Fidelity under isospectral perturbations: A random matrix study

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Abstract. The set of Hamiltonians generated by all unitary transformations from a single Hamiltonian is the largest set of isospectral Hamiltonian we can form. Taking advantage of the fact that the unitary group can be generated from hermitian matrices we can take the ones generated by the Gaussian unitary ensemble with a small parameter as small perturbations. Similarly the transformations generated by hermitian antisymmetric matrices form orthogonal matrices, form isospectral transformations among symmetric matrices. Based on this concept we can obtain the fidelity decay of a system that decays under a random isospectral perturbation with well defined properties regarding time reversal invariance. If we choose the Hamiltonian itself also from a classical random matrix ensemble we obtain solutions in terms of form factors in the limit of large matrices.

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1. Introduction

Isospectral perturbations of states where studied in [1], where simple space rotations were used as an example. Later this concept was used in a semi-classical context [2] by considering a phase space translation as the typical perturbation. The semi-classical solution for the fidelity amplitude was given there for chaotic systems. The possibility of a random matrix approach was mentioned but not implemented. This is the purpose of the present paper, following to a large extent the master thesis of one of the authors [4]. We shall see that there are various options to describe infinitesimal random isospectral perturbations, which are exactly determined by the symmetry groups of the classical random matrix ensembles [5, 6]. For the case of a time reversal breaking Hamiltonian the calculation is straightforward and we shall see the general structure of the solution emerge. In a next step we can consider time reversal invariant Hamiltonians which can have either the orthogonal invariance or the unitary symplectic one in the rather particular case of time reversal conservation, semi-integer spin and breaking of rotational invariance. The latter case will be of marginal interest in the present context, both because of limited applications and because the argument for the unitary case can be trivially extended to this case. Yet the former, i.e. the case associated to symmetric Hamiltonians poses an additional challenge, as the orthogonal group is not one of the classical ensembles [6]. Yet this is precisely the invariance we have to consider if the isospectral perturbation is to conserve the symmetry of the Hamiltonian. It so turns out that the hermitian antisymmetric generators of $O(N)$ cannot be diagonalized by the group $O(N)$ itself, but only by the full unitary group. Yet a block diagonalization is possible and we shall show, that a careful bookkeeping of the corresponding phases combined with the result of [7], which states that the form factors of $O(N)$ and $U(N)$ are the same, allows us to obtain again an analytical solution. We shall, in the next section pose the problem and and in the following one solve the simplest case of unitary ensembles. In the fourth section we shall solve the case of symmetric Hamiltonians, and then proceed to a discussion

2. The general problem

We consider a Hamiltonians H i.e. a hermitian $N \times N$ matrix. Then the set UHU^{-1} forms the set of all matrices that have the same spectrum as H if U covers the entire unitary group in N dimensions $U(N)$. As the Haar measure for $U(N)$ is known we have a unique measure or with other words a clear and simple definition of a set of random isospectral Hamiltonians.

If we deal with fidelity we usually wish to have a parameter say α, that gives us a "small" perturbation. We find this easily by representing a unitary transformation as

$$U_\alpha = \exp(-i\alpha h) \tag{1}$$

where h is a hermitian $N \times N$ matrix which is a generator of the group $U(N)$. This

representation is not unique, except near the identity, but any unitary transformation can be represented this way.

We may now write the fidelity amplitude for a given Hamiltonian H and a given perturbation U_α as

$$f_\alpha(t) = (\psi | e^{-itH} U_\alpha e^{itH} U_\alpha^{-1} \psi) \tag{2}$$

In the following we shall always normalize all matrices in such a way that the so-called *Heisenberg time* is one, that is, the average level spacing at the center o fthe matrix spectrum should be $1/(2\pi)$. This sets a scale so that the value of α determines the "size" of the perturbation U_α. We shall see that results differ acceding to whether this size is of order one or of order $1/N$.

We can state that the transition from H_0 to H_α is a general isospectral perturbation of H_0 and we shall proceed to calculate the fidelity amplitude for any such pair of Hamiltonians and eventually average over the respective ensembles of H and h which in the first instance will be GUE for both. To simplify our argument we shall also average over the functions ψ, although that is known to be superfluous for the unitary case , as it is implicit in the invariance of the two ensembles. The average fidelity amplitude can be expressed by taking the trace:

$$f(t) = \frac{1}{N} \operatorname{tr} e^{iH_0 t} e^{-iH_\alpha t} \tag{3}$$

In the following section, we proceed to calculate the GUE case explicitly.

