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Singular dimensions of the $N = 2$ superconformal algebras. I

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ABSTRACT

Verma modules of superconfomal algebras can have singular vector spaces with dimensions greater than 1. Following a method developed for the Virasoro algebra by Kent, we introduce the concept of adapted orderings on superconformal algebras. We prove several general results on the ordering kernels associated to the adapted orderings and show that the size of an ordering kernel implies an upper limit for the dimension of a singular vector space. We apply this method to the topological $N = 2$ algebra and obtain the maximal dimensions of the singular vector spaces in the topological Verma modules: 0, 1, 2 or 3 depending on the type of Verma module and the type of singular vector. As a consequence we prove the conjecture of Gato-Rivera and Rosado on the possible existing types of topological singular vectors (4 in chiral Verma modules and 29 in complete Verma modules). Interestingly, we have found two-dimensional spaces of singular vectors at level 1. Finally, by using the topological twists and the spectral flows, we also obtain the maximal dimensions of the singular vector spaces for the Neveu-Schwarz $N = 2$ algebra (0, 1 or 2) and for the Ramond $N = 2$ algebra (0, 1, 2 or 3).

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1 Introduction

More than two decades ago, superconformal algebras were first constructed independently and almost at the same time by Kac[20] and by Ademollo et al.[1]. Whilst Kac[20] derived them for mathematical purposes along with his classification of Lie super algebras, Ademollo et al.[1] constructed the superconformal algebras for physical purposes in order to define supersymmetric strings. Since then the study of superconformal algebras has made much progress on both mathematics and physics. On the mathematical side Kac and van de Leuer[23] and Cheng and Kac[6] have classified all possible superconformal algebras and Kac recently has proved that their classification is complete (see footnote in Ref. 22). As far as the physics side is concerned, superconformal models are gaining increasing importance. Many areas of physics make use of superconformal symmetries but the importance is above all due to the fact that superconformal algebras supply the underlying symmetries of Superstring Theory.

The classification of the irreducible highest weight representations of the superconformal algebras is of interest to both, mathematicians and physicists. After more than two decades, only the simpler superconformal highest weight representations have been fully understood. Namely, only the representations of $N = 1$ are completely classified and proven[2,3]. For $N = 2$ remarkable efforts have been taken by several research groups[5,13,28,8,10,19]. Already the $N = 2$ superconformal algebras contain several surprising features regarding their representation theory, most of them related to the rank 3 of the algebras, making them more difficult to study than the N=1 superconformal algebras. The rank of the superconformal algebras keeps growing with N and therefore even more difficulties can be expected for higher N.

The standard procedure of finding all possible irreducible highest weight representations starts off with defining freely generated modules over a highest weight vector, denoted as *Verma modules*. A Verma module is in general not irreducible, but the corresponding irreducible representation is obtained as the quotient space of the Verma module divided by all its proper submodules. Therefore, the task of finding irreducible highest weight representations can be reduced to the classification of all submodules of a Verma module. Obviously, every proper submodule needs to have at least one highest weight vector different from the highest weight vector of the Verma module. These vectors are usually called singular vectors of the Verma module. Conversely, a module generated on such a singular vector defines a submodule of the Verma module. Thus, singular vectors play a crucial rôle in finding submodules of Verma modules. However, the set of singular vectors may not generate all the submodules. The quotient space of a Verma module divided by the submodules generated by all singular vectors may still be reducible and may hence contain further submodules that again contain singular vectors. But this time they are singular vectors of the quotient space, known as subsingular vectors of the Verma module. Repeating this division procedure successively would ultimately lead to an irreducible quotient space.

On the Verma modules one introduces a hermitian contravariant form. The vanishing of the corresponding determinant indicates the existence of a singular vector. Therefore, a crucial step towards analysing irreducible highest weight representations is to compute the inner product determinant. This has been done for $N = 1$[21,32,33], $N = 2$[5,33,24,18,12], $N = 3$[26], and $N = 4$[27,31]. Once the determinant vanishes we can conclude the existence of a singular vector Ψ_l at a certain *level* l, although there may still be other singular vectors at higher levels even outside the submodule generated by Ψ_l, the so called *isolated singular vectors*. Thus the determinant may not give all singular vectors neither does it give the dimension of the space of singular vectors at a given level l, since at levels where the determinant predicts one singular vector, of a given type, there could in fact be more than one linearly independent singular vectors, as it happens for the $N = 2$ superconformal algebras[9,19]. Therefore, the construction of specific singular vectors at levels given by the determinant formula may not be enough. One needs in addition information about the dimension of the space of singular vectors, apart from the (possible) existence of isolated singular vectors .

The purpose of this paper is to give a simple procedure that derives necessary conditions on the space of dimensions of singular vectors of the N=2 superconformal algebras. This will result in an upper limit for the dimension of the spaces of singular vectors at a given level. For most weight spaces of a Verma module these upper limits on the dimensions will be trivial and we obtain a rigorous proof that there cannot exist any singular vectors for these weights. For some weights, however, we will find necessary conditions that allow one-dimensional singular vector spaces, as is the case for the Virasoro algebra, or even higher dimensional spaces. The method shown in this paper for the superconformal algebras originates from the method used by Kent[25] for the Virasoro algebra[a]. Kent analytically continued the Virasoro Verma modules to generalised Verma modules. In these generalised Verma modules he constructed generalised singular vector expressions in terms of analytically continued Virasoro operators. Then he proved that if a generalised singular vector exists at level 0 in a generalised Verma module, then it is proportional to the highest weight vector. And consequently, if a generalised singular vector exists at a given level in a generalised Verma module, then it is unique up to proportionality. This uniqueness can therefore be used in order to show that the generalised singular vector expressions for the analytically continued modules are actually singular vectors of the Virasoro Verma module, whenever the Virasoro Verma module has a singular vector. As every Virasoro singular vector is at the same time a generalised singular vector, this implies that Virasoro singular vectors also have to be unique up to proportionality.

In this paper we focus on the uniqueness proof of Kent and show that similar ideas can be applied directly to the superconformal algebras. Our procedure does not require any analytical continuation of the algebra, however, and therefore gives us a powerful method that can easily be applied to a vast number of algebras without the need of constructing singular vectors. We shall define the underlying idea as *the concept of adapted orderings*. For pedagogical reasons we will first apply Kent's ordering directly to the Virasoro Verma modules. Then we will present adapted orderings for the topological $N = 2$ superconformal algebra, which is the most interesting $N = 2$ algebra for current research in this field. The results obtained will be translated finally to the Neveu-Schwarz and to the Ramond $N = 2$ algebras. In a future publication we will further apply these ideas to the twisted $N = 2$ superconformal algebra.

The paper is structured as follows. In section 2 we explain the concept of adapted orderings for the case of the Virasoro algebra, which will also serve to illustrate Kent's proof in our setting. In section 3, we prove some general results on adapted orderings for superconformal algebras, which justify the use of this method. In section 4 we review some basic results concerning the topological $N = 2$ superconformal algebra. Section 5 introduces adapted orderings on generic Verma modules of the topological $N = 2$ superconformal algebra (those built on G_0-closed or Q_0-closed highest weight vectors). This procedure is extended to *chiral* Verma modules in section 6 and to *no-label* Verma modules in section 7. Section 8 summarises the implications of the adapted orderings on the dimensions of the singular vector spaces for the corresponding topological Verma modules. Section 9 translates these results to the singular vector spaces of the Neveu-Schwarz and the Ramond $N = 2$ superconformal algebras. Section 10 is devoted to conclusions and prospects. The proof of theorem

[a]Besides the later application to the Neveu-Schwarz $N = 2$ algebra in Ref. 9, only one further application is known to us which has been achieved by Bajnok[4] for the WA_2 algebra.

5.C fills several pages and readers that are not interested in the details of this proof can simply continue with theorem 5.E. In this case, the preliminary remarks to theorems 6.A and 7.B should also be skipped. Nevertheless, the main idea of the concept can easily be understood from the introductory example of the Virasoro Verma modules in section 2.

2 Virasoro algebra

It is a well-known fact that at a given level of a Verma module of the Virasoro algebra there can only be one singular vector which is unique up to proportionality. This is an immediate consequence of the proof of the Virasoro embedding diagrams by Feigin and Fuchs[14]. Using an analytically continued algebra of the Virasoro algebra, Kent constructed in Ref. 25 all Virasoro singular vectors in terms of products of analytically continued operators. Although similar methods had already been used earlier on Verma modules over Kac-Moody algebras[30], the construction by Kent not only shows the existence of analytically continued singular vectors for any complex level but also their uniqueness[b]. This issue is our main interest in this paper. We shall therefore concentrate on the part of Kent's proof that shows the uniqueness of Virasoro singular vectors rather than the existence of analytically continued singular vectors. It turns out that the extension of the Virasoro algebra to an analytically continued algebra, although needed for the part of Kent's proof showing the existence claim, is however not necessary for the uniqueness claim on which we will focus in this paper. We will first motivate and define our concept of *adapted orderings* for the Virasoro algebra and will then prove some first results for the implications of adapted orderings on singular vectors. Following Kent[25] we will then introduce an ordering on the basis of a Virasoro Verma module and describe it in our framework. If we assume that a singular vector exists at a fixed level, then this total ordering will show that this singular vector has to be unique up to proportionality.

The Virasoro algebra V is generated by the operators L_m with $m \in \mathbf{Z}$ and the *central extension* C satisfying the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}\,, \quad [C, L_m] = 0\,, \quad m, n \in \mathbf{Z}\,. \tag{1}$$

V can be written in its *triangular decomposition* $\mathsf{V} = \mathsf{V}^- \oplus \mathsf{V}^0 \oplus \mathsf{V}^+$, with $\mathsf{V}^+ = \mathrm{span}\{L_m : m \in \mathbf{N}\}$, the *positive Virasoro operators*, and $\mathsf{V}^- = \mathrm{span}\{L_{-m} : m \in \mathbf{N}\}$, the *negative Virasoro operators*. The *Cartan subalgebra* is given by $\mathsf{V}^0 = \mathrm{span}\{L_0, C\}$. For elements Y of V that are eigenvectors of L_0 with respect to the adjoint representation we call the L_0-eigenvalue the *level* of Y and denote it by[c] $|Y|$: $[L_0, Y] = |Y|Y$. The same shall be used for the universal enveloping algebra $U(\mathsf{V})$. In particular, elements of $U(\mathsf{V})$ of the form $Y = L_{-p_I} \ldots L_{-p_1}$, $p_q \in \mathbf{Z}$ for $q = 1, \ldots, I$, $I \in \mathbf{N}$, are at level $|Y| = \sum_{q=1}^{I} p_q$ and we furthermore define them to be of *length* $\|Y\| = I$. Finally, for the identity operator we set $\|1\| = |1| = 0$. For convenience we define the graded class of subsets of operators in $U(\mathsf{V})$ at positive level:

$$\mathcal{S}_m \;=\; \{S = L_{-m_I} \ldots L_{-m_1} :\; |S| = m\,;\; m_I \geq \ldots m_1 \geq 2\,;\; m_1, \ldots, m_I, I \in \mathbf{N}\}\,, \tag{2}$$

for $m \in \mathbf{N}$, $\mathcal{S}_0 = \{1\}$, and also

$$\mathcal{C}_n \;=\; \{X = S_m L_{-1}^{n-m} :\; S_m \in \mathcal{S}_m,\, m \in \mathbf{N}_0,\, m \leq n\}\,, \tag{3}$$

[b]The exact proof of Kent showed that generalised Virasoro singular vectors at level 0 are scalar multiples of the identity.

[c]Note that positive generators L_m have negative level $|L_m| = -m$. Therefore, any positive operators $\Gamma \in \mathsf{V}^+$ have a negative level $|\Gamma|$.

for $n \in \mathbf{N}_0$, which will serve to construct a basis for Virasoro *Verma modules* later on.

We consider representations of V for which the Cartan subalgebra V^0 is diagonal. Furthermore, C commutes with all operators of V and can hence be taken to be constant $c \in \mathbf{C}$ (in an irreducible representation). A representation with L_0-eigenvalues bounded from below contains a vector with L_0-eigenvalue Δ which is annihilated by V^+, a *highest weight vector* $|\Delta, c\rangle$:

$$\mathsf{V}^+ |\Delta, c\rangle = 0\,, \quad L_0 |\Delta, c\rangle = \Delta |\Delta, c\rangle\ , \quad C |\Delta, c\rangle = c |\Delta, c\rangle\ . \tag{4}$$

The *Verma module* $\mathcal{V}_{\Delta,c}$ is the left-module $\mathcal{V}_{\Delta,c} = U(\mathsf{V}) \otimes_{\mathsf{V}^0 \oplus \mathsf{V}^+} |\Delta, c\rangle$. For $\mathcal{V}_{\Delta,c}$ we choose the standard basis $\mathcal{B}^{\Delta,c}$ as:

$$\mathcal{B}^{\Delta,c} \;\; = \;\; \{S_m L_{-1}^n |\Delta, c\rangle : S_m \in \mathcal{S}_m\,, m, n \in \mathbf{N}_0\}\,. \tag{5}$$

$\mathcal{V}_{\Delta,c}$ and $\mathcal{B}^{\Delta,c}$ are L_0-graded in a natural way. The corresponding L_0-eigenvalue is called the *conformal weight* and the L_0-eigenvalue relative to Δ is the *level*. Let us introduce

$$\mathcal{B}_k^{\Delta,c} \;\; = \;\; \{X_k |\Delta, c\rangle : X_k \in \mathcal{C}_k\}\ , \quad k \in \mathbf{N}_0\,. \tag{6}$$

Thus, $\mathcal{B}_k^{\Delta,c}$ has conformal weight k and span$\{\mathcal{B}_k^{\Delta,c}\}$ is the grade space of $\mathcal{V}_{\Delta,c}$ at level k. For $x \in \text{span}\{\mathcal{B}_k^{\Delta,c}\}$ we again denote the level by $|x| = k$.

Verma modules may not be irreducible. In order to obtain physically relevant irreducible highest weight representations one thus needs to trace back the proper submodules of $\mathcal{V}_{\Delta,c}$ and divide them out. This finally leads to the notion of singular vectors as any proper submodule of $\mathcal{V}_{\Delta,c}$ needs to contain a vector Ψ_l that is not proportional to the highest weight vector $|\Delta, c\rangle$ but still satisfies the *highest weight vector conditions*[d] with conformal weight[e] $\Delta + l$ for some $l \in \mathbf{N}_0$:

$$\mathsf{V}^+ \Psi_l = 0\,, \quad L_0 \Psi_l = (\Delta + l)\Psi_l\,, \quad C\Psi_l = c\Psi_l \tag{7}$$

l is the level of Ψ_l, denoted by $|\Psi_l|$. An eigenvector Ψ_l of L_0 at level l in $\mathcal{V}_{\Delta,c}$, in particular a singular vector, can thus be written using the basis (6):

$$\Psi_l \;\; = \;\; \sum_{m=0}^{l} \sum_{S_m \in \mathcal{S}_m} c_{S_m}\, S_m L_{-1}^{l-m} |\Delta, c\rangle\ , \tag{8}$$

with coefficients $c_{S_m} \in \mathbf{C}$. The basis decomposition (8) of an L_0-eigenvector in $\mathcal{V}_{\Delta,c}$ will be denoted the *normal form* of Ψ_l, where $S_m L_{-1}^{l-m} \in \mathcal{C}_l$ and c_{S_m} will be referred to as the *terms* and *coefficients* of Ψ_l, respectively. A *non-trivial term* $Y \in \mathcal{C}_l$ of Ψ_l refers to a term Y in Eq. (8) with non-trivial coefficient c_Y.

Let $\mathcal{O}$ denote a total ordering on $\mathcal{C}_l$ with global minimum. Thus Ψ_l in Eq. (8) needs to contain an *$\mathcal{O}$-smallest* $X_0 \in \mathcal{C}_l$ with $c_{X_0} \neq 0$ and $c_Y = 0$ for all $Y \in \mathcal{C}_l$ with $Y <_{\mathcal{O}} X_0$ and $Y \neq X_0$. Let us assume that the ordering $\mathcal{O}$ exists with global minimum $L_{-1}^l \in \mathcal{C}_l$ and is such that for any term $X \in \mathcal{C}_l$, $X \neq L_{-1}^l$ of a vector Ψ_l at level $l \in \mathbf{N}_0$, the action of at least one positive operator Γ of V^+ on $X |\Delta, c\rangle$ contains a non-trivial basis term of $\mathcal{B}_{l'}^{\Delta,c}$, with $l' = l + |\Gamma|$, that cannot be obtained from any other term of Ψ_l $\mathcal{O}$-larger than X. However, for a singular vector Ψ_l the action of any positive operator of V^+ needs to vanish and thus we find immediately that any given non-trivial

[d] Note that the trivial vector 0 satisfies the highest weight conditions Eq. (4) at any level l. However, 0 is proportional to $|\Delta, c\rangle$ and is therefore not a singular vector.

[e] An eigenvector of L_0 with eigenvalue $\Delta + l$ in the Verma module $\mathcal{V}_{\Delta,c}$ is usually labeled by the level l.

singular vector Ψ_l contains in its normal form the non-trivial term $L_{-1}^l \in \mathcal{C}_l$. Otherwise there must exist a smallest non-trivial term X_0 of Ψ_l different from L_{-1}^l. However, by assumption we can find a positive operator Γ annihilating Ψ_l but also creating a term that can only be generated from the term X_0 and from no other terms of Ψ_l. Thus the coefficient c_{X_0} of X_0 is trivial and therefore Ψ_l is trivial. This motivates our definition of *adapted orderings* for Virasoro Verma modules:

Definition 2.A *A total ordering $\mathcal{O}$ on $\mathcal{C}_l$ $(l \in \mathbf{N}_0)$ with global minimum is called adapted to the subset $\mathcal{C}_l^A \subset \mathcal{C}_l$ in the Verma module $\mathcal{V}_{|\Delta,c\rangle}$ if for any element $X_0 \in \mathcal{C}_l^A$ at least one positive operator $\Gamma \in \mathsf{V}^+$ exists for which*

$$\Gamma\ X_0 \,|\Delta, c\rangle \;=\; \sum_{m=0}^{l+|\Gamma|} \sum_{S_m \in \mathcal{S}_m} c_{S_m}^{\Gamma X_0}\, S_m L_{-1}^{l+|\Gamma|-m}\,|\Delta, c\rangle \tag{9}$$

contains a non-trivial term $\tilde{X} \in \mathcal{C}_{l+|\Gamma|}$ (i.e. $c_{\tilde{X}}^{\Gamma X_0} \neq 0$) such that for all $Y \in \mathcal{C}_l$ with $X_0 <_{\mathcal{O}} Y$ and $X_0 \neq Y$ the coefficient $c_{\tilde{X}}^{\Gamma Y}$ in

$$\Gamma\ Y \,|\Delta, c\rangle \;=\; \sum_{m=0}^{l+|\Gamma|} \sum_{S_m \in \mathcal{S}_m} c_{S_m}^{\Gamma Y}\, S_m L_{-1}^{l+|\Gamma|-m}\,|\Delta, c\rangle \tag{10}$$

is trivial: $c_{\tilde{X}}^{\Gamma Y} = 0$. The complement of $\mathcal{C}_l^A$, $\mathcal{C}_l^K = \mathcal{C}_l \setminus \mathcal{C}_l^A$, is the kernel with respect to the ordering $\mathcal{O}$ in the Verma module $\mathcal{V}_{\Delta,c}$.

