Cosmological Models from Quintessence

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A generalized quintessence model is presented which corresponds to a richer vacuum structure that, besides a time-dependent, slowly varying scalar field, contains a varying cosmological term. From first principles we determine a number of scalar-field potentials that satisfy the constraints imposed by the field equations and conservation laws, both in the conventional and generalized quintessence models. Besides inverse-power law solutions, these potentials are given in terms of hyperbolic functions or the twelve Jacobian elliptic functions, and are all related to the luminosity distance by means of a integral equation. Integration of this equation for the different solutions leads to a large family of cosmological models characterized by luminosity distance-redshift relations. Out of such models, only four appear to be able to predict a required accelerating universe conforming to observations on supernova Ia, at large or moderate redshifts.

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I. ALLOWED POTENTIALS FOR QUINTESSENCE

Quintessence has been recently invoked [1] as an advantageous alternative to the cosmological constant in order to explain the apparent accelerating expansion of the universe which has stirred cosmologists after observations and measurements of distant supernovae [2-4]. The existence of a quintessential field has been related to supersymmetric models [5], the problem of fine-tuning of the cosmological constant [6], or supergravity models [7]. The bare standard cosmological model (BSCM), without any constant cosmological term or vacuum fields, predicts the existence of an expanding universe which can be closed, open or flat, but always decelerating. However, if a positive cosmological constant is added to the field equations, then the expansion of the universe may become accelerating. Actually, as early as 1975, Gunn and Tinsley, while discussing observations on the Hubble diagram and constraints on the matter density of the universe and ages of galaxies, found [8] a series of allowable, if not compelling, cosmological models with nonzero, positive cosmological constant, which were accelerating. Recent observations [2,3] on distant supernovas have resurrected the spirit of these early conclusions and led to the strong suspect that, in spite of the fact that the BSCM gives satisfactory explanations to many other observations, it probably is incomplete or even incorrect [9].

When the cosmological constant, Λ, is interpreted as the energy density of vacuum for an equation of state $p = -\rho$, if it is positive, then most inflationary models are suitably pinpointed. However, it is largely known that Λ is not free of fundamental problems [10]. Actually, the so-called cosmological constant problem is one of the most challenging questions in fundamental physics, as it is very hard to envisage any consistent mechanism that dynamically explains how the vacuum energy density can be lowered from its most natural value at around the Planck scale down to its observationally allowed value, $\epsilon \leq 10^{-47}$ GeV^4. Although quintessence models do not solve this problem, they may improve the related fine-tuning problem in the sense that [6] they can explain a tiny value for the vacuum energy density with a scale comparable with the scales of high energy physics. Besides, these models give rise to an accelerating universe by using a vacuum dynamically adjustable, time-dependent scalar field that is spatially (in-)homogeneous and evolves slowly enough so that the kinetic term of the energy density is always smaller than the potential energy term. It is worth noticing that this is not necessarily required in tracker models of quintessence (see e.g Ref. [15]). Indeed, in the case of an overshoot [15] the kinetic energy dominates at high redshift. If we disregard such tracker models, the resulting negative pressure will then correspond to an equation of state [1] $p = \omega\rho$ where the free-parameter ω can take on any values $0 \geq \omega > -1$. Thus, the cosmological constant will correspond to the extreme case $\omega = -1$.

Recently, however, di Prieto and Demaret have shown [11] that, if we restrict ourselves to a constant equation of state none of the vacuum scalar-field potentials, $V(\phi)$, currently used in quintessence models, such as the exponential [12], cosine [13] and inverse power-law [12] potentials can satisfy the constraint on $V(\phi)$ implied by field equations and conservation laws, i.e. [11]

$$\frac{V'}{V'_0} =$$

$$\pm \left[\Omega_\phi \left(\frac{V}{V_0} \right)^2 + \Omega_M \left(\frac{V}{V_0} \right)^{\frac{\omega+2}{\omega+1}} + \Omega_k \left(\frac{V}{V_0} \right)^{\frac{3\omega+5}{3(\omega+1)}} \right]^{\frac{1}{2}},$$

$$(1.1)$$

where V_0 and V_0' are the current values of the scalar-field potential, $V(\phi)$, and its derivative with respect to the field, $V' = dV(\phi)/d\phi$, respectively, and the Ω_i's (with $i = \phi, M, k$) are the dimensionless density parameters for the scalar field, ordinary matter and topological curvature. Apart from a solution for any ω in the flat case, Pietro and Demaret were nonetheless able to find [11] some solutions to constraint (1.1) for particular values of the parameter ω. Thus, for $\omega = -1/3$, they obtained

$$V(\phi) = V_0 \left\{ \frac{\sqrt{2}}{4\epsilon_0} \sinh \left[\pm \epsilon_0 (\phi - \phi_0) + \nu_0 \right] \right\}^{-4}, \quad (1.2)$$

where

$$\epsilon_0 = \frac{1}{2} \sqrt{\frac{\Omega_\phi + \Omega_k}{2\Omega_M}}, \quad \nu_0 = \arcsin \left(2\sqrt{2}\epsilon_0 \right),$$

and, for $\omega = -2/3$,

$$V(\phi) = V_0 \left\{ \frac{\Omega_M}{\Omega_\phi} \sinh \left[\pm (\phi - \phi_0) + \delta_0 \right] \right.$$

$$\left. + \frac{\Omega_k}{4\Omega_\phi} \left(\frac{\Omega_k}{\Omega_M} e^{\mp(\phi-\phi_0)-\sigma_0} - 2 \right) \right\}^{-\frac{1}{2}}, \quad (1.3)$$

where

$$e^{\sigma_0} = \frac{2\Omega_\phi + 2\sqrt{\Omega_\phi} + \Omega_k}{\Omega_M}$$

$$\delta_0 = \sigma_0 + \frac{1}{2} \ln \left(\frac{\Omega_M}{4\Omega_\phi} \right).$$