3. The GUE case

In the following we want to evaluate (3) for the case in which H and h are averaged over the GUE ensemble. We have two unitary matrices $\mathcal{U}_1$ and $\mathcal{U}_2$ such that

$$\mathcal{U}_1 H \mathcal{U}_1^{-1} = E \tag{4a}$$
$$\mathcal{U}_2 h \mathcal{U}_2^{-1} = \Lambda \tag{4b}$$

where E and Λ are diagonal matrices with eigenvalues E_k and Λ_k respectively. If we now define

$$U = \mathcal{U}_2^{-1} \mathcal{U}_1 \tag{5}$$

we have

$$f(t) = \operatorname{tr} \left(U \, e^{iEt} U^{-1} \, e^{i\alpha\Lambda} U \, e^{-iEt} U^{-1} \, e^{-i\alpha\Lambda} . \right) \tag{6}$$

We may now rewrite the r.h.s. of (7) as

$$U_{jk} U_{lk}^* U_{lm} U_{jm}^* e^{i(E_k - E_m)t} e^{i(\Lambda_l - \Lambda_j)\alpha} \tag{7}$$

Denoting the average over the unitary group by an overline, it is well-known that

$$\overline{U_{jk} U_{lk}^* U_{lm} U_{jm}^*} = \frac{1}{\Delta} \left[N \left(\delta_{jl} + \delta_{km} \right) - \delta_{jl} \delta_{km} - 1 \right] \tag{8}$$

where N is the space dimension and $\Delta = (N-1)N(N+1)$.

Substituting (8) in (6) and averaging over H_0 and h, one eventually obtains for $f(t)$:

$$f(t) = \frac{1}{N\Delta}\left[N^2\left(\left\langle e^{i(E_k - E_m)t}\right\rangle + \left\langle e^{i(\Lambda_l - \Lambda_j)\alpha}\right\rangle - 1\right) - \right.$$
$$\left. \left\langle e^{i(E_k - E_m)t}\right\rangle \cdot \left\langle e^{i(\Lambda_l - \Lambda_j)\alpha}\right\rangle\right] \tag{9}$$

Since both H_0 and H belong to the GUE, the form factors are given by the usual expressions:

$$\frac{1}{N}\sum_{k,m=1}^{N}\left\langle e^{i(E_k - E_m)t}\right\rangle = 1 + \delta(t) + b_{2,2}(t) \tag{10a}$$

$$\frac{1}{N}\sum_{j,l=1}^{N}\left\langle e^{i(\Lambda_l - \Lambda_j)\alpha}\right\rangle = 1 + \delta(t) + b_{2,2}(\alpha) \tag{10b}$$

This finally yields for $f(t)$ the result

$$f(t) = \frac{1}{N^2}\left[(N - 1 - b_{2,2}(\alpha))\,\delta(t) + (N - 1 - b_{2,2}(t))\,\delta(\alpha)\right]$$
$$- \frac{\delta(t)\delta(\alpha)}{N^2} + \frac{2 + b_{2,2}(t) + b_{2,2}(\alpha)}{N} \tag{11}$$

In the case of finite N, we should replace the δ functions by functions $\delta_N(t)$ defined by

$$\delta_N(t) = N\Phi(t/N), \tag{12}$$

where $\Phi(x)$ is related to the eigenvalue density and depends on the specific problem at hand. It is thus of order N, but only in a very small interval around zero. We therefore see that, if both α and t are outside that interval, then $f(t)$ is of order $1/N$. If, on the other hand, either t or α lies within that interval, the δ functions must be taken into account.

4. The GOE case

The fidelity we want to compute is now given by

$$f(t) = \left(\psi, e^{iH_0 t}e^{-iH_\alpha t}\psi\right) \tag{13}$$

where the quantity is averaged over all ψ as well as over all H_0 and all h. Now the first average is either over all real or all complex vectors, the second is over the GOE and the third is over the ensemble of antisymmetric matrices.