Obviously, any total ordering on $\mathcal{C}_l$ is always adapted to the subset $\emptyset \subset \mathcal{C}_l$ with ordering kernel $\mathcal{C}_l^K = \mathcal{C}_l$, what does not give much information. For our purposes we need to find suitable ordering restrictions in order to obtain the smallest possible ordering kernels (which is not a straightforward task). In the Virasoro case we will give an ordering such that the ordering kernel for each $l \in \mathbf{N}$ has just one element: L_{-1}^l. As indicated in our motivation, it is then fairly simple to show that a singular vector at level l needs to have a non-trivial coefficient for the term L_{-1}^l in its normal form. If two singular vectors have the same coefficient for this term, then their difference is either trivial or a singular vector with trivial L_{-1}^l term. Again the latter is not allowed and hence all singular vectors at level l are unique up to proportionality. This will be summarised in the following theorem:

Theorem 2.B *Let $\mathcal{O}$ denote an adapted ordering in $\mathcal{C}_l^A$ at level $l \in \mathbf{N}$ with a kernel $\mathcal{C}_l^K$ consisting of just one term K for a given Verma module $\mathcal{V}_{\Delta,c}$. If two vectors Ψ_l^1 and Ψ_l^2 at level l in $\mathcal{V}_{\Delta,c}$, both satisfying the highest weight conditions* Eq. (7), *have $c_K^1 = c_K^2$, then*

$$\Psi_l^1 \;\equiv\; \Psi_l^2\,. \tag{11}$$

Proof: Let us consider $\tilde{\Psi}_l = \Psi_l^1 - \Psi_l^2$. The normal form of $\tilde{\Psi}_l$ does not contain the term K as $c_K^1 = c_K^2$. As $\mathcal{C}_l$ is a totally ordered set with respect to $\mathcal{O}$, the non-trivial terms of $\tilde{\Psi}_l$, provided $\tilde{\Psi}_l$ is non-trivial, need to have an $\mathcal{O}$-minimum $X_0 \in \mathcal{C}_l$. By construction, the coefficient $\tilde{c}_{X_0}$ of X_0 in $\tilde{\Psi}_l$ is non-trivial, hence, X_0 is contained in $\mathcal{C}_l^A$. As $\mathcal{O}$ is adapted to $\mathcal{C}_l^A$ we can find a positive generator $\Gamma \in \mathsf{V}^+$ such that $\Gamma X_0\,|\Delta, c\rangle$ contains a non-trivial term that cannot be created from any other term of $\tilde{\Psi}_l$ which is $\mathcal{O}$-larger than X_0. But X_0 was chosen to be the $\mathcal{O}$-minimum of the non-trivial terms of $\tilde{\Psi}_l$. Therefore, $\Gamma X_0\,|\Delta, c\rangle$ contains a non-trivial term that cannot be created from any other term of $\tilde{\Psi}_l$. The coefficient of this term is obviously given by $a\tilde{c}_{X_0}$ with a non-trivial complex number a. Like Ψ_l^1 and Ψ_l^2, $\tilde{\Psi}_l$ is also annihilated by any positive generator, in particular by Γ. It follows that $\tilde{c}_{X_0}=0$ which leads to contradiction. Thus, the set of non-trivial terms of $\tilde{\Psi}_l$ is empty and therefore $\tilde{\Psi}_l=0$. This results in $\Psi_l^1=\Psi_l^2$. □

Equipped with definition 2.A and theorem 2.B we can now easily prove the well-known[25, 14] uniqueness of Virasoro singular vectors. Let us first review the total ordering on $\mathcal{C}_l$ defined by Kent[25]. Whilst Kent used the following ordering to show that in his generalised Virasoro Verma modules vectors at level 0 satisfying the highest weight conditions are actually proportional to the highest weight vector, we will use theorem 2.B to show that, furthermore, already the ordering implies that all Virasoro singular vectors are unique at their levels up to proportionality.

Definition 2.C *On the set $\mathcal{C}_l$ of Virasoro operators we introduce the total ordering $\mathcal{O}_{\mathsf{V}}$ for $l\in\mathbf{N}$. For two elements $X_1,X_2\in\mathcal{C}_l$, $X_1\neq X_2$, with $X_i=L_{-m^i_{I_i}}\dots L_{-m^i_1}L_{-1}^{n^i}$, $n^i=l-m^i_{I_i}\dots-m^i_1$, or $X_i=L_{-1}^l$, $i=1,2$ we define*

$$X_1 <_{\mathcal{O}_{\mathsf{V}}} X_2 \quad \text{if} \quad n^1>n^2\,. \tag{12}$$

If, however, $n^1=n^2$ we compute the index $j_0=\min\{j: m_j^1-m_j^2\neq 0, j=1,\dots,\min(I_1,I_2)\}$. We then define

$$X_1 <_{\mathcal{O}_{\mathsf{V}}} X_2 \quad \text{if} \quad m^1_{j_0}<m^2_{j_0}\,. \tag{13}$$

For $X_1=X_2$ we set $X_1 <_{\mathcal{O}_{\mathsf{V}}} X_2$ and $X_2 <_{\mathcal{O}_{\mathsf{V}}} X_1$.

In order to show that definition 2.C is well-defined, we need to prove that the set of indices $J=\{j: m_j^1-m_j^2\neq 0, j=1,\dots,\min(I_1,I_2)\}$ is non-trivial for the cases that $n^1=n^2$ but $X_1\neq X_2$ and thus the minimum j_0 is well-defined. Indeed, trivial J either implies that at least one of X_1 or X_2 is equal to L_{-1}^l, or that the positive numbers m_j^i agree for $i=1$ and $i=2$ for all j from 1 to $\min(I_1,I_2)$. In the first case, however, as L_{-1}^l is the only element of $\mathcal{C}_l$ with l operators L_{-1} and as we assumed $n^1=n^2$ we find that both X_1 and X_2 must be L_{-1}^l. In the second case, let us assume that $I_1<I_2$, then obviously $n^1=n^2$ and $|X_1|=|X_2|=l$ imply $\sum_{j=1}^{I_1}m_j^1=\sum_{j=1}^{I_1}m_j^2+\sum_{j=I_1+1}^{I_2}m_j^2=\sum_{j=1}^{I_1}m_j^1+\sum_{j=I_1+1}^{I_2}m_j^2$ and thus $\sum_{j=I_1+1}^{I_2}m_j^2=0$. As all the numbers m_j^2 are strictly positive we obtain $I_1=I_2$ and thus again $X_1=X_2$.

The index j_0 is therefore defined for all pairs X_1,X_2 with $n^1=n^2$ ($X_1\neq X_2$). j_0 describes the first index, read from the right to the left, for which the generators in X_1 and X_2 (L_{-1} excluded) are different. For example $L_{-3}L_{-2}L_{-2}L_{-1}^3$ is $\mathcal{O}_{\mathsf{V}}$-smaller than $L_{-5}L_{-2}L_{-1}^3$ with index $j_0=2$. Before proceeding we ought to remark that $L_{-1}^l\in\mathcal{C}_l$ is obviously the global $\mathcal{O}_{\mathsf{V}}$-minimum in $\mathcal{C}_l$. The following theorem combines our results so far and shows the significance of $\mathcal{O}_{\mathsf{V}}$.

Theorem 2.D *The ordering $\mathcal{O}_{\mathsf{V}}$ is adapted to $\mathcal{C}_l^A=\mathcal{C}_l\setminus\{L_{-1}^l\}$ for each level $l\in\mathbf{N}$ and for all Verma modules $\mathcal{V}_{\Delta,c}$. The ordering kernel is given by the single element set $\mathcal{C}_l^K=\{L_{-1}^l\}$.*

Proof: The idea of the proof is a generalisation of Kent's proof in Ref. 25. Let us consider $X_0=L_{-m_I}\dots L_{-m_1}L_{-1}^{n_0}\in\mathcal{C}_l^A$, $n_0=l-m_I\dots-m_1$, $m_I\geq\dots\geq m_1\geq 2$ for $I\in\mathbf{N}$. We then construct a vector $\Psi_l=X_0|\Delta,c\rangle$ at level l in the Verma module $\mathcal{V}_{\Delta,c}$. We apply the positive operator L_{m_1-1} to Ψ_l and write the result in its normal form

$$L_{m_1-1}\Psi_l = \sum_{m=0}^{l-m_1+1}\sum_{S_m\in\mathcal{S}_m} c_{S_m}S_m L_{-1}^{l-m_1+1-m}|\Delta,c\rangle\,, \tag{14}$$

following Eq. (9). Eq. (14) contains a non-trivial contribution of $\tilde{S} = L_{-m_I} \dots L_{-m_2}$, simply by commuting L_{m_1-1} with L_{-m_1} in X_0 which creates another operator L_{-1} but lets $L_{-m_I} \dots L_{-m_2}$ unchanged and thus creates the term $\tilde{X} = L_{-m_I} \dots L_{-m_2} L_{-1}^{n_0+1}$. In the case $m_1 = m_2 = \dots = m_j$ we simply obtain multiple copies of this term. However, for any other term $Y = L_{-m_J^Y} \dots L_{-m_1^Y} L_{-1}^{n^Y}$ with $Y \in \mathcal{C}_l$ ($n^Y = l - m_J^Y \dots - m_1^Y$, $J \in \mathbf{N}_0$) producing the term $\tilde{X}$ under the action of L_{m_1-1}, either the term Y needs to have already at least one L_{-1} more than X_0, and would consequently be $\mathcal{O}$-smaller than X_0 due to Eq. (12), or L_{m_1-1} needs to create L_{-1} by commuting through $L_{-m_J^Y} \dots L_{-m_1^Y}$. The latter, however, is only possible if $m_1^Y < m_1$. Otherwise the commutation relations would not allow L_{-1} being created from $L_{m_1^Y-1}$ and for $m_1^Y = m_1$ we would ultimately find $X_0 = Y$, as both terms need to create $\tilde{X}$. Hence $m_1^Y < m_1$ and therefore one finds $Y <_{\mathcal{O}_V} X_0$. Consequently, there is no term $\mathcal{O}_V$-bigger than X_0 producing the term $\tilde{X}$ under the action of the positive generator L_{m_1-1}. □

Theorem 2.D implies as an immediate consequence the following theorem about the uniqueness of Virasoro singular vectors.

Theorem 2.E *If the Virasoro Verma module $\mathcal{V}_{\Delta,c}$ contains a singular vector Ψ_l at level l, $l \in \mathbf{N}$, then Ψ_l is unique up to proportionality. The coefficient of the term $L_{-1}^l \in \mathcal{C}_l$ in the normal form of Ψ_l, i.e. the coefficient of $L_{-1}^l |\Delta, c\rangle$, is non-trivial.*

Proof: We first show that the $L_{-1}^l |\Delta, c\rangle$ component in the normal form of Ψ_l is non-trivial. Let us assume this component is trivial. The trivial vector 0 also satisfies the highest weight conditions Eq. (7) for any level l and has trivial $L_{-1}^l |\Delta, c\rangle$ component. According to theorem 2.D, $\{L_{-1}^l\}$ is an ordering kernel for the ordering $\mathcal{O}_V$ on $\mathcal{C}_l$. Therefore, from theorem 2.B we know that $\Psi_l = 0$ and therefore Ψ_l is not a singular vector, which is a contradiction. Hence, we obtain that the component of $L_{-1}^l |\Delta, c\rangle$ in Ψ_l has to be non-trivial. Let us now assume Ψ_l' is another singular vector in $\mathcal{V}_{\Delta,c}$ at the same level l as Ψ_l. We know that the coefficients $c'_{L_{-1}^l}$ and $c_{L_{-1}^l}$ of Ψ_l' and Ψ_l respectively are both non-trivial. Therefore $c'_{L_{-1}^l} \Psi_l$ and $c_{L_{-1}^l} \Psi_l'$ are two singular vectors at the same level which agree in their L_{-1}^l coefficient and according to theorem 2.B are identical. Thus, Ψ_l and Ψ_l' are proportional. □

Feigin and Fuchs[14] have proven for which Verma modules these unique Virasoro singular vectors do exist.

3 Superconformal algebras and adapted orderings

A superconformal algebra is a Lie super algebra that contains the Virasoro algebra as a subalgebra. Therefore superconformal algebras are also known as *super extensions of the Virasoro algebra*. Thanks to Kac[20,22], Cheng and Kac[6], and Kac and van de Leur[23] all superconformal algebras are known by now. Let $\mathcal{A}$ denote a superconformal algebra and let $U(\mathcal{A})$ be the universal enveloping algebra of $\mathcal{A}$. We want that the *energy operator* L_0 of the Virasoro subalgebra is contained in our choice of the Cartan subalgebra $\mathcal{H}_{\mathcal{A}}$ of $\mathcal{A}$. $\mathcal{A}$ thus decomposes in L_0-grade spaces which we want to group according to the sign of the grade: $\mathcal{A} = \mathcal{A}^- \oplus \mathcal{A}^0 \oplus \mathcal{A}^-$ where the L_0-grades of $\mathcal{A}^-$, $\mathcal{A}^0$, or

$\mathcal{A}^+$ are positive, zero, or negative respectively[f]. Consequently, $U(\mathcal{A})$ also decomposes in L_0-grade spaces: $U(\mathcal{A}) = U(\mathcal{A})^- \oplus U(\mathcal{A})^0 \oplus U(\mathcal{A})^+$. Obviously, the Cartan subalgebra $\mathcal{H}_\mathcal{A}$ is contained in $\mathcal{A}^0$ but does not need to be identical to $\mathcal{A}^0$. The L_0-grade is just one component of the roots $\mu \in \mathcal{H}^*_\mathcal{A}$ (the dual space of $\mathcal{H}_\mathcal{A}$) of $\mathcal{A}$. For simplicity, let us fix a basis for $\mathcal{H}_\mathcal{A}$ that contains L_0: $\{L_0, H^2, \ldots, H^r\}$, and hence let us denote the roots as (Δ, μ) where Δ indicates the L_0 component and $\mu = (\mu_2, \ldots, \mu_r)$ the vector of all other components.

Physicists are mainly interested in *positive energy representations*. One thus defines a *highest weight vector* $|\Delta, \mu\rangle$ as a simultaneous eigenvector of $\mathcal{H}_\mathcal{A}$ with eigenvalues, the *weights*, (Δ, μ) and vanishing $\mathcal{A}^+$ action: $\mathcal{A}^+ |\Delta, \mu\rangle = 0$. The L_0-weight $\Delta = \mu(L_0)$ is the conformal weight, which is for convenience always denoted explicitlty in addition to the other weights μ. Depending on the algebra $\mathcal{A}$, physical as well as mathematical applications may require highest weight vectors that satisfy additional vanishing conditions with respect to operators of $\mathcal{A}^0$ (the *zero modes*) with $\mathcal{H}_\mathcal{A}$ normally excluded. Later on, this shall be further explained in section 4 for the topological $N = 2$ algebra.

Definition 3.A *For a subalgebra $\mathcal{N}$ of $\mathcal{A}^0$ that includes the Cartan subalgebra $\mathcal{H}_\mathcal{A}$ we define a highest weight vector $|\Delta, \mu\rangle^\mathcal{N}$ with weight (Δ, μ):*

$$L_0 \, |\Delta, \mu\rangle^\mathcal{N} = \Delta \, |\Delta, \mu\rangle^\mathcal{N} \,, \tag{15}$$

$$H^i \, |\Delta, \mu\rangle^\mathcal{N} = \mu_i \, |\Delta, \mu\rangle^\mathcal{N} \,, \; i = 2, \ldots, r \,, \tag{16}$$

$$\mathcal{A}^+ \, |\Delta, \mu\rangle^\mathcal{N} = 0 \,, \tag{17}$$

$$\Gamma^0 \, |\Delta, \mu\rangle^\mathcal{N} = 0 \,, \; \forall \, \Gamma^0 \in \mathcal{N}/\mathcal{H}_\mathcal{A} \,. \tag{18}$$

A Verma module is then defined analogously to the Virasoro case as the left module $\mathcal{V}^\mathcal{N}_{\Delta,\mu} = U(\mathcal{A}) \otimes_{\mathcal{N} \oplus \mathcal{A}^+} |\Delta, \mu\rangle^\mathcal{N}$ where we use the representation Eqs. (15)-(17) *to act with $\mathcal{N} \oplus \mathcal{A}^+$ on $|\Delta, \mu\rangle^\mathcal{N}$. If $\mathcal{N} = \mathcal{H}_\mathcal{A}$ we shall simply write $\mathcal{V}_{\Delta,\mu}$ and $|\Delta, \mu\rangle$.*

The Verma module $\mathcal{V}^\mathcal{N}_{\Delta,\mu}$ is again graded with respect to $\mathcal{H}_\mathcal{A}$ into weight spaces $\mathcal{V}^{\mathcal{N},(l,q)}_{\Delta,\mu}$ with weights $(\Delta + l, \mu + q)$. For convenience we shall only use the relative weights (l, q) with $q = (q_2, \ldots, q_r)$ whenever we want to refer to a weight. The L_0 relative weight l is again called the *level*. Also for the universal enveloping algebra an element Y with well-defined L_0-grade l is said to be at level $|Y| = l$, i.e. $[L_0, Y] = |Y| Y$. As for the Virasoro case we shall define a *singular vector* of a Verma module $\mathcal{V}^\mathcal{N}_{\Delta,\mu}$ to be a vector which is not proportional to the highest weight vector but satisfies the highest weight conditions Eqs. (15)-(17) with possibly different weights.

Definition 3.B *A vector $\Psi^{\mathcal{N}'}_{l,q} \in \mathcal{V}^{\mathcal{N},(l,q)}_{\Delta,\mu}$ is said to satisfy the highest weight conditions if*

$$L_0 \; \Psi^{\mathcal{N}'}_{l,q} = (\Delta + l) \Psi^{\mathcal{N}'}_{l,q} \,, \tag{19}$$

$$H^i \; \Psi^{\mathcal{N}'}_{l,q} = (\mu_i + q_i) \Psi^{\mathcal{N}'}_{l,q} \,, \; i = 2, \ldots, r \,, \tag{20}$$

$$U(\mathcal{A})^+ \; \Psi^{\mathcal{N}'}_{l,q} = 0 \,, \tag{21}$$

$$\Gamma^0 \; \Psi^{\mathcal{N}'}_{l,q} = 0 \,, \; \forall \, \Gamma^0 \in \mathcal{N}'/U(\mathcal{H}_\mathcal{A}) \,, \tag{22}$$

[f] Note that for historical reasons the elements of $\mathcal{A}^-$ have positive L_0-grade.

for a subalgebra[g] $\mathcal{N}'$ *of* $U(\mathcal{A})^0$ *that contains* $U(\mathcal{H}_\mathcal{A})$ *and may or may not be equal to* $\mathcal{N}$. $\Psi_{l,q}^{\mathcal{N}'}$ *is called a singular vector if in addition* $\Psi_{l,q}^{\mathcal{N}'}$ *is not proportional to the highest weight vector* $|\Delta,\mu\rangle^{\mathcal{N}}$.

Each weight space $\mathcal{V}_{\Delta,\mu}^{\mathcal{N},(l,q)}$ can be generated by choosing the subset of the root space of the universal enveloping algebra $U(\mathcal{A})$ with root (l,q) that only consists of generators not taken from $\mathcal{N}\oplus\mathcal{A}^+$. From this set we choose elements such that the vectors generated by acting on the highest weight vector $|\Delta,\mu\rangle^{\mathcal{N}}$ are all linearly independent and thus form a basis for $\mathcal{V}_{\Delta,\mu}^{\mathcal{N},(l,q)}$. Let $\mathcal{C}_{l,q}^{\mathcal{N}}$ denote such a *basis set*, analogous to the basis set for the Virasoro algebra, to be specified later. Further, let $\mathcal{B}_{\Delta,\mu}^{\mathcal{N}}$ denote any basis for the Verma module $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$. The *standard basis* for the weight space $\mathcal{V}_{\Delta,\mu}^{\mathcal{N},(l,q)}$ shall be the basis $\tilde{\mathcal{B}}_{\Delta,\mu}^{\mathcal{N},(l,q)}$ generated by the sets $\mathcal{C}_{l,q}^{\mathcal{N}}$ on the highest weight vector and the basis decomposition of $\Psi_{l,q}$ with respect to $\tilde{\mathcal{B}}_{\Delta,\mu}^{\mathcal{N},(l,q)}$ is called *its normal form*. As in the Virasoro case, an element X of $\mathcal{C}_{l,q}^{\mathcal{N}}$ that generates a particular basis vector in $\tilde{\mathcal{B}}_{\Delta,\mu}^{\mathcal{N},l}$ is called a *term* of a vector $\Psi_{l,q}\in\mathcal{V}_{\Delta,\mu}^{\mathcal{N},(l,q)}$ and the corresponding coefficient c_X in its basis decomposition is simply called a *coefficient* of $\Psi_{l,q}$. Finally, we call the term X of $\Psi_{l,q}$ a *non-trivial term* if $c_X\neq 0$. This completes the necessary notation to define adapted orderings on $\mathcal{C}_{l,q}^{\mathcal{N}}$ just like in the Virasoro case.

Definition 3.C *A total ordering* $\mathcal{O}$ *on* $\mathcal{C}_{l,q}^{\mathcal{N}}$ *with global minimum is called adapted to the subset* $\mathcal{C}_{l,q}^{\mathcal{N},A}\subset\mathcal{C}_{l,q}^{\mathcal{N}}$ *in the Verma module* $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$ *with annihilation operators* $\mathcal{K}\subset U(\mathcal{A})^+\oplus U(\mathcal{A})^0/U(\mathcal{H}_\mathcal{A})$ *if for any element* $X_0\in\mathcal{C}_{l,q}^{\mathcal{N},A}$ *at least one annihilation operator* $\Gamma\in\mathcal{K}$ *exists for which*

$$\Gamma\, X_0\,|\Delta,\mu\rangle \;=\; \sum_{X\in\mathcal{B}_{\Delta,\mu}^{\mathcal{N}}} c_X^{\Gamma X_0}\, X \tag{23}$$

contains a non-trivial term $\tilde{X}\in\mathcal{B}_{\Delta,\mu}^{\mathcal{N}}$ *(i.e.* $c_{\tilde{X}}^{\Gamma X_0}\neq 0$*) such that for all* $Y\in\mathcal{C}_{l,q}^{\mathcal{N}}$ *with* $X_0 <_{\mathcal{O}} Y$ *and* $X_0\neq Y$ *the coefficient* $c_{\tilde{X}}^{\Gamma Y}$ *in*

$$\Gamma\, Y\,|\Delta,\mu\rangle \;=\; \sum_{X\in\mathcal{B}_{\Delta,\mu}^{\mathcal{N}}} c_X^{\Gamma Y}\, X \tag{24}$$

is trivial: $c_{\tilde{X}}^{\Gamma Y}=0$. *The complement of* $\mathcal{C}_{l,q}^{\mathcal{N},A}$, $\mathcal{C}_{l,q}^{\mathcal{N},K}=\mathcal{C}_{l,q}^{\mathcal{N}}\setminus\mathcal{C}_{l,q}^{\mathcal{N},A}$ *is the kernel with respect to the ordering* $\mathcal{O}$ *in the Verma module* $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$. *Here* $\mathcal{B}_{\Delta,\mu}^{\mathcal{N}}$ *represents a basis that can be chosen suitably for each* X_0 *and may or may not be the standard basis*[h].