We furthermore note that, besides solutions (1.2) and (1.3), there are a whole family of scalar potentials $V(\phi)$ defined in terms of the Jacobian elliptic functions [11], J_e, which satisfy the constraint (1.1) for $\omega = -1/6$. Such solution can be generally written as

$$V(\phi) = V_0 \left\{ J_e \left[\alpha_0 (\phi - \phi_0), m \right] \right\}^{-10}, \quad (1.4)$$

in which $\alpha_0 = \alpha_0(\Omega_i)$ is a given constant whose form depends on the particular elliptic function being considered, and $m = m(\Omega_i) \leq 1$ is the characteristic parameter (modulus) [14] of the corresponding elliptic function. For example, taking for J_e the function cn, we have

$$\alpha_0 = \sqrt{\frac{7\Omega_k(\Omega_\phi + 2\Omega_k)}{200\Omega_\phi(\Omega_k + \Omega_\phi)}},$$

$$m = \frac{\Omega_k}{\Omega_\phi + 2\Omega_k},$$

or for the function sd

$$\alpha_0 = \sqrt{\frac{7\Omega_M}{200\Omega_\phi}},$$

$$m = \frac{1}{2} \left(1 + \frac{2\Omega_\phi^2}{7} \right),$$

and similar, but distinct expressions of α_0 and m for the remaining 10 Jacobian elliptic functions.

It appears clearly of interest to investigate whether the new potentials (1.2)-(1.4) are actually able to predict accelerating cosmological models which can match recently obtained data from observations on distant supernova Ia, discussing their physical relevance as well. It is the aim of this paper to carry out such an investigation, incorporating other possible new solutions from a generalized quintessence model which simultaneously accommodates both a vacuum scalar field ϕ and a varying cosmological term Λ. In this paper, we shall restrict ourselves to equations of state with a constant ω, both in conventional and generalized quintessence models, disregarding tracker models [15], where time varying equations of state are invoked and a general inverse-power law potential for the quintessence field is assumed.

II. GENERALIZED QUINTESSENCE MODEL

The field equations corresponding to a Friedmann-Robertson-Walker spacetime with ordinary matter which is not coupled to a homogeneous (quintessence) scalar field ϕ, to which we add a varying cosmological term Λ can be written as

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{1}{3}\kappa^2 \left(\rho_\phi + \rho_M + \rho_\Lambda \right) \quad (2.1)$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \kappa^2 \left(p_\Lambda - p_\phi \right) \quad (2.2)$$

$$\ddot{\phi} + 3\dot{\phi}\frac{\dot{R}}{R} = -V', \quad (2.3)$$

where the overhead dot means time derivative, $' \equiv d/d\phi$, $\kappa^2 = 8\pi G_N$, k is the topological curvature and we have defined the scalar field such that

$$\kappa^2 \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (2.4)$$

$$\kappa^2 p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (2.5)$$

As usual, the ordinary matter is assumed to obey the equation of state for an ordinary fluid, $p_M = 0$. As pointed out before, the scalar quintessence field will be assumed to behave like a perfect fluid with equation of state [1]

$$p_\phi = \omega \rho_\phi, \quad -1 < \omega \le 0. \tag{2.6}$$

The generalization implied by the quintessence field with respect to the case of a pure cosmological constant can be manisfested by noting that the particular value of the constant parameter $\omega = -1$ corresponds to the cosmological constant case when the field ϕ becomes a constant as well [16].

The conservation laws that the involved fields are here assumed to satisfy are as follows. First of all, we note that, since there is no interaction between the scalar field and the other fields involved, we can take all these laws as separable from each other. For ordinary matter, M, and scalar field, ϕ, we should then have for all values of ω, except $\omega = -1$ [11]

$$\rho_M = \rho_{M0} \left(\frac{R_0}{R}\right)^3, \quad \rho_\phi = \rho_{\phi 0} \left(\frac{R_0}{R}\right)^{3(1+\omega)}, \tag{2.7}$$

with the subscript 0 taken to always mean current value. As to the varying cosmological term Λ, we generally assume $\kappa^2 \rho_\Lambda = \Lambda = \Lambda_0 (R_0/R)^n$, where n can, in principle, take on the values 1, 2 and 3. However, only the value $n = 1$ corresponds to a model with additional dynamical content relative to the constant-ω usual models, the cases $n = 2$ and $n = 3$ just reducing to the Pietro-Demaret model [11] with the cosmological dimensionless parameter (see below) Ω_k replaced for $\Omega_k + \Omega_\Lambda$ and Ω_M replaced for $\Omega_M + \Omega_\Lambda$, respectively. We then take for the most general conservation law for Λ

$$\kappa^2 \rho_\Lambda \equiv \Lambda = \Lambda_0 \left(\frac{R_0}{R}\right). \tag{2.8}$$