As shown in [3], it does not matter whether we average over all real or all complex vectors ψ and in either case, the average can be expressed by taking the trace

$$f(t) = \frac{1}{N}\,\text{tr}\left(e^{iH_0 t}e^{-iH_\alpha t}\right) \tag{14}$$

where N is the dimension of the space. Note that the results of [3] imply that this is only correct for quantities bilinear in the wave function. But this is of course the case we are considering here.

From this we see that we may simply evaluate (14). This gives

$$f(t) = \frac{1}{N} \operatorname{tr}\left(e^{iH_0 t} e^{\alpha h} e^{-iH_0 t} e^{-\alpha h}\right) \tag{15}$$

Now we have two orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$\mathcal{O}_1 H \mathcal{O}_1^{-1} = E \tag{16a}$$
$$\mathcal{O}_2 h \mathcal{O}_2^{-1} = \Lambda \tag{16b}$$

Here E is a diagonal matrix with the (real) eigenvalues of H, whereas Λ is an antisymmetric matrix in a block diagonal form, all of the entries of which are of the form

$$\Lambda_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix} \tag{17}$$

If we now define

$$O = \mathcal{O}_2^{-1} \mathcal{O}_1, \tag{18}$$

we have

$$f(t) = \frac{1}{N} \operatorname{tr}\left(O e^{iEt} O^{-1} e^{\alpha\Lambda} O e^{-iEt} O^{-1} e^{-\alpha\Lambda}\right) \tag{19}$$

Using Greek indices for the λ and Latin for the E, together with the Einstein summation convention throughout and distinguishing by a prime indices which are restricted to belong to the same 2×2 block, we may rewrite the r.h.s. of (19) as

$$O_{\tau' k} e^{iE_k t} O_{\sigma k} \left[e^{\alpha\Lambda}\right]_{\sigma\sigma'} O_{\sigma' l} e^{-iE_l t} O_{\tau l} \left[e^{-\alpha\Lambda}\right]_{\tau\tau'} \tag{20}$$

or equivalently

$$O_{\tau' k} O_{\sigma k} O_{\sigma' l} O_{\tau l} e^{i(E_k - E_l)t} \left[e^{\alpha\Lambda}\right]_{\sigma\sigma'} \left[e^{-\alpha\Lambda}\right]_{\tau\tau'} \tag{21}$$

Now we consider the pairs of indices τ and τ', as well as σ and σ'. These belong to the same block, which implies that $\lambda_\tau = \pm\lambda_{\tau'}$ and similarly for σ.

Denoting the orthogonal average by an overline, and denoting by $\hat{\rho}$ the index belonging to the same block as ρ, but different from ρ and similarly for σ, we have, using the formulae at the end of Andrés' thesis:

$$\overline{O_{\tau k} O_{\sigma k} O_{\sigma l} O_{\tau l}} = \frac{1}{\Delta}\Big[(N+1)\left(\delta_{kl} + \delta_{\sigma\tau} + \delta_{kl}\delta_{\sigma\tau}\right) -$$
$$(1 + \delta_{kl} + \delta_{\sigma\tau} + 3\delta_{kl}\delta_{\sigma\tau})\Big]$$
$$= \frac{1}{\Delta}\left[-1 + N\delta_{kl} + N\delta_{\sigma\tau} + (N-2)\delta_{\sigma\tau}\delta_{kl}\right] \tag{22a}$$

$$\overline{O_{\hat{\tau} k} O_{\sigma k} O_{\hat{\sigma} l} O_{\tau l}} = \frac{1}{\Delta}\Big[(N+1)\left(\delta_{\sigma\hat{\tau}} + \delta_{kl}\delta_{\sigma\tau}\right) -$$
$$(\delta_{\sigma\tau} + \delta_{kl}\delta_{\sigma\tau} + 2\delta_{kl}\delta_{\sigma\hat{\tau}})\Big] \tag{22b}$$

where N is the space dimension and $\Delta = (N-1)N(N+2)$. Here we have used the fact that if $\tau = \sigma$, then $\hat{\tau} = \hat{\sigma}$ and viceversa, whereas always $\sigma \neq \hat{\sigma}$ and $\tau \neq \hat{\tau}$.