In the motivation to theorem 2.B we assumed the existence of an ordering with the smallest kernel consisting of one element only. For the $N=2$ algebras we will find ordering kernels which contain more than one element and also ordering kernels that are trivial. We saw in theorem 2.E that if the ordering kernel has only one element (the global minimum L_{-1}^l in the case of the Virasoro algebra) then any singular vector needs to have a non-trivial coefficient for this element. The following theorems reveal what can be implied if the ordering kernel consists of more than one element or of none at all.

[g]Note that this time, for convenience, we have chosen the universal enveloping algebra to define the highest weight conditions which is equivalent to the earlier definition Eqs. (15)-(17).

[h]In the $N=2$ case we will choose for most X_0 the standard basis with only very few but important exceptions.

Theorem 3.D *Let $\mathcal{O}$ denote an adapted ordering on $\mathcal{C}_{l,q}^{\mathcal{N},A}$ at weight (l,q) with kernel $\mathcal{C}_{l,q}^{\mathcal{N},K}$ for a given Verma module $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$ and annihilation operators $\mathcal{K}$. If two vectors $\Psi_{l,q}^{\mathcal{N}',1}$ and $\Psi_{l,q}^{\mathcal{N}',2}$ at the same level l and weight q, satisfying the highest weight conditions* Eqs. (19)-(22) *with $\mathcal{N}' = \mathcal{K}$, have $c_X^1 = c_X^2$ for all $X \in \mathcal{C}_{l,q}^{\mathcal{N},K}$, then*

$$\Psi_{l,q}^{\mathcal{N}',1} \equiv \Psi_{l,q}^{\mathcal{N}',2} . \tag{25}$$

Proof: Let us consider $\tilde{\Psi}_{l,q} = \Psi_{l,q}^{\mathcal{N}',1} - \Psi_{l,q}^{\mathcal{N}',2}$. The normal form of $\tilde{\Psi}_{l,q}$ does not contain any terms of the ordering kernel $\mathcal{C}_{l,q}^{K}$, simply because $c_X^1 = c_X^2$ for all $X \in \mathcal{C}_{l,q}^{\mathcal{N},K}$. As $\mathcal{C}_{l,q}^{\mathcal{N}}$ is a totally ordered set with respect to $\mathcal{O}$ which has a global minimum, the non-trivial terms of $\tilde{\Psi}_{l,q}$, provided $\tilde{\Psi}_{l,q}$ is non-trivial, need to have an $\mathcal{O}$-minimum $X_0 \in \mathcal{C}_{l,q}^{\mathcal{N}}$. By construction, the coefficient $\tilde{c}_{X_0}$ of X_0 in $\tilde{\Psi}_{l,q}$ in its normal form is non-trivial, hence, X_0 is also contained in $\mathcal{C}_{l,q}^{\mathcal{N},A}$. As $\mathcal{O}$ is adapted to $\mathcal{C}_{l,q}^{\mathcal{N},A}$ one can find an annihilation operator $\Gamma \in \mathcal{K}$ such that $\Gamma X_0 \,|\Delta,\mu\rangle^{\mathcal{N}}$ contains a non-trivial term (for a suitably chosen basis depending on X_0) that cannot be created by any other term of $\tilde{\Psi}_{l,q}$ which is $\mathcal{O}$-larger than X_0. But X_0 was chosen to be the $\mathcal{O}$-minimum of $\tilde{\Psi}_{l,q}$. Therefore, $\Gamma X_0 \,|\Delta,\mu\rangle^{\mathcal{N}}$ contains a non-trivial term that cannot be created from any other term of $\tilde{\Psi}_{l,q}$. The coefficient of this term is obviously given by $a\tilde{c}_{X_0}$ with a non-trivial complex number a. Together with $\Psi_{l,q}^{\mathcal{N}',1}$ and $\Psi_{l,q}^{\mathcal{N}',2}$, $\tilde{\Psi}_{l,q}$ is also annihilated by any annihilation operator, in particular by Γ. It follows that $\tilde{c}_{X_0} = 0$, contrary to our original assumption. Thus, the set of non-trivial terms of $\tilde{\Psi}_{l,q}$ is empty and therefore $\tilde{\Psi}_{l,q} = 0$. This results in $\Psi_{l,q}^{\mathcal{N}',1} = \Psi_{l,q}^{\mathcal{N}',2}$. □

Theorem 3.D states that if two singular vectors at the same level and weight agree on the ordering kernel, then they are identical. The coefficients of a singular vector with respect to the ordering kernel are therefore sufficient to distinguish singular vectors. If the ordering kernel is trivial we consequently find 0 as the only vector that can satisfy the highest weight conditions.

Theorem 3.E *Let $\mathcal{O}$ denote an adapted ordering on $\mathcal{C}_{l,q}^{\mathcal{N},A}$ at weight (l,q) with trivial kernel $\mathcal{C}_{l,q}^{\mathcal{N},K} = \emptyset$ for a given Verma module $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$ and annihilation operators $\mathcal{K}$. A vector $\Psi_{l,q}^{\mathcal{N}'}$ at level l and weight q satisfying the highest weight conditions* Eqs. (19)-(22) *with $\mathcal{N}' = \mathcal{K}$, is therefore trivial. In particular, this shows that there are no singular vectors.*

Proof: We again make use of the fact that the trivial vector 0 satisfies any vanishing conditions for any level l and weight q. As the ordering kernel is trivial the components of the vectors 0 and $\Psi_{l,q}^{\mathcal{N}'}$ agree on the ordering kernel and using theorem 3.D we obtain $\Psi_{l,q}^{\mathcal{N}'} = 0$. □

We now know that singular vectors can be classified by their components on the ordering kernel. As we shall see if the ordering kernel has n elements, then the space of singular vectors for this weight is at most n-dimensional. Conversely, one could ask if there are singular vectors corresponding to all possible combinations of elements of the ordering kernel. In general this will not be the case, however, for the Virasoro algebra[25] and for the Neveu-Schwarz $N = 2$ algebra[9] it has been shown that for each element of the ordering kernel there exists a singular vector for suitably defined analytically continued Verma modules. Some of these generalised singular vectors lie in the embedded original non-continued Verma module and are therefore singular in the above sense. We finally conclude with the following theorem summarising all our findings so far.

Theorem 3.F *Let $\mathcal{O}$ denote an adapted ordering on $\mathcal{C}_{l,q}^{\mathcal{N},A}$ at weight (l,q) with kernel $\mathcal{C}_{l,q}^{\mathcal{N},K}$ for a given Verma module $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$ and annihilation operators $\mathcal{K}$. If the ordering kernel $\mathcal{C}_{l,q}^{\mathcal{N},K}$ has n elements, then there are at most n linearly independent singular vectors $\Psi_{l,q}^{\mathcal{N}'}$ in $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$ with weight (l,q) and $\mathcal{N}'=\mathcal{K}$.*

Proof: Suppose there were more than n linearly independent singular vectors $\Psi_{l,q}^{\mathcal{N}'}$ in $\mathcal{V}_{\Delta,\mu}^{\mathcal{N}}$ with weight (l,q). We choose $n+1$ linearly independent singular vectors among them $\Psi_1,\ldots,\Psi_{n+1}$. The ordering kernel $\mathcal{C}_{l,q}^{\mathcal{N},K}$ has the n elements $K_1,\ldots,K_n$. Let c_{jk} denote the coefficient of the term K_j in the vector Ψ_k in its standard basis decomposition. The coefficients c_{jk} thus form a n by $n+1$ matrix C. The homogeneous system of linear equations $C\lambda=0$ thus has a non-trivial solution $\lambda^0=(\lambda_1^0,\ldots,\lambda_{n+1}^0)^T$ for the vector λ. We then form the linear combination $\Psi=\sum_{i=1}^{n+1}\lambda_i^0\Psi_i$. Obviously, the coefficient of K_j in the vector Ψ in its normal form is just given by the j-th component of the vector $C\lambda$ which is trivial for $j=1,\ldots,n$. Hence, the coefficients of Ψ are trivial on the ordering kernel. On the other hand, Ψ is a linear combination of singular vectors and therefore also satisfies the highest weight conditions with $\mathcal{N}'=\mathcal{K}$ just like the trivial vector 0. Due to theorem 3.D one immediately finds that $\Psi\equiv 0$ and therefore $\sum_{i=1}^{n+1}\lambda_i\Psi_i=0$. This, however, is a non-trivial decomposition of 0 contradicting the assumption that $\Psi_1,\ldots,\Psi_{n+1}$ are linearly independent. □

4 Topological $N=2$ superconformal Verma modules

We will now apply the construction developed in the previous section to the topological $N=2$ superconformal algebra. We first introduce an adapted ordering on the basis of the $N=2$ Verma modules. Consequently the size of the ordering kernel will reveal a maximum for the degrees of freedom of the singular vectors in the same $N=2$ grade space. As the representation theory of the $N=2$ superconformal algebras has different types of Verma modules we will see that the corresponding ordering kernels also allow different degrees of freedom.

The topological $N=2$ superconformal algebra T_2 is a super Lie algebra which contains the Virasoro generators $\mathcal{L}_m$ with trivial central extension[i], a Heisenberg algebra $\mathcal{H}_m$ corresponding to the U(1) current, and the fermionic generators $\mathcal{G}_m$ and $\mathcal{Q}_m$, $m\in\mathbf{Z}$ corresponding to two anti-commuting fields with conformal weights 2 and 1 respectively. T_2 satisifies the (anti-)commutation relations[7]

$$\begin{aligned}
&[\mathcal{L}_m,\mathcal{L}_n]=(m-n)\mathcal{L}_{m+n}\,, && [\mathcal{H}_m,\mathcal{H}_n]=\tfrac{C}{3}m\delta_{m+n}\,,\\
&[\mathcal{L}_m,\mathcal{G}_n]=(m-n)\mathcal{G}_{m+n}\,, && [\mathcal{H}_m,\mathcal{G}_n]=\mathcal{G}_{m+n}\,,\\
&[\mathcal{L}_m,\mathcal{Q}_n]=-n\mathcal{Q}_{m+n}\,, && [\mathcal{H}_m,\mathcal{Q}_n]=-\mathcal{Q}_{m+n}\,,\\
&[\mathcal{L}_m,\mathcal{H}_n]=-n\mathcal{H}_{m+n}+\tfrac{C}{6}(m^2+m)\delta_{m+n}\,, &&\\
&\{\mathcal{G}_m,\mathcal{Q}_n\}=2\mathcal{L}_{m+n}-2n\mathcal{H}_{m+n}+\tfrac{C}{3}(m^2+m)\delta_{m+n}\,, &&\\
&\{\mathcal{G}_m,\mathcal{G}_n\}=\{\mathcal{Q}_m,\mathcal{Q}_n\}=0\,, && m,\ n\in\mathbf{Z}\,.
\end{aligned}\tag{26}$$

The central term C commutes with all other operators and can therefore be fixed again as $c\in\mathbf{C}$. $\mathcal{H}_{\mathsf{T}_2}=\text{span}\{\mathcal{L}_0,\mathcal{H}_0,C\}$ defines a commuting subalgebra of T_2, which can therefore be diagonalised simultaneously. Generators with positive index span the set of *positive operators* T_2^+

[i]Note our slightly different notation for the Virasoro generators $\mathcal{L}_n$ in the topological $N=2$ case.

of T_2 and likewise generators with negative index span the set of *negative operators* T_2^- of T_2:

$$\mathsf{T}_2^+ = \text{span}\{\mathcal{L}_m, \mathcal{H}_m, \mathcal{G}_n, \mathcal{Q}_n : m, n \in \mathbf{N}\}\,, \tag{27}$$

$$\mathsf{T}_2^- = \text{span}\{\mathcal{L}_{-m}, \mathcal{H}_{-m}, \mathcal{G}_{-n}, \mathcal{Q}_{-n} : m, n \in \mathbf{N}\}\,. \tag{28}$$

The *zero modes* are spanned by $\mathsf{T}_2^0 = \text{span}\{\mathcal{L}_0, \mathcal{H}_0, C, \mathcal{G}_0, \mathcal{Q}_0\}$ such that the generators $\{\mathcal{G}_0, \mathcal{Q}_0\}$ classify the different choices of Verma modules. $\mathcal{Q}_0$ has the properties of a BRST-charge[7] so that the energy-momentum tensor is BRST-exact: $\mathcal{L}_m = 1/2\,\{\mathcal{G}_m, \mathcal{Q}_0\}$.

Using definition 3.A a simultaneous eigenvector $|\Delta, q, c\rangle^{\mathcal{N}}$ of $\mathcal{H}_{\mathsf{T}_2}$ with $\mathcal{L}_0$ eigenvalue Δ, $\mathcal{H}_0$ eigenvalue q, C eigenvalue[j] c, and vanishing T_2^+ action is called a *highest weight vector*. Each representation with lower bound for the eigenvalues of $\mathcal{L}_0$ needs to contain a highest weight vector. Additional vanishing conditions $\mathcal{N}$ are possible only with respect to the operators $\mathcal{G}_0$ and $\mathcal{Q}_0$ which may or may not annihilate a highest weight vector. The different types of annihilation conditions have been analysed in Ref. 19 resulting as follows. One can distinguish 4 different types of highest weight vectors $|\Delta, q\rangle^{\mathcal{N}}$ labeled by a superscript $N \in \{G, Q, GQ\}$, or no superscript at all: highest weight vectors $|\Delta, q\rangle^{GQ}$ annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$ (*chiral*)[k], highest weight vectors $|\Delta, q\rangle^{G}$ annihilated by $\mathcal{G}_0$ but not by $\mathcal{Q}_0$ ($\mathcal{G}_0$-closed), highest weight vectors $|\Delta, q\rangle^{Q}$ annihilated by $\mathcal{Q}_0$ but not by $\mathcal{G}_0$ ($\mathcal{Q}_0$-closed), and finally highest weight vectors $|\Delta, q\rangle$ that are neither annihilated by $\mathcal{G}_0$ nor by $\mathcal{Q}_0$ (*no-label*).

Since $2\mathcal{L}_0 = \mathcal{G}_0\mathcal{Q}_0 + \mathcal{Q}_0\mathcal{G}_0$, a chiral vector, annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$, necessarily has vanishing $\mathcal{L}_0$-eigenvalue. On the other hand, any highest weight vector $|\Delta, q\rangle$ that is neither annihilated by $\mathcal{G}_0$ nor by $\mathcal{Q}_0$ can be decomposed into $\frac{1}{2\Delta}\mathcal{G}_0\mathcal{Q}_0\,|\Delta, q\rangle + \frac{1}{2\Delta}\mathcal{Q}_0\mathcal{G}_0\,|\Delta, q\rangle$, provided $\Delta \neq 0$. In this case the whole representation decomposes into a direct sum of two submodules one of them containing the $\mathcal{G}_0$-closed highest weight vector $\mathcal{G}_0\mathcal{Q}_0\,|\Delta, q\rangle$ and the other one containing the $\mathcal{Q}_0$-closed highest weight vector $\mathcal{Q}_0\mathcal{G}_0\,|\Delta, q\rangle$. Therefore, for *no-label* highest weight vectors, annihilated neither by $\mathcal{G}_0$ nor by $\mathcal{Q}_0$, we only need to consider the cases with $\Delta = 0$; i.e. the highest weight vectors that cannot be expressed as linear combinations of $\mathcal{G}_0$-closed and $\mathcal{Q}_0$-closed highest weight vectors. From now on *no-label* will refer exclusively to such highest weight vectors with $\Delta = 0$.

$\|0, q\rangle$	$\mathcal{G}_0\,\|0, q\rangle \neq 0$ and $\mathcal{Q}_0\,\|0, q\rangle \neq 0$	no-label
$\|\Delta, q\rangle^{G}$	$\mathcal{G}_0\,\|\Delta, q\rangle = 0$ and $\mathcal{Q}_0\,\|\Delta, q\rangle \neq 0$	$\mathcal{G}_0$-closed
$\|\Delta, q\rangle^{Q}$	$\mathcal{G}_0\,\|\Delta, q\rangle \neq 0$ and $\mathcal{Q}_0\,\|\Delta, q\rangle = 0$	$\mathcal{Q}_0$-closed
$\|0, q\rangle^{GQ}$	$\mathcal{G}_0\,\|0, q\rangle = 0$ and $\mathcal{Q}_0\,\|0, q\rangle = 0$	chiral

TAB. a Topological highest weight vectors.

Hence, according to definition 3.A, we have the following 4 different types of topological Verma modules, as shown in TAB. a:

$$\mathcal{V}_{0,q} = U(\mathsf{T}_2) \otimes_{\mathcal{H}_{\mathsf{T}_2} \oplus \mathsf{T}_2^+} |0, q\rangle\ , \tag{29}$$

$$\mathcal{V}_{\Delta,q}^{G} = U(\mathsf{T}_2) \otimes_{\mathcal{H}_{\mathsf{T}_2} \oplus \mathsf{T}_2^+ \oplus \text{span}\{\mathcal{G}_0\}} |\Delta, q\rangle^{G}\ , \tag{30}$$

$$\mathcal{V}_{\Delta,q}^{Q} = U(\mathsf{T}_2) \otimes_{\mathcal{H}_{\mathsf{T}_2} \oplus \mathsf{T}_2^+ \oplus \text{span}\{\mathcal{Q}_0\}} |\Delta, q\rangle^{Q}\ , \tag{31}$$

[j] For simplicity from now on we will supress the eigenvalue of C in $|\Delta, q, c\rangle^{\mathcal{N}}$ and simply write $|\Delta, q\rangle^{\mathcal{N}}$.
[k] Chirality conditions are important for physics[29, 7, 18].

$$\mathcal{V}_{0,q}^{GQ} \quad = \quad U(\mathsf{T}_2) \otimes_{\mathcal{H}_{\mathsf{T}_2} \oplus \mathsf{T}_2^+ \oplus \mathrm{span}\{\mathcal{G}_0, \mathcal{Q}_0\}} |0,q\rangle^{GQ} \; . \tag{32}$$

The Verma modules of types $\mathcal{V}_{\Delta,q}^{G}$ and $\mathcal{V}_{\Delta,q}^{Q}$, based on $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed highest weight vectors, are called[19] *generic* Verma modules, whereas the Verma modules of types $\mathcal{V}_{0,q}$ and $\mathcal{V}_{0,q}^{GQ}$ are called *no-label* and *chiral* Verma modules, respectively, for obvious reasons[1].

