# Faster Approximate Pattern Matching: A Unified Approach 

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#### Abstract

In the approximate pattern matching problem, given a text $T$, a pattern $P$, and a threshold $k$, the task is to find (the starting positions of) all substrings of $T$ that are at distance at most $k$ from $P$. We consider the two most fundamental string metrics: Under the Hamming distance, we search for substrings of $T$ that have at most $k$ mismatches with $P$, while under the edit distance, we search for substrings of $T$ that can be transformed to $P$ with at most $k$ edits.

Exact occurrences of $P$ in $T$ have a very simple structure: If we assume for simplicity that $|P|<|T| \leq \frac{3}{2}|P|$ and that $P$ occurs both as a prefix and as a suffix of $T$, then both $P$ and $T$ are periodic with a common period. However, an analogous characterization for occurrences with up to $k$ mismatches was proved only recently by Bringmann et al. [SODA'19]: Either there are $\mathcal{O}\left(k^{2}\right) k$-mismatch occurrences of $P$ in $T$, or both $P$ and $T$ are at Hamming distance $\mathcal{O}(k)$ from strings with a common string period of length $\mathcal{O}(m / k)$. We tighten this characterization by showing that there are $\mathcal{O}(k) k$-mismatch occurrences in the non-periodic case, and we lift it to the edit distance setting, where we tightly bound the number of $k$-error occurrences by $\mathcal{O}\left(k^{2}\right)$ in the non-periodic case. Our proofs are constructive and let us obtain a unified framework for approximate pattern matching with respect to both considered distances. In particular, we provide meta-algorithms that only rely on a small set of primitive operations. We showcase the generality of our meta-algorithms with results for the following settings: - The fully compressed setting, where both $T$ and $P$ are given as straight-line programs of sizes $n$ and $m$, respectively. Here, we obtain an $\tilde{\mathcal{O}}\left((n+m) k^{2}\right)$-time and an $\tilde{\mathcal{O}}\left((n+m) k^{4}\right)$-time algorithm for pattern matching with mismatches and edits, respectively. Note that while our algorithms are the first to work in the fully compressed setting (that is, without first decompressing the input), they also improve the state of the art for the setting where only the text is compressed: For pattern matching with mismatches, we improve the dependency on $k$ from $\tilde{\mathcal{O}}\left((n+|P|) k^{4}\right)$ [Bringmann et al. SODA'19]; for pattern matching with edits, we improve the overall running time from $\tilde{\mathcal{O}}\left(n \sqrt{|P|} k^{3}\right)$ [Gawrychowski, Straszak, ISAAC'13]. - The dynamic setting, where we maintain a collection of strings $\mathcal{X}$ of total length $N$ using the data structure of Gawrychowski et al. [SODA'18], which supports each of the operations "split", "concatenate" and "insert a length- 1 string" in $\mathcal{O}(\log N)$ time with high probability. Here, for any two strings $T, P \in \mathcal{X}$, we can compute all occurrences of $P$ in $T$ with up to $k$ mismatches in time $\tilde{\mathcal{O}}\left(|T| /|P| \cdot k^{2}\right)$ or up to $k$ edits in time $\tilde{\mathcal{O}}\left(|T| /|P| \cdot k^{4}\right)$. - The standard setting, where $T$ and $P$ are given explicitly. Here, we obtain an $\mathcal{O}\left(|T|+|T| /|P| \cdot k^{2} \log \log k\right)$-time algorithm for the Hamming distance case (improving polylog $|T|$ factors compared to the deterministic algorithm by Clifford et al. [SODA'18] and matching, up to the $\log \log k$ factor, the randomized algorithm by Chan et al. [STOC'20], the state of the art for $k \leq \sqrt{|P|}$ ), and an $\mathcal{O}\left(|T|+|T| /|P| \cdot k^{4}\right)$-time algorithm for the edit distance case (matching the algorithm by Cole and Hariharan [J. Comput.'02], the state of the art for $k \leq \sqrt[3]{|P|}$ ).


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## 1 Introduction

The pattern matching problem is perhaps the most fundamental problem on strings: Given a pattern $P$ and a text $T$, the task is to find all occurrences of $P$ in $T$. However, in most applications, finding all exact occurrences of a pattern is not enough: Think of human spelling mistakes or DNA sequencing errors, for example. In this work, we focus on approximate pattern matching, where we are interested in finding substrings of the text that are "similar" to the pattern. While various similarity measures are imaginable, we study the two most commonly encountered metrics in this context: the Hamming distance and the edit distance.

