrid 28006, Spain*

[c] *NIKHEF, Kruislaan 409, NL-1098 SJ Amsterdam, The Netherlands*

arXiv:hep-th/9712085v2 13 Dec 1999

ABSTRACT

We present subsingular vectors of the N=2 superconformal algebras other than the ones which become singular in chiral Verma modules, reported recently by Gato-Rivera and Rosado. We show that two large classes of singular vectors of the Topological algebra become subsingular vectors of the Antiperiodic NS algebra under the topological untwistings. These classes consist of BRST- invariant singular vectors with relative charges $q = -2, -1$ and zero conformal weight, and no-label singular vectors with $q = 0, -1$. In turn the resulting NS subsingular vectors are transformed by the spectral flows into subsingular and singular vectors of the Periodic R algebra. We write down these singular and subsingular vectors starting from the topological singular vectors at levels 1 and 2.

March 1999

*e-mail addresses: matthias@feynman.harvard.edu, bgato@imaff.cfmac.csic.es

1 Introduction

The N=2 Superconformal algebras provide the symmetries underlying the N=2 strings [1] [2]. These seem to be related to M-theory since many of the basic objects of M-theory are realized in the heterotic (2,1) N=2 strings [3]. In addition, the topological version of the algebra is realized in the world-sheet of the bosonic string [4], as well as in the world-sheet of the superstrings [5].

Recently, subsingular vectors of the N=2 Superconformal algebras were discovered in refs. [6][7]. These are states which become singular vectors (i.e. highest weight null vectors) only in the quotient of a Verma module by a submodule generated by singular vectors. Using a different language subsingular vectors can be viewed as becoming singular in incomplete Verma modules with constraints.

The only explicit examples of N=2 subsingular vectors reported so far [6][7] become singular in chiral Verma modules. That is, they are singular either in chiral Verma modules of the Topological algebra, or in chiral and antichiral Verma modules of the Antiperiodic NS algebra, or in Verma modules of the Periodic R algebra built on the Ramond ground states. The chiral Verma modules are incomplete, constrained Verma modules which correspond to the quotient of complete Verma modules by submodules generated by singular vectors (level-zero singular vectors in the case of the Topological and R algebras, and level-1/2 singular vectors in the case of the NS algebra).

In this paper we present N=2 subsingular vectors other than the ones which become singular in chiral Verma modules. In section 2 we review briefly some basic concepts and results related to the N=2 Superconformal algebras in order to facilitate the reading of the main text. In section 3 we show that two large classes of singular vectors of the Topological algebra become subsingular vectors of the NS algebra under the topological untwistings. These in turn are mapped to subsingular as well as to singular vectors of the R algebra by the spectral flows. In section 4 we write down these singular and subsingular vectors starting from the topological singular vectors at levels 1 and 2. Section 5 is devoted to conclusions and final remarks.

2 Basic Concepts

2.1 N=2 Superconformal algebras

The (non-topological) N=2 Superconformal algebras [1][8][9][10][11] can be expressed as

$$
\begin{array}{lcllcl}
[L_m, L_n] & = & (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}\,, & [H_m, H_n] & = & \frac{c}{3}m\delta_{m+n,0}\,, \\
[L_m, G_r^{\pm}] & = & \left(\frac{m}{2}-r\right)G_{m+r}^{\pm}\,, & [H_m, G_r^{\pm}] & = & \pm G_{m+r}^{\pm}\,, \\
[L_m, H_n] & = & -\,nH_{m+n} & & & \\
\{G_r^-, G_s^+\} & = & 2L_{r+s} - (r-s)H_{r+s} + \frac{c}{3}(r^2-\frac{1}{4})\delta_{r+s,0}\,, & & &
\end{array}
\tag{2.1}
$$

where L_m and H_m are the spin-2 and spin-1 bosonic generators corresponding to the stress-energy momentum tensor and the U(1) current, respectively, and G_r^+ and G_r^- are the spin-3/2 fermionic generators. These are half-integer moded for the case of the Antiperiodic NS algebra, and integer moded for the case of the Periodic R algebra. The eigenvalues of the bosonic zero modes (L_0, H_0) are the conformal weight and the U(1) charge of the states. These are split conveniently as ($\Delta + l$, $h + q$) for secondary states, where l and q are the level and the relative charge of the state and (Δ, h) are the conformal weight and U(1) charge of the primary on which the secondary is built.

Observe that we unify the notation for the $U(1)$ charge of the states of the NS algebra and the states of the R algebra since the $U(1)$ charges of the R states will be denoted by h, instead of $h \pm \frac{1}{2}$ which was commonly used in the past [12], and their relative charges q are defined to be integer, like for the NS states.

There is also the twisted N=2 algebra for which the generators of the U(1) current are half-integer moded. As a consequence there is no U(1) charge for this algebra.

The Topological N=2 Superconformal algebra reads [13]

$$
\begin{array}{lcllcl}
[\mathcal{L}_m, \mathcal{L}_n] & = & (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m, \mathcal{H}_n] & = & \frac{c}{3}m\delta_{m+n,0}\,, \\
[\mathcal{L}_m, \mathcal{G}_n] & = & (m-n)\mathcal{G}_{m+n}\,, & [\mathcal{H}_m, \mathcal{G}_n] & = & \mathcal{G}_{m+n}\,, \\
[\mathcal{L}_m, \mathcal{Q}_n] & = & -n\mathcal{Q}_{m+n}\,, & [\mathcal{H}_m, \mathcal{Q}_n] & = & -\mathcal{Q}_{m+n}\,, \qquad m,\ n \in \mathbf{Z}\,. \\
[\mathcal{L}_m, \mathcal{H}_n] & = & -n\mathcal{H}_{m+n} + \frac{c}{6}(m^2+m)\delta_{m+n,0}\,, & & & \\
\{\mathcal{G}_m, \mathcal{Q}_n\} & = & 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{3}(m^2+m)\delta_{m+n,0}\,, & & &
\end{array}
\tag{2.2}
$$

where the fermionic generators $\mathcal{Q}_m$ and $\mathcal{G}_m$ correspond to the spin-1 BRST current and the spin-2 fermionic current, respectively, $\mathcal{Q}_0$ being the BRST-charge. The eigenvalues of ($\mathcal{L}_0$, $\mathcal{H}_0$) are split, as before, as ($\Delta + l$, $h + q$). This algebra is topological because the Virasoro generators can be expressed as $\mathcal{L}_m = \frac{1}{2}\{\mathcal{G}_m, \mathcal{Q}_0\}$. This implies, as is well known, that the correlators of the fields do not depend on the metric.

2.2 Topological twists

The Topological algebra (2.2) can be viewed as a rewriting of the algebra (2.1) using one of the two topological twists:

$$\begin{aligned}
\mathcal{L}^{(1)}_m &= L_m + \tfrac{1}{2}(m+1)H_m\,, \\
\mathcal{H}^{(1)}_m &= H_m\,, \\
\mathcal{G}^{(1)}_m &= G^+_{m+\frac{1}{2}}\,, \qquad \mathcal{Q}^{(1)}_m = G^-_{m-\frac{1}{2}}\,,
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
\mathcal{L}^{(2)}_m &= L_m - \tfrac{1}{2}(m+1)H_m\,, \\
\mathcal{H}^{(2)}_m &= -H_m\,, \\
\mathcal{G}^{(2)}_m &= G^-_{m+\frac{1}{2}}\,, \qquad \mathcal{Q}^{(2)}_m = G^+_{m-\frac{1}{2}}\,,
\end{aligned} \tag{2.4}$$

which we denote as T_{W1} and T_{W2}, respectively. Observe that the two twists are mirrored under the exchange $H_m \rightarrow -H_m$, $G^+_r \leftrightarrow G^-_r$. In particular $(G^+_{1/2}, G^-_{-1/2})$ results in $(\mathcal{G}^{(1)}_0, \mathcal{Q}^{(1)}_0)$, while $(G^-_{1/2}, G^+_{-1/2})$ gives $(\mathcal{G}^{(2)}_0, \mathcal{Q}^{(2)}_0)$, so that the zero mode $\mathcal{G}_0$ corresponds to the positive modes $G^{\pm}_{1/2}$ of the NS algebra. As a result, all the highest weight (h.w.) states of the NS algebra (primary states and singular vectors) are transformed, under T_{W1} and T_{W2}, into states of the Topological algebra annihilated by $\mathcal{G}_0$ which are also h.w. states, as the reader can easily verify. The other way around, all the h.w. states of the Topological algebra annihilated by $\mathcal{G}_0$ are transformed under T_{W1} and T_{W2} into h.w. states of the NS algebra. The zero mode $\mathcal{Q}_0$, in turn, corresponds to the negative modes $G^{\mp}_{-1/2}$ of the NS algebra. Therefore the topological states annihilated by $\mathcal{Q}_0$ become antichiral and chiral states of the NS algebra (annihilated by $G^-_{-1/2}$ and by $G^+_{-1/2}$, respectively) under the twists T_{W1} and T_{W2}.

