

Dividing Line between Quantum and Classical Trajectories in a Measurement Problem: Bohmian Time Constant

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Abstract

This work proposes an answer to a challenge posed by Bell on the lack of clarity in regards to the line between the quantum and classical regimes in a measurement problem. To this end, a generalized logarithmic nonlinear Schrödinger equation is proposed to describe the time evolution of a quantum dissipative system under continuous measurement. Within the Bohmian mechanics framework, a solution to this equation reveals a novel result: it displays a time constant which should represent the dividing line between the quantum and classical trajectories. It is shown that continuous measurements and damping not only disturb the particle but compel the system to converge in time to a Newtonian regime. While the width of the wave packet may reach a stationary regime, its quantum trajectories converge exponentially in time to classical trajectories. In particular, it is shown that damping tends to suppress further quantum effects on a time scale shorter than the relaxation time of the system. If the initial wave packet width is taken to be equal to $2.8 \times 10^{-15}m$ (the approximate size of an electron), the Bohmian time constant is found to have an upper limit, i. e., $\tau_{B\max} = 10^{-26}s$.

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As pointed out by Bell,[1] the lack of clarity in regards to where the transition between the classical and quantum regimes is located is one aspect of the measurement problem. This problem represents one of the most important conceptual difficulties in quantum mechanics. Consequently, this topic of research has gained considerable interest in the last decades.[2, 3, 4] The presence of a classical apparatus considerably affects the behavior of the observed quantum system through continuous measurement [5, 6, 7, 8] which typically fail to have outcomes of the sort the theory was created to explain. These frequent measurements are also at the origin of the so-called Zeno effect.

Another conceptual difficulty is that in a system under observation there are many degrees of freedom such that information can be lost in the couplings which may account for dissipation. One possible approach that has often been used to answer this question is to introduce all degrees of freedom for the bath and solve a number of coupled equations in various limits of some approximation. In fact, by using the influence-functional method, it has been shown[9] that dissipation tends to destroy quantum interference in a time scale shorter than the relaxation time of the system. This result has given justification for the use of logarithmic nonlinear wave equations[10]-[15] to describe quantum dissipation. These equations have been validated as an appropriate, practical bath functional in time-dependent density functional theory for open quantum systems.[15]

This work addresses both conceptual difficulties mentioned above. In particular, an answer to a challenge posed by Bell [1] on the dividing line between the quantum and classical regimes in a measurement problem is given here. To this end, a generalized logarithmic nonlinear Schrödinger equation is proposed to describe the time evolution of a quantum dissipative system under continuous measurement. Thus, these two basic existing decoherence mechanisms are put on equal footing. Nowadays, there are several routes to deal with this classical-quantum divide. The main three routes were originally opened up by Bohm [16] with his Bohmian mechanics in 1952, many-worlds interpretation by Everett in 1957 [17], and wave function collapse models established on firm grounds by Ghirardi, Rimini and Weber in 1986 [18]. These last authors proposed a unified dynamics (which has to be stochastic) for microscopic and macroscopic systems, including the mesoscopic scale. Our approach is much more restricted by now and follows the first route. Our concerns are about the time dividing line between classical and quantum trajectories for quantum processes in presence of different decoherence mechanisms. The concept of trajectory used in our context is much more standard. It is limited to a dissipative (zero temperature) dynamics in presence of continuous measurements. Furthermore, for the dissipative case, the effective Hamiltonian we are implicitly considering is of the type of Caldirola-Kanai Hamiltonian or similar [19]. It is worth noting that we do not propose here a universal behavior. Physical processes we have in mind are, for example, electronic transport in materials, diffusion of adsorbates on surfaces or particles in bulk, motion of particles in quantum viscid media, friction in qubits, spectral lines under high pressure where the collisions among gas phase particles can be replaced by a collisional friction and so on. If, in all cases and any circumstance, an electron would converge to the classical regime in at most 10^{-26} s, and this time scale would depend on the mass as $1/\sqrt{m}$ (for negligible friction), one could never, contrary to a host evidence, observe interference phenomena, with electrons, neutrons, C_{60} , and so on. Such interference patterns are not considered in our framework of applications. In such experiments, the mea-

surement is carried out only at the Fraunhofer or far field region through a screen. Along the way to the screen, particles are not perturbed by any measurement at any time.