## Hamming Distance

Recall that the Hamming distance of two (equal-length) strings is the number of positions where the strings differ. Now, given a text $T$ of length $n$, a pattern $P$ of length $m$, and an integer threshold $k>0$, we want to compute the $k$-mismatch occurrences of $P$ in $T$, that is, all length- $m$ substrings of $T$ that are at Hamming distance at most $k$ from $P$. This pattern matching with mismatches problem has been extensively studied. In the late 1980s, Abrahamson [2] and Kosaraju [33] independently proposed an FFT-based $\mathcal{O}(n \sqrt{m \log m})$-time algorithm for computing the Hamming distance of $P$ and all the length- $m$ fragments of $T$. While their algorithms can be used to solve the pattern matching with mismatches problem, the first algorithm to benefit from the threshold $k$ was given by Landau and Vishkin [34] and slightly improved by Galil and Giancarlo [20]: Based on so-called "kangaroo jumping", they obtained an $\mathcal{O}(n k)$-time algorithm, which is faster than $\mathcal{O}(n \sqrt{m \log m})$ even for moderately large $k$. Amir et al. [4] developed two algorithms with running time $\mathcal{O}(n \sqrt{k \log k})$ and $\tilde{\mathcal{O}}\left(n+k^{3} n / m\right)$, respectively; the latter algorithm was then improved upon by Clifford et al. [15], who presented an $\tilde{\mathcal{O}}\left(n+k^{2} n / m\right)$-time solution. Subsequently, Gawrychowski and Uznański [22] provided a smooth trade-off between the running times $\tilde{\mathcal{O}}(n \sqrt{k})$ and $\tilde{\mathcal{O}}\left(n+k^{2} n / m\right)$ by designing an $\tilde{\mathcal{O}}(n+k n / \sqrt{m})$-time algorithm. Very recently, Chan et al. 13] removed most of the polylog $n$ factors in the latter solution at the cost of (Monte-Carlo) randomization. Furthermore, Gawrychowski and Uznański [22] showed that a significantly faster "combinatorial" algorithm would have (unexpected) consequences for the complexity of Boolean matrix multiplication. Pattern matching with mismatches on strings is thus well understood in the standard setting. Nevertheless, in the settings where the strings are not given explicitly, a similar understanding is yet to be obtained. One of the main contributions of this work is to improve the upper bounds for two such settings, obtaining algorithms with running times analogous to the algorithm of Clifford et al. 15.

## Edit Distance

Recall that the edit distance (also known as the Levenshtein distance) of two strings $S$ and $T$ is the minimum number of edits required to transform $S$ into $T$. Here, an edit is an insertion, a substitution, or a deletion of a single character. In the pattern matching with edits problem, we are given a text $T$, a pattern $P$, and an integer threshold $k>0$, and the task is to find the starting positions of all the $k$-edit (or $k$-error) occurrences of $P$ in $T$. Formally, we are to find all positions $i$ in $T$ such that the edit distance between $T[i . . j]$ and $P$ is at most $k$ for some position $j$. Again, a classic algorithm by Landau and Vishkin [35] runs in $\mathcal{O}(n k)$ time. Subsequent research 44, 17] resulted in an $\mathcal{O}\left(n+k^{4} n / m\right)$-time algorithm (which is faster for $k \leq \sqrt[3]{m})$. From a lower-bound perspective, we can benefit from the discovery that the classic quadratic-time algorithm for computing the edit distance of two strings is essentially optimal: Backurs and Indyk [6] recently proved that a significantly faster algorithm would yield a major breakthrough for the satisfiability problem. For pattern matching with edits, this means that there is no hope for an algorithm that is significantly faster than $\mathcal{O}\left(n+k^{2} n / m\right)$; however, apart from that "trivial" lower bound
and the 20-year-old conjecture of Cole and Hariharan [17] that an $\mathcal{O}\left(n+k^{3} n / m\right)$-time algorithm should be possible, nothing is known that would close this gap. While we do not manage to tighten this gap, we do believe that the structural insights we obtain may be useful for doing so. What we do manage, however, is to significantly improve the running time of the known algorithms in two settings where $T$ and $P$ are not given explicitly, thereby obtaining running times that can be seen as analogous to the running time of Cole and Hariharan's algorithm [17].