The expression (21) then separates in two parts

$$\overline{O_{\tau k} O_{\sigma k} O_{\sigma l} O_{\tau l}} \, e^{i(E_k - E_l)t} \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau \tag{23}$$

and

$$\overline{O_{\hat{\tau} k} O_{\sigma k} O_{\hat{\sigma} l} O_{\tau l}} \, e^{i(E_k - E_l)t} \sin \alpha \lambda_\sigma \sin \alpha \lambda_\tau \tag{24}$$

There are further two terms of the form

$$\overline{O_{\hat{\tau} k} O_{\sigma k} O_{\sigma l} O_{\tau l}} \, e^{i(E_k - E_l)t} \cos \alpha \lambda_\sigma \sin \alpha \lambda_\tau \tag{25}$$

which vanish, however, upon averaging over h, since the distribution of λ is even.

Substituting (22a) in (23) one obtains

$$\frac{1}{\Delta} \Big[- e^{i(E_k - E_l)t} \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau + N^2 e^{i(E_k - E_l)t} \cos^2 \alpha \lambda_\sigma +$$
$$N^2 \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau + N^2(N-2) \cos^2 \alpha \lambda_\sigma \Big] \tag{26}$$

If instead one substitutes (22b) in (24)

$$\frac{1}{\Delta} \Big[N^2 e^{i(E_k - E_l)t} \sin^2 \alpha \lambda_\sigma + N^2(N-2) \sin^2 \alpha \lambda_\sigma \Big] \tag{27}$$

and hence as a final result one obtains for $f(t)$

$$f(t) = \frac{1}{N\Delta} \Big[- e^{i(E_k - E_l)t} \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau + N^2 e^{i(E_k - E_l)t} +$$
$$N^2 \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau + N^2(N-2) \Big] \tag{28}$$

And averaging over the H_0 and the h one obtains

$$f(t) = \frac{1}{N\Delta} \Big[- \big\langle e^{i(E_k - E_l)t} \big\rangle \big\langle \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau \big\rangle + N^2 \big\langle e^{i(E_k - E_l)t} \big\rangle +$$
$$N^2 \big\langle \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau \big\rangle + N^2(N-2) \Big] \tag{29}$$

Since the H_0 are GOE, we have the usual form factor:

$$\frac{1}{N} \sum_{k,l=1}^{N} \big\langle e^{i(E_k - E_l)t} \big\rangle = 1 + \delta(t) + b_{2,1}(t) \tag{30}$$

where the $\delta(t)$ term is an expression for a term proportional to N related to the one-particle distribution function. For the other correlation, we note the trigonometric identity

$$\cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau = \frac{1}{2} \big[\cos \alpha \left(\lambda_\sigma - \lambda_\tau \right) - \cos \alpha \left(\lambda_\sigma + \lambda_\tau \right) \big] \tag{31}$$

However, the distribution of the λ is symmetric, that is, if λ_σ is an eigenphase, so is $-\lambda_\sigma$, so that the second term contributes exactly the same amount as the first. We thus find

$$\frac{1}{N} \sum_{\sigma,\tau=1}^{N} \big\langle \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau \big\rangle = \frac{1}{N} \sum_{\sigma,\tau=1}^{N} \big\langle \cos \alpha \left(\lambda_\sigma - \lambda_\tau \right) \big\rangle \tag{32}$$

We thus find

$$\frac{1}{N} \sum_{\sigma,\tau=1}^{N} \langle \cos \alpha \lambda_\sigma \cos \alpha \lambda_\tau \rangle = 1 + b_{2,2}(\alpha) + \delta(\alpha) \tag{33}$$

where we have used the fact that for antisymmetric matrices the form factor of the phases is the GUE form factor.

As we already know from the GUE case, there are terms which are of order one when *both* α and t are close to zero, that is, when both are of order $1/N$ with respect to their respective Heisenberg times, which we have both implicitly set to one. All other terms are of order $1/N$, so we must clearly collect both orders of magnitude.