For elements Y of T_2 which are eigenvectors of $\mathcal{H}_{\mathsf{T}_2}$ with respect to the adjoint representation we define similarly to the Virasoro case the *level* $|Y|_L$ as $[\mathcal{L}_0, Y] = |Y|_L Y$ and in addition the *charge* $|Y|_H$ as $[\mathcal{H}_0, Y] = |Y|_H Y$. In particular, elements of the form

$$Y \quad = \quad \mathcal{L}_{-l_L} \ldots \mathcal{L}_{-l_1} \mathcal{H}_{-h_H} \ldots \mathcal{H}_{-h_1} \mathcal{Q}_{-q_Q} \ldots \mathcal{Q}_{-q_1} \mathcal{G}_{-g_G} \ldots \mathcal{G}_{-g_1} \tag{33}$$

and any reorderings of Y have level $|Y|_L = \sum_{j=1}^{L} l_j + \sum_{j=1}^{H} h_j + \sum_{j=1}^{Q} q_j + \sum_{j=1}^{G} g_j$ and charge $|Y|_H = G - Q$. For these elements we shall also define their *length* $\|Y\| = L + H + G + Q$. Again, we shall set $|1|_L = |1|_H = \|1\| = 0$. For convenience we define the following sets of negative operators for $m \in \mathbf{N}$

$$\mathsf{L}_m \quad = \quad \{Y = \mathcal{L}_{-l_L} \ldots \mathcal{L}_{-l_1} : \; l_L \geq \ldots \geq l_1 \geq 2, \; |Y|_L = m\} \, , \tag{34}$$

$$\mathsf{H}_m \quad = \quad \{Y = \mathcal{H}_{-h_H} \ldots \mathcal{H}_{-h_1} : \; h_H \geq \ldots \geq h_1 \geq 1, \; |Y|_L = m\} \, , \tag{35}$$

$$\mathsf{G}_m \quad = \quad \{Y = \mathcal{G}_{-g_G} \ldots \mathcal{G}_{-g_1} : \; g_G > \ldots > g_1 \geq 2, \; |Y|_L = m\} \, , \tag{36}$$

$$\mathsf{Q}_m \quad = \quad \{Y = \mathcal{Q}_{-q_Q} \ldots \mathcal{Q}_{-q_1} : \; q_Q > \ldots > q_1 \geq 2, \; |Y|_L = m\} \, , \tag{37}$$

$$\mathsf{L}_0 \quad = \quad \mathsf{H}_0 \; = \; \mathsf{G}_0 \; = \; \mathsf{Q}_0 \; = \; \{1\} \, . \tag{38}$$

We are now able to define a graded basis for the Verma modules as described in the previous section. We choose $l \in \mathbf{N}_0$, $n \in \mathbf{Z}$ and define:

$$\begin{aligned} \mathcal{S}_{m,n}^{G} \quad = \quad & \{Y = LHGQ : \; L \in \mathsf{L}_l, \; H \in \mathsf{H}_h, \; G \in \mathsf{G}_g, \; Q \in \mathsf{Q}_q, \\ & |Y|_L = m = l + h + g + q, \; |Y|_H = n = |G|_H + |Q|_H, \; l,h,g,q \in \mathbf{N}_0\} \; , \end{aligned} \tag{39}$$

$$\begin{aligned} \mathcal{S}_{m,n}^{Q} \quad = \quad & \{Y = LHQG : \; L \in \mathsf{L}_l, \; H \in \mathsf{H}_h, \; Q \in \mathsf{Q}_q, \; G \in \mathsf{G}_g, \\ & |Y|_L = m = l + h + g + q, \; |Y|_H = n = |Q|_H + |G|_H, \; l,h,g,q \in \mathbf{N}_0\} \; . \end{aligned} \tag{40}$$

And finally for $m \in \mathbf{N}_0$, $n \in \mathbf{Z}$:

$$\begin{aligned} \mathcal{C}_{m,n}^{G} \quad = \quad & \Big\{S_{p,q} \, \mathcal{L}_{-1}^{m-p-r_1-r_2} \mathcal{G}_{-1}^{r_1} \mathcal{Q}_{-1}^{r_2} \mathcal{Q}_0^{r_3} : \; S_{p,q} \in \mathcal{S}_{p,q}^{G}, \; p \in \mathbf{N}_0, \; r_1, r_2, r_3 \in \{0,1\}, \\ & m - p - r_1 - r_2 \geq 0, \; n = q + r_1 - r_2 - r_3\} \; , \end{aligned} \tag{41}$$

$$\begin{aligned} \mathcal{C}_{m,n}^{Q} \quad = \quad & \Big\{S_{p,q} \, \mathcal{L}_{-1}^{m-p-r_1-r_2} \mathcal{Q}_{-1}^{r_1} \mathcal{G}_{-1}^{r_2} \mathcal{G}_0^{r_3} : \; S_{p,q} \in \mathcal{S}_{p,q}^{Q}, \; p \in \mathbf{N}_0, \; r_1, r_2, r_3 \in \{0,1\}, \\ & m - p - r_1 - r_2 \geq 0, \; n = q - r_1 + r_2 + r_3\} \; . \end{aligned} \tag{42}$$

Thus, a typical element of $\mathcal{C}_{m,n}^{G}$ is of the form

$$Y \quad = \quad \mathcal{L}_{-l_L} \ldots \mathcal{L}_{-l_1} \mathcal{H}_{-h_H} \ldots \mathcal{H}_{-h_1} \mathcal{G}_{-g_G} \ldots \mathcal{G}_{-g_1} \mathcal{Q}_{-q_Q} \ldots \mathcal{Q}_{-q_1} \mathcal{L}_{-1}^{m'} \mathcal{G}_{-1}^{r_1} \mathcal{Q}_{-1}^{r_2} \mathcal{Q}_0^{r_3} \, , \tag{43}$$

[1] As explained in Ref. 19, the chiral Verma modules $\mathcal{V}_{0,q}^{GQ}$, built on chiral highest weight vectors, are not complete Verma modules because the chirality constraint is not required (just allowed) by the algebra.

$r_1, r_2, r_3 \in \{0,1\}$, such that $|Y|_L = m$ and $|Y|_H = n$. $S_{p,q}$ of $Y \in \mathcal{C}^G_{m,n}$ or of $Y \in \mathcal{C}^Q_{m,n}$ is called the *leading part of* Y and is denoted by Y^*. Hence, one can define the following standard bases:

$$\begin{aligned}\mathcal{B}_{|\Delta,q\rangle^G} &= \left\{ Y\,|\Delta,q\rangle^G :\ Y \in \mathcal{C}^G_{m,n}\ m \in \mathbf{N}_0,\ n \in \mathbf{Z} \right\} , \\ \mathcal{B}_{|\Delta,q\rangle^Q} &= \left\{ Y\,|\Delta,q\rangle^Q :\ Y \in \mathcal{C}^Q_{m,n}\ m \in \mathbf{N}_0,\ n \in \mathbf{Z} \right\} ,\end{aligned} \tag{44}$$

obtaining finally $\mathcal{V}^G_{\Delta,q} = \text{span}\{\mathcal{B}_{|\Delta,q\rangle^G}\}$ and $\mathcal{V}^Q_{\Delta,q} = \text{span}\{\mathcal{B}_{|\Delta,q\rangle^Q}\}$. For Verma modules built on no-label and chiral highest weight vectors, $\mathcal{V}_{0,q}$ and $\mathcal{V}^{G,Q}_{0,q}$ both with $\Delta = 0$, one defines in exactly the same way:

$$\begin{aligned}\mathcal{S}_{m,n} &= \{ Y = LHGQ :\ L \in \mathsf{L}_l,\ H \in \mathsf{H}_h,\ G \in \mathsf{G}_g,\ Q \in \mathsf{Q}_q, \\ &\quad |Y|_L = m = l+h+g+q,\ |Y|_H = n = |G|_H + |Q|_H,\ l,h,g,q \in \mathbf{N}_0 \}\end{aligned} \tag{45}$$

$$\begin{aligned}\mathcal{C}_{m,n} &= \Big\{ S_{p,q}\,\mathcal{L}_{-1}^{m-p-r_1-r_2}\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2}\mathcal{G}_0^{r_4}\mathcal{Q}_0^{r_3} :\ S_{p,q} \in \mathcal{S}_{p,q},\ p \in \mathbf{N}_0,\ r_1,r_2,r_3,r_4 \in \{0,1\}, \\ &\quad m-p-r_1-r_2 \geq 0,\ n = q + r_1 - r_2 + r_4 - r_3 \Big\} ,\end{aligned} \tag{46}$$

$$\begin{aligned}\mathcal{C}^{GQ}_{m,n} &= \Big\{ S_{p,q}\,\mathcal{L}_{-1}^{m-p-r_1-r_2}\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2} :\ S_{p,q} \in \mathcal{S}_{p,q},\ p \in \mathbf{N}_0,\ r_1,r_2 \in \{0,1\}, \\ &\quad m-p-r_1-r_2 \geq 0,\ n = q + r_1 - r_2 \Big\} .\end{aligned} \tag{47}$$

$S_{p,q}$ of $Y \in \mathcal{C}_{m,n}$ or of $Y \in \mathcal{C}^{GQ}_{m,n}$ will be also called the *leading part* Y^* *of* Y. One obtains the following standard bases for the modules $\mathcal{V}_{0,q}$ and $\mathcal{V}^{G,Q}_{0,q}$:

$$\begin{aligned}\mathcal{B}_{|0,q\rangle} &= \{ Y\,|0,q\rangle :\ Y \in \mathcal{C}_{m,n},\ m \in \mathbf{N}_0,\ n \in \mathbf{Z} \} , \\ \mathcal{B}_{|0,q\rangle^{GQ}} &= \left\{ Y\,|0,q\rangle^{GQ} :\ Y \in \mathcal{C}^{GQ}_{m,n},\ m \in \mathbf{N}_0,\ n \in \mathbf{Z} \right\} .\end{aligned} \tag{48}$$

The bases Eq. (44) and Eq. (48) are naturally $\mathbf{N}_0 \times \mathbf{Z}$ graded with respect to their $\mathcal{H}_{\mathsf{T}_2}$ eigenvalues relative to the eigenvalues (Δ, q) of the highest weight vector. For an eigenvector $\Psi_{l,p}$ of $\mathcal{H}_{\mathsf{T}_2}$ in $\mathcal{V}^N_{\Delta,q}$ the $\mathcal{L}_0$-eigenvalue is $\Delta + l$ and the $\mathcal{H}_0$-eigenvalue is $q + p$ with $l \in \mathbf{N}_0$ and $p \in \mathbf{Z}$. We define the *level* $|\Psi_{l,p}|_L = l$ and *charge* $|\Psi_{l,p}|_H = p$.

Like for the Virasoro case, we shall use $\mathcal{C}^G_{m,n}$, $\mathcal{C}^Q_{m,n}$, $\mathcal{C}_{m,n}$ and $\mathcal{C}^{GQ}_{m,n}$ in order to define the *normal form* of an eigenvector $\Psi_{l,p}$ of $\mathcal{H}_{\mathsf{T}_2}$. It is defined to be the basis decomposition with respect to the corresponding standard bases Eq. (44) and Eq. (48). Again, we call the operators $X \in \mathcal{C}^G_{l,p}$, or $X \in \mathcal{C}^Q_{l,p}$, or $X \in \mathcal{C}_{l,p}$, or $X \in \mathcal{C}^{GQ}_{l,p}$ simply the *terms* of $\Psi_{l,p}$ and the coefficients c_X its *coefficients*.

We introduce topological singular vectors according to definition 3.B as $\mathcal{H}_{\mathsf{T}_2}$ eigenvectors that are not proportional to the highest weight vector but are annihilated by T_2^+ and may also satisfy additional vanishing conditions with respect to the operators $\mathcal{G}_0$ and $\mathcal{Q}_0$. Therefore one also distinguishes singular vectors $\Psi_{l,p}$, $\Psi^G_{l,p}$, $\Psi^Q_{l,p}$ and $\Psi^{GQ}_{l,p}$ carrying the superscript G and/or Q depending on whether the singular vector is annihilated by $\mathcal{G}_0$ and/or $\mathcal{Q}_0$. Obviously one obtains similar restrictions on the eigenvalues of $\Psi^{\mathcal{N}'}_{l,p} \in \mathcal{V}^{\mathcal{N}}_{\Delta,q}$ as for the highest weight vectors, as shown in Tab. b.

$\Psi_{l,p}$	$\mathcal{G}_0\Psi_{l,p} \neq 0$ and $\mathcal{Q}_0\Psi_{l,p} \neq 0$; $l+\Delta = 0$	*no-label*
$\Psi^G_{l,p}$	$\mathcal{G}_0\Psi_{l,p} = 0$ and $\mathcal{Q}_0\Psi_{l,p} \neq 0$	$\mathcal{G}_0$-closed
$\Psi^Q_{l,p}$	$\mathcal{G}_0\Psi_{l,p} \neq 0$ and $\mathcal{Q}_0\Psi_{l,p} = 0$	$\mathcal{Q}_0$-closed
$\Psi^{GQ}_{l,p}$	$\mathcal{G}_0\Psi_{l,p} = 0$ and $\mathcal{Q}_0\Psi_{l,p} = 0$; $l+\Delta = 0$	chiral

TAB. b Topological singular vectors.

As there are 4 types of topological Verma modules and 4 types of topological singular vectors one might think of 16 different combinations of singular vectors in Verma modules. However, as will be explained later, no-label and chiral singular vectors do not exist neither in no-label Verma modules nor in chiral Verma modules (with one exception: chiral singular vectors at level 0 in no-label Verma modules). Most of these types of singular vectors are connected via the $N = 2$ *topological spectral flow mappings*[17, 16, 19] which have been analysed in detail in Refs. 16, 19.

5 Adapted orderings on generic Verma modules $\mathcal{V}^G_{\Delta,q}$ and $\mathcal{V}^Q_{\Delta,q}$

We will now introduce total orderings $\mathcal{O}_G$ and $\mathcal{O}_Q$ on $\mathcal{C}^G_{m,n}$ and $\mathcal{C}^Q_{m,n}$ respectively. For convenience, however, we shall first give an ordering on the sets L_m, H_m, G_m and Q_m.

Definition 5.A *Let Y denote either L, H, G, or Q, (but the same throughout this definition) and take two elements $X_i \in Y_{m^i}$ for $m^i \in \mathbf{N}_0$, $i = 1, 2$, such that $X_i = Z^i_{-m^i_{\|X_i\|}} \dots Z^i_{-m^i_1}$, $|X_i|_L = m^i$ or $X_i = 1$, $i = 1, 2$, with $Z^i_{-m^i_j}$ being an operator of the type $\mathcal{L}_{-m^i_j}$, $\mathcal{H}_{-m^i_j}$, $\mathcal{G}_{-m^i_j}$, or $\mathcal{Q}_{-m^i_j}$ depending on whether Y denotes L, H, G or Q respectively. For $X_1 \neq X_2$ we compute the index*[m] *$j_0 = \min\{j : m^1_j - m^2_j \neq 0, j = 1, \dots, \min(\|X_1\|, \|X_2\|)\}$. j_0 is, if non-trivial, the index for which the level of the operators in X_1 and X_2 first disagree when read from the right to the left. For $j_0 > 0$ we then define*

$$X_1 <_Y X_2 \quad \text{if} \quad m^1_{j_0} < m^2_{j_0} \,. \tag{49}$$

If, however, $j_0 = 0$, we set

$$X_1 <_Y X_2 \quad \text{if} \quad \|X_1\| > \|X_2\| \,. \tag{50}$$

For $X_1 = X_2$ we set $X_1 <_Y X_2$ and $X_2 <_Y X_1$.

Some examples of definition 5.A are:

$$\begin{aligned} \mathcal{L}_{-4}\mathcal{L}_{-3}\mathcal{L}_{-2} &<_L \mathcal{L}_{-4}\mathcal{L}_{-2} \,, \\ \mathcal{H}_{-4}\mathcal{H}_{-3}\mathcal{H}_{-2} &<_H \mathcal{H}_{-3}\mathcal{H}_{-2} \,, \\ \mathcal{Q}_{-2} &<_Q 1 \,. \end{aligned} \tag{51}$$

We can now define an ordering on $\mathcal{C}^G_{m,n}$ which will turn out to be adapted with a very small kernel.

Definition 5.B *On the set $\mathcal{C}^G_{m,n}$ we introduce the total ordering $\mathcal{O}_G$. For two elements $X_1, X_2 \in \mathcal{C}^G_{m,n}$, $X_1 \neq X_2$ with $X_i = L^i H^i G^i Q^i \mathcal{L}^{k^i}_{-1} \mathcal{G}^{r^i_1}_{-1} \mathcal{Q}^{r^i_2}_{-1} \mathcal{Q}^{r^i_3}_0$, $L^i \in \mathsf{L}_{l_i}$, $H^i \in \mathsf{H}_{h_i}$, $G^i \in \mathsf{G}_{g_i}$, $Q^i \in \mathsf{Q}_{q_i}$ for some $l_i, h_i, g_i, q_i, k^i, r^i_1, r^i_2, r^i_3 \in \mathbf{N}_0$, $i = 1, 2$ such that $m \in \mathbf{N}_0$, $n \in \mathbf{Z}$ we define*

$$X_1 <_{\mathcal{O}_G} X_2 \quad \text{if} \quad k^1 > k^2 \,. \tag{52}$$

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[m] For subsets of $\mathbf{N}$ we define $\min \emptyset = 0$.

For $k^1 = k^2$ we set

$$X_1 <_{\mathcal{O}_G} X_2 \quad \text{if} \quad r_1^1 + r_2^1 > r_1^2 + r_2^2 \,. \tag{53}$$

If $r_1^1 + r_2^1 = r_1^2 + r_2^2$, then we set

$$X_1 <_{\mathcal{O}_G} X_2 \quad \text{if} \quad Q^1 <_Q Q^2 \,. \tag{54}$$

In the case where also $Q^1 = Q^2$ we define

$$X_1 <_{\mathcal{O}_G} X_2 \quad \text{if} \quad G^1 <_G G^2 \,. \tag{55}$$

If even $G^1 = G^2$ we then define

$$X_1 <_{\mathcal{O}_G} X_2 \quad \text{if} \quad L^1 <_L L^2 \,. \tag{56}$$

If further $L^1 = L^2$ we set

$$X_1 <_{\mathcal{O}_G} X_2 \quad \text{if} \quad H^1 <_H H^2 \,, \tag{57}$$

which finally has to give an answer. For $X_1 = X_2$ we define $X_1 <_{\mathcal{O}_G} X_2$ and $X_2 <_{\mathcal{O}_G} X_1$.

Definition 5.B is well-defined since one obtains an answer for any pair $X_1, X_2 \in \mathcal{C}^G_{m,n}$, $X_1 \neq X_2$ after going through Eqs. (52)-(57), and hence the ordering $\mathcal{O}_G$ proves to be a total ordering on $\mathcal{C}^G_{m,n}$. Namely, if Eqs. (52)-(57) do not give an answer on the ordering of X_1 and X_2, then obviously X_1 and X_2 are of the form $X_i = LHGQ\mathcal{L}^k_{-1}\mathcal{G}^{r^i_1}_{-1}\mathcal{Q}^{r^i_2}_{-1}\mathcal{Q}^{r^i_3}_0$, with common L, H, G, Q, k and also $r^1_1 + r^1_2 = r^2_1 + r^2_2$. The fact that both X_1 and X_2 has charge n implies $r^1_1 - r^1_2 - r^1_3 = r^2_1 - r^2_2 - r^2_3$, and using $r^1_1 + r^1_2 = r^2_1 + r^2_2$ one obtains $2r^1_2 + r^1_3 = 2r^2_2 + r^2_3$. But this equation has solutions from $\{0,1\}$ only for $r^1_2 = r^2_2$ and consequently also $r^1_3 = r^2_3$ and hence $X_1 = X_2$.

Obviously the $\mathcal{O}_G$-smallest element of $\mathcal{C}^G_{m,0}$ is $\mathcal{L}^m_{-1}$ followed by $\mathcal{L}^{m-1}_{-1}\mathcal{G}_{-1}\mathcal{Q}_0$ whilst the $\mathcal{O}_G$-smallest element of $\mathcal{C}^G_{m,-1}$ is $\mathcal{L}^m_{-1}\mathcal{Q}_0$ followed by $\mathcal{L}^{m-1}_{-1}\mathcal{Q}_{-1}$. Similarly, for $\mathcal{C}^G_{m,1}$ we find $\mathcal{L}^{m-1}_{-1}\mathcal{G}_{-1}$ as $\mathcal{O}_G$-smallest element followed by $\mathcal{L}^{m-2}_{-1}\mathcal{G}_{-1}$. We will now show that the ordering $\mathcal{O}_G$ is adapted and we will compute the ordering kernels. We do not give a theoretical proof that these kernels are the smallest possible ordering kernels. However, we shall refer later to explicit examples of singular vectors that show that for general values of Δ and q most of the ordering kernels presented here cannot be smaller.

Theorem 5.C *If the central extension satisfies $c \neq 3$, then the ordering $\mathcal{O}_G$ is adapted to $\mathcal{C}^G_{m,n}$ for all Verma modules $\mathcal{V}^G_{\Delta,q}$ and for all grades (m,n) with $m \in \mathbf{N}_0$, $n \in \mathbf{Z}$. Ordering kernels are given by the following tables for all levels*[n] *m, depending on the set of annihilation operators and depending on the charge n.*

n	*ordering kernel*
$+1$	$\{\mathcal{L}^{m-1}_{-1}\mathcal{G}_{-1}\}$
0	$\{\mathcal{L}^m_{-1}, \mathcal{H}_{-1}\mathcal{L}^{m-1}_{-1}, \mathcal{L}^{m-1}_{-1}\mathcal{G}_{-1}\mathcal{Q}_0\}$
-1	$\{\mathcal{L}^m_{-1}\mathcal{Q}_0, \mathcal{H}_{-1}\mathcal{L}^{m-1}_{-1}\mathcal{Q}_0, \mathcal{L}^{m-1}_{-1}\mathcal{Q}_{-1}\}$
-2	$\{\mathcal{L}^{m-1}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0\}$

Tab. c *Ordering kernels for $\mathcal{O}_G$, annihilation operators T^+_2.*

[n]Note that for levels $m=0$ and $m=1$ some of the kernel elements obviously do not exist.

n	*ordering kernel*
$+1$	$\{\mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\}$
0	$\{\mathcal{L}_{-1}^{m}, \mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\mathcal{Q}_0\}$
-1	$\{\mathcal{L}_{-1}^{m}\mathcal{Q}_0\}$

TAB. d *Ordering kernels for $\mathcal{O}_G$, annihilation operators T_2^+ and $\mathcal{G}_0$.*

n	*ordering kernel*
0	$\{\mathcal{L}_{-1}^{m}\}$
-1	$\{\mathcal{L}_{-1}^{m}\mathcal{Q}_0, \mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\}$
-2	$\{\mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\mathcal{Q}_0\}$

TAB. e *Ordering kernels for $\mathcal{O}_G$, annihilation operators T_2^+ and $\mathcal{Q}_0$.*

n	*ordering kernel*
0	$\{\mathcal{L}_{-1}^{m}\}$
-1	$\{\mathcal{L}_{-1}^{m}\mathcal{Q}_0\}$

TAB. f *Ordering kernels for $\mathcal{O}_G$, annihilation operators T_2^+, $\mathcal{Q}_0$, and $\mathcal{G}_0$.*

Charges that do not appear in the tables have trivial ordering kernels.

Like in the Virasoro case our strategy will be to find annihilation operators that are able to produce an additional $\mathcal{L}_{-1}$. Hence, we raise the term in question to the class of terms with one additional $\mathcal{L}_{-1}$ and try to prove that terms that can also be raised to this class of terms have to be $\mathcal{O}_G$-smaller. That we need to focus only on operators that create $\mathcal{L}_{-1}$ from the leading part of a term is a consequence of the following theorem which we therefore shall prove before starting the proof of theorem 5.C.