2.3 Spectral flows

The spectral flows $\mathcal{U}_\theta$ and $\mathcal{A}_\theta$ are one-parameter families of transformations providing a continuum of isomorphic N=2 Superconformal algebras. The 'usual' spectral flow $\mathcal{U}_\theta$ [9][11][15][16] is even, given by

$$
\begin{aligned}
\mathcal{U}_\theta\, L_m\, \mathcal{U}_\theta^{-1} &= L_m + \theta H_m + \tfrac{\mathsf{c}}{6}\theta^2 \delta_{m,0}\,, \\
\mathcal{U}_\theta H_m\, \mathcal{U}_\theta^{-1} &= H_m + \tfrac{\mathsf{c}}{3}\theta \delta_{m,0}\,, \\
\mathcal{U}_\theta\, G_r^+\, \mathcal{U}_\theta^{-1} &= G_{r+\theta}^+\,, \\
\mathcal{U}_\theta\, G_r^-\, \mathcal{U}_\theta^{-1} &= G_{r-\theta}^-\,,
\end{aligned}
\tag{2.5}
$$

satisfying $\mathcal{U}_\theta^{-1} = \mathcal{U}_{(-\theta)}$. For $\theta = 0$ it is just the identity operator, i.e. $\mathcal{U}_0 = \mathbf{1}$. It transforms the (L_0, H_0) eigenvalues, i.e. the conformal weight and the U(1) charge, (Δ, h) of a given state as $(\Delta - \theta h + \frac{\mathsf{c}}{6}\theta^2,\, h - \frac{\mathsf{c}}{3}\theta)$. From this one gets straightforwardly that the level l of any secondary state changes to $l - \theta q$, while the relative charge q remains equal.

The spectral flow $\mathcal{A}_\theta$ [15][16] is odd, given by

$$
\begin{aligned}
\mathcal{A}_\theta\, L_m\, \mathcal{A}_\theta^{-1} &= L_m + \theta H_m + \tfrac{\mathsf{c}}{6}\theta^2 \delta_{m,0}\,, \\
\mathcal{A}_\theta H_m\, \mathcal{A}_\theta^{-1} &= -H_m - \tfrac{\mathsf{c}}{3}\theta \delta_{m,0}\,, \\
\mathcal{A}_\theta\, G_r^+\, \mathcal{A}_\theta^{-1} &= G_{r-\theta}^-\,, \\
\mathcal{A}_\theta\, G_r^-\, \mathcal{A}_\theta^{-1} &= G_{r+\theta}^+\,,
\end{aligned}
\tag{2.6}
$$

satisfying $\mathcal{A}_\theta^{-1} = \mathcal{A}_\theta$. It is therefore an involution. The odd spectral flow $\mathcal{A}_\theta$ is 'quasi' mirror symmetric to the even spectral flow $\mathcal{U}_\theta$, under the exchange $H_m \to -H_m$, $G_r^+ \leftrightarrow G_r^-$ and $\theta \to -\theta$, and it is in fact the only fundamental spectral flow since it generates the latter [16]. For $\theta = 0$ it is the mirror map, i.e. $\mathcal{A}_0 = \mathcal{M}$. It transforms the (L_0, H_0) eigenvalues of the states as $(\Delta + \theta h + \frac{\mathsf{c}}{6}\theta^2, -h - \frac{\mathsf{c}}{3}\theta)$. The level l of the secondary states changes to $l + \theta q$, while the relative charge q reverses its sign.

For half-integer values of θ the two spectral flows interpolate between the NS algebra and the R algebra. In particular, for $\theta = \pm 1/2$ the h.w. states of the NS algebra are transformed into h.w. states of the R algebra with helicities $(\mp)$(i.e. annihilated by G_0^- and G_0^+, respectively). As a result the NS singular vectors are transformed into R singular vectors with helicities $(\mp)$ built on R primaries with the same helicities.

By performing the topological twists T_{W1} (2.3) and T_{W2} (2.4) on the spectral flows one obtains the topological spectral flows [14][16].

3 Transmutation between N=2 Singular and Subsingular Vectors

In what follows the singular vectors of the Topological algebra, the NS algebra and the R algebra will be denoted as $|\chi\rangle$, $|\chi_{NS}\rangle$ and $|\chi_R\rangle$, respectively. The states which are not singular, like the subsingular vectors, will be denoted as $|\Upsilon\rangle$, $|\Upsilon_{NS}\rangle$ and $|\Upsilon_R\rangle$, respectively.

3.1 Topological singular vectors

It has recently been shown [7][17] that the singular vectors of the Topological algebra can be classified in 29 different types in complete Verma modules, and in 4 different types in chiral Verma modules, regarding the relative charge q and the BRST-invariance properties of the singular vectors and of the primaries on which they are built. Namely, there are ten types of topological singular vectors built on $\mathcal{G}_0$-closed primaries $|\Delta,\, h\rangle^G$ (i.e. annihilated by $\mathcal{G}_0$), ten types of topological singular vectors built on $\mathcal{Q}_0$-closed primaries $|\Delta,\, h\rangle^Q$ (i.e. annihilated by $\mathcal{Q}_0$), four types built on chiral primaries $|0,\, h\rangle^{G,Q}$ (annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$), and nine types built on no-label primaries $|0,\, h\rangle$ (which cannot be expressed as linear combinations of $\mathcal{G}_0$-closed, $\mathcal{Q}_0$-closed and chiral primaries). The no-label and the chiral topological states (primaries as well as secondaries) have zero conformal weight, as deduced from the anticommutator $\{\mathcal{G}_0, \mathcal{Q}_0\} = 2\mathcal{L}_0$.

The $\mathcal{Q}_0$-closed and no-label h.w. states (primaries or singular vectors) are transformed, under T_{W1} (2.3) and T_{W2} (2.4), into states of the NS algebra which are not h.w. states since they are not annihilated by one of the modes $G^{\pm}_{1/2}$ (because the topological state is not annihilated by $\mathcal{G}_0$). The $\mathcal{G}_0$-closed and chiral h.w. states, however, are transformed into h.w. states of the NS algebra. In particular the $\mathcal{G}_0$-closed primaries $|\Delta,\, h\rangle^G$ give rise to the topological Verma modules which are transformed into the NS Verma modules, whereas the chiral primaries $|0,\, h\rangle^{G,Q}$ generate topological chiral (incomplete) Verma modules which are transformed into antichiral and chiral (incomplete) Verma modules of the NS algebra under T_{W1} and T_{W2}, respectively.

In what follows we will restrict our attention to the topological singular vectors built on $\mathcal{G}_0$-closed primaries $|\Delta,\, h\rangle^G$. The ten different types of singular vectors in this case are shown in the table* below [7][17]. The chiral and no-label singular vectors have zero conformal weight, therefore they satisfy $\Delta + l = 0$.

*The results of this table were conjectured in ref. [7] and have been rigorously proved in ref. [17].

$$
\begin{array}{r|llll}
 & q=-2 & q=-1 & q=0 & q=1 \\
\hline
 & & & & \\
\mathcal{G}_0\text{-closed} & - & |\chi\rangle_l^{(-1)G} & |\chi\rangle_l^{(0)G} & |\chi\rangle_l^{(1)G} \\
\mathcal{Q}_0\text{-closed} & |\chi\rangle_l^{(-2)Q} & |\chi\rangle_l^{(-1)Q} & |\chi\rangle_l^{(0)Q} & - \\
\text{chiral} & - & |\chi\rangle_l^{(-1)G,Q} & |\chi\rangle_l^{(0)G,Q} & - \\
\text{no-label} & - & |\chi\rangle_l^{(-1)} & |\chi\rangle_l^{(0)} & -
\end{array}
\qquad (3.1)
$$

An important observation is that chiral singular vectors can be viewed as particular cases of $\mathcal{G}_0$-closed singular vectors and/or as particular cases of $\mathcal{Q}_0$-closed singular vectors, which 'become' chiral when the conformal weight turns out to be zero. In particular, the singular vectors of types $|\chi\rangle_l^{(0)Q}$ and $|\chi\rangle_l^{(-1)G}$ in table (3.1) always become chiral when the conformal weight is zero [20][17].

The spectrum of $(\Delta,\, h)$ corresponding to the $\mathcal{G}_0$-closed primaries which contain singular vectors in their Verma modules has been derived in ref. [7] for the $\mathcal{G}_0$-closed, for the $\mathcal{Q}_0$-closed, and (partially) for the chiral singular vectors. For the latter it has been derived completely in ref. [18] together with the spectrum corresponding to no-label singular vectors.

3.2 Transmutation between topological and NS singular and subsingular vectors

In this subsection we will identify two large classes of topological singular vectors, built on $\mathcal{G}_0$-closed primaries, which become subsingular vectors of the NS algebra under the topological untwistings. These classes consist of $\mathcal{Q}_0$-closed (BRST-invariant) singular vectors with zero conformal weight, which only exist for relative charges $q=-2,-1$, and no-label singular vectors, which only exist for $q=0,-1$.

$\mathcal{Q}_0$-closed singular vectors

Let us consider a $\mathcal{Q}_0$-closed topological singular vector $|\chi\rangle_l^{(q)Q}$, from table (3.1), with zero conformal weight. The action of $\mathcal{G}_0$ on this singular vector produces another singular vector with zero conformal weight, at level l, with relative charge $q+1$, and annihilated by $\mathcal{G}_0$. Moreover, since $\Delta+l=0$ this singular vector is also annihilated by $\mathcal{Q}_0$:

$$
\mathcal{Q}_0\, \mathcal{G}_0\, |\chi\rangle_l^{(q)Q} = -\mathcal{G}_0\, \mathcal{Q}_0\, |\chi\rangle_l^{(q)Q} + 2\, \mathcal{L}_0\, |\chi\rangle_l^{(q)Q} = 2(\Delta+l)\, |\chi\rangle_l^{(q)Q} = 0\,. \qquad (3.2)
$$

As a result, $|\chi\rangle_l^{(q+1)G,Q} = \mathcal{G}_0\, |\chi\rangle_l^{(q)Q}$ is a chiral singular vector which cannot 'come back' to the singular vector $|\chi\rangle_l^{(q)Q}$ by acting with the algebra (the chiral singular vector is therefore a level-zero secondary singular vector with respect to this one). An important observation now is that chiral singular vectors of type $|\chi\rangle_l^{(1)G,Q}$ do not exist, according to table (3.1). As a consequence the singular vectors of type $|\chi\rangle_l^{(0)Q}$ with zero conformal weight, built on $\mathcal{G}_0$-closed primaries, are absent too, 'becoming' all of them chiral; that is, of type $|\chi\rangle_l^{(0)G,Q}$ instead (as $\mathcal{G}_0\,|\chi\rangle_l^{(0)Q} = 0$).