In any event, however, Λ can be taken to represent the energy density of the field ϕ corresponding to a particular value of parameter ω ($\omega = -2/3$ in the chosen conservation law (2.8)), so that if we would allow both signs for Λ and ρ_ϕ then keeping simultaneously Λ and ρ_ϕ in the field equations would just be redundant. Nevertheless, one still can consistently consider field equations with Λ and ρ_ϕ simultaneously provided that we restrict their values to be either (i) $\Lambda > 0$, $\rho_\phi < 0$, or (ii) $\Lambda < 0$, $\rho_\phi > 0$. In what follows, we shall confine ourselves to just case (i), looking at the quantity $\rho_v \equiv \rho_\Lambda + \rho_\phi$ as the total vacuum energy which will always be taken to be $\rho_v \ge 0$. Bearing in mind such a restriction, we choose Eqs. (2.7) and (2.8) as our conservation laws and introduce then the following dimensionless cosmological parameters [11]

$$\Omega_k = -\frac{k}{R_0^2 H_0^2}, \quad \Omega_\phi = \frac{\kappa^2 \rho_{\phi 0}}{3H_0^2}, \quad \Omega_M = \frac{\kappa^2 \rho_{M0}}{3H_0^2} \tag{2.9}$$

$$\Omega_\Lambda = \frac{\Lambda_0}{3H_0^2}, \tag{2.10}$$

which should satisfy the *quadrilateral* constraint

$$\Omega_0 = 1 - \Omega_k = \Omega_M + \Omega_\phi + \Omega_\Lambda, \tag{2.11}$$

rather than the constraint implied by the usual cosmological triangle. From the conservation laws and the first field equation, we obtain a differential constraint on the scale factor

$$\dot{R}^2 = R_0^2 H_0^2 \left[\Omega_M \frac{R_0}{R} \right.$$

$$\left. +\Omega_k + \Omega_\phi \left(\frac{R_0}{R}\right)^{1+3\omega} + \Omega_\Lambda \frac{R}{R_0}\right], \tag{2.12}$$

and from the relations between R and V, and $\dot{R}$ and V', derived by di Prieto and Demaret [11], the following generalized constraint on the scalar quintessence potential

$$\left(\frac{V'}{V_0'}\right)^2 = \Omega_M \left(\frac{V}{V_0}\right)^{\frac{\omega+2}{\omega+1}}$$

$$+ \Omega_\phi \left(\frac{V}{V_0}\right)^2 + \Omega_k \left(\frac{V}{V_0}\right)^{\frac{3\omega+5}{3(\omega+1)}} + \Omega_\Lambda \left(\frac{V}{V_0}\right)^{\frac{3\omega+4}{3(\omega+1)}}, \tag{2.13}$$

which has been obtained assuming that $\omega \ne -1$, with

$$V_0 = \frac{3}{2}(1-\omega)H_0^2 \Omega_\phi, \quad V_0' = \pm H_0 \sqrt{\frac{1}{2}(1-\omega^2) V_0}. \tag{2.14}$$

Constraint (2.13) is of course a generalization from constraint (1.1) and reduces to this when we set $\Omega_\Lambda = 0$.

Although again the exponential [12] and cosine [13] potentials cannot satisfy the constraint (2.13) even if we relax the condition of nonclosedness implied by nucleosynthesis and supernova observations, we note that there exist some inverse-power law potentials, similar to those used in the literature [12], which satisfy our field equations and conservation laws. Thus, if we set $\Omega_\phi + \Omega_k = \Omega_\Lambda = 0$, and $\Omega_M = 1$, we have as a solution to constraint (2.13)

$$V = V_0 \left(\frac{\phi_0}{\phi}\right)^4,$$

for $\omega = -1/3$. On the other hand, setting $\Omega_\phi + \Omega_\Lambda = \Omega_M = 0$, we obtain another solution

$$V = V_0 \left(\frac{\phi_0}{\phi}\right)^2,$$

for $\omega = -2/3$. Even though they correspond to particular constant values of ω, these two potentials could

still be implemented in the realm of high energy physics. They have the same general form as the Ratra-Peebles potential [12], though this does not actually require that ω be assumed constant. Indeed, the above two potentials may be regarded to belong to a potential family $V = V_0(\phi_0/\phi)^{6(1+\omega)}$ which can be related to the Ratra-Peebles potential by the field transformation $\phi_{RP} \rightarrow \alpha\phi^{1-3\omega}$, with α a suitable dimensional constant. In addition, there are other allowable potentials for the field ϕ which are solution to Eq. (2.13) for particular values of the quintessence parameter ω, without imposing any restriction on the cosmological parameters Ω_j. Thus, for $\omega = -1/3$, we get a family of generalized quintessence potentials which are given in terms of the Jacobian elliptic functions, J_e [14],

$$V(\phi) = V_0 \left\{ J_e \left[\beta_0(\phi - \phi_0) \right], m \right\}^{-4}, \qquad (2.15)$$

where $\beta_0 \equiv \beta_0(\Omega_i)$ and $m \equiv m(\Omega_i) \leq 1$ (with i some elements of the set $\{\phi, \Lambda, M, k\}$) are dimensionless quantities whose explicit shape will depend on the particular function J_e being considered. For reasons which will become clear in the next section, of particular interest for reproducing a suitable accelerating model of the universe are the elliptic functions $J_e = \mathrm{cn}$, for which

$$\beta_0 = \frac{1}{2}\sqrt{\frac{\Omega_M}{\Omega_\phi}\left(1 + \frac{\Omega_\Lambda}{1-\Omega_M}\right)}, \quad m = \frac{\Omega_\Lambda}{1+\Omega_\Lambda-\Omega_M},$$