## Grammar Compression

One of the settings that we consider in this paper is the fully compressed setting, where both the text $T$ and the pattern $P$ are given as straight-line programs. Compressing the text and the pattern is, in general, a natural thing to do - think of huge natural-language texts or genomic databases, which are easily compressible. While one approach to solve pattern matching in the fully compressed setting is to first decompress the strings and then run an algorithm for the standard setting, this voids most benefits of compression in the first place. Hence, there has been a long line of research with the goal of designing text algorithms directly operating on compressed strings. Naturally, such algorithms highly depend on the chosen compression method. In this work, we consider grammar compression, where a string $T$ is represented using a context-free grammar that generates the singleton language $\{T\}$; without loss of generality, such a grammar is a straight-line program (SLP).

Straight-line programs are popular due to mathematical elegance and equivalence [42, 30, 29] (up to logarithmic factors and moderate constants) to widely-used dictionary compression schemes, including the LZ77 parsing [50] and the run-length-encoded Burrows-Wheeler transform [12]. Many more schemes, such as byte-pair encoding [46], Re-Pair [36], Sequitur [39], and further members of the Lempel-Ziv family [51, 49, to name but a few, can be expressed as straight-line programs. We refer an interested reader to [43, 41, 37, 45] to learn more about grammar compression.

Working directly with a compressed representation of a text, intuitively at least, seems to be hard in general - in fact, Abboud et al. [1] showed that, for some problems, decompress-and-solve is the best we can hope for, under some reasonable assumptions from fine-grained complexity theory. Nevertheless, Jeż [27] managed to prove that exact pattern matching can be solved on grammar-compressed strings in near-linear time: Given an SLP of size $n$ representing a string $T$ and an SLP of size $m$ representing a string $P$, we can find all exact occurrences of $P$ in $T$ in $\mathcal{O}((n+m) \log |P|)$ time. For fully compressed approximate pattern matching, no such near-linear time algorithm is known, though. While the $\tilde{\mathcal{O}}\left((n+|P|) k^{4}\right)$-time algorithm by Bringmann et al. [11] for pattern matching with mismatches comes close, it works in an easier setting where only the text is compressed. We fill this void by providing the first algorithm for fully compressed pattern matching with mismatches that runs in near-linear time. Denote by $\operatorname{Occ}_{k}^{H}(P, T)$ the set of (starting positions of) $k$-mismatch occurrences of $P$ in $T$; then, our result reads as follows.
$\boldsymbol{r}$ Main Theorem 1. Let $\mathcal{G}_{T}$ denote an SLP of size $n$ generating a text $T$, let $\mathcal{G}_{P}$ denote an SLP of size $m$ generating a pattern $P$, let $k>0$ denote an integer threshold, and set $N:=|T|+|P|$.

Then, we can compute $\left|\operatorname{Occ}_{k}^{H}(P, T)\right|$ in time $\mathcal{O}\left(m \log N+n k^{2} \log ^{2} N \log \log N\right)$. The elements of $\mathrm{Occ}_{k}^{H}(P, T)$ can be reported within $\mathcal{O}\left(\left|\operatorname{Occ}_{k}^{H}(P, T)\right|\right)$ extra time.

For pattern matching with edits, near-linear time algorithms are not known even in the case that the pattern is given explicitly. Currently, the best pattern matching algorithms on an SLP-compressed text run in time $\mathcal{O}(n|P| \log |P|)$ [48] and $\mathcal{O}\left(n\left(\min \left\{|P| k, k^{4}+|P|\right\}+\log |T|\right)\right)$ 8. Moreover, an $\tilde{\mathcal{O}}\left(n \sqrt{|P|} k^{3}\right)$-time solution [21] is known for (weaker) LZW compression [49]. Again, we obtain a near-linear time algorithm for fully compressed pattern matching with edits. Denote by $\operatorname{Occ}_{k}^{E}(P, T)$ the set of all starting positions of $k$-error occurrences of $P$ in $T$; then, our result reads as follows.

- Main Theorem 2. Let $\mathcal{G}_{T}$ denote an SLP of size $n$ generating a string $T$, let $\mathcal{G}_{P}$ denote an SLP of size $m$ generating a string $P$, let $k>0$ denote an integer threshold, and set $N:=|T|+|P|$.