Theorem 5.D *Let us assume that there exists an annihilation operator Γ that creates a term X^Γ with $n+1$ operators $\mathcal{L}_{-1}$ by acting on*

$$X_0 \;=\; X_0^*\,\mathcal{L}_{-1}^{n}\,\mathcal{G}_{-1}^{r_1}\,\mathcal{Q}_{-1}^{r_2}\,\mathcal{Q}_0^{r_3} \quad \in \mathcal{C}_{m_0,n_0}^{G} \tag{58}$$

with $X_0^* = L^0\,H^0\,G^0\,Q^0$,

$$\begin{aligned} L^0 &= \mathcal{L}_{-m^L_{\|L^0\|}} \ldots \mathcal{L}_{-m^L_1} \in \mathsf{L}_l\,, \\ H^0 &= \mathcal{H}_{-m^H_{\|H^0\|}} \ldots \mathcal{H}_{-m^H_1} \in \mathsf{H}_h\,, \\ G^0 &= \mathcal{G}_{-m^G_{\|G^0\|}} \ldots \mathcal{G}_{-m^G_1} \in \mathsf{G}_g\,, \\ Q^0 &= \mathcal{Q}_{-m^Q_{\|Q^0\|}} \ldots \mathcal{Q}_{-m^Q_1} \in \mathsf{Q}_q\,, \end{aligned}$$

$l,h,g,q \in \mathbf{N}_0$ and $r_1,r_2,r_3 \in \{0,1\}$. Let us further assume that this additional $\mathcal{L}_{-1}$ is created by commuting Γ through X_0^*. Then any other term $Y \in \mathcal{C}_{m_0,n_0}^{G}$ with $X_0 <_{\mathcal{O}_G} Y$ for which the action of Γ also produces the term X^Γ will also create one additional $\mathcal{L}_{-1}$ and this by commuting Γ through the leading part Y^* of Y.

Theorem 5.D thus tells us that any $Y \in \mathcal{C}^G_{m_0,n_0}$ that does not generate an additional $\mathcal{L}_{-1}$ from Y^* in the described sense is automatically $\mathcal{O}_G$-smaller than X_0 and is therefore irrelevant for the adapted ordering.

Proof: Let us show that if Γ does not create one additional $\mathcal{L}_{-1}$ by commuting through Y^* but still satisfies that it also creates the term X^Γ acting on Y, then $Y <_{\mathcal{O}_G} X_0$. If Y has already $n+1$ or even more operators $\mathcal{L}_{-1}$, then Y is obviously $\mathcal{O}_G$-smaller than X_0. Thus the action of Γ on Y needs to create at least one $\mathcal{L}_{-1}$. However, the action of one operator Γ can create at most one $\mathcal{L}_{-1}$. This is an imediate consequence of the commutation relations Eq. (26): the action of one operator Γ can take several operators in X_0 away but it can at most create only one new operator. We can therefore concentrate on terms Y that have exactly n operators $\mathcal{L}_{-1}$. The additional $\mathcal{L}_{-1}$ is either created from Y^* or from $\mathcal{G}_{-1}^{r_1^Y}\mathcal{Q}_{-1}^{r_2^Y}\mathcal{Q}_0^{r_3^Y}$. For the latter case there are a few possibilities to create $\mathcal{L}_{-1}$: only under the action of $\mathcal{G}_0$, $\mathcal{Q}_0$, $\mathcal{L}_1$, or $\mathcal{H}_1$, and depending on the values of r_1^Y, r_2^Y, and r_3^Y. The operator Γ could be one of these operators or the commutation of Γ with Y^* could produce one of them. (Note that it is not possible to create one of these operators and to create in addition a $\mathcal{L}_{-1}$ from Y^*.) If now $\mathcal{G}_0$, $\mathcal{Q}_0$, $\mathcal{L}_1$, or $\mathcal{H}_1$ creates a $\mathcal{L}_{-1}$ from $\mathcal{G}_{-1}^{r_1^Y}\mathcal{Q}_{-1}^{r_2^Y}\mathcal{Q}_0^{r_3^Y}$, then we find in each case that at least one of r_1^Y or r_2^Y changes from 1 to 0 whilst r_3^Y remains unchanged. But X^Γ has the same r_i as X_0 for $i=1,2,3$. One therefore deduces that $r_1^Y+r_2^Y > r_1+r_2$ and thus $Y <_{\mathcal{O}_G} X_0$. Hence the additional $\mathcal{L}_{-1}$ must be created by commuting Γ through the leading part Y^* in order that $X_0 <_{\mathcal{O}_G} Y$. □

Equipped with theorem 5.D we can now proceed with the proof of theorem 5.C:

Let us consider the term

$$X_0 \;=\; L^0\, H^0\, G^0\, Q^0\, \mathcal{L}_{-1}^n\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{Q}_0^{r_3} \;\in \mathcal{C}^G_{m_0,n_0}\,, \tag{59}$$

with L^0, H^0, G^0 and Q^0 given above. We construct the vector $\Psi^0 = X_0\,|\Delta,q\rangle^G \in \mathcal{V}^G_{\Delta,q}$ at level $|X_0|_L = m_0$ and charge $|X_0|_H = n_0$.

Let us first consider the annihilation operators to be those in $U(\mathsf{T}_2)^+$ only. If $Q^0 \neq 1$ we act with $\mathcal{G}_{m_1^Q-1} \in \mathsf{T}_2^+$ on Ψ^0 and write the result again in its normal form. We will thus obtain a non-trivial term in $\mathcal{G}_{m_1^Q-1}\Psi^0$ with one additional $\mathcal{L}_{-1}$:

$$\begin{aligned} X^Q &= L^0\, H^0\, G^0\, \tilde{Q}^0\, \mathcal{L}_{-1}^{n+1}\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{Q}_0^{r_3}\,, \\ \tilde{Q}^0 &= \mathcal{Q}_{-m_q^Q}\ldots\mathcal{Q}_{-m_2^Q} \end{aligned} \tag{60}$$

or, if $\|Q^0\|=1$, $\tilde{Q}^0=1$ simply by commuting $\mathcal{G}_{m_1^Q-1}$ with $\mathcal{Q}_{-m_1^Q}$ which produces the additional operator $\mathcal{L}_{-1}$. Any other term $Y \in \mathcal{C}^G_{m_0,n_0}$ also producing X^Q under the action of $\mathcal{G}_{m_1^Q-1} \in \mathsf{T}_2^+$ and being $\mathcal{O}_G$-bigger than X_0 also needs to create one $\mathcal{L}_{-1}$ by commuting $\mathcal{G}_{m_1^Q-1}$ with Y^* due to theorem 5.D. We can therefore focus on terms $Y = L^Y H^Y G^Y Q^Y \mathcal{L}_{-1}^n \mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3}$. One finds that by commuting $\mathcal{G}_{m_1^Q-1}$ with operators in L^Y or H^Y one can only produce terms of the form $\mathcal{G}_{m'}$ with $m' < m_1^Q - 1$. Therefore, in order to create subsequently the operator $\mathcal{L}_{-1}$ from $\mathcal{G}_{m'}$ or directly from $\mathcal{G}_{m_1^Q-1}$, Y needs to contain an operator of the form $\mathcal{Q}_{-m^\star-1}$ that satisfies[o] $0 < m^\star+1 \leq m_1^Q$. If $m^\star+1 < m_1^Q$ one finds that Y is $\mathcal{O}_G$-smaller than X_0 as the equation deciding on the ordering of X_0 and Y would in this case be Eq. (54). If $m^\star+1 = m_1^Q$, on the other hand, Y must be necessarily

[o]Note that for $m^\star = 0, -1$ $\mathcal{Q}_{-m^\star-1}$ would not be in the leading part Y^*.

equal to X_0. Note that $\mathcal{G}_{m_1^Q-1}$ and $\mathcal{G}_{m'}$ simply anticommute with operators of G^Y and therefore cannot create any $\mathcal{L}_{-1}$. Hence there are no terms Y $\mathcal{O}_G$-bigger than the terms X_0 producing the same terms X^Q under the action of $\mathcal{G}_{m_1^Q-1} \in \mathsf{T}_2^+$. We have therefore shown that *the ordering $\mathcal{O}_G$ is adapted on the set of terms X_0 of the form given by* Eq. (59) *with $Q^0 \neq 1$ for all grades (m_0, n_0) and all central terms $c \in \mathbf{C}$.*

As the terms X_0 with $Q^0 \neq 1$ are now proven to be adapted, next we will consider the terms X_0 with $Q^0 = 1$ and $G^0 \neq 1$:

$$X_0 \quad = \quad L^0\, H^0\, G^0\, \mathcal{L}_{-1}^{n}\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{Q}_{0}^{r_3}\,. \tag{61}$$

If $G^0 \neq 1$ we act with the annihilation operator $\mathcal{Q}_{m_1^G-1}$ on $\Psi^0 = X_0\, |\Delta, q\rangle^G$. This produces the term

$$X^G \quad = \quad L^0\, H^0\, \tilde{G}^0\, \mathcal{L}_{-1}^{n+1}\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{Q}_{0}^{r_3} \tag{62}$$

with $\tilde{G}^0 = \mathcal{G}_{-m_g^G} \ldots \mathcal{G}_{-m_2^G}$ or, if $\|G^0\| = 1$, $\tilde{G}^0 = 1$. Again, any other term Y with $X_0 <_{\mathcal{O}_G} Y$ also producing X^G under the action of $\mathcal{Q}_{m_1^G-1} \in \mathsf{T}_2^+$ needs to create one $\mathcal{L}_{-1}$ by commuting $\mathcal{Q}_{m_1^G-1}$ through the leading part Y^* due to theorem 5.D. Thus we focus on operators Y of the form $Y = L^Y H^Y G^Y Q^Y \mathcal{L}_{-1}^{n} \mathcal{G}_{-1}^{r_1} \mathcal{Q}_{-1}^{r_2} \mathcal{Q}_{0}^{r_3}$ with $Q^Y = 1$ as otherwise $Y <_{\mathcal{O}_G} X_0$. Commuting $\mathcal{Q}_{m_1^G-1}$ with operators in L^Y or H^Y can only create operators of the form $\mathcal{Q}_{m'}$ with $m' < m_1^G - 1$. The operators $\mathcal{Q}_{m_1^G-1}$ and $\mathcal{Q}_{m'}$ can create $\mathcal{L}_{-1}$ from G^Y only if G^Y contains $\mathcal{G}_{-m'-1}$ with $m'+1 \leq m_1^G$ and therefore Y is again $\mathcal{O}_G$-smaller or equal than X_0. Commuting $\mathcal{Q}_m$ through G^Y can also give rise to operators of the form $\mathcal{L}_p$, $\mathcal{H}_p$ and consequently even to $\mathcal{G}_p$ with $p < m_1^G - 1$. In order to create $\mathcal{L}_{-1}$ from Q^Y it would require that $Q^Y \neq 1$ so that one again finds $Y <_{\mathcal{O}_G} X_0$. This shows that *the ordering $\mathcal{O}_G$ is adapted on the set of terms X_0 of the form given by* Eq. (59) *with $Q^0 \neq 1$ or $G^0 \neq 1$ for all grades (m_0, n_0) and all central terms $c \in \mathbf{C}$.*

Next we will consider the terms X_0 with $Q^0 = 1$, $G^0 = 1$ and $L^0 \neq 1$:

$$X_0 \quad = \quad L^0\, H^0\, \mathcal{L}_{-1}^{n}\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{Q}_{0}^{r_3}\,. \tag{63}$$

If $L^0 \neq 1$ we act with the annihilation operator $L_{m_1^L-1} \in \mathsf{T}_2^+$ on $\Psi^0 = X_0\, |\Delta, q\rangle^G$. This produces a term of the form

$$X^L \quad = \quad \tilde{L}^0\, H^0\, \mathcal{L}_{-1}^{n+1}\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{Q}_{0}^{r_3} \tag{64}$$

with $\tilde{L}^0 = \mathcal{L}_{-m_l^L} \ldots \mathcal{L}_{-m_2^L}$ or, if $\|L^0\| = 1$, $\tilde{L}^0 = 1$. If $m_2^L = m_1^L$ we may simply obtain multiple copies of the same term X^L. Again theorem 5.D allows us to focus on $\mathcal{O}_G$-bigger terms Y of the form $Y = L^Y H^Y \mathcal{L}_{-1}^{n} \mathcal{G}_{-1}^{r_1} \mathcal{Q}_{-1}^{r_2} \mathcal{Q}_{0}^{r_3} \in \mathcal{C}_{m_0,n_0}^{G}$ (in addition, $G^Y \neq 1$ or $Q^Y \neq 1$ would lead to $Y <_{\mathcal{O}_G} X_0$). The operator $\mathcal{L}_{m_1^L-1}$ commuted with operators in H^Y cannot create any $\mathcal{L}_{-1}$ and obviously, following the arguments of the Virasoro case (proof of theorem 2.D), terms Y that produce X^L creating $\mathcal{L}_{-1}$ out of L^Y would again be $\mathcal{O}_G$-smaller than X_0. Therefore we can state that *the ordering $\mathcal{O}_G$ is adapted on the set of terms X_0 of the form given by* Eq. (59) *with $Q^0 \neq 1$, or $G^0 \neq 1$, or $L^0 \neq 1$ for all grades (m_0, n_0) and all central terms $c \in \mathbf{C}$.*

We are thus left with terms X_0 of the form

$$X_0 \quad = \quad H^0\, \mathcal{L}_{-1}^{n}\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{Q}_{0}^{r_3}\,. \tag{65}$$

At this stage it is not possible to create operators $\mathcal{L}_{-1}$ by acting directly with positive operators of T_2^+ on H^0. Therefore we cannot use further theorem 5.D, which has proven to be very fruitful so far, and a different strategy must be applied. Let us first assume that H^0 contains operators other than $\mathcal{H}_{-1}$ and let $j_0 \in \mathbf{N}$ be the smallest index of H^0 such that $m^H_{j_0} \neq 1$. There are four different cases to study depending on the values of r_1, r_2 and r_3. Let us start with the cases where $r_2 = 0$. Acting with[p] $\mathcal{G}_{m^H_{j_0}} \mathcal{Q}_{-1} \in U(\mathsf{T}_2)^+$ on $\Psi^0 = X_0 \,|\Delta, q\rangle^G$ and writing the result in its normal form, one obtains a non-trivial term

$$\begin{aligned} X^H &= \tilde{H}^0 \, \mathcal{L}_{-1}^{n+1} \, \mathcal{G}_{-1}^{r_1} \, \mathcal{Q}_0^{r_3} \,, \\ \tilde{H}^0 &= \mathcal{H}_{-m^H_h} \ldots \mathcal{H}_{-m^H_{j+1}} \mathcal{H}_{-1}^{j_0-1} \end{aligned} \qquad (66)$$

or, if $\|H^0\| = j_0$, $\tilde{H}^0 = \mathcal{H}_{-1}^{j_0-1}$ (commuting $\mathcal{Q}_{-1}$ with $\mathcal{H}_{-m^H_{j_0}}$ produces $\mathcal{Q}_{-m^H_{j_0}-1}$ and subsequently the commutation with $\mathcal{G}_{m^H_{j_0}}$ produces $\mathcal{L}_{-1}$). If $m^H_{j_0+1} = m^H_{j_0}$ one simply obtains multiple copies of X^H. Now we must show that any other term $Y \in \mathcal{C}^G_{m_0,n_0}$ also producing X^H under the action of $\mathcal{G}_{m^H_{j_0}} \mathcal{Q}_{-1}$ is $\mathcal{O}_G$-smaller than X_0. Just like in theorem 5.D, if Y already has $n+1$ or more operators $\mathcal{L}_{-1}$, then $Y <_{\mathcal{O}_G} X_0$. But unlike in theorem 5.D, we act now with two operators and could therefore also produce two new operators. However $\mathcal{G}_{m^H_{j_0}} \mathcal{Q}_{-1}$ cannot produce two $\mathcal{L}_{-1}$ as there are no operators $\mathcal{G}_0$ in Y. We therefore take first $Y = H^Y \mathcal{L}^n_{-1} \mathcal{G}^{r_1}_{-1} \mathcal{Q}^{r_2}_{-1} \mathcal{Q}^{r_3}_0 \in \mathcal{C}^G_{m_0,n_0}$. If $\mathcal{G}_{m^H_{j_0}} \mathcal{Q}_{-1}$ produces one $\mathcal{L}_{-1}$ by commuting through H^Y and leaves r_1, r_2 and r_3 unchanged one finds $Y <_{\mathcal{O}_G} X_0$, because then H^Y needs to have more operators $\mathcal{H}_{-1}$ than X_0, so that $m^Y_{j_0} < m^H_{j_0}$. On the other hand, $\mathcal{Q}_{-1}$ acting on $\mathcal{G}^{r^Y_1}_{-1} \mathcal{Q}^{r^Y_2}_{-1} \mathcal{Q}^{r^Y_3}_0$ cannot create any $\mathcal{L}_{-1}$ but it could change r^Y_2 from 0 to 1 or r^Y_1 from 1 to 0. The first case would not produce X^H as we assumed $r_2 = 0$ and in addition $\mathcal{G}_{m^H_{j_0}}$ cannot create $\mathcal{L}_{-1}$ from H^Y. The latter case can produce X^H but only for $r_1 = 0$ and $r^Y_1 = r^Y_2 = 1$ ($\mathcal{G}_2 \mathcal{Q}_{-1}$ creating $\mathcal{L}_{-1}$ from $\mathcal{G}^{r^Y_1}_{-1} \mathcal{Q}^{r^Y_2}_{-1} \mathcal{Q}^{r^Y_3}_0$). Thus $r^Y_1 + r^Y_2 = 2 > 0 = r_1 + r_2$ resulting in $Y <_{\mathcal{O}_G} X_0$.

Now let us take $r_2 = 1$ and let us assume $r_1 = 0$. In this case we proceed analogously as in the previous case by acting with $\mathcal{Q}_{m^H_{j_0}} \mathcal{G}_{-1} \in U(\mathsf{T}_2)^+$. Again one cannot produce the term X^H from a term Y $\mathcal{O}_G$-bigger than X_0. In particular, the only way $\mathcal{G}_{-1}$ could change the triple (r^Y_1, r^Y_2, r^Y_3) is by changing r^Y_1 from 0 to 1, which is not allowed as $r_1 = 0$, and in addition $\mathcal{L}_{-1}$ could not be created by $\mathcal{Q}_{m^H_{j_0}}$ acting on H^Y. For the case that both $r_1 = r_2 = 1$ let us first assume that $r_3 = 0$. By acting with $\mathcal{G}_{m^H_{j_0}-1} \mathcal{Q}_0 \in U(\mathsf{T}_2)^+$ one produces $\mathcal{L}_{-1}$ from $\mathcal{H}_{-m^H_{j_0}}$ in a similar way as before. $\mathcal{Q}_0$ can change the triple (r^Y_1, r^Y_2, r^Y_3) in two ways: it can change r^Y_3 from 0 to 1 or it could change r^Y_1 from 1 to 0. The first case, however, would not lead to the term X^H as $r_3 = 0$. The latter case can only lead to X^H if $\mathcal{G}_{m^H_{j_0}-1}$ can be converted into $\mathcal{G}_{-1}$ which requires $H^Y <_H H^0$. Therefore we find $Y <_{\mathcal{O}_G} X_0$. Finally, if $r_1 = r_2 = r_3 = 1$ we act with $\mathcal{Q}_{m^H_{j_0}-1} \mathcal{G}_0$. In this case r^Y_3 can only be changed from 1 to 0, which does not lead to X^H, and r^Y_2 can only be changed from 1 to 0, which requires $\mathcal{Q}_{m^H_{j_0}-1}$ to be converted into $\mathcal{Q}_{-1}$ in order to obtain X^H, resulting again in $Y <_{\mathcal{O}_G} X_0$. We can thus summarise that *the ordering $\mathcal{O}_G$ is adapted on the set of terms X_0 of the form given by* Eq. (59) *with $Q^0 \neq 1$, or $G^0 \neq 1$, or $L^0 \neq 1$, or $H^0 \neq \mathcal{H}^k_{-1}$ for some $k \in \mathbf{N}_0$ for all grades (m_0, n_0) and all central terms $c \in \mathbf{C}$.*

[p] At this stage of the proof we need the annihilation operators to be from the universal enveloping algebra. Therefore, definition 3.C is slighlty modified compared to definition 2.A.

Next let us consider $X_0 = \mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n}\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3}$. For the ordering we have chosen, $\mathcal{H}_1$ is not capable to rule out elements of the ordering kernel containing $\mathcal{H}_{-1}$. The action of $\mathcal{H}_1$ rather concerns $\mathcal{G}_{-1}\mathcal{Q}_{-1}$ as this combination of operators is necessarily needed in order to create $\mathcal{L}_{-1}$ under the action of $\mathcal{H}_1$. If we take $r_1=r_2=1$, i.e. $X_0=\mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n}\mathcal{G}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0^{r_3}$, then the action of $\mathcal{H}_1$ creates an additional $\mathcal{L}_{-1}$ in the only possible way that $\mathcal{H}_1$ can create $\mathcal{L}_{-1}$, producing a term $\mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n+1}\mathcal{Q}_0^{r_3}$ that cannot be obtained from any other term Y $\mathcal{O}_G$-bigger than X_0. *Thus elements of the ordering kernel are of the form* $\mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n}\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3}$ *with* $r_1+r_2<2$ *for all grades and all central terms* $c\in\mathbf{C}$.