Let us repeat this procedure from the viewpoint of the NS algebra. That is, let us untwist every step using T_{W1} (2.3) and T_{W2} (2.4). To start, the $\mathcal{Q}_0$-closed topological singular vector $|\chi\rangle_l^{(q)Q}$, built on the primary $|-l,\,h\rangle^G$, is not transformed into singular vectors of the NS algebra but into mirrored antichiral and chiral states $|\Upsilon_{NS}\rangle_{l-q/2}^{(\pm q)\mp}$, built on the mirrored NS primaries $|-l-h/2, \pm h\rangle$, which are not annihilated by $G_{1/2}^{\mp}$ (the upper signs using T_{W1} and the lower signs using T_{W2}, the superscript $(-)$ for antichiral and $(+)$ for chiral). These states are annihilated, however, by all other positive modes of the NS algebra, so that $G_{1/2}^{\pm}\,|\Upsilon_{NS}\rangle_{l-q/2}^{(\pm q)\mp}$ produces truly singular vectors of the NS algebra which are antichiral and chiral respectively. That is, $|\chi_{NS}\rangle_{l-(q+1)/2}^{(q+1)-} = G_{1/2}^{+}\,|\Upsilon_{NS}\rangle_{l-q/2}^{(q)-}$ is an antichiral NS singular vector (annihilated by $G_{-1/2}^{-}$), in the Verma module $V_{NS}(|-l-h/2,\,h\rangle)$, whereas $|\chi_{NS}\rangle_{l-(q+1)/2}^{(-q-1)+} = G_{1/2}^{-}\,|\Upsilon_{NS}\rangle_{l-q/2}^{(-q)+}$ is the mirrored chiral NS singular vector (annihilated by $G_{-1/2}^{+}$), in the mirrored Verma module $V_{NS}(|-l-h/2,\,-h\rangle)$. From these singular vectors it is not possible to reach back the states $|\Upsilon_{NS}\rangle_{l-q/2}^{(\pm q)\mp}$, respectively, which are located therefore outside the (incomplete) Verma modules built on the singular vectors.

Hence the antichiral and chiral NS states $|\Upsilon_{NS}\rangle_{l-q/2}^{(\pm q)\mp}$, obtained by untwisting the $\mathcal{Q}_0$-closed topological singular vectors with zero conformal weight, are mirrored subsingular vectors of the NS algebra in the mirrored Verma modules $V_{NS}(|-l-h/2,\,\pm h\rangle)$. Once the quotients of the Verma modules by the corresponding singular vectors are performed, by setting these to zero: $G_{1/2}^{\pm}\,|\Upsilon_{NS}\rangle_{l-q/2}^{(\pm q)\mp} = 0$, the subsingular vectors become singular because they recover the only missing h.w. condition.

The spectrum of U(1) charges h corresponding to the $\mathcal{G}_0$-closed primaries $|-l,\,h\rangle^G$ which contain $\mathcal{Q}_0$-closed singular vectors at level l in their Verma modules can be deduced from the general formulae given in refs. [7][18]. As to the chiral singular vectors, the corresponding spectrum has been given in ref. [18]. One obtains the following results. The topological Verma module $V_T(|-l,\,h\rangle^G)$ contains two singular vectors at level l of the types $|\chi\rangle_l^{(-2)Q}$ and $|\chi\rangle_l^{(-1)G,Q}$ for

$$h = \frac{t}{2}(1+l) + 1\ , \tag{3.3}$$

and two singular vectors at level l of the types $|\chi\rangle_l^{(-1)Q}$ and $|\chi\rangle_l^{(0)G,Q}$ for

$$h = \frac{t(1-r)-s}{2}\ , \tag{3.4}$$

where $t = \frac{3-\mathtt{c}}{3}$ and $(r,\, s)$ are two positive integers (s even) such that $l = \frac{rs}{2}$. For the discrete values

$$t = -\frac{s}{n}\,, \qquad n = 1, ..., r\,, \tag{3.5}$$

however, $|\chi\rangle_l^{(-1)Q}$ also becomes chiral, i.e. of type $|\chi\rangle_l^{(-1)G,Q}$ instead.

No-label singular vectors

Now let us consider a no-label singular vector $|\chi\rangle_l^{(q)}$ from table (3.1). This vector cannot be expressed as a linear combination of $\mathcal{G}_0$-closed, $\mathcal{Q}_0$-closed and chiral singular vectors, and has zero conformal weight necessarily, i.e. $\Delta + l = 0$. The action of $\mathcal{G}_0$ on $|\chi\rangle_l^{(q)}$ produces a $\mathcal{G}_0$-closed singular vector $|\chi\rangle_l^{(q+1)G}$ with zero conformal weight at the same level. The action of $\mathcal{Q}_0$ on $|\chi\rangle_l^{(q+1)G}$ now produces a chiral singular vector $|\chi\rangle_l^{(q)G,Q} = \mathcal{Q}_0\mathcal{G}_0|\chi\rangle_l^{(q)}$, for the same reasons as in the previous case.

From the viewpoint of the NS algebra $|\chi\rangle_l^{(q)}$ does not correspond to a singular vector since it is not annihilated by $\mathcal{G}_0$ and therefore is not annihilated by one of the modes $G^{\pm}_{1/2}$ under the untwistings, as happened with the $\mathcal{Q}_0$-closed topological singular vectors in the previous case. Let us denote as $|\Upsilon_{NS}\rangle^{(\pm q)}_{l-q/2}$ the mirrored states, built on the mirrored primaries $|{-l}-h/2, \pm h\rangle$, obtained under the untwistings of $|\chi\rangle_l^{(q)}$. The action of $G^{\pm}_{1/2}$ on these states produce NS singular vectors $|\chi_{NS}\rangle^{(\pm(q+1))}_{l-(q+1)/2} = G^{\pm}_{1/2}|\Upsilon_{NS}\rangle^{(\pm q)}_{l-q/2}$. From these, however, it is not possible to reach the previous non-singular states $|\Upsilon_{NS}\rangle^{(\pm q)}_{l-q/2}$ because the action of $G^{\mp}_{-1/2}$ on the singular vectors produces antichiral and chiral NS singular vectors instead (the untwistings of $|\chi\rangle_l^{(q)G,Q}$). That is, $G^{\mp}_{-1/2}|\chi_{NS}\rangle^{(\pm(q+1))}_{l-(q+1)/2} = |\chi_{NS}\rangle^{(\pm q)\mp}_{l-q/2}$.

Hence the untwistings of the no-label topological singular vectors $|\chi\rangle_l^{(q)}$ produce mirrored subsingular vectors $|\Upsilon_{NS}\rangle^{(\pm q)}_{l-q/2}$ of the NS algebra which become singular when the singular vectors $|\chi_{NS}\rangle^{(\pm(q+1))}_{l-(q+1)/2} = G^{\pm}_{1/2}\,|\Upsilon_{NS}\rangle^{(\pm q)}_{l-q/2}$ are set to zero. Observe that the antichiral and chiral NS singular vectors $|\chi_{NS}\rangle^{(\pm q)\mp}_{l-q/2}$ also "go away" because they are descendant, secondary singular vectors of the singular vectors $|\chi_{NS}\rangle^{(\pm(q+1))}_{l-(q+1)/2}$.

There is also the possibility of acting with $\mathcal{Q}_0$ on $|\chi\rangle_l^{(q)}$. In this case one obtains a $\mathcal{Q}_0$-closed topological singular vector $|\chi\rangle_l^{(q-1)Q}$ with zero conformal weight, like the ones we analyzed before. Therefore $|\chi\rangle_l^{(q)}$ and $|\chi\rangle_l^{(q-1)Q} = \mathcal{Q}_0\,|\chi\rangle_l^{(q)}$, both in the topological Verma module $V_T(|{-l},\, h\rangle^G)$, are transformed into subsingular vectors of the NS algebra under the topological untwistings. Namely, $|\chi\rangle_l^{(q)}$ is transformed into non-chiral mirrored NS subsingular vectors of the types $|\Upsilon_{NS}\rangle^{(\pm q)}_{l-q/2}$, in the mirrored Verma modules $V_{NS}(|{-l}-h/2,\, \pm h\rangle)$, while $|\chi\rangle_l^{(q-1)Q}$ is transformed into antichiral and chiral mirrored NS subsingular vectors of the types $|\Upsilon_{NS}\rangle^{(\pm(q-1))\mp}_{l-(q-1)/2} = G^{\mp}_{-1/2}\,|\Upsilon_{NS}\rangle^{(\pm q)}_{l-q/2}$, in the same mirrored Verma modules.