Then, we can compute $\left|\operatorname{Occ}_{k}^{E}(P, T)\right|$ in time $\mathcal{O}\left(m \log N+n k^{4} \log ^{2} N \log \log N\right)$. The elements of $\mathrm{Occ}_{k}^{E}(P, T)$ can be reported within $\mathcal{O}\left(\left|\operatorname{Occ}_{k}^{E}(P, T)\right|\right)$ extra time.

Note that our algorithms also improve the state of the art when the pattern is given in an uncompressed form; this is because any string $P$ admits a trivial SLP of size $\mathcal{O}(|P|)$.

## Dynamic Strings

While compression handles large static data sets, a different approach is available if the data changes frequently. Several works on pattern matching in dynamic strings considered the indexing problem, assuming that the text is maintained subject to updates and the pattern is given explicitly at query time; we refer an interested reader to [24, 19, 44, 3, 40] and references therein.

Recently, Clifford et al. [16] considered the problem of maintaining a data structure for a text $T$ and a pattern $P$, both of which undergo character substitutions, in order to be able to efficiently compute the Hamming distance between $P$ and any given fragment of $T$. Among other results, for the case where $|T| \leq 2|P|$ and constant alphabet size, they presented a data structure with $\mathcal{O}(\sqrt{m \log m})$ time per operation, and they proved that, conditioned on the Online Boolean Matrix-Vector Multiplication (OMv) Conjecture [25], one cannot simultaneously achieve $\mathcal{O}\left(m^{1 / 2-\varepsilon}\right)$ for the query and update time for any constant $\varepsilon>0$.

We consider the following more general setting: We maintain an initially empty collection of strings $\mathcal{X}$ that can be modified via the following "update" operations:

- makestring $(U)$ : Insert a string $U$ to $\mathcal{X}$.
- concat $(U, V):$ Insert $U V$ to $\mathcal{X}$, for $U, V \in \mathcal{X}$.
- $\operatorname{split}(U, i):$ Insert $U[0 \ldots i)$ and $U[i \ldots|U|)$ in $\mathcal{X}$, for $U \in \mathcal{X}$ and $i \in[1 \ldots|U|)$.

The strings in $\mathcal{X}$ are persistent, meaning that concat and split do not destroy their arguments.
The main goal in this model (and for the dynamic setting in general) is to provide algorithms that are faster than recomputing the answer from scratch after every update. Specifically for dynamic strings, already the task of testing equality of strings in $\mathcal{X}$ is challenging. After a long line of research 47, 38, 3, Gawrychowski et al. [23] proved the following: There is a data structure that supports equality queries in $\mathcal{O}(1)$ time, while each of the update operations takes $\mathcal{O}(\log N)$ time, where $N$ is an upper bound on the total length of all strings in $\mathcal{X}]^{\text {a }}$ In fact, as shown in [23], one can also support $\mathcal{O}(1)$-time queries for the longest common prefix of two strings in $\mathcal{X}$ with no increase in the update times. The data structure of [23] is Las-Vegas randomized: the answers are correct, but the update times are guaranteed only with high probability. Randomization can be avoided at the cost of extra logarithmic factors in the running times (see [38, 40]), and the same is true for our results.

We extend the data structure of Gawrychowski et al. [23] with approximate pattern matching queries:

- Main Theorem 3. A collection $\mathcal{X}$ of non-empty persistent strings of total length $N$ can be maintained subject to makestring $(U)$, concat $(U, V)$, and $\operatorname{split}(U, i)$ operations requiring $\mathcal{O}(\log N+|U|), \mathcal{O}(\log N)$, and $\mathcal{O}(\log N)$ time, respectively, so that given two strings $P, T \in \mathcal{X}$ with $|P|=m$ and $|T|=n$ and an integer threshold $k>0$, we can compute $\left|\operatorname{Occ}_{k}^{H}(P, T)\right|$ in time $\mathcal{O}\left(n / m \cdot k^{2} \log ^{2} N\right)$ and $\left|\operatorname{Occ}_{k}^{E}(P, T)\right|$ in time $\mathcal{O}\left(n / m \cdot k^{4} \log ^{2} N\right) \bigsqcup^{\text {b }}$ The elements of $\mathrm{Occ}_{k}^{H}(P, T)$ and $\operatorname{Occ}_{k}^{E}(P, T)$ can be reported in $\mathcal{O}\left(\left|\operatorname{Occ}_{k}^{H}(P, T)\right|\right)$ and $\mathcal{O}\left(\left|\mathrm{Occ}_{k}^{E}(P, T)\right|\right)$ extra time, respectively.