$J_e = \mathrm{nc}$, for which

$$\beta_0 = \frac{1}{2}\sqrt{\frac{\Omega_M}{\Omega_\phi}\left(-1 + \frac{\Omega_\Lambda}{1-\Omega_M}\right)}, \quad m = \frac{1-\Omega_M}{1+\Omega_\Lambda-\Omega_M},$$

and $J_e = \mathrm{sd}$, for which

$$\beta_0 = \frac{1}{2}\sqrt{\frac{\Omega_M}{\Omega_\phi}}, \quad m = \frac{1}{2}\left(\frac{1-\Omega_\Lambda}{\Omega_M}\right).$$

Note, furthermore, that in the limiting case that $m \rightarrow 1$, the cn-solution becomes a sech-solution for an open universe, the nc-solution becomes a cosh-solution for $\Omega_\Lambda = 0$, whereas the sd-solution reduces to a sinh-solution for $\Omega_\Lambda = 0$ (i.e. the potential first found by Di Prieto and Demaret [11], as it should be expected.)

On the other hand, for $\omega = -2/3$, we obtain the potentials satisfying constraint (2.13):

$$V(\phi) = V_0 \left[A_\pm \sinh(\phi - \phi_0) + B \cosh(\phi - \phi_0) - C \right]^{-1}, \tag{2.16}$$

where

$$A_\pm = \pm\sqrt{\frac{\Omega_M}{\Omega_\phi + \Omega_\Lambda}} \cosh\delta_0' \mp \frac{\Omega_k^2 e^{-\sigma_0'}}{4\Omega_M(\Omega_\phi + \Omega_\Lambda)} \quad (2.17)$$

$$B = \sqrt{\frac{\Omega_M}{\Omega_\phi + \Omega_\Lambda}} \sinh\delta_0' + \frac{\Omega_k^2 e^{-\sigma_0'}}{4\Omega_M(\Omega_\phi + \Omega_\Lambda)} \qquad (2.18)$$

and

$$C = \frac{1}{2}\frac{\Omega_k}{\Omega_\phi + \Omega_\Lambda}, \qquad (2.19)$$

with δ_0' and σ_0' as given by δ_0 and σ_0 in Eq. (1.3), but with Ω_ϕ replaced for $\Omega_\phi + \Omega_\Lambda$. Clearly, for $\Omega_\Lambda = 0$, solutions (2.16) reduce to solutions (1.3).

Finally, we can also have solutions to (2.13) for any ω, satisfying $-1 < \omega \leq 0$. These solution are in turn obtained for particular values of the cosmological parameters Ω_j. Thus, setting $\Omega_k = \Omega_\Lambda = 0$, we have

$$V = V_0 \sinh^{\frac{2(\omega+1)}{\omega}}\left[\pm\left(\frac{\sqrt{\Omega_M}\,\omega}{2(\omega+1)}\right)(\phi - \phi_0)\right], \quad (2.20)$$

and for $\Omega_k = \Omega_M = 0$,

$$V = V_0 \sinh^{\frac{6(\omega+1)}{3\omega+2}}\left[\pm\sqrt{\frac{3\omega+2}{2}}(\phi - \phi_0)\right]. \qquad (2.21)$$

Potentials (2.20) and (2.21) and at least some of the potentials in the family (2.15) can actually be regarded as generalizations from inverse-power law potentials which hold only as approximations for small values of $\phi - \phi_0$, at large values of the redshift (see Sec. III). Let us for example consider the case $Je = sd$ in the family (2.15). For most of its cosmological evolution $\phi - \phi_0$ remains very small at large values of the redshift, so that we can approximate $V \propto (\phi - \phi_0)^{-4}$, except when the potential approaches current values. Moreover, though for quintessence the most interesting models are those where ω is not constant and, in particular, the tracker models [15], one can see that at least some of the good properties of these models may be somehow shared by the potential considered in this paper. Since at least some of our potentials can be approximated as inverse-power law functions of the field containing at least one free parameter, along their primordial evolution these potentials can be implemented in the realm of high energy physics [18] and linked to particle models with dynamical symmetry breaking or nonperturbative effects [19]. On the other hand, it appears that such potentials can also help solving the cosmic coincidence problem [20]. Taking again as an illustrative example the solution $Je = sd$ in Eq. (2.15) we see (see Eq. (3.14)) that if one sets the initial conditions inmediately after inflation, i.e. at a redshift $z \sim 10^{28}$, then $\phi \simeq \phi_0$ initially, and $\phi - \phi_0 \sim 1$ only now, so explaining why the quintessence field begins to dominate now. Tracker models [15] also seem to improve the fine-tuning problem; we hope this to be the case with some of our potentials as well, in particular in those models where the total vacuum energy, $\Omega\phi + \Omega_\Lambda$, is zero, or generally for the reasons discussed in Sec. IV.

At first glance studying the cases $\omega = -1/6, -1/3$ could seem without any physical motivation, as several authors have already shown [21] that quintessence models based on such values should be ruled out. However,

our quintessence approach is based on the idea that the vacuum field is splitted into two parts, one manifested as a varying cosmological constant with positive energy density, and the other, the quintessence field, always having negative energy density. This splitting appears to enlarge the allowed domain of ω-values which are physically relevant and reproduce the wanted accelerating expansion of the universe (see Sec. III). This translates, in particular, in a generalized expression for the deceleration parameter (Eq. (3.7)), according to which no value of ω can be ruled out from the onset.

