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Lorentzian wormholes generalizes thermodynamics still further.

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This paper deals with some thermodynamical aspects of Lorentzian wormholes, including the formulation of the three main laws and the consideration of a possible thermal emission made up of some sort of phantom radiation coming out from the wormhole at a negative temperature. In order for these topics to be consistently developed we have used a 2+2 formalism first advanced by Hayward for spherically symmetric space-times, where a generalized surface gravity is defined on the trapping horizon. Our results generalize still further those of the already generalized gravitational thermodynamics.

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Twenty-one years ago, Morris, Thorne and Yurtsever [1] presented the first classically consistent solution for a wormhole that was stabilized by exotic matter and became furthermore convertible into a time machine. That paper meant a real breakthrough in that it inaugurated the history of these space-time tunnels as real scientific objects, rather than as science-fiction toys. In spite of that, wormholes remain still belonging to the framework of technology fiction. However, recent cosmological observations are compatible with the dominance in the universal vacuum of a so called cosmic phantom energy, which has been shown to play the same stabilizing role as the conventional exotic matter with respect to wormholes [2].

It is well known that the thermodynamical description of gravitational vacuum and black holes has provided these objects with quite a more robust consistency, allowing moreover for a deeper understanding of their space-time structure and properties. We think that, since exotic matter can be viewed as the time-reversed version of ordinary matter and hence one may well think of wormholes as time-reversed black holes, if a similar thermodynamical representation of wormholes would be feasible, then the physical status of such tunneling would likewise greatly improve, entitling us by the way to get a more comprehensible account for the exotic matter which, on the other hand, would thus become the largest energy source in the universe.

The aim of this paper is to construct a complete and physically consistent wormhole thermodynamics, following the 2+2 formalism developed by Hayward for dynamical black holes [3, 4, 5], as that formalism depends on local variables and can be in this way applicable to space-times having no event horizon, such as that for wormholes is. In fact, Hayward himself already realized that this formalism may also be applied to wormholes [6], as wormholes have an actual generalized surface gravity [7], too. However, the important contribution by Hayward to that subject lacks of a precise definition of dynamic wormholes and of the formulation of the laws of their thermodynamics, restricting himself to study the thermal radiation just for the case of black holes. Also contemplated in the present paper is the study of all these latter issues.

It is well known that in spherically symmetric space-times the metric can generally be written as

$$ds^2 = 2g_{+-}d\xi^+ d\xi^- + r^2 d\Omega^2, \qquad (1)$$

where $\xi^\pm$ are the double-null coordinates and r is the areal radius. In terms of such an areal radius, the expansion becomes

$$\Theta_\pm = \frac{2}{r}\partial_\pm r, \qquad (2)$$

with $\partial_\pm \equiv \partial/\partial\xi^\pm$ the two preferred normal directions, which we shall consider to be future-pointing.

In a spherically symmetric space-time one can introduce the Kodama vector [8], which is defined by

$$k = \mathrm{curl}_2 r \qquad (3)$$

where the subscript 2 means referring to the two-dimensional space normal to the spheres of symmetry. An interesting property of the Kodama vector, which eventually turns out to be similar to that of the Killing vector, is [4]

$$k \cdot \left(\nabla \wedge k^b\right) = \pm\kappa k^b \qquad (4)$$

on a trapping horizon, that is an hypersurface which can be foliated by marginal spheres ($\Theta_+\Theta_- = 0$) [3], at which k vanishes. In Eq. (4) κ is the generalized surface gravity

$$\kappa = \frac{1}{2}\mathrm{div}_2\mathrm{grad}_2 r, \qquad (5)$$

implying (choosing e.g. $\Theta_+ = 0$) that the outer ($\partial_-\Theta_+ < 0$), degenerated ($\partial_-\Theta_+ = 0$) and inner ($\partial_-\Theta_+ > 0$) trapping horizons, respectively have $\kappa > 0$, $\kappa = 0$ and $\kappa < 0$.