[^0]In the Hamming distance case，for $k<\sqrt{m} / \log N$ ，the data structure of Main Theorem 3 is faster than recomputing the occurrences from scratch after each update：Recall that，in the standard setting，the fastest known algorithm for pattern matching with mismatches costs $\tilde{\mathcal{O}}(n+k n / \sqrt{m})=\tilde{\mathcal{O}}(n+k \sqrt{m} \cdot n / m)$ time；in particular，the additive $\tilde{\mathcal{O}}(n)$ term dominates the time complexity for the considered parameter range．Observe further that，for any $k$ ，Main Theorem 3 is not slower（ignoring polylog factors）than running the $\tilde{\mathcal{O}}\left(n+k^{2} n / m\right)$－time algorithm by Clifford et al．［15］after every update．

In the edit distance case，for $k<\left(m / \log ^{2} N\right)^{1 / 3}$ ，the data structure of Main Theorem 3 is faster than running the $\mathcal{O}(n k)$－time Landau－Vishkin algorithm for the standard setting．Note further that，for any $k$ ，Main Theorem 3 is not slower（ignoring polylog $N$ factors）than running after every update the $\mathcal{O}\left(n+k^{4} n / m\right)$－time Cole－Hariharan algorithm，whose bottleneck for $k<m^{1 / 4}$ is the additive $\mathcal{O}(n)$ term．

## Structure of Pattern Matching with Mismatches

As in［11，we obtain our algorithms by exploiting new structural insights for approximate pattern matching． Our contribution in this area is two－fold：We strengthen the structural result of［11］for pattern matching with mismatches，and we prove a similar result for pattern matching with edits．

Before we describe our new structural insights，let us recall the structure of exact pattern matching． Let $P$ denote a pattern of length $m$ and let $T$ denote a text of length $n \leq 3 / 2 m$ ．Assume that $T$ is trimmed so that $P$ occurs both at the beginning and at the end of $T$ ，that is，$P=T[0 \ldots m)=T[n-m \ldots n)$ ． By the length constraints，we have $P[n-m \ldots m)=P[0 . .2 m-n)$ ．Repeating this argument for the overlapping parts of $P$ ，we obtain the following well－known characterization（where $X^{\infty}$ denotes the concatenation of infinitely many copies of a string $X$ ）：
Fact 1.1 （folklore［10］）．Let $P$ denote a pattern of length $m$ and let $T$ denote a text of length $n \leq 3 / 2 m$ ．If $T[0 \ldots m)=T[n-m \ldots m)=P$ ，then there is a string $Q$ such that $P=Q^{\infty}[0 \ldots m)$ and $T=Q^{\infty}[0 \ldots n)$ ， that is，both the text and the pattern are periodic with a common string period $Q$ ，and the starting positions of all exact occurrences of $P$ in $T$ form an arithmetic progression with difference $|Q|$ ．

Hence，it is justified to say that the structure of exact pattern matching is fully understood．Surprisingly， a similar characterization for approximate pattern matching was missing for a long time．Only recently， Bringmann et al．［11］proved a similar result for pattern matching with mismatches（we write $\delta_{H}(S, T)$ for the Hamming distance of $S$ and $T$ ）：
『 Theorem 1.2 （［11 Theorem 1．2］，simplified）．Given a pattern $P$ of length $m$ ，a text $T$ of length $n \leq 3 / 2 m$ ，and a positive integer threshold $k \leq m$ ，at least one of the following holds：
－The number of $k$－mismatch occurrences of $P$ in $T$ is bounded by $\mathcal{O}\left(k^{2}\right)$ ．
－There is a primitive string $Q$ of length $\mathcal{O}(m / k)$ such that $\delta_{H}\left(P, Q^{\infty}[0 \ldots m)\right) \leq 6 k$ ．』
Motivated by the absence of examples proving the tightness of their result，Bringmann et al．11 conjectured that the bound on the number of $k$－mismatch occurrences in Theorem 1.2 can be improved to $\mathcal{O}(k)$ ．We resolve their conjecture positively by proving the following stronger variant of Theorem 1.2