III. PREDICTING COSMOLOGICAL MODELS FROM QUINTESSENCE POTENTIALS

In this section we are going to check whether the considered solutions satisfying the constraints on the scalar-field potentials (1.1) and (2.13) are suitable potentials to fullfil the quintessence's aim, that is, whether such solutions are able to predict cosmological models matching observations from supernova Ia. More precisely, we want to compute the luminosity distance(D_L)-redshift(z) relations from the quintessence potentials satisfying the constraints. Bringing then that relation, $D_L(z)$, into the magnitude-redshift relation [2],

$$m_B^{\text{eff}} = \check{M}_B + 5\log[D_L(z)], \qquad (3.1)$$

where $\check{M}_B = M_B - 5\log H_0 + 25$ is the Hubble-constant-free B-band absolute magnitude at the maximum of a Ia supernova whose values have to be suitably corrected [2], we could thus directly compare the predictions from the cosmological quintessence potentials with the conveniently corrected observed magnitude. The luminosity distance D_L depends only on the theory we are working on, and can be written as [22]

$$D_L H_0 = \frac{(1+z)}{\sqrt{|\Omega_k|}} S\left\{ \sqrt{|\Omega_k|} \times \right.$$

$$\left. \int_0^z dz' \left[\sum_j \Omega_j (1+z')^{3(1+\alpha_j)} + \Omega_k (1+z')^2 \right]^{-\frac{1}{2}} \right\}, \qquad (3.2)$$

in which $S\{x\} = \sin x$ for $k = +1$, $S\{x\} = x$ for $k = 0$ and $S\{x\} = \sinh x$ for $k = -1$, and the parameter α_i is defined from the energy density so that

$$\rho_j \propto \left(\frac{R_0}{R} \right)^{3(1+\alpha_j)},$$

with the subscript j labelling the distinct nongeometrical contributions, namely $j = M, \phi$ and Λ, in such a way that $\alpha_M = 0$, $\alpha_\phi = \omega$ and $\alpha_\Lambda = -2/3$. Using the definition of

the redshift in terms of the scale factor R, we then attain for the argument of the squared root in the integrand of Eq. (3.2) in case of the generalized quintessence model,

$$\Pi \equiv \Omega_M \left(\frac{R_0}{R} \right)^3$$

$$+ \Omega_\phi \left(\frac{R_0}{R} \right)^{3(1+\omega)} + \Omega_\Lambda \left(\frac{R_0}{R} \right) + \Omega_k \left(\frac{R_0}{R} \right)^2. \qquad (3.3)$$

Inserting now the relation [8] $V/V_0 = (R_0/R)^{3(1+\omega)}$, we obtain for Π:

$$\Pi \equiv \Omega_M \left(\frac{V}{V_0} \right)^{\frac{1}{\omega+1}}$$

$$+ \Omega_\phi \left(\frac{V}{V_0} \right) + \Omega_\Lambda \left(\frac{V}{V_0} \right)^{\frac{1}{3(\omega+1)}} + \Omega_k \left(\frac{V}{V_0} \right)^{\frac{2}{3(\omega+1)}}. \qquad (3.4)$$

It is now readily realized that $\Pi^{-1/2}$ is the same as $V_0' V^{1/2}/V' V_0^{1/2}$ as obtained from the constraint (2.13). Hence,

$$\int_0^z \frac{dz'}{\sqrt{\Pi}} =$$

$$\frac{V_0'}{3(\omega+1)V_0^{\frac{1}{2}+\frac{1}{3(\omega+1)}}} \int_{\phi(0)}^{\phi(z)} \frac{d\phi}{V(\phi)^{\frac{3\omega+1}{6(\omega+1)}}}, \qquad (3.5)$$

and therefore we have the following relation between the luminosity distance and the quintessence potential

$$D_L H_0 =$$

$$\frac{(1+z)}{\sqrt{|\Omega_k|}} S\left\{ \frac{V_0' \sqrt{|\Omega_k|}}{3(\omega+1)V_0^{\frac{1}{2}+\frac{1}{3(\omega+1)}}} \int_{\phi(0)}^{\phi(z)} \frac{d\phi}{V(\phi)^{\frac{3\omega+1}{6(\omega+1)}}} \right\}. \qquad (3.6)$$

On the other hand, the deceleration parameter q_0 can also be expressed in terms of the quintessence parameter ω as follows [22]

$$q_0 = \frac{1}{2} \sum_j \Omega_j (1 + 3\alpha_j) =$$

$$\frac{1}{2} \left[\Omega_M + \Omega_\phi (1 + 3\omega) - \Omega_\Lambda \right]. \qquad (3.7)$$

Thus, in order to reproduce the wanted accelerating behaviour of the universe we must have a suitable combination for the values of the parameters Ω_j and ω, such that the resulting value of q_0 be negative. We note that

for pure quintessence models with $\Omega_\Lambda = 0$, any scalar-field potentials defined for $\omega > -1/3$ could only give a topological accelerating behaviour if $\Omega_\phi < 0$. Clearly, for $\omega = -1/3$, irrespective of the value of Ω_ϕ, we always have $q_0 > 0$, provided $\Omega_M > 0$. In generalized quintessence models with $\Omega_\Lambda > 0$, all the above situations predicting deceleration could still predict acceleration for sufficiently high positive values of Ω_Λ.