Within the Bohmian mechanics framework, a solution to this equation reveals a novel result: it displays a time constant which establishes the dividing line between the quantum and classical trajectories. As in RC circuits, the time constant is the key measure of how quickly the capacitor becomes charged or discharged; in electronic pacemakers, the pulsing rate of the heart's contractions is controlled by a RC circuit in which the time constant represents the most important dividing line between normal and abnormal heartbeats.[20] It is shown below that continuous measurements and damping not only disturb the particle but compel the system to converge in time to a Newtonian regime without any assumption of collapse. While the width of the wave packet may reach a stationary regime, its quantum trajectories converge exponentially in time to classical trajectories. In particular, it is shown that damping tends to suppress further quantum effects on a time scale shorter than the relaxation time of the system. For example, experiments to measure the size of the electron consist on colliding two beams of electrons against each other and counting how many are scattered and altered their trajectories. By counting the collisions, and knowing how many particles we have thrown, we can estimate the average size of each particle in the beam [21]. If the initial wave packet width is taken to be equal to $2.8 \times 10^{-15}m$ (the approximate size of an electron), the Bohmian time constant is found to have an upper limit, i. e., $\tau_{B\max} = 10^{-26}s$.

Bohmian mechanics has recently attracted increasing attention from researchers.[22]-[24] Despite the uncertainty principle, the predictions of nonrelativistic quantum mechanics permit particles to have precise positions at all times. The simplest theory demonstrating that this is so is indeed Bohmian mechanics. One of the fundamental aspects of this mechanics is its ability to tackle more clearly the quantum measurement problem. The wave function plays a dual role in this framework; it determines the probability of the actual location of the particle and monitors its motion. As pointed out by Bell,[1] in physics the only observations we must consider are position observations - a definite outcome in an individual measurement is determined by the relevant position variable associated with the apparatus. It is a great merit of the Bohmian picture to force us to consider this fact.

For simplicity, let us consider a one-dimensional problem. The time evolution of the wave function of a quantum dissipative system $\psi(x, t)$ under continuous measurement can be described in terms of a nonlinear Schrödinger equation. This equation combines two types of logarithmic nonlinearities: (1) For the description of a system under continuous measurement, Nassar[4] has recently proposed a Schrödinger-type equation with the nonlinear logarithmic term $-i\hbar\kappa \ln |\psi|^2$, along the lines of the pioneering work of Mensky[2] and Bialynicki-Birula and Mycielski [3], and where the coefficient κ characterizes the resolution of the continuous measurement. However, it is fundamentally different from such an equation due to the imaginary coefficient in front the logarithmic term. A remarkable feature of this equation is the existence of exact soliton-like solutions of Gaussian shape. Hefter [25] has given physical grounds for the use of this logarithmic nonlinear equation by applying it to nuclear physics and obtaining qualitative and quantitative positive results. He argues that this type of equation can be applied to extended objects such as nucleons and alpha particles. Furthermore, the origin of the non-linearity can also be understood coming from an

energy dissipation operator in an effective Hamiltonian due to the continuous measurement or by quantizing the corresponding Hamilton-Jacobi equation for a linear damped system using the so-called Schrödinger method of quantization [19, 24]. (2) For the description of quantum dissipative systems, Kostin [10] constructed a Schrödinger-type equation with the nonlinear logarithmic term $(i\nu\hbar/2)\ln(\psi/\psi^*)$ with ν being the friction coefficient. This equation has the very interesting property that, at the level of observables, it satisfies the dissipative Langevin equation at $T = 0$. This equation has subsequently been derived by Skagerstam[12] and Yasue[13] and has found extensive use in many applications.[14] The Kostin nonlinear logarithmic term has recently been suggested by Yuen-Zhou et al. [15] as an appropriate, practical bath functional in time-dependent density functional theory for open quantum systems with unitary propagation. So, by combining both nonlinearities, the generalized logarithmic nonlinear Schrödinger equation reads