The no-label singular vectors are very scarce since they only exist for $t = \frac{2}{r}$, i.e. $\mathsf{c} = \frac{3r-6}{r}$ [18]. Thus at level 1 they only exist for $\mathsf{c} = -3$ and at level 2 they only exist for $\mathsf{c} = -3$ and $\mathsf{c} = 0$. The spectrum of U(1) charges h corresponding to the $\mathcal{G}_0$-closed primaries $|-l,\, h\rangle^G$ which contain no-label singular vectors at level $l = \frac{rs}{2}$ in their Verma modules is [18]

$$h = \frac{1}{r} - \frac{s}{2} - 1\,, \quad t = \frac{2}{r}\,. \tag{3.6}$$

3.3 Transmutation between NS and R singular and subsingular vectors

The spectral flows $\mathcal{U}_{1/2}$ and $\mathcal{A}_{1/2}$, (2.5) and (2.6), transform the h.w. states of the NS algebra (primaries and singular vectors) into h.w. states of the R algebra annihilated by G_0^-, denoted as helicity $(-)$ states. The spectral flows $\mathcal{U}_{-1/2}$ and $\mathcal{A}_{-1/2}$, on the contrary, transform the h.w. states of the NS algebra into h.w. states of the R algebra annihilated by G_0^+, denoted as helicity $(+)$ states.

Now we will investigate the spectral flow transformations, into R states, of the NS subsingular vectors analyzed in the last subsection. For this purpose let us inspect the transformations of the "missing" h.w. conditions $G^{\pm}_{1/2}\,|\Upsilon_{NS}\rangle \neq 0$ and the transformations of the (anti)chirality conditions $G^{\mp}_{-1/2}\,|\Upsilon_{NS}\rangle = 0$, which correspond to the NS subsingular vectors obtained under the untwisting of the $\mathcal{Q}_0$-closed topological singular vectors with zero conformal weight (whereas the untwisting of the no-label topological singular vectors gives non-chiral NS subsingular vectors). The transformations of $G^{\pm}_{1/2}$ and $G^{\mp}_{-1/2}$ under $\mathcal{U}_{\pm 1/2}$ and $\mathcal{A}_{\pm 1/2}$ are:

$$\begin{array}{rclcrcl} \mathcal{U}_{1/2}\, G^+_{1/2}\, \mathcal{U}_{-1/2} &=& G^+_1 & \qquad & \mathcal{U}_{1/2}\, G^-_{-1/2}\, \mathcal{U}_{-1/2} &=& G^-_{-1} \\ \mathcal{U}_{1/2}\, G^-_{1/2}\, \mathcal{U}_{-1/2} &=& G^-_0 & & \mathcal{U}_{1/2}\, G^+_{-1/2}\, \mathcal{U}_{-1/2} &=& G^+_0 \end{array} \tag{3.7}$$

$$\begin{array}{rclcrcl} \mathcal{A}_{1/2}\, G^+_{1/2}\, \mathcal{A}_{1/2} &=& G^-_0 & \qquad & \mathcal{A}_{1/2}\, G^-_{-1/2}\, \mathcal{A}_{1/2} &=& G^+_0 \\ \mathcal{A}_{1/2}\, G^-_{1/2}\, \mathcal{A}_{1/2} &=& G^+_1 & & \mathcal{A}_{1/2}\, G^+_{-1/2}\, \mathcal{A}_{1/2} &=& G^-_{-1} \end{array} \tag{3.8}$$

$$\begin{array}{rclcrcl} \mathcal{U}_{-1/2}\, G^+_{1/2}\, \mathcal{U}_{1/2} &=& G^+_0 & \qquad & \mathcal{U}_{-1/2}\, G^-_{-1/2}\, \mathcal{U}_{1/2} &=& G^-_0 \\ \mathcal{U}_{-1/2}\, G^-_{1/2}\, \mathcal{U}_{1/2} &=& G^-_1 & & \mathcal{U}_{-1/2}\, G^+_{-1/2}\, \mathcal{U}_{1/2} &=& G^+_{-1} \end{array} \tag{3.9}$$

$$\begin{array}{rclcrcl} \mathcal{A}_{-1/2}\, G^+_{1/2}\, \mathcal{A}_{-1/2} &=& G^-_1 & \qquad & \mathcal{A}_{-1/2}\, G^-_{-1/2}\, \mathcal{A}_{-1/2} &=& G^+_{-1} \\ \mathcal{A}_{-1/2}\, G^-_{1/2}\, \mathcal{A}_{-1/2} &=& G^+_0 & & \mathcal{A}_{-1/2}\, G^+_{-1/2}\, \mathcal{A}_{-1/2} &=& G^-_0 \end{array} \tag{3.10}$$

From the expressions on the left one "reads" the missing conditions (h.w. or helicity) of the corresponding R states, whereas the expressions on the right give the constraints associated to each case (when they apply).

We see that under $\mathcal{U}_{1/2}$ the missing h.w. condition $G^+_{1/2}\,|\Upsilon_{NS}\rangle \neq 0$ on the NS subsingular vectors results in the missing h.w. condition $G^+_1\,|\Upsilon_R\rangle^- \neq 0$ on the corresponding helicity $(-)$ R states. The missing h.w. condition $G^-_{1/2}\,|\Upsilon_{NS}\rangle \neq 0$, however, results in the "missing helicity" $(-)$ on the R states. The chirality constraint $G^+_{-1/2}\,|\Upsilon_{NS}\rangle = 0$, when it applies, compensates the loss of the helicity $(-)$ providing the helicity $(+)$ to the corresponding states. The antichirality constraint $G^-_{-1/2}\,|\Upsilon_{NS}\rangle = 0$ is just transformed into the constraint $G^-_{-1}\,|\Upsilon_R\rangle^- = 0$ on the R states.

Under $\mathcal{A}_{1/2}$ the missing h.w. condition $G^+_{1/2}\,|\Upsilon_{NS}\rangle \neq 0$ leads to the "missing helicity" $(-)$, although the antichirality constraint compensates (when it applies) providing a helicity $(+)$ to the corresponding R state. The missing h.w. condition $G^-_{1/2}\,|\Upsilon_{NS}\rangle \neq 0$, on the other hand, results in the missing h.w. condition $G^+_1\,|\Upsilon_R\rangle^- \neq 0$ and the chirality constraint gives the constraint $G^-_{-1}\,|\Upsilon_R\rangle^- = 0$ on the R states.

Similar remarks apply to the transformations of the missing h.w. conditions and the transformations of the (anti)chirality constraints under $\mathcal{U}_{-1/2}$ and $\mathcal{A}_{-1/2}$. In these cases the resulting missing h.w. condition on the helicity $(+)$ R states is $G^-_1\,|\Upsilon_R\rangle^+ \neq 0$, the resulting "missing helicity" is $(+)$, and the resulting constraint on the R states (when it applies) is $G^+_{-1}\,|\Upsilon_R\rangle^+ = 0$.

From this analysis one deduces straightforwardly the following. First, the two mirrored antichiral and chiral NS subsingular vectors obtained by untwisting the $\mathcal{Q}_0$-closed topological singular vectors are mapped, under all four spectral flows $\mathcal{U}_{\pm 1/2}$ and $\mathcal{A}_{\pm 1/2}$, to one R singular vector, with $(+)$ or $(-)$ helicity, and one state which is not singular and satisfies a constraint. Second, the two mirrored non-chiral NS subsingular vectors obtained by untwisting the no-label topological singular vectors are mapped, under all four spectral flows $\mathcal{U}_{\pm 1/2}$ and $\mathcal{A}_{\pm 1/2}$, to one R singular vector without any helicity (annihilated only by the positive modes of the R algebra), and one state which is not singular.

Now let us show that the R state which is not singular, in each case, is actually a subsingular vector of the R algebra. Let us take the spectral flow $\mathcal{U}_{1/2}$, eq. (2.5). The NS subsingular vectors with the missing h.w. condition $G^+_{1/2}\,|\Upsilon_{NS}\rangle \neq 0$ are of the types $|\Upsilon_{NS}\rangle^{(q)-}_{l-q/2}$ and $|\Upsilon_{NS}\rangle^{(q)}_{l-q/2}$, depending on whether they come from the untwisting of the $\mathcal{Q}_0$-closed topological singular vectors or from the untwisting of the no-label topological singular vectors. In both cases they are located in the Verma modules $V_{NS}(|{-l}-h/2,\,h\rangle)$ (with different spectrum of h for each type). These NS subsingular vectors are transformed by $\mathcal{U}_{1/2}$ into R states of the types $|\Upsilon_R\rangle^{(q)-}_{l-q}$ (some of them with the constraint $G^-_{-1}\,|\Upsilon_R\rangle = 0$), in the Verma modules $V_R(|{-l}-h+\mathsf{c}/24,\,h-\mathsf{c}/6\rangle^-)$, which satisfy all but one of the h.w. conditions: $G^+_1\,|\Upsilon_R\rangle^{(q)-}_{l-q} \neq 0$. But $G^+_1\,|\Upsilon_R\rangle^{(q)-}_{l-q}$ is a singular vector of type $|\chi_R\rangle^{(q+1)-}_{l-q-1}$, as one can easily verify, which cannot reach back the state $|\Upsilon_R\rangle^{(q)-}_{l-q}$.

To see this let us analyze the action of G^-_{-1} on the singular vector $|\chi_R\rangle^{(q+1)-}_{l-q-1} = G^+_1 |\Upsilon_R\rangle^{(q)-}_{l-q}$:

$$G^-_{-1}G^+_1 |\Upsilon_R\rangle^{(q)-}_{l-q} = -G^+_1 G^-_{-1} |\Upsilon_R\rangle^{(q)-}_{l-q} + 2(L_0 + H_0 + \mathsf{c}/8) |\Upsilon_R\rangle^{(q)-}_{l-q} = -G^+_1 G^-_{-1} |\Upsilon_R\rangle^{(q)-}_{l-q} \tag{3.11}$$

This result vanishes for the states with the constraint $G^-_{-1} |\Upsilon_R\rangle^{(q)-}_{l-q} = 0$, which originate from the $\mathcal{Q}_0$-closed topological singular vectors. For the states without this constraint, which originate from the no-label topological singular vectors, $G^-_{-1}G^+_1 |\Upsilon_R\rangle^{(q)-}_{l-q}$ is a singular vector of type $|\chi_R\rangle^{(q)-}_{l-q}$ annihilated by G^-_{-1}, as one can easily verify.