At this stage all restrictions on the ordering kernel arising from operators in T_2^+ which create an additional $\mathcal{L}_{-1}$ have been used. One might think that the smallest ordering kernel has been found. However we will now show that, considering the action of two annihilation operators at the same time, we can still reduce the ordering kernel at least for central terms $c\neq 3$. Let us consider the case $X_0=\mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n}\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3}$. with $m^H\neq 0$ and $r_1+r_2<2$. The action of $\mathcal{H}_1\in\mathsf{T}_2^+$ on $\Psi^0 = X_0\,|\Delta,q\rangle^G$ creates, provided $c\neq 0$, a non-trivial term of the form $X^H=\mathcal{H}_{-1}^{m^H-1}\mathcal{L}_{-1}^{n}\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3}$ with one $\mathcal{H}_{-1}$ removed but no new $\mathcal{L}_{-1}$ created. Furthermore, as $r_1+r_2<2$ $\mathcal{H}_1$ cannot create any $\mathcal{L}_{-1}$. Now, depending on r_1, r_2 and r_3 it may be possible to find terms Y $\mathcal{O}_G$-bigger than X_0 that do not create $\mathcal{L}_{-1}$ but still generate X^H under $\mathcal{H}_1$. In the case $r_1=1$, $r_2=0$ one finds that $Y=\mathcal{H}_{-1}^{m^H-1}\mathcal{G}_{-2}\mathcal{L}_{-1}^{n}\mathcal{Q}_0^{r_3}$ is the only such term. Thus, X_0 and Y both create the same term $X^H=\mathcal{H}_{-1}^{m^H-1}\mathcal{L}_{-1}^{n}\mathcal{G}_{-1}\mathcal{Q}_0^{r_3}$ under the action of $\mathcal{H}_1$ with coefficients $m^H\frac{c}{3}$ and 1 respectively[q], with $X_0 <_{\mathcal{O}_G} Y$. On the other hand, acting with $\mathcal{Q}_1$ again shows that Y is the only term $\mathcal{O}_G$-bigger than X_0, both of them generating[r] $\tilde{X}^H=\mathcal{H}_{-1}^{m^H-1}\mathcal{L}_{-1}^{n+1}\mathcal{Q}_0^{r_3}$. Let us therefore take the combination $\mathcal{H}_1+\mathcal{Q}_1$ acting on X_0 and on Y. This results in

$$\begin{aligned}(\mathcal{H}_1+\mathcal{Q}_1)X_0 &= m^H\frac{c}{3}X^H+2m^H\tilde{X}^H+\ldots, \\ (\mathcal{H}_1+\mathcal{Q}_1)Y &= X^H+2\tilde{X}^H+\ldots,\end{aligned} \tag{67}$$

where "..." denotes terms that are irrelevant for us[s]. We now alter the standard basis of the normal form by defining new basis terms $X^1=m^H\frac{c}{3}X^H+2m^H\tilde{X}^H$ and $X^2=X^H+2\tilde{X}^H$. If this change of basis is possible the action of $\mathcal{H}_1+\mathcal{Q}_1$ on X_0 thus yields a term X^1 that cannot be produced from any other term Y unless $Y<_{\mathcal{O}_G}X_0$. In order for this basis transformation to be allowed, the determinant of the transformation coefficients must be non-trivial: $2m^H\frac{c}{3}-2m^H\neq 0$ with $m^H>0$ and $c\neq 3$. In the case $r_1=0$, $r_2=1$ we can repeat exactly the same procedure with $Y=\mathcal{H}_{-1}^{m^H-1}\mathcal{Q}_{-2}\mathcal{L}_{-1}^{n}\mathcal{Q}_0^{r_3}$, $\tilde{X}^H=\mathcal{H}_{-1}^{m^H-1}\mathcal{L}_{-1}^{n+1}\mathcal{Q}_0^{r_3}$ and $\mathcal{Q}_1$ replaced by $\mathcal{G}_1$. Eqs. (67) turn in this case into

$$\begin{aligned}(\mathcal{H}_1+\mathcal{G}_1)X_0 &= m^H\frac{c}{3}X^H-2m^H\tilde{X}^H+\ldots, \\ (\mathcal{H}_1+\mathcal{G}_1)Y &= -X^H+2\tilde{X}^H+\ldots,\end{aligned} \tag{68}$$

and thus result in exactly the same conditions from the determinant: $m^H>0$ and $c\neq 3$.

Finally in the case $r_1=r_2=0$ we are left with X_0 of the form

$$X_0 = \mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n}\mathcal{Q}_0^{r_3}. \tag{69}$$

[q]There are certainly other terms that are $\mathcal{O}_G$-smaller than X_0 and create X^H such as $\mathcal{H}_{-1}^{m^H-1}\mathcal{L}_{-1}^{n}\mathcal{G}_{-1}\mathcal{Q}_{-1}$ for $r_2=0$ and $r_3=1$. But as before, $\mathcal{O}_G$-smaller terms are not relevant due to definition 3.C.

[r]Note that we have used the action of $\mathcal{Q}_1$ before to rule out Y in the ordering kernel.

[s]Note that this is consistent for $c=0$.

For $m^H \geq 1$ one finds that the action of $\mathcal{H}_1$ on X_0 and on $Y = \mathcal{H}_{-1}^{m^H-1}\mathcal{L}_{-1}^{n-1}\mathcal{G}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0^{r_3}$ produces a term $X^H = \mathcal{H}_{-1}^{m^H-1}\mathcal{L}_{-1}^{n}\mathcal{Q}_0^{r_3}$ with coefficients $\frac{c}{3}m^H$ and 2 respectively. As $X_0 <_{\mathcal{O}_G} Y$ one again needs to find a suitable second operator creating from both X_0 and Y a common term $\tilde{X}^H$ that cannot be created from any other terms that are $\mathcal{O}_G$-bigger than X_0 and Y. For $m^H \geq 2$ we act with $\mathcal{H}_1\mathcal{G}_0\mathcal{Q}_0$ on X_0 and Y. In both cases one obtains a term $\tilde{X} = \mathcal{H}_{-1}^{m^H-2}\mathcal{L}_{-1}^{n+1}\mathcal{Q}_0^{r_3}$ with coefficients $2m^H(m^H-1)(\frac{c}{3}-1)$ and $4(m^H-1)(\frac{c}{3}-1)$ respectively. As above, the change of basis is possible if the determinant of these coefficients is non-trivial:

$$\begin{vmatrix} m^H\frac{c}{3} & 2 \\ 2m^H(m^H-1)(\frac{c}{3}-1) & 4(m^H-1)(\frac{c}{3}-1) \end{vmatrix} \quad = \quad 4m^H(m^H-1)(\frac{c}{3}-1)^2\,, \tag{70}$$

and thus result in the conditions: $m^H \geq 2$ and $c \neq 3$.

Therefore the kernel of the ordering $\mathcal{O}_G$, is given by

$$\mathcal{C}_{m_0,n_0}^K \quad = \quad \{\mathcal{L}_{-1}^n\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3},\ \mathcal{H}_{-1}\mathcal{L}_{-1}^n\mathcal{Q}_0^{r_3}\ :\ r_1+r_2<2,\ m_0=n+r_1+r_2,\ n_0=r_1-r_2-r_3\}\,,$$

for all grades (m_0, n_0) and all central terms $c \in \mathbf{C}$ with $c \neq 3$. This proves the results shown in TAB. c.

For $\mathcal{G}_0$-closed vectors $\mathcal{G}_0$ is also in the set of annihilation operators. In this case the action of $\mathcal{G}_0$ on X_0 of the form $\mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^n\mathcal{G}_{-1}^{r_1}\mathcal{Q}_{-1}\mathcal{Q}_0^{r_3}$ produces the term $\mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n+1}\mathcal{G}_{-1}^{r_1}\mathcal{Q}_0^{r_3}$ that cannot be obtained from any other term Y $\mathcal{O}_G$-bigger than X_0 (commuting with $\mathcal{Q}_{-1}$ is the only way to produce $\mathcal{L}_{-1}$ acting with $\mathcal{G}_0$). Thus, the ordering kernel contains no terms with $\mathcal{Q}_{-1}$ for all complex values of c.

If we now take $X_0 = \mathcal{H}_{-1}\mathcal{L}_{-1}^n\mathcal{Q}_0^{r_3}$ we find the (unique) $\mathcal{O}_G$-bigger term $Y = \mathcal{L}_{-1}^{n-1}\mathcal{G}_{-1}\mathcal{Q}_{-1}\mathcal{Q}_0^{r_3}$, both producing the terms $\mathcal{L}_{-1}^n\mathcal{G}_{-1}\mathcal{Q}_0^{r_3}$ and $\mathcal{L}_{-1}^n\mathcal{Q}_0^{r_3}$ under the action of $\mathcal{G}_0$ and $\mathcal{H}_1$ respectively[t]. As a result $\mathcal{H}_{-1}$ can also be removed by changing the basis suitably provided $c \neq 3$. The determinant of the coefficients results in:

$$\begin{vmatrix} -1 & -2 \\ \frac{c}{3} & 2 \end{vmatrix} \quad = \quad 2(\frac{c}{3}-1)\,, \tag{71}$$

which is again non-trivial for $c \neq 3$. This proves the results shown in TAB. d.

In a completely analogous way one can remove the terms containing $\mathcal{G}_{-1}$ or $\mathcal{H}_{-1}$ in the case of $\mathcal{Q}_0$-closed singular vectors. The former can be done by acting with $\mathcal{Q}_0$ on $X_0 = \mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^n\mathcal{G}_{-1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3}$, creating the term $\mathcal{H}_{-1}^{m^H}\mathcal{L}_{-1}^{n+1}\mathcal{Q}_{-1}^{r_2}\mathcal{Q}_0^{r_3}$, whilst the latter is achieved from the action of $\mathcal{Q}_0$ and $\mathcal{H}_1$ on the same X_0 and Y as above. The determinant of the coefficients is again non-trivial for $c \neq 3$:

$$\begin{vmatrix} 1 & 2 \\ \frac{c}{3} & 2 \end{vmatrix} \quad = \quad -2(\frac{c}{3}-1)\,, \tag{72}$$

The results are shown in TAB. e.

Finally, combining our considerations for $\mathcal{G}_0$-closed singular vectors and $\mathcal{Q}_0$-closed singular vectors we obtain that the ordering kernel for the case of chiral singular vectors does not contain any operators of the form $\mathcal{Q}_{-1}$, $\mathcal{G}_{-1}$ or $\mathcal{H}_{-1}$. This proves the results shown in TAB. f and finally completes the proof of theorem 5.C. □

[t]The basis we have to choose in this case is not L_0 graded.

By replacing the rôles of the operators $\mathcal{G}_n$ and $\mathcal{Q}_n$ for all $n \in \mathbf{Z}$ we can define analogously an ordering $\mathcal{O}_Q$ on $\mathcal{C}^Q_{m,n}$ which is adapted for $c \neq 3$ for all levels. The corresponding ordering kernels are as follows.

Theorem 5.E *If the central extension satisfies $c \neq 3$ then the ordering $\mathcal{O}_Q$ is adapted to $\mathcal{C}^Q_{m,n}$ for all Verma modules $\mathcal{V}^Q_{\Delta,q}$ and for all grades (m,n) with $m \in \mathbf{N}_0$, $n \in \mathbf{Z}$. Ordering kernels are given by the following tables for all levels m, depending on the set of annihilation operators and depending on the charge n.*

n	*ordering kernel*
$+2$	$\{\mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\mathcal{G}_0\}$
$+1$	$\{\mathcal{L}_{-1}^{m}\mathcal{G}_0, \mathcal{H}_{-1}\mathcal{L}_{-1}^{m-1}\mathcal{G}_0, \mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\}$
0	$\{\mathcal{L}_{-1}^{m}, \mathcal{H}_{-1}\mathcal{L}_{-1}^{m-1}, \mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\mathcal{G}_0\}$
-1	$\{\mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\}$

TAB. g *Ordering kernels for $\mathcal{O}_Q$, annihilation operators T_2^+.*

n	*ordering kernel*
$+1$	$\{\mathcal{L}_{-1}^{m}\mathcal{G}_0\}$
0	$\{\mathcal{L}_{-1}^{m}, \mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\mathcal{G}_0\}$
-1	$\{\mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\}$

TAB. h *Ordering kernels for $\mathcal{O}_Q$, annihilation operators T_2^+ and $\mathcal{Q}_0$.*

n	*ordering kernel*
$+2$	$\{\mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\mathcal{G}_0\}$
$+1$	$\{\mathcal{L}_{-1}^{m}\mathcal{G}_0, \mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\}$
0	$\{\mathcal{L}_{-1}^{m}\}$

TAB. i *Ordering kernels for $\mathcal{O}_Q$, annihilation operators T_2^+ and $\mathcal{G}_0$.*

n	*ordering kernel*
$+1$	$\{\mathcal{L}_{-1}^{m}\mathcal{G}_0\}$
0	$\{\mathcal{L}_{-1}^{m}\}$

TAB. j *Ordering kernels for $\mathcal{O}_Q$, annihilation operators T_2^+, $\mathcal{Q}_0$, and $\mathcal{G}_0$.*

Charges that do not appear in the tables have trivial ordering kernels.

Proof: The proof of theorem 5.E is completely analogous to the proof of theorem 5.C. We just need to swap the rôles of the operators $\mathcal{G}_n$ and $\mathcal{Q}_n$ for all $n \in \mathbf{Z}$. □

6 Adapted orderings on chiral Verma modules $\mathcal{V}^{GQ}_{0,q}$

We saw in section 4 that for both chiral and no-label highest weight vectors the conformal weight is zero. This applies to Verma modules as well as to singular vectors. Thus a chiral singular vector in the chiral Verma module $\mathcal{V}_{0,q}^{GQ}$ needs to have level 0 and the same is true for no-label singular vectors in $\mathcal{V}_{0,q}^{GQ}$. But at level 0 there are no singular vectors in $\mathcal{V}_{0,q}^{GQ}$, as the only state at level 0 is the highest weight state $|0,q\rangle^{GQ}$ itself. For chiral Verma modules $\mathcal{V}_{0,q}^{GQ}$ we shall therefore only consider adapted orderings with additional annihilation conditions corresponding to $\mathcal{G}_0$ or $\mathcal{Q}_0$, but not to both.

In section 4 we also introduced the set $\mathcal{C}_{m,n}^{GQ}$, Eq. (48), defining the standard basis $\mathcal{B}_{|0,q\rangle^{GQ}}$ of the chiral Verma modules $\mathcal{V}_{0,q}^{GQ}$. $\mathcal{C}_{m,n}^{GQ}$ can be obtained by setting $r_3 \equiv 0$ in $\mathcal{C}_{m,n}^{G}$, Eq. (41). Therefore, $\mathcal{O}_G$ is also defined on $\mathcal{C}_{m,n}^{GQ}$, a subset of $\mathcal{C}_{m,n}^{G}$. This suggests that the ordering kernels for $\mathcal{O}_G$ on $\mathcal{C}_{m,n}^{GQ}$ may simply be appropriate subsets of the ordering kernels of $\mathcal{O}_G$ on $\mathcal{C}_{m,n}^{G}$, given in theorem 5.C. This can easily be shown by considering the fact that r_3 is never a deciding element of the ordering $\mathcal{O}_G$ in Eqs. (52)-(57). Furthermore, during the proof of theorem 5.C it happens in each case that the considered term X^{Γ}, constructed from X_0 under the action of a suitable annihilation operator Γ, has always the same exponent r_3 of $\mathcal{Q}_0$ as X_0 itself. Therefore, the whole proof of theorem 5.C can also be applied to $\mathcal{O}_G$ defined on $\mathcal{C}_{m,n}^{GQ}$ simply by imposing $r_3 \equiv 0$ in every step. As a result the new ordering kernels are simply the intersections of $\mathcal{C}_{m,n}^{GQ}$ with the ordering kernels for $\mathcal{C}_{m,n}^{G}$. Hence, we have already proven the following theorem.

Theorem 6.A *For the set of annihilation operators that contains $\mathcal{G}_0$ or $\mathcal{Q}_0$ but not both and for $c \neq 3$ the ordering $\mathcal{O}_G$ is adapted to $\mathcal{C}_{m,n}^{GQ}$ for all chiral Verma modules $\mathcal{V}_{0,q}^{GQ}$ and for all grades (m,n) with $m \in \mathbf{N}_0$, $n \in \mathbf{Z}$. Depending on the set of annihilation operators and depending on the charge n, ordering kernels are given by the following tables for all levels m:*

n	*ordering kernel*
$+1$	$\{\mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\}$
0	$\{\mathcal{L}_{-1}^{m}\}$

Tab. k *Ordering kernels for $\mathcal{O}_G$ on $\mathcal{C}_{m,n}^{GQ}$, annihilation operators T_2^+ and $\mathcal{G}_0$.*

n	*ordering kernel*
0	$\{\mathcal{L}_{-1}^{m}\}$
-1	$\{\mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\}$

Tab. l *Ordering kernels for $\mathcal{O}_G$ on $\mathcal{C}_{m,n}^{GQ}$, annihilation operators T_2^+ and $\mathcal{Q}_0$.*

Charges that do not appear in the tables have trivial ordering kernels.

7 Adapted orderings on no-label Verma modules $\mathcal{V}_{0,q}$

We will now consider adapted orderings for no-label Verma modules $\mathcal{V}_{0,q}$. In section 4, the standard basis for $\mathcal{V}_{0,q}$ is defined using $\mathcal{C}_{m,n}$ of Eq. (48). No-label Verma modules have zero conformal weight, like chiral Verma modules. Consequently chiral singular vectors as well as no-label singular vectors in $\mathcal{V}_{0,q}$ can only exist at level 0. The space of states in $\mathcal{V}_{0,q}$ at level 0 is spanned by $\{|0,q\rangle, \mathcal{G}_0|0,q\rangle, \mathcal{Q}_0|0,q\rangle, \mathcal{G}_0\mathcal{Q}_0|0,q\rangle\}$. Therefore, there are no no-label singular vectors in $\mathcal{V}_{0,q}$ and there is exactly one chiral singular vector in $\mathcal{V}_{0,q}$ for all q and for all central extensions c, namely $\mathcal{G}_0\mathcal{Q}_0|0,q\rangle$. Hence, our main interest focuses on the $\mathcal{G}_0$-closed singular vectors and the $\mathcal{Q}_0$-closed singular vectors in $\mathcal{V}_{0,q}$ and we shall therefore investigate adapted orderings with the corresponding vanishing conditions. The states $\mathcal{G}_0\,|0,q\rangle$ and $\mathcal{Q}_0\,|0,q\rangle$ satisfy $\mathcal{Q}_0\mathcal{G}_0\,|0,q\rangle = -\mathcal{G}_0\mathcal{Q}_0\,|0,q\rangle$. Consequently the norms of these states have opposite signs and can be set to zero.

Clearly, $\mathcal{V}^G_{0,q}$ is isomorphic to the quotient module of $\mathcal{V}_{0,q}$ divided by the submodule generated by the singular vector $\mathcal{G}_0\,|0,q\rangle$, i.e. $\mathcal{V}^G_{0,q} = \frac{\mathcal{V}_{0,q}}{\mathcal{G}_0|0,q\rangle}$ and likewise $\mathcal{V}^Q_{0,q} = \frac{\mathcal{V}_{0,q}}{\mathcal{Q}_0|0,q\rangle}$. If we consider a singular vector Ψ of $\mathcal{V}_{0,q}$ then the canonical projection of Ψ into $\mathcal{V}^G_{0,q}$ is either trivial or a singular vector in $\mathcal{V}^G_{0,q}$ and similarly for $\mathcal{V}^Q_{0,q}$. The converse, however, is not true, a singular vector in $\mathcal{V}^G_{0,q}$ or $\mathcal{V}^Q_{0,q}$ do not necessarily correspond to a singular vector in $\mathcal{V}_{0,q}$, it may only be subsingular in $\mathcal{V}_{0,q}$. One may also ask whether all singular vectors in $\mathcal{V}_{0,q}$ correspond to singular vectors in either $\mathcal{V}^G_{0,q}$ or $\mathcal{V}^Q_{0,q}$, in which case the investigation of no-label Verma modules would not give us more information than what we already know, or rather there can also be singular vectors in $\mathcal{V}_{0,q}$ that vanish for both canonical projections into the generic Verma modules $\mathcal{V}^G_{0,q}$ and $\mathcal{V}^Q_{0,q}$. This is indeed the case, as was shown by the explicit examples at level 1 given in Ref. 19 (we will come back to this point at the end of next section).

Unlike for chiral Verma modules, the no-label Verma modules are not simply a subcase of the $\mathcal{G}_0$-closed Verma modules with respect to the adapted ordering $\mathcal{O}_G$. In fact, rather than $\mathcal{C}_{m,n}$ being a subset of $\mathcal{C}^G_{m,n}$, we find that $\mathcal{C}^G_{m,n}$ is a subset of $\mathcal{C}_{m,n}$ and we hence need to extend the ordering $\mathcal{O}_G$ suitably.