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = [H(x,t) + i\hbar(W_c(x,t) + W_f(x,t))]\psi(x,t), \quad (1)$$

with

$$W_c(x,t) = -\kappa \left[\ln |\psi(x,t)|^2 - \left\langle \ln |\psi(x,t)|^2 \right\rangle \right] \quad (2)$$

and

$$W_f(x,t) = \frac{\nu}{2} \left[\ln \frac{\psi(x,t)}{\psi^*(x,t)} - \left\langle \ln \frac{\psi(x,t)}{\psi^*(x,t)} \right\rangle \right], \quad (3)$$

The terms in $\langle \rangle$ arise from the requirement that the integration of Equation (1) with respect to the variable x must be equal to the expectation values of the kinetic and potential energies through the Hamiltonian H [10]. The expectation value of the energy $\langle E(t) \rangle$ is defined as in its standard form

$$\langle E(t) \rangle \equiv \int_{-\infty}^{+\infty} \psi^*(x,t)E(t)\psi(x,t)dx. \quad (4)$$

For the system studied here, no external potential is assumed (i.e., $V = 0$).

Equation (1) has several interesting and unique properties. First of all, it guarantees the separability of noninteracting subsystems. Other nonlinear modifications can introduce interactions between two subsystems even when there are no real forces acting between them. Second, the stationary states can always be normalized. For other nonlinearities, stationary solutions have their norms fully determined and after multiplication by a constant they cease to satisfy the equation. And third, the logarithmic nonlinear Schrödinger equation (1) possesses simple analytic solutions in a number of dimensions – especially nonspreading wave-packet solutions. It is fundamentally different from the equations proposed

by Bialynicki-Birula and Mycielski [3] due to (i) the imaginary coefficient in front of the logarithmic terms and (ii) the last term $\langle \ln |\psi(x, t)|^2 \rangle$. Equation (1) also generalizes the equation proposed by Kostin in order to account for continuous observation.

Equation (1) can now be solved via the Bohmian formalism.[16, 22] To this end, the wave function is first expressed in polar form:

$$\psi(x, t) = \phi(x, t) \exp(iS(x, t)/\hbar). \quad (5)$$

Now, after substitution of Equation (5) into Equation (1), we obtain

$$\begin{aligned} i\hbar \left[\frac{\partial \phi}{\partial t} + \frac{i}{\hbar} \frac{\partial S}{\partial t} \phi \right] = \\ = -\frac{\hbar^2}{2m} \left\{ \left[\frac{\partial^2 \phi}{\partial x^2} - \frac{\phi}{\hbar^2} \left(\frac{\partial S}{\partial x} \right)^2 \right] + \frac{i}{\hbar} \left[2 \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial^2 S}{\partial x^2} \right] \right\} \\ -i\hbar\kappa \left[\ln \phi^2 - \langle \ln \phi^2 \rangle \right] \phi + \nu [S - \langle S \rangle] \phi. \end{aligned} \quad (6)$$

Equation (6) can be separated into real and imaginary parts. By defining the quantum hydrodynamical density ρ , velocity v and quantum potential V_{qu} respectively as

$$\rho(x, t) = \phi^2(x, t), \quad (7)$$

$$v = \frac{1}{m} \frac{\partial S}{\partial x}, \quad (8)$$

$$V_{qu} = -\frac{\hbar^2}{2m\phi} \frac{\partial^2 \phi}{\partial x^2}, \quad (9)$$

we reach

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \nu v = -\frac{1}{m} \frac{\partial V_{qu}}{\partial x} \quad (10)$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) + \kappa [\ln \rho - \langle \ln \rho \rangle] \rho = 0. \quad (11)$$