Hence the spectral flow $\mathcal{U}_{1/2}$ transforms the NS subsingular vectors of the types $|\Upsilon_{NS}\rangle^{(q)-}_{l-q/2}$ and $|\Upsilon_{NS}\rangle^{(q)}_{l-q/2}$, in the Verma modules $V_{NS}(|-l-h/2,\, h\rangle)$, into R subsingular vectors of the types $|\Upsilon_R\rangle^{(q)-}_{l-q}$, in the Verma modules $V_R(|-l-h+\mathsf{c}/24,\, h-\mathsf{c}/6\rangle^-)$.

Similar reasonings and results apply to the spectral flow mappings $\mathcal{U}_{-1/2}$ and $\mathcal{A}_{\pm 1/2}$ on the NS subsingular vectors; that is, the R state which is not singular, in each case, is a subsingular vector of the R algebra.

4 Results from the Topological Singular Vectors at Levels 1 and 2

In this section we will present the explicit expressions for the singular and subsingular vectors that we have analyzed in the previous section, starting from the topological singular vectors at levels 1 and 2. We will follow the usual parametrization $\mathsf{c} = 3 - 3t$.

4.1 Level 1

$\mathcal{Q}_0$-closed topological singular vectors versus NS subsingular vectors

According to table (3.1), the $\mathcal{Q}_0$-closed topological singular vectors $|\chi\rangle^{(q)Q}_l$ built on $\mathcal{G}_0$-closed primaries $|\Delta,\, h\rangle^G$ only exist for relative charges $q = 0, -1, -2$. At zero conformal weight $\Delta + l = 0$, however, there are no singular vectors of the type $|\chi\rangle^{(0)Q}_l$ because all of them "become" chiral, i.e. of type $|\chi\rangle^{(0)G,Q}_l$, as we explained in section 3. The negatively charged singular vectors exist for both allowed values of q. As one can deduce from expressions (3.3) and (3.4), at level 1 these vectors can be found for $h = -1$ in the case $q = -1$ and for $h = 1 + t$ in the case $q = -2$:

$$
\begin{aligned}
|\chi\rangle_1^{(-1)Q} &= \Big\{t\mathcal{L}_{-1}\mathcal{Q}_0 - 2\mathcal{Q}_{-1} - 2\mathcal{H}_{-1}\mathcal{Q}_0\Big\}|{-1},-1\rangle^G \\
|\chi\rangle_1^{(-2)Q} &= \mathcal{Q}_{-1}\mathcal{Q}_0|{-1},1+t\rangle^G
\end{aligned} \tag{4.1}
$$

Under the action of $\mathcal{G}_0$, these vectors are mapped to chiral singular vectors $|\chi\rangle_1^{(0)G,Q}$ and $|\chi\rangle_1^{(-1)G,Q}$ respectively:

$$
\begin{aligned}
|\chi\rangle_1^{(0)G,Q} &= \mathcal{G}_0|\chi\rangle_1^{(-1)Q} = (t+2)\Big\{\mathcal{G}_{-1}\mathcal{Q}_0 - 2\mathcal{L}_{-1}\Big\}|{-1},-1\rangle^G \\
|\chi\rangle_1^{(-1)G,Q} &= \mathcal{G}_0|\chi\rangle_1^{(-2)Q} = 2\Big\{\mathcal{L}_{-1}\mathcal{Q}_0 + \mathcal{H}_{-1}\mathcal{Q}_0 + \mathcal{Q}_{-1}\Big\}|{-1},1+t\rangle^G
\end{aligned} \tag{4.2}
$$

Observe that for $t=-2$ $|\chi\rangle_1^{(-1)Q}$ is annihilated by $\mathcal{G}_0$, i.e. it becomes chiral of type $|\chi\rangle_1^{(-1)G,Q}$ instead, as predicted by eq. (3.5) (level 1 corresponds to $r=1$, $s=2$). Untwisting $|\chi\rangle_1^{(-1)Q}$ and $|\chi\rangle_1^{(-2)Q}$ using T_{W1} (2.3) leads to the antichiral NS subsingular vectors:

$$
\begin{aligned}
|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)-} &= \Big\{-2G^-_{-\frac{3}{2}} + tL_{-1}G^-_{-\frac{1}{2}} - 2H_{-1}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{1}{2}},-1\rangle\,, \\
|\Upsilon_{NS}\rangle_2^{(-2)-} &= G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}}|{-\tfrac{3+t}{2}},1+t\rangle\,.
\end{aligned} \tag{4.3}
$$

Using T_{W2} (2.4) instead one gets mirrored chiral NS subsingular vectors i.e. which differ from the antichiral ones by the exchange $H_m \to -H_m$, $G^\pm_r \to G^\mp_r$, $h\to -h$ and $q\to -q$.

The action of $G^+_{1/2}$ maps $|\Upsilon_{NS}\rangle_{3/2}^{(-1)-}$ and $|\Upsilon_{NS}\rangle_2^{(-2)-}$ to the antichiral NS singular vectors $|\chi_{NS}\rangle_1^{(0)-}$ (for $t\neq -2$) and $|\chi_{NS}\rangle_{3/2}^{(-1)-}$, respectively:

$$
\begin{aligned}
|\chi_{NS}\rangle_1^{(0)-} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)-} = (t+2)\Big\{-2L_{-1} + G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{1}{2}},-1\rangle\,, \\
|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)-} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_2^{(-2)-} = 2\Big\{G^-_{-\frac{3}{2}} + L_{-1}G^-_{-\frac{1}{2}} + H_{-1}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{3+t}{2}},1+t\rangle\,.
\end{aligned} \tag{4.4}
$$

The action of any other positive modes on the vectors $|\Upsilon_{NS}\rangle_{3/2}^{(-1)-}$ and $|\Upsilon_{NS}\rangle_2^{(-2)-}$ vanishes identically. Thus these vectors are subsingular with respect to the corresponding singular vectors which, being antichiral, cannot reach the subsingular vectors back. As $|\chi_{NS}\rangle_1^{(0)-} = 0$ for $t=-2$ ($\mathsf{c}=9$), the vector $|\Upsilon_{NS}\rangle_{3/2}^{(-1)-}$ turns out to be singular in this particular case.

It is worth mentioning that the NS algebra does not have singular vectors with relative charges different from $|q|=0,1$ [19]. This fact is important for understanding the $N=2$ character formulae. Most surprisingly is therefore that there are subsingular vectors with relative charges $|q|=2$.

No-label topological singular vectors versus NS subsingular vectors

Let us now investigate the NS subsingular vectors obtained by untwisting the topological no-label singular vectors at level 1. As table (3.1) shows, the no-label singular vectors on $\mathcal{G}_0$-closed primaries only exist for relative charges $q = 0, -1$. As explained in section 3, the restrictions on this kind of vectors are very strict, such that they exist just for some discrete values of t: $t = \frac{2}{r}$, i.e. $\mathsf{c} = \frac{3r-6}{r}$.

At level 1 there are no-label singular vectors only for $t = 2$ ($\mathsf{c} = -3$) as $r = 1$. These vectors and the $\mathcal{G}_0$-closed singular vectors obtained by the action of $\mathcal{G}_0$ are given by:

$$
\begin{aligned}
|\chi\rangle_1^{(0)} &= \Big\{\mathcal{H}_{-1} - \mathcal{L}_{-1}\Big\}|{-1}, -1, t = 2\rangle^G\,, \\
|\chi\rangle_1^{(1)G} = \mathcal{G}_0|\chi\rangle_1^{(0)} &= -2\mathcal{G}_{-1}|{-1}, -1, t = 2\rangle^G\,, \\
|\chi\rangle_1^{(-1)} &= \Big\{\mathcal{L}_{-1}\mathcal{Q}_0 + 2\mathcal{H}_{-1}\mathcal{Q}_0\Big\}|{-1}, 3, t = 2\rangle^G\,, \\
|\chi\rangle_1^{(0)G} = \mathcal{G}_0|\chi\rangle_1^{(-1)} &= -\Big\{\mathcal{G}_{-1}\mathcal{Q}_0 + 2\mathcal{L}_{-1} + 4\mathcal{H}_{-1}\Big\}|{-1}, 3, t = 2\rangle^G\,.
\end{aligned}
\tag{4.5}
$$

By untwisting the no-label singular vectors using T_{W1} one finds the NS subsingular vectors

$$
\begin{aligned}
|\Upsilon_{NS}\rangle_1^{(0)} &= \Big\{H_{-1} - L_{-1}\Big\}|{-\tfrac{1}{2}}, -1, t = 2\rangle \\
|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)} &= \Big\{L_{-1}G^-_{-\frac{1}{2}} + 2H_{-1}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{5}{2}}, 3, t = 2\rangle\,.
\end{aligned}
\tag{4.6}
$$

This kind of subsingular vector is always accompanied by an antichiral singular vector at the same level and with the same charge, which corresponds to the untwisting of the chiral topological singular vector. Thus, the NS subsingular vectors obtained by untwisting the no-label topological singular vectors also have different representations, like the latter. The vectors $|\Upsilon_{NS}\rangle_1^{(0)}$ and $|\Upsilon_{NS}\rangle_{3/2}^{(-1)}$ are subsingular with respect to the singular vectors

$$
\begin{aligned}
|\chi_{NS}\rangle_{\frac{1}{2}}^{(1)} = \; G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_1^{(0)} &= -2G^+_{-\frac{1}{2}}|{-\tfrac{1}{2}}, -1, t = 2\rangle\,, \\
|\chi_{NS}\rangle_1^{(0)} = \; G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)} &= -\Big\{2L_{-1} + 4H_{-1} + G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{5}{2}}, 3, t = 2\rangle\,,
\end{aligned}
\tag{4.7}
$$

respectively, obtained by untwisting the $\mathcal{G}_0$-closed singular vectors in (4.5).