Let us consider in what follows the cosmological predictions from the permissible quintessence potentials dealt with in Secs. I and II for the case that $\Omega_M > 0$. We shall start with the family of potentials (1.4) for $\omega = -1/6$ and $\Omega_\Lambda = 0$, in which case Eqs. (3.5) and (3.6) become

$$D_L H_0 =$$

$$\frac{1+z}{\sqrt{|\Omega_k|}} S \left\{ \frac{\sqrt{14|\Omega_k|}}{5\sqrt{\Omega_\phi}} \int_{\phi(0)}^{\phi(z)} d\phi \, J_e \left[\alpha_0 (\phi - \phi_0), m \right] \right\}, \quad (3.8)$$

where we have used the definitions (2.14), and

$$q_0 = \frac{1}{2} \left(\Omega_M + \frac{1}{2}\Omega_\phi \right). \quad (3.9)$$

Substituting the different Jacobian elliptic functions in the luminosity distance expression (3.7) and integrating the resulting expression, it turns out that the only of such functions for which we obtain a consistent, real $D_L - z$ relation predicting a nonclosed universe with positive vacuum energy density which is dynamically accelerating is $J_e = \mathrm{cn}$. In this case the integral in Eq. (3.8) gives

$$\frac{1}{\alpha_0 \sqrt{m}} \arccos \sqrt{1 - m + \frac{m}{\sqrt{1+z'}}} \Bigg|_0^z,$$

where

$$\alpha_0 = \sqrt{\frac{7\Omega_k}{(1-m)\Omega_\phi}}$$

$$m = \frac{\Omega_k}{\Omega_\phi + 2\Omega_k}.$$

For whichever combinations of values of the cosmological parameters Ω_M, Ω_k and Ω_ϕ satisfying the triangular constraint $\Omega_k + \Omega_M + \Omega_\phi = 1$ we then obtain a dynamically accelerating universe which, according to Eq. (3.8), is however topologically decelerating if the vacuum energy density is positive. On the other hand, although all possible resulting $5 \log D_L H_0 - z$ plots give a nearly straight line between $z \simeq 0.01$ and $z \simeq 0.5$ which appear to slightly accelerate thereafter, the full $5 \log D_L H_0$-interval that corresponds to the z-interval of available Ia supernova observations ($\simeq (0.01-1)$) is always around 6, quite smaller than the observed value $\triangle m_B^{\mathrm{eff}} \simeq 13$ [2,3]. Thus, the scalar-field potentials (1.4) cannot conform to the observations on supernovae Ia.

We consider next the potentials for $\omega = -1/3$ which are given by the general expression (2.15) for $\Omega_M > 0$ and total vacuum energy density $\Omega_\phi + \Omega_\Lambda \geq 0$. In these cases, Eqs. (3.5) and (3.6) give

$$D_L H_0 = \frac{1+z}{\sqrt{|\Omega_k|}} S \left\{ \sqrt{\frac{|\Omega_k|}{\Omega_\phi}} \phi(z') \Bigg|_0^z \right\} \quad (3.10)$$

$$q_0 = \frac{1}{2} \left(\Omega_M - \Omega_\Lambda \right). \quad (3.11)$$

For the 12 different Jacobian elliptic functions involved in solutions (2.15) we have derived the expressions of the scalar field in terms of the redshift, $\phi(z)$, in the form of elliptic integrals of the first kind [14]. It turns out that only the elliptic functions cn, nc and sd can generate non closed universes which are both topologically and dynamically accelerating. In the case that $J_e = \mathrm{cn}$, we have for the luminosity distance

$$D_L H_0 = \frac{1+z}{\sqrt{|\Omega_k|}} \times$$

$$S \left\{ 2\sqrt{\frac{|\Omega_k|}{\Omega_M \left(1 + \frac{\Omega_k}{1-\Omega_M} \right)}} \, F \left[\arcsin \sqrt{\frac{z'}{1+z'}}, m \right] \Bigg|_0^z \right\}, \quad (3.12)$$

with

$$m = \frac{\Omega_\Lambda}{1 - \Omega_M + \Omega_\Lambda},$$

and for $J_e = \mathrm{nc}$,

$$D_L H_0 = \frac{1+z}{\sqrt{|\Omega_k|}} \times$$

$$S \left\{ 2\sqrt{\frac{|\Omega_k|}{\Omega_M \left(1 + \frac{\Omega_k}{1-\Omega_M} \right)}} \, F \left[\arcsin(i\sqrt{z'}), m \right] \Bigg|_0^z \right\}, \quad (3.13)$$

with

$$m = \frac{1 - \Omega_M}{1 - \Omega_M + \Omega_\Lambda}.$$

In expressions (3.11) and (3.12) the symbol F denotes elliptic integral of the first kind [14]. These expressions give $5 \log D_L H_0 - z$ plots for different combinations of cosmological parameters satisfying the quadrilateral constraint $\Omega_k + \Omega_M + \Omega_\phi + \Omega_\Lambda = 1$ and conditions $\Omega_k, \Omega_\phi + \Omega_\Lambda \geq 0$ which represent accelerating expansion

with suitably slight deviations from straight lines occurring at $z \geq 0.5$ only. However, again as for the case $\omega = -1/6$, the full variations of $5 \log D_L H_0$ along the z-observation interval $\simeq 6$, are quite smaller than the corresponding value observed in supernovae.