Equation (10) is an Euler-type equation describing trajectories of a fluid particle, with momentum $p = mv$, whereas Equation (11) describes the evolution of the quantum fluid density ρ . This density is interpreted as the probability density of a particle being actually present within a specific region. Such a particle follows a definite space-time trajectory that is determined by its wave function through an equation of motion in accordance with the initial position, formulated in a way that is consistent with the Schrödinger time evolution. An essential and unique feature of the quantum potential is that the force arising from it is unlike a mechanical force of a wave pushing on a particle with a pressure proportional to the wave intensity. By assuming that the wave packet is initially centered at $x = 0$ and $\rho(x, 0) = [2\pi\delta^2(0)]^{-1/2} \exp[-x^2/2\delta^2(0)]$ and ρ vanishes for $|x| \rightarrow \infty$ at any time we may rewrite

$$\rho(x, t) = |\psi(x, t)|^2 = [2\pi\delta^2(t)]^{-1/2} \exp\left(-\frac{[x - \bar{x}(t)]^2}{2\delta^2(t)}\right), \quad (12)$$

where $\delta(t)$ is the total width of the Gaussian wave packet and $\bar{x}(t)$ a classical trajectory. Equation (12) can be readily used to demonstrate that

$$\int_{-\infty}^{+\infty} ([x - \bar{x}(t)]^2) \rho(x, t) dx = \delta^2(t). \quad (13)$$

Substitution of Equation (12) into Equation (11) yields

$$\frac{\partial \rho}{\partial t} = \left[-\frac{\dot{\delta}}{\delta} + \frac{(x - \bar{x})}{\delta^2} \dot{x} + \frac{1}{\delta^3} (x - \bar{x})^2 \dot{\delta} \right] \rho, \quad (14)$$

and

$$\frac{\partial(\rho v)}{\partial x} = \left(\frac{\dot{\delta}}{\delta} - \kappa \right) \rho + \left[\left(\frac{\dot{\delta}}{\delta} - \kappa \right) (x - \bar{x}) + \dot{x} \right] \left(-\frac{(x - \bar{x})}{\delta^2} \right) \rho, \quad (15)$$

which implies that

$$v(x, t) = \left(\frac{\dot{\delta}}{\delta} - \kappa \right) (x - \bar{x}) + \dot{x}. \quad (16)$$

Analogously, substitution of Equation (16) into Equation (10) yields

$$\left(\ddot{\delta}(t) + (\nu - 2\kappa)\dot{\delta}(t) + (\kappa^2 - \kappa\nu)\delta(t) - \frac{\hbar^2}{4m^2\delta^3(t)} \right) (x - \bar{x})^1 + (\ddot{x} + \nu\dot{x})(x - \bar{x})^0 = 0, \quad (17)$$

which implies that

$$\ddot{\delta}(t) + (\nu - 2\kappa)\dot{\delta}(t) + (\kappa^2 - \kappa\nu)\delta(t) = \frac{\hbar^2}{4m^2\delta^3(t)} \quad (18)$$

and

$$\ddot{\bar{x}} + \nu \dot{\bar{x}} = 0. \quad (19)$$

Equations (18) and (19) show that continuous measurement of a quantum dissipative wave packet gives specific features to its evolution: the appearance of distinct classical and quantum elements, respectively. This measurement consists of monitoring the position of the quantum dissipative system and the result is the measured classical trajectory $\bar{x}(t)$ for t within a quantum uncertainty $\delta(t)$.

The associated Bohmian trajectories [23, 24] of an evolving i^{th} particle of the ensemble with an initial position x_{oi} can be calculated by first substituting

$$\dot{x}_i(t) = v_i(x, t) \quad (20)$$

into Equation (16) to obtain

$$x_i(t) = \bar{x}(t) + x_{oi} \frac{\delta(t)}{\delta_o} e^{-\kappa t}. \quad (21)$$

where $\delta_o = \delta(0)$ is the initial width. As said above, the position of the center of mass of the wave packet (the classical trajectory) is represented by $\bar{x}(t)$, while x_{oi} is the initial position of the i^{th} individual particle in the Gaussian ensemble corresponding to the wave function given by Equation (5). Now, Equation (18) admits analytic Gaussian-shaped soliton-like solutions (Gaussons) when

$$\kappa = \frac{\nu}{2} + \sqrt{\frac{\nu^2}{4} + \frac{\hbar^2}{4m^2\delta_o^4}}. \quad (22)$$