By untwisting the no-label singular vectors in (4.5) using T_{W2} one obtains the mirror-symmetric NS subsingular vectors which are subsingular with respect to the mirror-symmetric NS singular vectors.

NS subsingular vectors versus R singular and subsingular vectors

We now use the spectral flows $\mathcal{U}_{\pm 1/2}$ and $\mathcal{A}_{\pm 1/2}$ in order to transform the NS subsingular vectors $|\Upsilon_{NS}\rangle_{3/2}^{(-1)-}$ $(t \neq -2)$, $|\Upsilon_{NS}\rangle_2^{(-2)-}$, $|\Upsilon_{NS}\rangle_1^{(0)}$, and $|\Upsilon_{NS}\rangle_{3/2}^{(-1)}$ into R singular and subsingular vectors. Using $\mathcal{U}_{1/2}$ and $\mathcal{A}_{-1/2}$ we obtain subsingular vectors in the R algebra as well. Using $\mathcal{U}_{-1/2}$ and $\mathcal{A}_{1/2}$, however, we obtain R singular vectors instead. The R subsingular vectors obtained using $\mathcal{U}_{1/2}$ are

$$\begin{aligned}
|\Upsilon_R\rangle_2^{(-1)-} &= \mathcal{U}_{1/2}|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)-} = \Big\{-2G^-_{-2} + tL_{-1}G^-_{-1} + (\tfrac{t}{2}-2)H_{-1}G^-_{-1}\Big\}|\tfrac{1-t}{8}, \tfrac{t-3}{2}\rangle^- \\
|\Upsilon_R\rangle_3^{(-2)-} &= \mathcal{U}_{1/2}|\Upsilon_{NS}\rangle_2^{(-2)-} = G^-_{-2}G^-_{-1}|-\tfrac{15+9t}{8}, \tfrac{1+3t}{2}\rangle^- \\
|\Upsilon_R\rangle_1^{(0)-} &= \mathcal{U}_{1/2}|\Upsilon_{NS}\rangle_1^{(0)} = \Big\{\tfrac{1}{2}H_{-1} - L_{-1}\Big\}|-\tfrac{1}{8}, -\tfrac{1}{2}, t=2\rangle^- \\
|\Upsilon_R\rangle_2^{(-1)-} &= \mathcal{U}_{1/2}|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)} = \Big\{L_{-1}G^-_{-1} + \tfrac{5}{2}H_{-1}G^-_{-1}\Big\}|-\tfrac{33}{8}, \tfrac{7}{2}, t=2\rangle^- \ ,
\end{aligned} \tag{4.8}$$

where the helicity $(-)$ of the R vectors (subsingular and primaries) indicate that they are annihilated by G_0^-. $|\Upsilon_R\rangle_2^{(-1)-}$ and $|\Upsilon_R\rangle_3^{(-2)-}$ are in addition also annihilated by G^-_{-1}.

Using the spectral flow $\mathcal{U}_{-1/2}$ we obtain the helicity $(-)$ and the no-helicity R singular vectors:

$$\begin{aligned}
|\chi_R\rangle_1^{(-1)-} &= \mathcal{U}_{-1/2}|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)-} = \Big\{-2G^-_{-1} + tL_{-1}G^-_0 - (\tfrac{t}{2}+2)H_{-1}G^-_0\Big\}|\tfrac{-7-t}{8}, \tfrac{-1-t}{2}\rangle^+ \\
|\chi_R\rangle_1^{(-2)-} &= \mathcal{U}_{-1/2}|\Upsilon_{NS}\rangle_2^{(-2)-} = G^-_{-1}G^-_0|-\tfrac{7+t}{8}, \tfrac{3+t}{2}\rangle^+ \\
|\chi_R\rangle_1^{(0)} &= \mathcal{U}_{-1/2}|\Upsilon_{NS}\rangle_1^{(0)} = \Big\{\tfrac{3}{2}H_{-1} - L_{-1}\Big\}|-\tfrac{9}{8}, -\tfrac{3}{2}, t=2\rangle^+ \\
|\chi_R\rangle_1^{(-1)} &= \mathcal{U}_{-1/2}|\Upsilon_{NS}\rangle_{\frac{3}{2}}^{(-1)} = \Big\{L_{-1}G^-_0 + \tfrac{3}{2}H_{-1}G^-_0\Big\}|-\tfrac{9}{8}, \tfrac{5}{2}, t=2\rangle^+ \ ,
\end{aligned} \tag{4.9}$$

where the helicity $(+)$ of the R primaries indicate that they are annihilated by G_0^+. This result is due to the fact that $\mathcal{U}_{-1/2}$ transforms $G^+_{1/2}$ into G^+_0, so that instead of a missing h.w. condition one has the missing helicity $(+)$ for the resulting R singular vectors. The antichirality of the NS subsingular vectors $|\Upsilon_{NS}\rangle_{3/2}^{(-1)-}$ and $|\Upsilon_{NS}\rangle_2^{(-2)-}$ compensates this loss as it is converted into helicity $(-)$ for the resulting R singular vectors.

The corresponding results for $\mathcal{A}_{\pm 1/2}$ are simply mirror-symmetric to the above expressions. To be precise, $\mathcal{A}_{1/2}$ produces helicity $(+)$ and no-helicity R singular vectors mirrored to the ones in eq. (4.9), whereas $\mathcal{A}_{-1/2}$ produces helicity $(+)$ R subsingular vectors mirrored to the ones in eq. (4.8).

Two important remarks are now in order. First, the R singular vectors $|\chi_R\rangle_l^{(-2)-}$ with relative charge $q=-2$, built on the helicity $(+)$ primaries $|\Delta, h\rangle^+$, can be regarded equivalently, provided $\Delta \neq \mathsf{c}/24$, as singular vectors with $q=-1$ built on the helicity $(-)$ primaries $|\Delta, h-1\rangle^- = G_0^-|\Delta, h\rangle^+$. Second, 'no-helicity' R singular vectors have never been considered in the literature before[†]. These R singular vectors, which are analogous to the no-label topological singular vectors, are permitted by the R algebra provided the conformal weight satisfies $\Delta + l = \mathsf{c}/24$, as the anticommutator $\{G_0^+, G_0^-\} = 2L_0 - \mathsf{c}/12$ shows. Like the no-label topological singular vectors, the no-helicity R singular vectors

[†]Between the first version and the present version of this paper we have considered 'no-helicity' R singular vectors also in refs. [17] and [18].

only exist for $t = \frac{2}{r}$. The existence of no-helicity R singular vectors does not contradict the determinant formula because they are always accompanied by helicity (+) and helicity (−) singular vectors at the same level, obtained by the action of G_0^+ and G_0^-. These two issues are explained in detail in ref. [18].

4.2 Level 2

$\mathcal{Q}_0$-closed topological singular vectors versus NS subsingular vectors

At level 2 the $\mathcal{Q}_0$-closed topological singular vectors with zero conformal weight, built on $\mathcal{G}_0$-closed primaries, are the following. As deduced from eqs. (3.3) and (3.4), there are singular vectors with $q = -1$ for $h = -2$ and $h = -1 - \frac{t}{2}$, which we label by a and b respectively, and there are singular vectors with $q = -2$ for $h = 1 + \frac{3}{2}t$. They are given by:

$$\begin{aligned}
|\chi\rangle_2^{(-1)Q,a} &= \Big\{8t\mathcal{L}_{-1}\mathcal{Q}_{-1} + 6t\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_0 - 4(2+t)\mathcal{H}_{-2}\mathcal{Q}_0 - 8\mathcal{H}_{-1}^2\mathcal{Q}_0 - 4(t+4)\mathcal{Q}_{-2} \\
&\quad + t\mathcal{G}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0 - 16\mathcal{H}_{-1}\mathcal{Q}_{-1} - t^2\mathcal{L}_{-1}^2\mathcal{Q}_0\Big\}|-2,-2\rangle^G \\
|\chi\rangle_2^{(-1)Q,b} &= \Big\{-(3t+2)\mathcal{L}_{-1}\mathcal{Q}_{-1} - t(2+\tfrac{t}{2})\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_0 + (1+\tfrac{t}{2})(2+t)\mathcal{H}_{-2}\mathcal{Q}_0 \\
&\quad + (1-\tfrac{t^2}{4})(2+t)\mathcal{L}_{-2}\mathcal{Q}_0 + \tfrac{t^2}{2}\mathcal{L}_{-1}^2\mathcal{Q}_0 - \tfrac{4+2t-t^2}{4}\mathcal{G}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0 \\
&\quad + (2+t)\mathcal{H}_{-1}^2\mathcal{Q}_0 + 2(2+t)\mathcal{H}_{-1}\mathcal{Q}_{-1} + (2+\tfrac{t}{2})(2+t)\mathcal{Q}_{-2}\Big\}|-2,-1-\tfrac{t}{2}\rangle^G \\
|\chi\rangle_2^{(-2)Q} &= \Big\{2\mathcal{L}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0 + 4\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0 + (2-t)\mathcal{Q}_{-2}\mathcal{Q}_0\Big\}|-2,1+\tfrac{3}{2}t\rangle^G
\end{aligned} \tag{4.10}$$