If we take for J_e the function sd, then the z-dependence of the scalar field can be expressed in the form

$$\phi(z') = \phi_0 + 2\sqrt{\frac{\Omega_\phi}{\Omega_M}} F\left[\arcsin \frac{1}{\sqrt{1 + m + z'}}, m\right], \quad (3.14)$$

where

$$m = \frac{1}{2}\left(\sqrt{1 + \frac{4}{\Omega_M}} - 1\right).$$

Inserting the scalar field (3.13) in Eq. (3.10) we get the wanted $D_L - z$ relation. In this case, this relation gives plots which show the required accelerating expansion after $z \geq 0.6$, both for flat and open universes, with a full variation of $5 \log D_L H_0$ along the observed z-interval of the order 13, fitting well with the observations [2,3], along all available z-values.

Finally, for the case $\omega = -2/3$, the relation (3.5) reduces to

$$D_L H_0 = \frac{1 + z}{\sqrt{|\Omega_k|}} S\left\{\sqrt{\frac{5|\Omega_k|}{\Omega_\phi}} \times \right.$$

$$\left. \int_{\phi(0)}^{\phi(z)} \frac{d\phi}{\sqrt{A_\pm \sinh(\phi - \phi_0) B \cosh(\phi - \phi_0) - C}} \right\}, \quad (3.15)$$

with the constants $A_\pm$, B and C as defined by expressions (2.17)-(2.19), and the deceleration parameter given now by

$$q_0 = \frac{1}{2}\left[\Omega_M - (\Omega_\phi + \Omega_\Lambda)\right]. \quad (3.16)$$

In order to obtain cosmological models described by real $D_L - z$ relations the conditions of integration [23] in expression (3.14) must be such that the cosmological parameters satisfy the following conditions

$$\Omega_\phi + \Omega_\Lambda + \frac{1}{2}\left(\Omega_k \pm \sqrt{2\Omega_k}\right) = 0 \quad (3.17)$$

$$Q^4 - 4(4 - \Omega_M)(\Omega_\phi + \Omega_\Lambda) = \Omega_k^2 \quad (3.18)$$

and, either

$$\Omega_k = -1 \pm \sqrt{1 + \frac{1}{4}(\Omega_\phi + \Omega_\Lambda)(\Omega_M - 16)}, \quad (3.19)$$

or,

$$\Omega_k - 2(\Omega_\phi + \Omega_\Lambda) = 0. \quad (3.20)$$

The parameter Q in Eq. (3.18) has been introduced to simplify the equation. It is defined in terms of the Ω's only, as:

$$Q^2 = 4(\Omega_\phi + \Omega_\Lambda) + [2(\Omega_\phi + \Omega_\Lambda) + \Omega_k]^2. \quad (3.21)$$

There are two cosmological models which satisfy all these conditions. They are: *Model I*, a flat universe defined by the parameters $\Omega_M = 1$, $\Omega_\phi + \Omega_\Lambda = \Omega_k = 0$, and *Model II*, an open universe defined by the parameters $\Omega_M = \Omega_\phi + \Omega_\Lambda = 1/4$, $\Omega_k = 1/2$. In both cases $B = 0$ and $A_\pm \equiv A$ and C become indeterminate and real. For Model I we obtain $q_0 = +1/2$ and, from expression (3.15),

$$D_L H_0 = \sqrt{\frac{2}{|A|}}(1 + z)\sqrt{|A|^2 - \left(|C|^2 - \frac{1}{1 + z}\right)^2} \times$$

$$F\left[\arcsin \sqrt{\frac{|A| + |C| - \frac{1}{1+z'}}{|A| + |C|}}, \sqrt{\frac{|A| + |C|}{2|A|}}\right]\Bigg|_0^z, \quad (3.22)$$

which always gives rise to a topologically and dynamically decelerating universe for any combinations of constants A and C satisfying $|A| + |C| = 1$ and the integration condition $|A| > |C| > 0$.

More interesting is Model II, for which one obtains a topologically uniform expansion, $q_0 = 0$, and again the integration condition $|A| > |C| > 0$. Taking e.g. $|A| = 2$ and $|C| = 1$, it follows from expression (3.15)

$$D_L H_0 = \sqrt{2}(1 + z)\sin\left\{\frac{(z + 1)\sqrt{2\left[4 - \left(1 + \frac{1}{1+z}\right)^2\right]}}{z + 2} \times \right.$$

$$\left. F\left[\arcsin \sqrt{2 - \left(1 + \frac{1}{1 + z}\right)}, \frac{1}{2}\right]\right\}. \quad (3.23)$$

Eq. (3.23) gives rise to a $D_L - z$ plot which, in spite of being associated with a topologically uniform universe, starts accelerating after $z \simeq 0.5$ in a way that matches the behaviour observed in distant supernovas. That plot, on the other hand, consistently shows a variation $\triangle(5 \log D_L H_0) \simeq 13$ along the observed z-interval, fitting well with the observations at all available values of the redshift. Therefore, it could be thought that Model II and the solution in terms of the Jacobian elliptic function sd for $\omega = -1/3$ dealt with above, correspond to "good" quintessence potentials.