Definition 7.A *On the set $\mathcal{C}_{m,n}$ we introduce the total ordering $\mathcal{O}_{GQ}$. For two elements $X_1, X_2 \in \mathcal{C}_{m,n}$, $X_1 \neq X_2$ with $X_i = L^i H^i G^i Q^i \mathcal{L}_{-1}^{k^i} \mathcal{G}_{-1}^{r_1^i} \mathcal{Q}_{-1}^{r_2^i} \mathcal{G}_0^{r_4^i} \mathcal{Q}_0^{r_3^i}$, $L^i \in \mathsf{L}_{l_i}$, $H^i \in \mathsf{H}_{h_i}$, $G^i \in \mathsf{G}_{g_i}$, $Q^i \in \mathsf{Q}_{q_i}$ for some $l_i, h_i, g_i, q_i, k^i, r_1^i, r_2^i, r_3^i, r_4^i \in \mathbf{N}_0$, $i = 1, 2$ such that $m \in \mathbf{N}_0$, $n \in \mathbf{Z}$ we define*

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad k^1 > k^2\,. \tag{73}$$

For $k^1 = k^2$ we set

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad r_1^1 + r_2^1 > r_1^2 + r_2^2\,. \tag{74}$$

If $r_1^1 + r_2^1 = r_1^2 + r_2^2$ we set

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad Q^1 <_Q Q^2\,. \tag{75}$$

In the case $Q^1 = Q^2$ we define

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad G^1 <_G G^2\,. \tag{76}$$

If also $G^1 = G^2$ we then define

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad L^1 <_L L^2\,. \tag{77}$$

If further $L^1 = L^2$ we set

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad H^1 <_H H^2\,, \tag{78}$$

unless $H^1 = H^2$ in which case we set

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad r_1^1 > r_1^2\,. \tag{79}$$

If also $r_1^1 = r_1^2$ we finally define

$$X_1 <_{\mathcal{O}_{GQ}} X_2 \quad \text{if} \quad r_3^1 + r_4^1 > r_3^2 + r_4^2\,, \tag{80}$$

which necessarily has to give an answer. For $X_1 = X_2$ we define $X_1 <_{\mathcal{O}_{GQ}} X_2$ and $X_2 <_{\mathcal{O}_{GQ}} X_1$.

If Eqs. (73)-(80) do not give an answer on the ordering of X_1 and X_2, then X_1 and X_2 are of the form $X_i = LHGQ\mathcal{L}_{-1}^k \mathcal{G}_{-1}^{r_1} \mathcal{Q}_{-1}^{r_2} \mathcal{G}_0^{r_4^i} \mathcal{Q}_0^{r_3^i}$, with $r_3^1 + r_4^1 = r_3^2 + r_4^2$. Since X_1 and X_2 both have charge n one has $r_3^1 - r_4^1 = r_3^2 - r_4^2$ and thus $X_1 = X_2$. Hence, definition 7.A is a total ordering well-defined on $\mathcal{C}_{m,n}$.

We will now argue that the proof of theorem 5.C can easily be modified in such a way that exactly the same restrictions on the ordering kernels of $\mathcal{O}_G$ extend to the ordering kernels of $\mathcal{O}_{GQ}$. As a first step we see that theorem 5.D extends straightforwardly to $\mathcal{C}_{m,n}$ simply by replacing $X_0 = X_0^* \mathcal{L}_{-1}^n \mathcal{G}_{-1}^{r_1} \mathcal{Q}_{-1}^{r_2} \mathcal{Q}_0^{r_3} \in \mathcal{C}_{m,n}^G$ by $X_0 = X_0^* \mathcal{L}_{-1}^n \mathcal{G}_{-1}^{r_1} \mathcal{Q}_{-1}^{r_2} \mathcal{G}_0^{r_4} \mathcal{Q}_0^{r_3} \in \mathcal{C}_{m,n}$. Note that in the proof r_4^Y would behave exactly like r_3^Y which does not interfere with any arguments. As theorem 5.D turned out to be the key tool to remove operators of the form $\mathcal{L}_{-n}$, $\mathcal{G}_{-n}$, or $\mathcal{Q}_{-n}$ from the ordering kernel of $\mathcal{C}_{m,n}^G$, we can in exactly the same way already state that *the ordering $\mathcal{O}_{GQ}$ is for all grades (m,n) and all central extensions $c \in \mathbf{C}$ adapted to the set of terms*

$$X_0 \quad = \quad L^0\, H^0\, G^0\, Q^0\, \mathcal{L}_{-1}^n\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{G}_0^{r_4}\, \mathcal{Q}_0^{r_3} \quad \in \mathcal{C}_{m,n}\,, \tag{81}$$

with $L^0 \neq 1$, or $G^0 \neq 1$, or $Q^0 \neq 1$. We can thus focus on terms X_0 of the form

$$X_0 \quad = \quad \mathcal{H}_{-m_I^H} \ldots \mathcal{H}_{-m_{j_0}^H}\, \mathcal{H}_{-1}^{j_0-1}\, \mathcal{L}_{-1}^n\, \mathcal{G}_{-1}^{r_1}\, \mathcal{Q}_{-1}^{r_2}\, \mathcal{G}_0^{r_4}\, \mathcal{Q}_0^{r_3}\,, \tag{82}$$

with $m_{j_0}^H > 1$. In the proof of theorem 5.C we dealt with these terms by acting with $\mathcal{G}_{m_{j_0}^H-1}\mathcal{Q}_{-1}$ for $r_2 = 0$. At first, the existence of $\mathcal{G}_0$ in the no-label case seems to interfere with this argument. However, the ordering $\mathcal{O}_{GQ}$ has been defined in such a way that $\mathcal{Q}_{-1}$ does never interact with $\mathcal{G}_0$ as it would simply be stuck on the left of $\mathcal{G}_0$. Therefore, one easily sees that the same arguments as in the proof of theorem 5.C hold for $r_2 = 0$. In the case of $r_2 = 1$ and $r_1 = 0$ the proof even holds without any modification. For the cases $r_1 = r_2 = 1$, we act with $\mathcal{G}_{m_{j_0}^H}\mathcal{Q}_0$ or $\mathcal{Q}_{m_{j_0}^H}\mathcal{G}_0$ for $r_3 = 0$ or $r_3 = 1$ respectively. In these cases we have to consider the additional possibility that r_4^Y changes from 1 to 0 or from 0 to 1 respectively. However, as $\mathcal{G}_{m_{j_0}^H}$ or $\mathcal{Q}_{m_{j_0}^H}$ still needs to create a $\mathcal{L}_{-1}$ we easily see that any term Y also satisfying the conditions of proof 5.C for this case must contain operators L^Y, G^Y, Q^Y, or H^Y with $H^Y <_H H^0$ and therefore $Y <_{GQ} X_0$ as the r_4 term decides very last in the ordering Eqs. (73)-(80). For the final part of proof 5.C where we found restrictions on $\mathcal{H}_{-1}^{j_0}$ in X_0, we can simply note that $\mathcal{G}_0$ cannot interfere with the arguments of the proof for $\mathcal{H}_1$ (terms that could interfere to produce $\mathcal{G}_0$ would be $\mathcal{O}_{GQ}$-smaller than X_0) and $\mathcal{Q}_1$ and $\mathcal{G}_1$ are used in the proof to create $\mathcal{L}_{-1}$ which is neither possible with $\mathcal{G}_0$. Similar arguments show that the proof also holds for $r_1 = r_2 = 0$. It is easy to see that in the $\mathcal{G}_0$-closed or $\mathcal{Q}_0$-closed cases all considerations of proof 5.C extend to $\mathcal{C}_{m,n}$. We have thus shown that the proof of theorem 5.C extends to prove the following theorem regarding no-label Verma modules $\mathcal{V}_{0,q}$.

Theorem 7.B *For the set of annihilation operators that contains $\mathcal{G}_0$ or $\mathcal{Q}_0$ but not both and for $c \neq 3$ we find that the ordering $\mathcal{O}_{GQ}$ is adapted to $\mathcal{C}_{m,n}$ for all Verma modules $\mathcal{V}_{0,q}$ and for all grades (m,n) with $m \in \mathbf{N}_0$, $n \in \mathbf{Z}$. Depending on the set of annihilation operators and depending on the charge n, ordering kernels are given by the following tables for all levels m:*

n	*ordering kernel*
$+2$	$\{\mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\mathcal{G}_0\}$
$+1$	$\{\mathcal{L}_{-1}^{m}\mathcal{G}_0, \mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}, \mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\mathcal{G}_0\mathcal{Q}_0\}$
0	$\{\mathcal{L}_{-1}^{m}, \mathcal{L}_{-1}^{m}\mathcal{G}_0\mathcal{Q}_0, \mathcal{L}_{-1}^{m-1}\mathcal{G}_{-1}\mathcal{Q}_0\}$
-1	$\{\mathcal{L}_{-1}^{m}, \mathcal{Q}_0\}$

TAB. m *Ordering kernels for $\mathcal{O}_{GQ}$ on $\mathcal{C}_{m,n}$, annihilation operators T_2^+ and $\mathcal{G}_0$.*

n	*ordering kernel*
$+1$	$\{\mathcal{L}_{-1}^{m}\mathcal{G}_0\}$
0	$\{\mathcal{L}_{-1}^{m}, \mathcal{L}_{-1}^{m}\mathcal{G}_0\mathcal{Q}_0, \mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\mathcal{G}_0\}$
-1	$\{\mathcal{L}_{-1}^{m}\mathcal{Q}_0, \mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}, \mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\mathcal{G}_0\mathcal{Q}_0\}$
-2	$\{\mathcal{L}_{-1}^{m-1}\mathcal{Q}_{-1}\mathcal{Q}_0\}$

TAB. n *Ordering kernels for $\mathcal{O}_{GQ}$ on $\mathcal{C}_{m,n}$, annihilation operators T_2^+ and $\mathcal{Q}_0$.*

Charges that do not appear in the tables have trivial ordering kernels.

8 Dimensional analysis

In previous sections we argued, following Ref. 19, that a naive estimate would give 16 types of singular vectors in $N = 2$ topological Verma modules, depending on whether the highest weight vector or the singular vector itself satisfy additional vanishing conditions with respect to the zero modes $\mathcal{G}_0$ or $\mathcal{Q}_0$, each of these types coming with different charges. Three of these types can be ruled out, however, simply by taking into account that *chiral* highest weight conditions and *no-label* highest weight conditions apply only to states with zero conformal weight. In chiral Verma modules this rules out chiral singular vectors as well as no-label singular vectors, whilst in no-label Verma modules the no-label singular vectors are ruled out. A fourth type of singular vectors, chiral singular vectors in no-label Verma modules, turns out to consist of only the level zero singular vector $\mathcal{G}_0\mathcal{Q}_0\,|0, q\rangle$. In this section we will use theorem 3.F, together with the results for the ordering kernels of the previous sections, as the main tools to give upper limits for the dimensions of the remaining 12 types of topological singular vectors. For most charges this procedure will even show that there are no singular vectors corresponding to them.

The dimension of the singular vector spaces in $N = 2$ superconformal Verma modules can be larger than one. This fact was discovered for the Neveu-Schwarz $N = 2$ algebra in Ref. 9. In particular, sufficient conditions were found (and proved) to guarantee the existence of two-dimensional spaces of uncharged singular vectors. Before this had been shown, it was a false common belief that singular vectors at the same level and with the same charge would always be linearly dependent. Later some of the results in Ref. 9 were extended[19] to the topological $N = 2$ algebra. As a consequence two-dimensional spaces for four different types of topological singular vectors were shown to exist (those given in TAB. o below). However, as we will discuss, the Neveu-Schwarz counterpart of most topological singular vectors are not singular vectors themselves, but either descendants of singular vectors or subsingular vectors[11, 19, 15], for which very little is known. As a consequence, in order to compute the maximal dimensions for the singular vector spaces of the topological $N = 2$ algebra, an independent method, like the one presented in this paper, was needed.

Let us first proceed with a clear definition of what we mean by *singular vector spaces.*

Definition 8.A *A $\mathcal{G}_0$-closed singular vector space of the topological Verma module $\mathcal{V}^M_{\Delta,q}$ is a subspace of $\mathcal{V}^M_{\Delta,q}$ of vectors at the same level and with the same charge for which each non-trivial element is a $\mathcal{G}_0$-closed singular vector. M stands for $\mathcal{G}_0$-closed, $\mathcal{Q}_0$-closed, chiral, or no-label. Analogously we define $\mathcal{Q}_0$-closed singular vector spaces, chiral singular vector spaces, and no-label singular vector spaces.*

Let us denote by $\Psi^{K,n}_{m,|\Delta,q\rangle^M}$ a singular vector in the topological Verma module $\mathcal{V}^M_{\Delta,q}$ at level m and with charge n. K denotes the additional vanishing conditions of the singular vector, whith respect to $\mathcal{G}_0$ and $\mathcal{Q}_0$, whilst M denotes the additional vanishing conditions of the highest weight vector, as introduced in section 4. The ordering kernels of theorems 5.C and 5.E together with theorem 3.F allow us to write down an upper limit for the dimensions of the singular vector spaces simply by counting the number of elements of the ordering kernels.

Theorem 8.B *For singular vectors with additional vanishing conditions in $\mathcal{V}^G_{\Delta,q}$ or in $\mathcal{V}^Q_{\Delta,q}$, $c \neq 3$, we find the following upper limits for the number of linearly independent singular vectors at the same level $m \in \mathbf{N}_0$ and with the same charge $n \in \mathbf{Z}$ ($\Delta = -m$ for chiral singular vectors).*

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$\Psi^{G,n}_{m,\|\Delta,q\rangle^G}$	0	1	2	1	0
$\Psi^{Q,n}_{m,\|\Delta,q\rangle^G}$	1	2	1	0	0
$\Psi^{GQ,n}_{m,\|-m,q\rangle^G}$	0	1	1	0	0
$\Psi^{Q,n}_{m,\|\Delta,q\rangle^Q}$	0	1	2	1	0
$\Psi^{G,n}_{m,\|\Delta,q\rangle^Q}$	0	0	1	2	1
$\Psi^{GQ,n}_{m,\|-m,q\rangle^Q}$	0	0	1	1	0

TAB. o *Maximal dimensions for singular vectors spaces annihilated by $\mathcal{G}_0$ and/or $\mathcal{Q}_0$ in $\mathcal{V}^G_{\Delta,q}$ or in $\mathcal{V}^Q_{\Delta,q}$.*

Singular vectors can only exist if they contain in their normal form at least one non-trivial term of the corresponding ordering kernel of theorem 5.C or 5.E. Charges n that are not given have dimension 0 and hence do not allow any singular vectors.

The ordering kernels for the vanishing conditions T_2^+, given in tables TAB. c and TAB. g, do not include any conditions requiring the action of $\mathcal{G}_0$ and $\mathcal{Q}_0$ not to be trivial. As a result, the ordering kernels of tables TAB. c and TAB. g include not only the no-label cases but also the cases of $\mathcal{G}_0$-closed singular vectors, $\mathcal{Q}_0$-closed singular vectors, and chiral singular vectors. However, for no-label singular vectors in $\mathcal{V}^G_{\Delta,q}$ or in $\mathcal{V}^Q_{\Delta,q}$ we can find in addition the following restrictions. If $\Psi^n_{m,|\Delta,q\rangle^M}$ is a no-label singular vector, then $\mathcal{G}_0\Psi^n_{m,|\Delta,q\rangle^M}$ must be a singular vector of type $\Psi^{G,n+1}_{m,|\Delta,q\rangle^M}$. Consequently, the dimension for the space of the no-label singular vector $\Psi^n_{m,|\Delta,q\rangle^M}$ cannot be larger than the dimension for the space of the $\mathcal{G}_0$-closed singular vector[u] $\Psi^{G,n+1}_{m,|\Delta,q\rangle^M}$. This can easily be

[u]If Ψ is a no-label singular vector and Ξ is a $\mathcal{G}_0$-closed, or $\mathcal{Q}_0$-closed, or chiral singular vector, both at the same level and with the same charge, then $\Psi+\Xi$ is again a no-label singular vector (in the sense of not being annihilated by $\mathcal{G}_0$ or $\mathcal{Q}_0$) which is linearly idenpendent of Ψ. However, the space spanned by Ψ and $\Psi+\Xi$ is not considered to be a two-dimensional no-label singular vector space as it decomposes into a one-dimensional no-label singular vector space and a one-dimensional $\mathcal{G}_0$-closed, or $\mathcal{Q}_0$-closed, or chiral singular vector space.

seen as follows. Assume that Ψ_1 and Ψ_2 are two no-label linearly independent singular vectors at the same level and with the same charge, and suppose $\mathcal{G}_0\Psi_1$ and $\mathcal{G}_0\Psi_2$ are linearly dependent. Then obviously there exist numbers α, β ($\alpha\beta \neq 0$) such that $\mathcal{G}_0(\alpha\Psi_1+\beta\Psi_2)=0$ and thus the $\mathcal{G}_0$-closed singular vector $\alpha\Psi_1+\beta\Psi_2$ is contained in the space spanned by Ψ_1 and Ψ_2, which is therefore not a no-label singular vector space. Therefore, linearly independent singular vectors of type $\Psi^n_{m,|\Delta,q\rangle^M}$ imply linearly independent singular vectors of type $\Psi^{G,n+1}_{m,|\Delta,q\rangle^M}$. The converse is not true, however, since most $\mathcal{G}_0$-closed singular vectors are not generated by the action of $\mathcal{G}_0$ on a no-label singular vector (in fact there are many more $\mathcal{G}_0$-closed singular vectors than no-label singular vectors, as was shown in Ref. 19). Hence, the dimension for the space of singular vectors $\Psi^n_{m,|\Delta,q\rangle^M}$ is limited by the dimension for the space of singular vectors $\Psi^{G,n+1}_{m,|\Delta,q\rangle^M}$. Similarly, $\mathcal{Q}_0\Psi^n_{m,|\Delta,q\rangle^M}$ is a singular vector of type $\Psi^{Q,n-1}_{m,|\Delta,q\rangle^M}$. This again restricts the dimension for $\Psi^n_{m,|\Delta,q\rangle^M}$ to be less or equal to the dimension of $\Psi^{Q,n-1}_{m,|\Delta,q\rangle^M}$.

Theorem 8.C *For no-label singular vectors in $\mathcal{V}^G_{\Delta,q}$ or in $\mathcal{V}^Q_{\Delta,q}$, $c\neq 3$, we find the following upper limits for the dimensions of singular vector spaces at level $m\in\mathbf{N}$ and with charge $n\in\mathbf{Z}$ ($\Delta=-m$ for no-label singular vectors).*

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$\Psi^n_{m,\|-m,q\rangle^G}$	0	1	1	0	0
$\Psi^n_{m,\|-m,q\rangle^Q}$	0	0	1	1	0

TAB. p *Maximal dimensions for spaces of no-label singular vectors in $\mathcal{V}^G_{\Delta,q}$ or in $\mathcal{V}^Q_{\Delta,q}$.*

Singular vectors can only exist if they contain in their normal form at least one non-trivial term of the corresponding ordering kernel of theorem 5.C or 5.E. Charges n that are not given have dimension 0 and hence do not allow any singular vectors.

We now use the ordering kernels of section 6 and section 7 for chiral and no-label Verma modules. Again, simply by counting the number of elements in the ordering kernels one obtains the corresponding dimensions, given in the tables that follow.

Theorem 8.D *For singular vectors in chiral Verma modules $\mathcal{V}^{GQ}_{0,q}$ or in no-label Verma modules $\mathcal{V}_{0,q}$, $c\neq 3$, we find the following upper limits for the number of linearly independent singular vectors at the same level $m\in\mathbf{N}_0$ and with the same charge $n\in\mathbf{Z}$.*

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$\Psi^{G,n}_{m,\|0,q\rangle^{GQ}}$	0	0	1	1	0
$\Psi^{Q,n}_{m,\|0,q\rangle^{GQ}}$	0	1	1	0	0
$\Psi^{GQ,n}_{0,\|0,q\rangle^{GQ}}$	0	0	0	0	0
$\Psi^{n}_{0,\|0,q\rangle^{GQ}}$	0	0	0	0	0

TAB. q *Maximal dimensions for singular vectors spaces in $\mathcal{V}^{GQ}_{0,q}$.*

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$\Psi^{G,n}_{m,\|0,q\rangle}$	0	1	3	3	1
$\Psi^{Q,n}_{m,\|0,q\rangle}$	1	3	3	1	0
$\Psi^{GQ,n}_{0,\|0,q\rangle}$	0	0	1	0	0
$\Psi^{n}_{0,\|0,q\rangle}$	0	0	0	0	0

TAB. r *Maximal dimensions for singular vectors spaces in $\mathcal{V}_{0,q}$.*

Singular vectors can only exist if they contain in their normal form at least one non-trivial term of the corresponding ordering kernel of theorem 6.A *or* 7.B. *Charges n that are not given have dimension* 0 *and hence do not allow any singular vectors.*

Tables TAB. o, TAB. p, TAB. q and TAB. r prove the conjecture made in Ref. 19 about the possible existing types of topological singular vectors. Namely, using the algebraic mechanism denoted *the cascade effect* it was deduced (although not rigorously) the existence of 4 types of singular vectors in chiral Verma modules (the ones given in TAB. q), and 29 types in complete Verma modules (the ones given in tables TAB. o, TAB. p and TAB. r). In addition, low level examples were constructed for all these types of singular vectors what proves that all these types do exist (already at level 1, in fact, except the type $\Psi^{GQ,n}_{0,|0,q\rangle}$ in no-label Verma modules that only exists at level 0).