In the case of a Morris-Thorne wormhole we would have a bifurcating, i. e. $\Theta_\pm = 0$, outer trapping horizon at the wormhole throat r_0. Although this is a static

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case, no Killing horizon is present. However one can then obtain a generalized surface gravity given by

$$\kappa|_H = \frac{1-K'(r_0)}{4r_0} = -2\pi r_0 (p+\rho)|_H, \tag{6}$$

where $K(r)$ is the shape-function, p the radial pressure, ρ the energy density and "$|_H$" means evaluation at the horizon. The outward flaring condition implies $K'(r_0) < 1$, therefore the surface gravity is positive ($\kappa|_H > 0$ because $p+\rho < 0$, equivalently), as it should be since we have an outer horizon.

On the other hand, it is known that in a spherically symmetric space-time a generalized first law of thermodynamics can be written as [4],

$$L_z E = \frac{\kappa L_z A}{8\pi} + \omega L_z V, \tag{7}$$

where A is the surface area, $L_z = z \cdot \nabla$ and $z = z^+\partial_+ + z^-\partial_-$ is tangent to the trapping horizon, E is the Misner-Sharp energy given by

$$E = \frac{1}{2} r \left(1 - \partial^a r \partial_a r\right), \tag{8}$$

and

$$\omega = -\frac{1}{2}\mathrm{trace}_2 T. \tag{9}$$

Eq. (6) allows then to introduce an expression for the geometric entropy, with the familiar dependence on the surface area

$$S \propto A. \tag{10}$$

In order to see how the area of a trapping horizon evolves, we can express the surface area in terms of the 2-form area μ as $A = \int_S \mu$, with $\mu = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ in the spherically symmetric case. It follows that the evolution of a trapping horizon area is governed by the integral expression

$$L_z A_H = \int_H \mu z^- \Theta_-, \tag{11}$$

in which we have chosen $\Theta_+ = 0$ for the trapping horizon and from the very definition of such a horizon $L_z \Theta_+ = 0$ should keep taking on vanishing values along the entire horizon, so implying

$$\frac{z^+}{z^-} = -\frac{\partial_- \Theta_+|_H}{\partial_+ \Theta_+|_H}. \tag{12}$$

The ++ component of the Einstein equations, corresponding to metric (1), the definition (2) and an energy momentum tensor component T_{++}, can be written as [9]

$$\partial_+ \Theta_+ = -\frac{1}{2}\Theta_+^2 - \Theta_+ \partial_+ \log\left(-g_{+-}\right) - 8\pi T_{++}, \tag{13}$$

which, when evaluated at the trapping horizon, yields

$$\partial_+ \Theta_+|_H = -8\pi T_{++}|_H. \tag{14}$$

It can be seen that by considering an energy-momentum tensor of type I according to the Hawking-Ellis classification [10], i. e. an energy momentum tensor which takes a diagonal form in an orthonormal basis, we have $T_{++} \propto p+\rho$ as expressed in our basis, which actually is a very natural and consistent assumption[1].

Now, for a dynamical black hole which is defined by a future ($\Theta_- < 0$) outer trapping horizon [3] and surrounded by ordinary matter with $p+\rho > 0$, Eqs. (12) and (14) will impose that the signs of non-vanishing z^+ and z^- ought to be different, so that the horizon is space-like. Taking $z^+ > 0$, i. e. choosing z with a positive component along the future-pointing null direction of vanishing expansion, it then follows from Eq.(11) that $L_z A_H > 0$ [5]. A similar line of reasoning starting with $p+\rho < 0$ would finally lead to $L_z A_H < 0$.

Babichev et al. [11] used a test-fluid approach to study the evolution of the horizon area of a Schwarzschild black hole induced by the accretion of dark energy, and showed similar results to those that we have just derived. If such results are consistently assumed to be originated from a flow of the surrounding matter into the hole, then one can regard both methods to actually describe just the same single, physical process. In what follows we shall consider that the above coincidence is more than a mere analogy, so that it must also hold in the case of wormholes. In fact, when applied to wormholes [12], the Babichev et al. procedure leads to an increase (decrease) of the size of the wormhole throat in case that $p+\rho < 0$ ($p+\rho > 0$). Such results can only be recovered by using the 2+2 formalism in terms of trapping horizons whereas the outer trapping horizon of the wormhole is past ($\Theta_- > 0$ for $\Theta_+ = 0$, in which case ξ^+ would be ingoing and ξ^- outgoing). Or, in other words, in order to recover the same result following the 2+2 formalism obtained by the mentioned procedure, i. e.

$$L_z A_H \geq 0, \tag{15}$$

for $p+\rho < 0$ dominating the environment that surrounds a wormhole, the wormhole must be characterized by a past outer trapping horizon[2]. We would use in what follows this characterization.