It appears also relevant to perform a similar computation for the inverse-power law potentials obtained in Sec. II, which are of the type already proposed in the literature [12] and that might be justified from high energy physics. For the first of these potentials, $V = V_0(\phi_0/\phi)^4$, one obtains

$$D_L H_0 = (1+z)\sinh\left[\pm\sqrt{\frac{1}{2\Omega_\phi}}\left(1-\frac{1}{\sqrt{1+z}}\right)\right], \quad (3.24)$$

and for $V = V_0(\phi_0/\phi)^2$,

$$D_L H_0 = (1+z)\sin\left[\pm\sqrt{\frac{1}{\Omega_\phi}}\ln\left(\frac{1}{\sqrt{1+z}}\right)\right]; \quad (3.25)$$

these two equations are, of course, not valid for $k = 0$. It is interesting to note that in the two cases, we reproduce a $D_L - z$ plot which nearly matches the observed results, producing a distinguishable accelerating pattern starting at an expected $z \simeq 0.5$, and a variation $\triangle(5\log D_L H_0)$ between 11 and 12, only slightly smaller than what has been measured, along the observed z-interval. Moreover, our analytical formulae for the luminosity distance-redshift relation can also be applied to potentials which are defined for any value of ω, as those given in Eqs. (2.20) and (2.21), for sufficiently large values of the redshift. Thus, for potential (2.20) we obtain

$$D_L H_0 \simeq 2(1+z)\sqrt{\frac{1+\omega}{3\Omega_\phi\Omega_M}}\left(1-\frac{1}{\sqrt{1+z}}\right), \quad (3.26)$$

at large z, and for the potential (2.21),

$$D_L H_0 \simeq (1+z)\sqrt{\frac{2(3\omega+2)}{3(\omega+1)\Omega_\phi}}\left(\sqrt{1+z}-1\right), \quad (3.27)$$

at large z for $\omega < -2/3$. It can be checked that these functions give $D_L - z$ plots which show a nearly uniform expansion at the allowed sufficiently large values of the redshift.

It is worth noticing that for the limiting expressions from solutions (2.15) and (2.16), obtained by restricting $\Omega_\Lambda = 0$ and $\Omega_\phi \geq 0$, we either cannot even obtain a topologically accelerating universe ($\omega = -1/3$), or have no consistent integration procedure along the complete range of allowed z-values that leads to a definite real luminosity distance ($\omega = -2/3$). Thus, at least for the particular potentials considered in this work, if we want to consistently predict cosmological models compatible with observations on Ia supernovae at large and moderate redshifts, it appears that quintessence should be generalized in a way that allows for a more complicate vacuum structure made up of (i) a time-dependent, "axionic" [18] (as it is pure imaginary classically) scalar field, $\phi(t)$, with positive pressure and negative energy density, and (ii) a time-varying positive cosmological term, $\Lambda(t)$, whose current value Λ_0 can be quite smaller than that with which it started the cosmological evolution, in such a way that the full vacuum energy density is restricted to be $\rho_\phi + \rho_\Lambda \geq 0$.

IV. SUMMARY AND DISCUSSION

In this paper we have considered the problem of the quintessence potential, restricting ourselves to a constant equation of state, that is: what are the permissible potentials for a vacuum, time-dependent scalar field predicting cosmological models that conform to recent observations and, at the same time, satisfy the constraints imposed by the field equations and conservation laws, discussing their physical relevance in the case that the quintessence field corresponds to a non-tracking constant equation of state. We have generalized the usual quintessence model, introducing a positive cosmological varying term, while restricting the scalar-field energy density to be definite negative and its pressure definite positive in such a way that the overall vacuum energy density is necessarily positive or vanishing. Quintessence potentials that satisfy the above-alluded constraint, both for the usual models and for models with a varying cosmological term, have been obtained for particular values of the constant parameter defining the state equation for the scalar field in the two kinds of models. None of these potentials have hitherto been used in quintessence, except those which are given as an inverse-power law. We also obtain potentials which are given in terms of either hyperbolic functions or Jacobian elliptic functions, the latter generally reducing to the former in the limit when the cosmological term tends to zero. We have also obtained a general expression relating the luminosity distance with the quintessence potential, and this has been integrated for all particular solutions expressed in terms of the redshift. It turned out that only some of such solutions with nonzero cosmological term are able to produce cosmological models that conform to an acceleratingly expanding universe and agree with recent observations on Ia supernovae.

The cosmological term Λ we have used in our generalized quintessence model depends on the cosmological time through a linear dependence on the scale factor R. This varying character of Λ could *a priori* be an useful property to help solving the known cosmological constant problem, though not by itself only. Actually, one could not justify how an initial vacuum energy density of the order M_p^4 may be lowered down to a value smaller than 10^{-47} GeV invoking such a dependence; instead, if the R-dependence of Λ would be assumed to be the same along the entire cosmological evolution, then one would need a conservation law for Λ given by $\Lambda = \Lambda_0(R_0/R)^\gamma$, with $\gamma \geq 123/50$. However, the conservation law chosen in this paper, $\Lambda = \Lambda_0(R_0/R)$, is assumed to hold only in the late classical cosmological regime; it could well be that during primordial expansion γ had taken on values larger than $123/50$. For example, taking $\gamma \simeq 3$ during the primeval expansion up to $R \geq 10^4$ cm, and $\gamma = 1$ thereafter, would solve the cosmological constant problem. The price one would pay to get such a big reward would just be the allowance for the dynamical content

of the quintessence field to be that of the conventional models, with $\gamma = 3$ and $\gamma = 2$ (see Sec. II) during its early evolution.

The conclusions obtained in this work are not general. They just refer to the solutions that correspond to the particular values of the quintessence parameter $\omega = -1/6, -1/3, -2/3$ and -1, and some special cases for any ω. Possibly there will be other potentials corresponding to different, intermediate values of ω that also reproduce the observed cosmological expansion within the generalized quintessence model. We do not believe however this to be the case in the realm of the conventional quintessence model, though more work is obviously needed to reach a final verdit on this.

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