For $\kappa \neq 0$, and no friction, a stationary regime can be reached and the width of the wave packet can be related to the resolution of measurement which means that if an initially free wave packet is kept under a certain continuous measurement, its width may not spread in time. Note that the inverse of $\frac{\hbar}{2m\delta_o^2}$ is associated with the relative spreading of a free Gaussian wave packet [26]; in other words, the corresponding spreading velocity is given by $\frac{\hbar}{2m\delta_o}$. Thus, the effective or renormalized friction given by κ takes into account the two spreading mechanisms present in this dynamics. Equation (22) displays a very similar structure to that found for the renormalized frequency of a damped harmonic oscillator. It is worth stressing that when the friction mechanism is added, the resolution of the apparatus is changed showing the intertwining role played by both mechanisms. Finally, the general procedure will thus be to add more and more independent decoherence mechanisms in order to take into account the global effect in the corresponding time evolution of the system, showing again this entanglement among the different mechanisms.

The transition from quantum to classical trajectories can then be defined as the Bohmian time constant to be $\tau_B \equiv \kappa^{-1}$ and Equation (21) can be further simplified to

$$x_i(t) = \bar{x}(t) + x_{oi}e^{-t/\tau_B}. \quad (23)$$

It follows from Equation (23) that if $x_{oi} = 0$, then the particle follows the Newtonian trajectory at any time. If, however, x_{oi} is positive, then the particles distributed in the right half of the initial ensemble are accelerated whereas the particles distributed in the left half of the initial ensemble are decelerated. Nevertheless, there is only a temporary asymmetry in the Bohmian velocities between any two symmetric particles since the rate of the asymmetry diminishes with time. After a short time, the distance in position space shifted by the particles initially lying at positive and negative x_{oi} 's converges to a constant value. So, continuous measurements not only disturb the particle but compel it to eventually converge to a classical position. It is also noticeable that damping tends to suppress further quantum effects on a time scale shorter than the relaxation time of the system. For a small friction coefficient ($\nu < \frac{\hbar}{m\delta_o^2}$), the Bohmian time constant can be expressed as

$$\tau_B \simeq \frac{2m\delta_o^2}{\hbar} \left(1 - \frac{\nu m\delta_o^2}{\hbar} \right). \quad (24)$$

Further, from Equations (9) and (21) we have that the quantum force is given by

$$F_{qu} = -\frac{\partial V_{qu}}{\partial x} = -\frac{\partial}{\partial x} \left[-\frac{\hbar^2}{8m\delta_o^4}(x - \bar{x})^2 + \frac{\hbar^2}{4m\delta_o^2} \right] = \frac{\hbar^2}{4m\delta_o^4} x_{oi} e^{-t/\tau_B}. \quad (25)$$

Thus, the convergence of the quantum particle trajectories to classical trajectories is due to the influence of the measuring apparatus and friction through the quantum force.[27] This quantum force is directly proportional to the initial position of the i^{th} particle and decays exponentially in time (it drops 63% of its initial value after a time constant τ_B). Likewise, the quantum position $x_i(t)$ - the initial position of the i^{th} individual particle in the Gaussian ensemble - approaches its classical value. So, friction and continuous observation of a wave packet may lead to a gradual freezing of the quantum features of the particle.

Finally, if the initial wave packet width for an electron is taken to be equal to $2.8 \times 10^{-15}m$ (the approximate size of an electron [21]) and the coefficient of friction is made very small ($\nu \ll \frac{\hbar}{m\delta_o^2}$), the Bohmian time constant is found to have an upper limit:

$$\tau_{B \max} = 10^{-26} s. \quad (26)$$

This result provides an answer to a challenge posed by Bell [1, 28] on the lack of clarity about the line between the quantum and classical regimes in a measurement problem: The Bohmian time constant above defined may establish that dividing line.[20]

By adopting the Bohmian framework, which is one of the three main routes mentioned at the beginning, the interpretational scheme is different but pointing in the same

direction as other works on decoherence. Furthermore, we also surmise that there is no single universal time scale, but several ones depending on the experimental situation. On the other hand, the modeling of effective collapse induced by non-linearity at the quantum level is scarcely in this route. In any case, further investigation is needed in order to better understand the dynamics of a system interacting with an environment, which is traced out, by considering stochasticity through additional noise terms.

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