The singular vector $|\chi\rangle_2^{(-1)Q,a}$ becomes chiral for $t = -4$ (it corresponds to $r = 1,\ s = 4$) and $|\chi\rangle_2^{(-1)Q,b}$ becomes chiral for $t = -1, -2$ (it corresponds to $r = 2,\ s = 2$), according to the result (3.5). The corresponding NS subsingular vectors, obtained by untwisting using T_{W1}, together with the NS singular vectors with respect to which they are subsingular, are the following:

$$\begin{aligned}
|\Upsilon_{NS}\rangle_{\frac{5}{2}}^{(-1)-,a} &= \Big\{-4(t+4)G^-_{-\frac{5}{2}} + 8tL_{-1}G^-_{-\frac{3}{2}} - 4(2+t)H_{-2}G^-_{-\frac{1}{2}} \\
&\quad + 6tL_{-1}H_{-1}G^-_{-\frac{1}{2}} - 16H_{-1}G^-_{-\frac{3}{2}} - 8H_{-1}^2G^-_{-\frac{1}{2}} - t^2L_{-1}^2G^-_{-\frac{1}{2}} \\
&\quad + tG^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}}\Big\}|-1,-2\rangle\,, \quad t \neq -4\,, \\
|\chi_{NS}\rangle_2^{(0)-,a} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_{\frac{5}{2}}^{(-1)-,a} = (t+4)\Big\{-8L_{-2} + 4H_{-2} - 8L_{-1}H_{-1} + 4tL_{-1}^2
\end{aligned}$$

$$+4G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} + 4H_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - 2tL_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-1},-2\rangle\,,$$

$$\begin{aligned}
|\Upsilon_{NS}\rangle^{(-1)-,b}_{\frac{5}{2}} &= \Big\{(2+t)(2+\tfrac{t}{2})G^-_{-\frac{5}{2}} - (3t+2)L_{-1}G^-_{-\frac{3}{2}} + \tfrac{2+t}{2}(1+t+\tfrac{t^2}{4})H_{-2}G^-_{-\frac{1}{2}} \\
&\quad +(2+t)(1-\tfrac{t^2}{4})L_{-2}G^-_{-\frac{1}{2}} + 2(2+t)H_{-1}G^-_{-\frac{3}{2}} - t(2-\tfrac{t}{2})L_{-1}H_{-1}G^-_{-\frac{1}{2}} \\
&\quad +(2+t)H^2_{-1}G^-_{-\frac{1}{2}} + \tfrac{t^2}{2}L^2_{-1}G^-_{-\frac{1}{2}} - (1+\tfrac{t}{2}-\tfrac{t^2}{2})G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}}\Big\} \\
&\quad |{-\tfrac{3}{2}}+\tfrac{t}{4},-1-\tfrac{t}{2}\rangle\,,\quad t\neq -1,-2\,, \qquad (4.11) \\
|\chi_{NS}\rangle^{(0)-,b}_{2} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle^{(-1)-,b}_{\frac{5}{2}} = (1+t)(2+t)\Big\{tL_{-2} - \tfrac{t}{2}H_{-2} - 2L^2_{-1} \\
&\quad +(1-\tfrac{t}{2})G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} - G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} + 2L_{-1}H_{-1} + L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \\
&\quad -H_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{3}{2}}+\tfrac{t}{4},-1-\tfrac{t}{2}\rangle\,,
\end{aligned}$$

$$\begin{aligned}
|\Upsilon_{NS}\rangle^{(-2)-}_{3} &= \Big\{(2-t)G^-_{-\frac{5}{2}}G^-_{-\frac{1}{2}} + 2L_{-1}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}} + 4H_{-1}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{5}{2}}-\tfrac{3}{4}t,1+\tfrac{3}{2}t\rangle, \\
|\chi_{NS}\rangle^{(-1)-}_{\frac{5}{2}} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle^{(-2)-}_{3} = \Big\{4(2-t)G^-_{-\frac{5}{2}} + 2(2-t)L_{-2}G^-_{-\frac{1}{2}} - (3t+2)H_{-2}G^-_{-\frac{1}{2}} \\
&\quad +8L_{-1}G^-_{-\frac{3}{2}} + 16H_{-1}G^-_{-\frac{3}{2}} - 2G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}} + 4L^2_{-1}G^-_{-\frac{1}{2}} + 8H^2_{-1}G^-_{-\frac{1}{2}} \\
&\quad +12L_{-1}H_{-1}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{5}{2}}-\tfrac{3}{4}t,1+\tfrac{3}{2}t\rangle\,.
\end{aligned}$$

Observe that for $t=-4$ the subsingular vector $|\Upsilon_{NS}\rangle^{(-1)-,a}_{5/2}$ becomes singular (since $|\chi_{NS}\rangle^{(0)-,a}_{2}=0$) and similarly for $t=-1,-2$ the subsingular vector $|\Upsilon_{NS}\rangle^{(-1)-,b}_{5/2}$ becomes singular (since $|\chi_{NS}\rangle^{(0)-,b}_{2}=0$). By untwisting using T_{W2} one finds the mirror-symmetric NS subsingular vectors which are subsingular with respect to the mirror-symmetric NS singular vectors.

No-label topological singular vectors versus NS subsingular vectors

At level 2 the no-label topological singular vectors exist only for $t=1$ and $t=2$ ($\mathsf{c}=0$ and $\mathsf{c}=-3$). They are the following:

$$\begin{aligned}
|\chi\rangle^{(0)}_{2} &= \Big\{4\mathcal{H}^2_{-1} - 6\mathcal{L}_{-1}\mathcal{H}_{-1} - 2\mathcal{L}_{-2} + \mathcal{L}_{-1}\mathcal{G}_{-1}\mathcal{Q}_0 - \mathcal{G}_{-1}\mathcal{Q}_{-1} + 8\mathcal{H}_{-2} \\
&\quad +\mathcal{G}_{-2}\mathcal{Q}_0\Big\}|{-2},-2,t=2\rangle^G \\
|\chi\rangle^{(0)}_{2} &= \Big\{24\mathcal{H}^2_{-1} - 10\mathcal{H}_{-1}\mathcal{G}_{-1}\mathcal{Q}_0 - 48\mathcal{L}^2_{-1} + 26\mathcal{L}_{-1}\mathcal{G}_{-1}\mathcal{Q}_0
\end{aligned}$$

$$
-20\mathcal{G}_{-1}\mathcal{Q}_{-1}+36\mathcal{H}_{-2}+19\mathcal{G}_{-2}\mathcal{Q}_0\Big\}|{-2},-\tfrac{3}{2},t=1\rangle^G \tag{4.12}
$$

$$
\begin{aligned}
|\chi\rangle_2^{(-1)} &= \Big\{12\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_0-4\mathcal{L}_{-1}\mathcal{Q}_{-1}-23\mathcal{H}_{-2}\mathcal{Q}_0-6\mathcal{L}_{-2}\mathcal{Q}_0+17\mathcal{H}^2_{-1}\mathcal{Q}_0 \\
&\quad +\mathcal{L}^2_{-1}\mathcal{Q}_0+\mathcal{G}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0-2\mathcal{H}_{-1}\mathcal{Q}_{-1}-6\mathcal{Q}_{-2}\Big\}|{-2},4,t=2\rangle^G \\
|\chi\rangle_2^{(-1)} &= \Big\{\mathcal{L}_{-1}\mathcal{Q}_{-1}+9\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_0-\tfrac{21}{2}\mathcal{H}_{-2}\mathcal{Q}_0-\tfrac{3}{2}\mathcal{L}_{-2}\mathcal{Q}_0+11\mathcal{H}^2_{-1}\mathcal{Q}_0+\mathcal{L}^2_{-1}\mathcal{Q}_0 \\
&\quad -\mathcal{G}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0+10\mathcal{H}_{-1}\mathcal{Q}_{-1}-\tfrac{5}{2}\mathcal{Q}_{-2}\Big\}|{-2},\tfrac{5}{2},t=1\rangle^G
\end{aligned}
$$

The untwisting of these topological singular vectors, using T_{W1}, leads to the following NS subsingular vectors, given together with the NS singular vectors with respect to which they are subsingular:

$$
\begin{aligned}
|\Upsilon_{NS}\rangle_2^{(0)} &= \Big\{-2L_{-2}+9H_{-2}+4H^2_{-1}-6L_{-1}H_{-1}-G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}+G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} \\
&\quad +L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-1},-2,t=2\rangle\,, \\
|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_2^{(0)}=12\Big\{L_{-1}G^+_{-\frac{1}{2}}-H_{-1}G^+_{-\frac{1}{2}}\Big\}|{-1},-2,t=2\rangle\,, \\
|\Upsilon_{NS}\rangle_2^{(0)} &= \Big\{36H_{-2}+24H^2_{-1}-48L^2_{-1}-20G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}+19G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} \\
&\quad -10H_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}+26L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{5}{4}},-\tfrac{3}{2},t=1\rangle\,, \qquad (4.13) \\
|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_2^{(0)}=24\Big\{G^+_{-\frac{3}{2}}+2L_{-1}G^+_{-\frac{1}{2}}-2H_{-1}G^+_{-\frac{1}{2}}\Big\}|{-\tfrac{5}{4}},-\tfrac{3}{2},t=1\rangle\,, \\
|\Upsilon_{NS}\rangle_{\frac{5}{2}}^{(-1)} &= \Big\{-6G^-_{-\frac{5}{2}}-4L_{-1}G^-_{-\frac{3}{2}}-20H_{-2}G^-_{-\frac{1}{2}}-6L_{-2}G^-_{-\frac{1}{2}}+12L_{-1}H_{-1}G^-_{-\frac{1}{2}} \\
&\quad -2H_{-1}G^-_{-\frac{3}{2}}+17H^2_{-1}G^-_{-\frac{1}{2}}+L^2_{-1}G^-_{-\frac{1}{2}}+G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}}\Big\}|{-4},4,t=2\rangle\,, \\
|\chi_{NS}\rangle_2^{(0)} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_{\frac{5}{2}}^{(-1)}=6\Big\{2L_{-2}+11H_{-2}-G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}-12H^2_{-1}-10L_{-1}H_{-1} \\
&\quad -2L^2_{-1}+3G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}-2L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}-4H_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-4},4,t=2\rangle\,, \\
|\Upsilon_{NS}\rangle_{\frac{5}{2}}^{(-1)} &= \Big\{-\tfrac{5}{2}G^-_{-\frac{5}{2}}+L_{-1}G^-_{-\frac{3}{2}}-\tfrac{39}{4}H_{-2}G^-_{-\frac{1}{2}}-\tfrac{3}{2}L_{-2}G^-_{-\frac{1}{2}}+9L_{-1}H_{-1}G^-_{-\frac{1}{2}} \\
&\quad +10H_{-1}G^-_{-\frac{3}{2}}+11H^2_{-1}G^-_{-\frac{1}{2}}+L^2_{-1}G^-_{-\frac{1}{2}}-G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{13}{4}},\tfrac{5}{2},t=1\rangle, \\
|\chi_{NS}\rangle_2^{(0)} &= G^+_{\frac{1}{2}}|\Upsilon_{NS}\rangle_{\frac{5}{2}}^{(-1)}=\Big\{\tfrac{23}{2}H_{-2}+L_{-2}+\tfrac{15}{2}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}-24H^2_{-1}-14L_{-1}H_{-1} \\
&\quad -2L^2_{-1}-5G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}-11H_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}-5L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\}|{-\tfrac{13}{4}},\tfrac{5}{2},t=1\rangle\,.
\end{aligned}
$$

By untwisting using T_{W2} one finds the mirror-symmetric NS subsingular vectors which are subsingular with respect to the mirror-symmetric NS singular vectors.

The spectral flow transformations of these NS subsingular vectors into singular and subsingular vectors of the R algebra we leave to the interested reader.

5 Conclusions and Final Remarks

We have considered singular and subsingular vectors of the N=2 Superconformal algebras. The latter are non-highest weight null vectors located outside the (incomplete) Verma submodules built on certain singular vectors, becoming singular (i.e. highest weight) after taking the quotient of the Verma module by the submodule generated by the singular vector.

We have shown that two large classes of singular vectors of the Topological algebra, built on $\mathcal{G}_0$-closed primaries, correspond to subsingular vectors of the NS algebra and to singular and subsingular vectors of the R algebra. These classes consist of Q_0-closed (BRST-invariant) singular vectors with relative charges $q = -2, -1$ and zero conformal weight, and no-label singular vectors with $q = 0, -1$. This provides one more step towards the derivation of the correct $N = 2$ embedding diagrams and towards understanding the $N = 2$ character formulae.

We have also shown the existence of subsingular vectors of the NS algebra with $|q| = 2$. This has significant implications for the embedding diagrams as subsingular vectors should be included in them. (It was believed so far that embedding diagrams for the $N = 2$ algebras contained only vectors of charge $|q| = 0, 1$).

In addition we have written down, for the first time, examples of R singular vectors with relative charge $|q| = 2$ and examples of R singular vectors without any helicity. These types of R singular vectors have been overlooked in the early literature and only recently they have been paid some attention [17][18].

Besides the classes of subsingular vectors of the $N = 2$ algebras presented in this work, there is, so far, only one other class of subsingular vectors of the $N = 2$ algebras known. These are subsingular vectors of the Topological, NS and R algebras, which become singular in chiral Verma modules [6][7]. A chiral Verma module is the quotient of a complete Verma module by a submodule generated by a lowest-level singular vector. One could think of constructing, from the beginning, quotient modules of Verma modules with higher-level singular vectors. These quotient spaces would possibly lead to subsingular vectors in exactly the same way as chiral Verma modules do. As a matter of fact we have already constructed further classes of subsingular vectors in this way, what will be presented in a forthcoming publication.

Acknowledgements

We would like to thank S. Hwang, A. Kent, J. Navarro-Salas and A. Tiemblo for valuable comments. M.D. is supported by a DAAD fellowship and in part by NSF grant PHY-92-18167.

Note

In the recent paper 'The Structure of Verma Modules over the N=2 Superconformal Algebra', Comm. Math. Phys. 195 (1998) 129, (hep-th/9704111) by A.M. Semikhatov and I.Yu. Tipunin, the authors claim to present a complete classification of N=2 subsingular vectors. However, the only explicit examples known at that time, due to Gato-Rivera and Rosado [6][7], do not fit into that classification. In addition, the classification is based on several misleading assumptions, the most relevant being the following. First, the authors claim that there are only two types of submodules, overlooking no-label singular vectors which do not fit into their submodules and create a different type of submodules themselves (no-label singular vectors were discovered and examples given in ref. [7]). Second, they claim to have constructed some null states, called non-conventional singular vectors, which generate maximal submodules such that there are not subsingular vectors outside of them. We have counterexamples that show that these states do not generate maximal submodules as one can find subsingular vectors outside of them. (In fact, the existence of no-label singular vectors already proves that the states given by the authors do not generate maximal submodules). Third, the authors confuse, all along the paper, the determinant formula for the Topological N=2 algebra (which was unknown at that time and has been written down just recently [18]) with the determinant formula for the Neveu-Schwarz N=2 algebra. In addition they believe that the determinant formula identify all possible submodules (what has never been proved for the N=2 algebras). Finally, the authors claim that all the results found for the Verma modules of the Topological N=2 algebra hold for the Verma modules of the Neveu-Schwarz and the Ramond N=2 algebras. The work presented by us in this paper shows precisely the contrary: two large classes of singular vectors of the Topological algebra do not correspond to singular vectors of the Neveu-Schwarz N=2 algebra, but to subsingular vectors. (As was analysed in refs. [6][7], only the topological singular vectors annihilated by $\mathcal{G}_0$, built on primaries annihilated also by $\mathcal{G}_0$, correspond to singular vectors of the Neveu-Schwarz N=2 algebra. Some other topological singular vectors correspond simply to null descendants of NS singular vectors).

References

[1] M. Ademollo et al., Phys. Lett. B62 (1976) 105; Nucl. Phys. B111 (1976) 77; Nucl. Phys. B114 (1976) 297.

[2] L. Brink and J.H. Schwarz, Nucl. Phys. B121 (1977) 285;
E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B106 (1981) 63; Phys. Lett. B162 (1985) 295;
S.D. Mathur and S. Mukhi, Phys. Rev. D 36 (1987) 465; Nucl. Phys. B302 (1988) 130;
H. Ooguri and C. Vafa, Nucl. Phys. B361 (1991) 469; Nucl. Phys. B367 (1991) 83;
N. Marcus, talk at the Rome String Theory Workshop (1992), hep-th/9211059.

[3] E. Martinec, *M-theory and N=2 Strings*, hep-th/9710122 (1997), and references there.

[4] B. Gato-Rivera and A. M. Semikhatov, Phys. Lett. B293 (1992) 72, Theor. Mat. Fiz. 95 (1993) 239, Theor. Math. Phys. 95 (1993) 536; Nucl. Phys. B408 (1993) 133.

[5] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Nucl. Phys. B401 (1993) 304.

[6] B. Gato-Rivera and J.I. Rosado, *Chiral Determinant Formulae and Subsingular Vectors for the N=2 Superconformal Algebras*, Nucl. Phys. B503 (1997) 447 (similar version in hep-th/9706041) . The first version of this paper where the N=2 subsingular vectors were discovered was presented in hep-th/9602166.

[7] B. Gato-Rivera and J.I. Rosado, *Families of Singular and Subsingular Vectors of the Topological N=2 Superconformal Algebra*, Nucl. Phys. B514 [PM] (1998) 477, (similar version in hep-th/9701041).

[8] P. Di Vecchia et al., Phys. Lett. B162 (1985) 327.

[9] A. Schwimmer and N. Seiberg, Phys. Lett. B184 (1987) 191.

[10] E.B. Kiritsis, Phys. Rev. D36 (1987) 3048.

[11] W. Lerche, C. Vafa and N. P. Warner, Nucl. Phys. B324 (1989) 427.

[12] W. Boucher, D. Friedan, A. Kent, Phys. Lett. B172 (1986) 316.

[13] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.

[14] B. Gato-Rivera and J.I. Rosado, *Spectral Flows and Twisted Topological Theories*, Phys. Lett. B369 (1996) 7.

[15] B. Gato-Rivera and J.I. Rosado, *The Other Spectral Flow*, Mod. Phys. Lett. A11 (1996) 423.

[16] B. Gato-Rivera, *The Even and the Odd Spectral Flows on the N=2 Superconformal Algebras*, Nucl. Phys. B512 (1998) 431 (similar version in hep-th/9707211).

[17] M. Dörrzapf and B. Gato-Rivera, *Singular Dimensions of the N=2 Superconformal Algebras. I*, Commun. Math. Phys. 206, 493 (1999).

[18] M. Dörrzapf and B. Gato-Rivera, *Determinant Formula for the Topological N=2 Superconformal Algebra*, Nucl. Phys. B 558 [PM] 503 (1999).

[19] M. Dörrzapf, Commun. Math. Phys. 180 (1996) 195.

[20] B. Gato-Rivera, *Construction Formulae for Singular Vectors of the Topological N=2 Superconformal Algebra*, IMAFF-FM-98/05 hep-th/9802204 (1998).