We ought to mention that the dimensions given in the previous three theorems are consistent with the *spectral flow box diagrams* analysed in Refs. 15, 18, 19. Namely, types of singular vectors that are connected by the *topological* spectral flow automorphism $\mathcal{A}$ always show the same singular vector space dimensions[v].

Finally let us consider the results of TAB. o in more detail for the case when the conformal weight Δ is a negative integer: $\Delta = -m \in -\mathbf{N}_0$. In this case, we easily find for each singular vector $\Psi^{G,n}_{m,|-m,q\rangle^G}$ (which has zero conformal weight) a companion $\mathcal{Q}_0\Psi^{G,n}_{m,|-m,q\rangle^G}$ which is of chiral type $\Psi^{GQ,n-1}_{m,|-m,q\rangle^G}$. Note that $\mathcal{Q}_0\Psi^{G,n}_{m,|-m,q\rangle^G}$ cannot be trivial. It is rather a *secondary* singular vector at level 0 with respect to the singular vector $\Psi^{G,n}_{m,|-m,q\rangle^G}$. Using the same arguments as for the no-label singular vectors of TAB. p we obtain that the dimension for $\Psi^{G,n}_{m,|-m,q\rangle^G}$ is restricted by the dimension for $\Psi^{GQ,n-1}_{m,|-m,q\rangle^G}$. Similarly, we can act with $\mathcal{G}_0$ on $\Psi^{Q,n}_{m,|-m,q\rangle^G}$ in order to obtain a secondary singular vector of chiral type $\Psi^{GQ,n+1}_{m,|-m,q\rangle^G}$. The same statements are true for Verma modules of type $\mathcal{V}^Q_{\Delta,q}$. We hence obtain the following theorem.

Theorem 8.E *For singular vectors at level $m \in \mathbf{N}$ and with charge $n \in \mathbf{Z}$ in $\mathcal{V}^G_{\Delta,q}$ or in $\mathcal{V}^Q_{\Delta,q}$, with $\Delta = -m$ and $c \neq 3$, we find the following maximum dimensions for singular vector spaces.*

[v] $\mathcal{A}$ is the universal odd spectral flow[17, 16, 19], discovered in Ref. 17, which transform any topological singular vector into another topological singular vector; in particular $\mathcal{A}$ transforms chiral singular vectors into chiral singular vectors and no-label singular vectors into no-label singular vectors.

	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
$\Psi^{G,n}_{m,\|-m,q\rangle^G}$	0	0	1	1	0
$\Psi^{Q,n}_{m,\|-m,q\rangle^G}$	1	1	0	0	0
$\Psi^{Q,n}_{m,\|-m,q\rangle^Q}$	0	1	1	0	0
$\Psi^{G,n}_{m,\|-m,q\rangle^Q}$	0	0	0	1	1

TAB. s *Maximal dimensions for spaces of singular vectors at level m in $\mathcal{V}^G_{\Delta,q}$ or in $\mathcal{V}^Q_{\Delta,q}$ with $\Delta = -m$. Charges n that are not given have dimension 0 and hence do not allow any singular vectors.*

The results of TAB. s imply that if, for example, there are two linearly independent singular vectors at level m with charge 0 in $\mathcal{V}^G_{\Delta,q}$, with $\Delta = -m$, both annihilated by $\mathcal{G}_0$, then there exists a non-trivial linear combination of the two singular vectors that turns out to be a chiral singular vector (annihilated by $\mathcal{G}_0$ *and* $\mathcal{Q}_0$). The space spanned by these two singular vectors hence decomposes into a one-dimensional $\mathcal{G}_0$-closed singular vector space and a one-dimensional chiral singular vector space (see Ref. 19 for examples at level 3).

Some remarks are now in order concerning the existence of the considered spaces of $N = 2$ singular vectors. First of all observe that the dimensions given by tables TAB. o - TAB. s are the maximal possible dimensions for the spaces generated by singular vectors of the corresponding types. That is, dimension 2 for a given type of singular vector in TAB. o does not mean that all the spaces generated by singular vectors of such type are two-dimensional. Rather, most of them are in fact one-dimensional and only under certain conditions one finds two-dimensional spaces. The same applies to the three-dimensional spaces in TAB. r. To be more precise, in Ref. 9 it was proved that for the Neveu-Schwarz $N = 2$ algebra two-dimensional spaces exist only for uncharged singular vectors and under certain conditions, starting at level 2. For the topological $N = 2$ algebra this implies, as was shown in Ref. 19, that the four types of two-dimensional singular vector spaces of TAB. o must also exist starting at level 2, provided the corresponding conditions are satisfied. (To see this[19, 15] one only needs to apply the topological twists to the singular vectors of the Neveu-Schwarz $N = 2$ algebra and then construct the box-diagrams using $\mathcal{G}_0$, $\mathcal{Q}_0$ and the odd spectral flow automorphism $\mathcal{A}$). As a matter of fact, also in Ref. 19 several examples of these two-dimensional spaces were constructed at level 3.

For the case of the three-dimensional singular vector spaces in no-label Verma modules in TAB. r, we do not know as yet of any conditions for them to exist. In fact these are the only spaces, among all the spaces given in tables TAB. o - TAB. s, which have not been observed so far, although the corresponding types of singular vectors have been constructed at level 1 generating one-dimensional[19] as well as two-dimensional spaces[w] (but not three-dimensional). The latter case is interesting, in addition, because the corresponding two-dimensional spaces exist already at level 1 (in contrast with the two-dimensional spaces given by the conditions of Ref. 9, which exist at levels 2 and higher). Namely, for $c = 9$ one can easily find two-dimensional spaces of singular vectors of types $\Psi^{G,0}_{1,|0,-3\rangle}$ and $\Psi^{Q,-1}_{1,|0,-3\rangle}$, in the no-label Verma module $\mathcal{V}_{0,-3}$, and two-dimensional spaces of singular vectors of types $\Psi^{G,1}_{1,|0,0\rangle}$ and $\Psi^{Q,0}_{1,|0,0\rangle}$, in the no-label Verma module $\mathcal{V}_{0,0}$, all four types of singular vectors belonging to the same box-diagram[15, 18, 19]. That is, the spectral flow automorphism $\mathcal{A}$, transforming the Verma modules $\mathcal{V}_{0,-3}$ and $\mathcal{V}_{0,0}$ into each other, map $\Psi^{G,0}_{1,|0,-3\rangle}$

[w] In Ref. 19 the existence of these two-dimensional spaces of singular vectors at level 1 was overlooked. We give examples of them here for the first time.

to $\Psi^{Q,0}_{1,|0,0\rangle}$ and $\Psi^{G,1}_{1,|0,0\rangle}$ to $\Psi^{Q,-1}_{1,|0,-3\rangle}$, and the other way around, whereas $\mathcal{G}_0$ and $\mathcal{Q}_0$ transform the singular vectors into each other inside a given Verma module. One of these two-dimensional spaces is, for example, the space spanned by the singular vectors of type $\Psi^{Q,0}_{1,|0,0\rangle}$:

$$\Psi^{Q,0}_{1,|0,0,c\rangle} \quad = \quad \mathcal{L}_{-1}\mathcal{Q}_0\mathcal{G}_0\,|0,0,c\rangle \tag{83}$$

and

$$\hat{\Psi}^{Q,0}_{1,|0,0,9\rangle} \quad = \quad [\mathcal{H}_{-1}\mathcal{Q}_0\mathcal{G}_0+\mathcal{Q}_{-1}\mathcal{G}_0-2\mathcal{Q}_0\mathcal{G}_{-1}]\,|0,0,9\rangle\ , \tag{84}$$

the latter existing only for $c=9$. As one can see, the canonical projections of the first singular vector into the generic Verma modules $\mathcal{V}^G_{0,0}$ and $\mathcal{V}^Q_{0,0}$ vanish. On the contrary, the canonical projections of the second singular vector into the generic Verma modules $\mathcal{V}^G_{0,0}$ and $\mathcal{V}^Q_{0,0}$ are different from zero, giving rise to the singular vectors

$$\Psi^{Q,0}_{1,|0,0,9\rangle^G}=\mathcal{Q}_0\mathcal{G}_{-1}]\,|0,0,9\rangle^G \quad , \quad \Psi^{Q,0}_{1,|0,0,9\rangle^Q}=[\mathcal{Q}_{-1}\mathcal{G}_0-4\mathcal{L}_{-1}]\,|0,0,9\rangle^Q\ , \tag{85}$$

respectively. These are, in turn, the particular cases for ($\Delta=0$, $q=0$, $c=9$) of the general expressions for $\Psi^{Q,0}_{1,|\Delta,q\rangle^G}$ and $\Psi^{Q,0}_{1,|\Delta,q\rangle^Q}$, given in Ref. 19.

9 Dimensions for the Neveu-Schwarz and the Ramond $N=2$ algebras

Transferring the dimensions we have found for the topological $N=2$ algebra to the Neveu-Schwarz and to the Ramond $N=2$ algebras is straightforward. The Neveu-Schwarz $N=2$ algebra is related to the topological $N=2$ algebra through the topological twists $T^{\pm}_W$: $\mathcal{L}_m=L_m\pm1/2H_m$, $\mathcal{H}_m=\pm H_m$, $\mathcal{G}_m=G^{\pm}_{m+1/2}$ and $\mathcal{Q}_m=G^{\mp}_{m-1/2}$. As a consequence, the (non-chiral) Neveu-Schwarz highest weight vectors correspond to $\mathcal{G}_0$-closed topological highest weight vectors (annihilated by $\mathcal{G}_0$), whereas the chiral and antichiral Neveu-Schwarz highest weight vectors (annihilated by $G^+_{-1/2}$ and by $G^-_{-1/2}$, respectively), correspond to chiral topological highest weight vectors (annihilated by $\mathcal{G}_0$ and $\mathcal{Q}_0$). This implies (see the details in Refs. 18, 19) that the standard Neveu-Schwarz singular vectors ('normal', chiral and antichiral) correspond to topological singular vectors of the types $\Psi^{G,n}_{m,|\Delta,q\rangle^G}$ and $\Psi^{GQ,n}_{m,|\Delta,q\rangle^G}$, whereas the Neveu-Schwarz singular vectors in chiral or antichiral Verma modules correspond to topological singular vectors of only the type $\Psi^{G,n}_{m,|\Delta,q\rangle^{GQ}}$ (as there are no chiral singular vectors in chiral Verma modules). To be precise, for the singular vectors of the Neveu-Schwarz $N=2$ algebra one finds the following results.

Theorem 9.A *The spaces of Neveu-Schwarz singular vectors $\Psi^n_{m,|\Delta,q\rangle}$, $\Psi^{ch,n}_{m,|\Delta,q\rangle}$ and $\Psi^{a,n}_{m,|\Delta,q\rangle}$, where the supercripts ch and a stand for chiral and antichiral, respectively, have the same maximal dimensions as the spaces of topological singular vectors $\Psi^{G,\pm n}_{m\pm n/2,|\Delta\pm q/2,\pm q\rangle^G}$ and $\Psi^{GQ,\pm n}_{m\pm n/2,|-m\mp n/2,\pm q\rangle^G}$, given in* TAB. o. *Therefore for singular vectors in Neveu-Schwarz Verma modules $\mathcal{V}^{NS}_{\Delta,q}$ we find the following upper limits for the number of linearly independent singular vectors at the same level m and with the same charge $n\in\mathbf{Z}$ ($m\in\mathbf{N}$ for $n=0$ while $m\in\mathbf{N}-\mathbf{1/2}$ for $n=\pm1$).*

	$n=-1$	$n=0$	$n=1$
$\Psi^{n}_{m,\|\Delta,q\rangle}$	1	2	1
$\Psi^{ch,n}_{m,\|\Delta,q\rangle}$	0	1	1
$\Psi^{a,n}_{m,\|-m,q\rangle}$	1	1	0

TAB. t *Maximal dimensions for singular vectors spaces in* $\mathcal{V}^{NS}_{\Delta,q}$.

(Chiral and antichiral singular vectors satisfy $\Delta + m = \pm 1/2(q+n)$, respectively). Charges n that are not given have dimension 0 and hence do not allow any singular vectors.

Theorem 9.B *The spaces of Neveu-Schwarz singular vectors $\Psi^{n}_{m,|\Delta,q\rangle^{ch}}$, $\Psi^{n}_{m,|\Delta,q\rangle^{a}}$ $\Psi^{a,n}_{m,|\Delta,q\rangle^{ch}}$ and $\Psi^{ch,n}_{m,|\Delta,q\rangle^{a}}$ in chiral or antichiral Verma modules, $\mathcal{V}^{NS,ch}_{\Delta,q}$ or $\mathcal{V}^{NS,a}_{\Delta,q}$ with $\Delta = \pm q/2$ respectively, have the same maximal dimensions as the spaces of topological singular vectors $\Psi^{G,\pm n}_{m\pm n/2,|0,\pm q\rangle^{GQ}}$ in chiral topological Verma modules, given in* TAB. q. *Therefore for Neveu-Schwarz singular vectors in chiral or antichiral Verma modules we find the following upper limits for the number of linearly independent singular vectors at the same level m and with the same charge $n \in \mathbf{Z}$ ($m \in \mathbf{N}$ for $n = 0$ while $m \in \mathbf{N} - \mathbf{1/2}$ for $n = \pm 1$). The supercripts ch and a stand for chiral and antichiral, respectively.*

	$n=-1$	$n=0$	$n=1$
$\Psi^{n}_{m,\|q/2,q\rangle^{ch}}$	1	1	0
$\Psi^{a,n}_{m,\|q/2,q\rangle^{ch}}$	1	1	0
$\Psi^{n}_{m,\|-q/2,q\rangle^{a}}$	0	1	1
$\Psi^{ch,n}_{m,\|-q/2,q\rangle^{a}}$	0	1	1

TAB. u *Maximal dimensions for singular vectors spaces in* $\mathcal{V}^{NS,ch}_{q/2,q}$ *and* $\mathcal{V}^{NS,a}_{-q/2,q}$.

($\Psi^{a,n}_{m,|q/2,q\rangle^{ch}}$ and $\Psi^{ch,n}_{m,|-q/2,q\rangle^{a}}$ satisfy in addition $q = \mp m - n/2$, respectively). Charges n that are not given have dimension 0 and hence do not allow any singular vectors.

Observe that there are no chiral singular vectors in chiral Verma modules, neither antichiral singular vectors in antichiral Verma modules; that is, there are no Neveu-Schwarz singular vectors of types $\Psi^{ch,n}_{m,|q/2,q\rangle^{ch}}$ and $\Psi^{a,n}_{m,|-q/2,q\rangle^{a}}$, which would correspond to the non-existing chiral topological singular vectors $\Psi^{GQ,\pm n}_{m\pm n/2,|0,\pm q\rangle^{GQ}}$ in chiral topological Verma modules.

The first row of TAB. t recovers the results already proven[x] in Refs. 8, 9, using adapted orderings in generalised (analytically continued) Verma modules. That is, in complete Verma modules of the Neveu-Schwarz $N = 2$ algebra singular vectors can only exist with charges $n = 0, \pm 1$ and, under certain conditions, there exist two-dimensional spaces of (only) uncharged singular vectors. TAB. u proves the conjecture, made in Refs. 18, 19, that in chiral Neveu-Schwarz Verma modules $\mathcal{V}^{NS,ch}_{q/2,q}$ the charged singular vectors are always negatively charged, with $n = -1$, whereas in antichiral Neveu-Schwarz Verma modules $\mathcal{V}^{NS,a}_{-q/2,q}$ the charged singular vectors are always positively charged,

[x]In Ref. 9 it had not been explicitly stated that the results do not hold for $c = 3$. Also, the necessity of the change of basis in the final consideration of the proof of theorem 5.C had been overlooked in Ref. 9. Nevertheless, theorem 9.A shows that all the results of Ref. 9 do hold.

with $n = 1$. In contrast to this, the chiral charged singular vectors in the Verma modules $\mathcal{V}_{\Delta,q}^{NS}$ and $\mathcal{V}_{-q/2,q}^{NS,a}$ are always positively charged, with $n = 1$, whereas the antichiral charged singular vectors in the Verma modules $\mathcal{V}_{\Delta,q}^{NS}$ and $\mathcal{V}_{q/2,q}^{NS,ch}$ are always negatively charged, with $n = -1$. This fact was observed also in Ref. 19 and can be deduced from the results of Ref. 9.

As to the Ramond $N = 2$ algebra, combining the topological twists $T_W^{\pm}$ and the spectral flows it is possible to construct a one-to-one mapping between every Ramond singular vector and every topological singular vector, at the same levels and with the same charges[y] (see the details in Ref. 12). As a consequence, the results of tables Tab.o - Tab.s can be transferred to the Ramond singular vectors simply by exchanging the labels $G \to (+)$, $Q \to (-)$, where the helicity $(+)$ denote the Ramond states annihilated by G_0^+ and the helicity $(-)$ denote the Ramond states annihilated by G_0^-. The *no-helicity* Ramond states, analogous to the *no-label* topological states, have been overlooked until recently in the literature (see Refs. 11, 12). They require conformal weight $\Delta + m = c/24$ in the same way that no-label states require zero conformal weight $\Delta + m = 0$.

10 Conclusions and prospects

For the study of the highest weight representations of a Lie algebra or a Lie super algebra, the determinant formula plays a crucial rôle. However, the determinant formula does not give the complete information about the submodules existing in a given Verma module. Exactly which Verma modules contain proper submodules and at which level can be found the lowest non-trivial grade space of the biggest proper submodule is the information that may easily be obtained from the determinant formula. But it does not give a proof that the singular vectors obtained in that way are all the existing singular vectors, i.e. generate the biggest proper submodules, neither does it give the dimensions of the singular vector spaces. However, it has been shown[9, 19] that singular vector spaces with more than one dimension exist already for the $N = 2$ superconformal algebra.

In this paper we have presented a method that can easily be applied to many Lie algebras and Lie super algebras. This method is based on the concept of adapted ordering, which implies that any singular vector needs to contain at least one non-trivial term included in the ordering kernel. The size of the ordering kernel therefore limits the dimension of the corresponding singular vector space. Weights for which the ordering kernel is trivial do not allow any singular vectors in the corresponding weight space. On the other hand, non-trivial ordering kernels give us the maximal dimension of a possible singular vector space. The framework can easily be understood using the example of the Virasoro algebra where the ordering kernel always has size one and therefore Virasoro singular vectors at the same level in the same Verma module are always proportional. In its original version, Kent[25] used the idea of an ordering for generalised (analytically extended) Virasoro Verma modules in order to show that all vectors satisfying the highest weight conditions at level 0 are proportional to the highest weight vector.

As an important application of this method, we have computed the maximal dimensions of the singular vector spaces for Verma modules of the topological $N = 2$ algebra, obtaining maximal dimensions 0, 1, 2 or 3, depending on the type of Verma module and the type of singular vector. The results are consistent with the topological spectral flow automorphisms and with all known examples of topological singular vectors. On the one hand, singular vector spaces with maximal

[y]We define the charges for the Ramond states in the same way as for the Neveu-Schwarz states, see the details in Refs. 18, 11.

dimension bigger than 1 agree with explicilty computed examples found before (and during) this work, although in the case of the three-dimensional spaces in no-label Verma modules, the singular vectors of the corresponding types known so far generate only one and two-dimensional spaces. These exist already at level 1, in contrast with the previously known two-dimensional spaces, which exist at levels 2 and higher. On the other hand, singular vector spaces with zero dimension imply that the 'would-be' singular vectors of the corresponding types do not exist. As a consequence, our results provide a rigorous proof to the conjecture made in Ref. 19 about the possible existing types of topological singular vectors: 4 types in chiral Verma modules and 29 types in complete Verma modules.

Finally we have transferred the results found for the topological $N = 2$ algebra to the Neveu-Schwarz and to the Ramond $N = 2$ algebras. In the first case we have recovered the results obtained in Ref. 9 for complete Verma modules: maximal dimensions 0, 1 or 2, the latter only for uncharged singular vector spaces, and allowed charges only 0 and ± 1. In addition, we have proved the conjecture made in Refs. 18, 19 on the possible existing types of Neveu-Schwarz singular vectors in chiral and antichiral Verma modules. In the case of the Ramond $N = 2$ algebra we have found a one-to-one mapping between the Ramond singular vectors and the topological singular vectors, so that the corresponding results are essentially the same.

The only exception for which the adapted orderings presented in this paper are not suitable is for central term $c = 3$. This case needs a separate consideration. The application of the adapted ordering method to the twisted $N = 2$ algebra will be the subject of a forthcoming paper.

The example of the $N = 2$ topological Verma modules is only one out of many cases where the concept of adapted orderings can be applied. For example, Bajnok[4] showed that the *analytic continuation* method of Kent[25] can be extended to generalised Verma modules of the WA_2 algebra. Not only will the concept of adapted orderings allow us to obtain information about superconformal Verma modules with $N > 2$, it should also be easily applicable to any other Lie algebra whenever an adapted ordering can be constructed with small ordering kernels.

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