The expression for $f(t)$ given in (29) leads to 4 different terms in leading order of N:

$$f(t) = T_1 + T_2 + T_3 + T_4 \tag{34a}$$

$$T_1 = \frac{1}{N^2} [1 + b_{2,2}(\alpha) + \delta(\alpha)] [1 + \delta(t) + b_{2,1}(t)] \tag{34b}$$

$$T_2 = \frac{1}{N} [1 + \delta(t) + b_{2,1}(t)] \tag{34c}$$

$$T_3 = \frac{1}{N} [1 + b_{2,2}(\alpha) + \delta(\alpha)] \tag{34d}$$

$$T_4 = \frac{1}{N} \tag{34e}$$

The terms can now be collected by order, yielding

$$f(t) = \frac{\delta(t)\delta(\alpha)}{N^2} + \frac{\delta(t) + \delta(\alpha)}{N} +$$
$$\frac{1}{N} \left[1 + \left(1 + \frac{\delta(t)}{N} \right) \left(1 + b_{2,2}(\alpha) \right) + \right.$$
$$\left. \left(1 + \frac{\delta(\alpha)}{N} \right) \left(1 + b_{2,1}(t) \right) \right] \tag{35}$$

where we have systematically treated the δ functions as having order N.

An additional possible case is the following: starting from a GOE Hamiltonian H_0, we perturb it via a general unitary transformation U_α as a perturbation. In this case we do not need the complicated bookkeeping described above, but, keeping the approach of the previous section, we obtain a result similar to that of (11), where we replace the GUE form factor $b_{2,2}(t)$ that refers to H by the corresponding GOE form factor $b_{2,1}(t)$. For reference:

$$f(t) = \frac{1}{N^2} \left[(N - 1 - b_{2,2}(\alpha)) \delta(t) + (N - 1 - b_{2,1}(t)) \delta(\alpha) \right]$$
$$- \frac{\delta(t)\delta(\alpha)}{N^2} + \frac{2 + b_{2,1}(t) + b_{2,2}(\alpha)}{N} \tag{36}$$

Again, the same remark holds for the δ functions, which, for finite N, should be expressed as indicated in (12)

5. Conclusions

In the previous Sections we evaluated the average fidelity amplitude under isospectral perturbations for the cases of a GUE Hamiltonian perturbed by a unitary transformation as well as the other cases of a OE perturbed by either an orthogonal transformation or a unitary one. Variations on the symplectic are left to the enthusiastic reader.

We found that the result is of order $1/N$ when all parameters are of order one in some appropriate normalization, whereas they show a complicated behaviour when one or both of these parameters are of order $1/N$. If time is extremely short, this corresponds to the Zeno regime, which for $N \gg 1$ ends very fast. For the parameter α, on the other hand, some fixed value may be of order $1/N$, and will introduce a plateau that limits possible fidelity decay for all times. The question whether this represents a freeze in the sense of Refs[8, 9, **?**] has given rise to some discussion, which may be clarified by the following consideration. For isospectral perturbations one can write the perturbed hamiltonian as

$$H_\alpha = e^{\mathrm{i}h\alpha} H_0 e^{-\mathrm{i}h\alpha} = H_0 + \mathrm{i}[h, H_0]\alpha - \frac{1}{2}[h, [h, H_0]]\alpha^2 + \mathcal{O}(\alpha^3). \tag{37}$$

Within the leading order in α, which is *all that contributes* to $1 - F(t)$ (one minus fidelity) to order α^2 (namely, subsequent quadratic correction to H_α will give lower contribution to $1 - F(t)$ in $\mathcal{O}(\alpha^3)$), we have the perturbation

$$V = \mathrm{i}[h, H_0] \tag{38}$$

which is exactly what is needed for applying the general theory of fidelity freeze by Prosen and Žnidarič [8, 9, 10]. For example, such a V has an exactly vanishing diagonal in the eigenbasis of H_0. Yet in higher order terms the plateau no longer corresponds to a freeze, as it extends to infinity rather than ending at some fixed time. Note that these considerations apply to the fidelity, whereas our results deal with the fidelity amplitude. It is, however, well-known [11] that both quantities only differ by quantities of order $1/N$.

As a final remark, we wish to emphasize the fact that we have derived finite N results, which is important for practical reasons, as it will give rise to a plateau for values of α of order $1/N$.

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