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[1] In general one would have $T_{++} \propto T_{00} + T_{11} - 2T_{01}$, where the components of energy-momentum tensor on the r.h.s. are expressed with respect to an orthonormal basis. In our case, we consider an energy-momentum tensor of type I [10], not just because it represents all observer fields with non-zero rest mass and zero rest mass fields, except in special cases when it is type II [10], but also because if this would be not the case either $T_{++} = 0$ (for types II and III) which at the end of the day would imply no horizon expansion, or we would be considering the case where the energy density vanishes (type IV)

[2] The same results can be obtained by using the trapping horizon

Let us now consider the possible emission process associated with the semi-classical effects, with the particle production rate being given by the WKB approximation for the tunneling probability Γ along a classically forbidden trajectory, $\Gamma \propto \exp\left[-2\mathrm{Im}\left(I\right)\right]$. If that probability would take on a thermal form at the horizon, $\Gamma \propto \exp\left(-\omega_\phi/T_H\right)$, then one could easily obtain an expression for the temperature of that radiation.

Following then a parallel reasoning to that of Ref. [13] for black holes, we can express the metric given by Eq.(1) in terms of the most convenient generalized retarded Eddington-Finkelstein coordinates, owing to the feature that a wormhole possesses a past outer trapping horizon. That is

$$\mathrm{d}s^2 = -e^{2\Psi} C \mathrm{d}u^2 - 2e^{\Psi}\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\Omega^2, \tag{16}$$

where we have again considered $u = \xi^+$ related to the ingoing direction, implying $\Theta_+|_H = 0$, $\Theta_-|_H > 0$ and $\partial_-\Theta_+|_H < 0$, $\mathrm{d}\xi^- = \partial_u\xi^-\mathrm{d}u + \partial_r\xi^-\mathrm{d}r$, $e^\Psi = -g_{+-}\partial_r\xi^- > 0$ and $e^{2\Psi}C = -2g_{+-}\partial_u\xi^-$, with $C = 1 - 2E/r$, E defined by Eq. (3). Ψ expresses all the gauge freedom contained in the choice of the null coordinate u. It can be noted that the use of retarded coordinates ensures that the marginal surfaces, for which $C = 0$, are past marginal surfaces. We want to emphasize that such a change of coordinates is general enough for our present purposes, assuming that the spacetime must possess a past outer trapping horizon; therefore it must be applicable to wormholes without any restriction about its traversability. On the other hand, a possible bad behaviour of the r coordinate could be expected taking into account that the more natural and well behaved coordinate to describe the radial coordinate of a wormhole is l (with l such that $g_{ll} = 1$ in orthogonal coordinates).

Let us also consider a massless scalar field in the eikonal approximation, $\phi = \phi_0 \exp\left(iI\right)$, with a slowly varying amplitude and being governed by a rapidly varying action given by [13]

$$I = \int \omega_\phi e^{\Psi}\mathrm{d}u - \int k_\phi \mathrm{d}r, \tag{17}$$

in which ω_ϕ and k_ϕ should be interpreted as the angular frequency and wave number for the scalar field ϕ, respectively; that is $\partial_u I = \omega_\phi e^\Psi$ and $\partial_r I = -k_\phi$. The field would then describe radially outgoing radiation. Since the wave equation $\nabla^2\phi = 0$ implies $g^{ab}\nabla_a I \nabla_b I = 0$, one can finally obtain

$$k_\phi^2 C + 2\omega_\phi k_\phi = 0. \tag{18}$$

This equation possesses two solutions: $k_\phi^{(1)} = 0$, which corresponds to the outgoing modes, and $k_\phi^{(2)} = -2\omega_\phi/C$ for the ingoing modes. $k_\phi^{(2)}$ produces a pole in the action (17), because $C|_H = 0$ on the horizon. Noting that $\kappa|_H = \partial_r C/2$, where we have taken $\Psi = 0$, without any loss of generality as $\kappa|_H$ should be gauge-invariant, with expanding C, one obtains $k_\phi \approx -\omega_\phi/\left[\kappa|_H(r-r_0)\right]$. Therefore the action has an imaginary contribution which is obtained by deforming the contour of integration in the upper r half-plane, i.e. $\mathrm{Im}\left(I\right)|_H = -\frac{\pi\omega_\phi}{\kappa|_H}$ and Γ has a thermal form for a temperature

$$T = -\frac{\kappa|_H}{2\pi}. \tag{19}$$

Of course, the radiation associated with temperature (19) has a semiclassical origin, being independent of any possible classically allowed path passing through the traversable wormhole throat. A rather key property of such a temperature is its characteristic of being always negative, a property stemming from the positiveness of the surface gravity on the outer horizon, i. e. $\kappa|_H > 0$. Some authors seem to be rather uncomfortable with the concept of negative temperatures in gravitational systems. While this attitude can well be understandable for classical systems, it is not definitively so in case of a quantum-mechanical system, where for sure one must not be afraid of negative temperatures. In fact, different experimentally checked devices which are inexorably interpreted in terms of quantum-mechanically governed phenomena, have shown the existence and properties of negative temperatures. On the other hand, it is known that phantom energy has associated a negative temperature [14], so manifesting its quite likely deep quantum nature. As we mentioned above, a phantom fluid is one of the kinds of exotic materials which can be used as the stuff to build up a traversable wormhole [2]. Therefore, Eq.(19) would imply that a wormhole radiates "particles" with the same properties as its surrounding matter, such as it already occurred with dynamical black holes in relation to ordinary matter.

Eq. (19) allows us to rewrite Eq.(7) in a more familiar form which is given by

$$L_z E = -T L_z S + \omega L_z V \tag{20}$$

on a trapping horizon, where

$$S = \frac{A}{4}. \tag{21}$$

The first term in the right hand side (r.h.s) of Eq. (20) can be interpreted as an energy-exchange term (in analogy with the heat term of usual thermodynamics) and the second one as a work-term. The negative sign in the energy-exchange term would imply that the exotic matter which supports this space-time gets energy from the space-time itself. Parallely, the term in the r.h.s. of Eq.(20) indicated that exotic matter carried up a work in order to support the space-time. So we are already prepared to formulate a *first law of wormhole thermodynamics* in the following terms: *the change in the gravitational energy of the wormhole is equal to the sum of the*

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defined by $\Theta_- = 0$, evolving according to $L_z\Theta_-|_H = 0$, and employing Einstein equations which, evaluated at the horizon produces, $\partial_-\Theta_-|_H = -8\pi T_{--}|_H$, since $T_{--} \propto \rho + p_r$, too.

energy removed from the wormhole and the work done in the wormhole. Even more, Eq. (21) confirms relation (10), specifying the involved proportionality constant which turns out to get the familiar numerical value 1/4, and, since $L_z A > 0$ in an exotic background, one has $L_z S > 0$. That is to say, we have now a proper *second law of wormhole thermodynamics expressible as: the entropy of a wormhole, which is given in terms of the throat surface area, can never decrease, when placed in its most natural dominant-energy-condition violating environment.*

On the other hand, Eq.(5) leads furthermore to a formulation of the *third law of wormhole thermodynamics*, as it implies that an outer horizon has always $\kappa > 0$. If we consider that no dynamical evolution is able to modify *the outer property of the horizon, then the generalized surface gravity would always remain being positive.* The above arguments can in fact be expressed by stating: *if no dynamical process can change the outer character of the trapping horizon, then it is impossible to reach the absolute zero for surface gravity by means of any dynamical process.*

It must be noted that if some dynamical process could change the outer character of a trapping horizon in such a way that it becomes an inner horizon, then the wormhole would converts itself into a different physical object and, therefore, the laws of wormhole thermodynamics would no longer be valid.

It has been argued [6] that by replacing the background energy from exotic to ordinary, one also changes the causal nature of an outer trapping horizon. It can then be also considered that caused by such a process, or by a subsequent one, a past outer trapping horizon (i. e. a dynamical wormhole) should change into a future outer trapping horizon (i.e. a dynamical black hole), and vice versa (check, for example, in the case of the Schwarzschild solution). The thermal radiation can be computed for a dynamical black hole surrounded by ordinary matter by using the advanced Eddington-Finkelstein coordinates, obtaining again Eq. (19) but with a plus sign, [13]. Therefore, if such a conversion would be at all possible, we expected the temperature to also change from negative (wormhole) to positive (black hole) in a way which is necessarily discontinuous due to the holding of the third law, without passing through the zero temperature.

Actually, black holes and wormholes can be viewed to be the time-reversed version of each other. It could seem that a white hole is the time-reversed version of a black hole, but if one considers also the time inversion of the surrounding material, then the surrounding ordinary matter would become exotic matter and vice versa, changing the causal nature of the horizon from space-like to time-like and vice versa. Such an idea about the relation between wormholes and black holes under time-inversion is supported not only by the fact that both are defined by an outer trapping horizon, which is past (wormhole) and future (black holes), being both bifurcating in the static case where no local dynamical properties can be rigorously derived. Really, the three laws of wormhole thermodynamics respectively become those of black hole thermodynamics under time inversion. This can be easily checked taking into account that the exotic matter is nothing but ordinary matter moving backward in time, so implying a change in the sign of temperature. Moreover, when one considers a time reversed first law of wormholes thermodynamics, the energy-exchange term in Eq. (20) changes sign and the temperature becomes positive, so that ordinary matter supplies energy to space-time. On the other hand, we should also take into account that a dynamical black hole emits radiation at a positive temperature. It is worth noticing nevertheless that the other two wormhole thermodynamical laws remain nevertheless unchanged under time inversion.

Finally, the growth of the wormhole area can be related to the accretion of the surrounded exotic matter, but the radiation process at the trapping horizon would produce a decrease of the wormhole size, decreasing thereby the wormhole entropy, too. Like in black holes thermodynamics such a violation of the second law of thermodynamics is only apparent, because it is the total entropy of the universe, composed by the wormhole, the surrounded matter and the thermal radiation, what should increase. In fact, one could perform an analysis similar to that followed in [15] where a box filled with a black hole and radiation was considered, in order to show that the entropy of the whole system should always increase until the thermal equilibrium is reached. The case of a box containing a wormhole just differs from what is discussed in Ref. [15], in that all the contents in the box are at negative temperature, so that any subsystem with lower temperature is hotter [16]. Carefully extending that analysis to the most general case of a box simultaneously containing a black hole, a wormhole and given amounts of ordinary and exotic radiations at different proportions would lead to the conclusion that the interwoven final effect from all possible involved thermal processes that can take place within the box inexorably implies the holding of a most generalized second law of thermodynamics for which the sum of all four involved entropies always increases.

After completion of this paper we became aware of a paper by Hayward [17] in which some part of the present work was also discussed following partly similar though somewhat divergent arguments. In work [17] he studies the thermodynamic of two-types of dynamic wormholes, characterized by past or future outer trapping horizon. Although these two types are completely consistent mathematical solutions, we have concentrated on the present work in the first one since we consider that they are the only physical consistent wormholes solution. One of the reasons which support the previous claim has already been mentioned in this work and is based on the possible equivalence between the 2+2 formalism and the Babichev et al. method. On the other hand, a traversable wormhole must be supported by exotic matter and it is known that it can collapse by accretion of normal matter. That is precisely the problem of how to traverse a traversable wormhole finding the mouth open for the back-travel, or at least avoiding a possible death by a strangulated wormhole throat during the trip. If the physical wormhole could be characterized by a future outer trapping horizon, by Eqs. 11, 12 and 14, then it would increase (decrease) its size by accretion of ordinary (exotic) matter and, therefore, it would be not problem to traverse it, even more, it would increase its size when a traveler would pass through the wormhole, contrary to what it is expected for the basis of the wormhole physics.

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