『 Main Theorem 4 （Compare Theorem 1．2．Given a pattern $P$ of length $m$ ，a text $T$ of length $n \leq \frac{3}{2} m$ ， and a positive integer threshold $k \leq m$ ，at least one of the following holds：
－The number of $k$－mismatch occurrences of $P$ in $T$ is bounded by $\mathcal{O}(k)$ ．
－There is a primitive string $Q$ of length $\mathcal{O}(m / k)$ that satisfies $\delta_{H}\left(P, Q^{\infty}[0 \ldots m)\right)<2 k$ ．
Examples from［11，illustrated in Figure 1］prove the asymptotic tightness of Main Theorem 4
As in the exact pattern matching case，we can also characterize the（approximately）periodic case in more detail．

(a) Consider a text $T:=\mathrm{a}^{3 m / 4} \mathrm{c}^{3 m / 4}$ and a pattern $P:=\mathrm{a}^{m / 2} \mathrm{c}^{m / 2}$, neither of which is approximately periodic. Then, shifting the exact occurrence of $P$ in $T$ by up to $k$ positions in either direction still yields a $k$-mismatch occurrence. Hence, we need $\Omega(k)$ distinct $k$-mismatch occurrences to derive approximate periodicity of $P$.

(b) Consider a text $T$ and a pattern $P$ obtained from $\mathrm{a}^{3 m / 2}$ and $\mathrm{a}^{m}$, respectively, by substituting a to c at $k / 2$ random positions. Then, all length $-m$ fragments of $T$ are $k$-mismatch occurrences of $P$, but, with high probability, neither $T$ nor $P$ is perfectly periodic. Hence, we need a relaxed periodicity notion allowing for $\Omega(k)$ mismatches.

Figure 1 Examples (1) and (2) from [11.

「 Main Theorem 5 (Compare [11, Claim 3.1]). Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n \leq \frac{3}{2} m$, and let $0 \leq k \leq m$ denote an integer threshold. Suppose that both $T[0 \ldots m)$ and $T[n-m \ldots n)$ are $k$-mismatch occurrences of $P$. If there is a positive integer $d \geq 2 k$ and a primitive string $Q$ with $|Q| \leq m / 8 d$ and $\delta_{H}\left(P, Q^{\infty}[0 \ldots m)\right) \leq d$, then each of following holds:
(a) Every $k$-mismatch occurrence of $P$ in $T$ starts at a position that is a multiple of $|Q|$.
(b) The string $T$ satisfies $\delta_{H}\left(T, Q^{\infty}[0 \ldots n)\right) \leq 3 d$.
(c) The set $\operatorname{Occ}_{k}^{H}(P, T)$ can be decomposed into $\mathcal{O}\left(d^{2}\right)$ arithmetic progressions with difference $|Q|$.

Note that Theorem 1.2 as originally formulated in [11, includes a weaker version of Main Theorem 5 We also observe that part (c) of the new characterization is asymptotically tight, as justified by modifying the example of Figure 1 b Let $P$ be obtained from $\mathrm{a}^{m}$ by placing c at $(k+1) / 2$ random positions, and let $T$ be obtained from $\mathrm{a}^{3 m / 2}$ by placing c at $(k+1) / 2$ random positions within the middle third of $\mathrm{a}^{3 m / 2}$. Then, each $k$-mismatch occurrence must align at least one c from $P$ with one c from $T$ and, conversely, each such alignment results in a $k$-mismatch occurrence. Hence, the number of $k$-mismatch occurrences is $\Theta\left(k^{2}\right)$. Furthermore, for every $q$, with high probability, $\operatorname{Occ}_{k}^{H}(P, T)$ can only be decomposed into $\Theta\left(k^{2}\right)$ progressions with difference $q$.

## Structure of Pattern Matching with Edits

Having understood the structure of pattern matching with mismatches, we turn to the more complicated situation of pattern matching with edits. First, observe that the examples of Figures 1a and 1b are still valid: Any $k$-mismatch occurrence is also a $k$-error occurrence. However, as the edit distance allows insertions and deletions of characters, we can construct an example where neither $P$ nor $T$ is approximately periodic, yet the number of $k$-error occurrences is $\Omega\left(k^{2}\right)$; see Figure 2 In the example of Figure 2 there are still only $\mathcal{O}(k)$ regions of size $\mathcal{O}(k)$ each where $k$-error occurrences start. In fact, we can show that this is the worst that can happen (we write $\delta_{E}(S, T)$ for the edit distance of $S$ and $T$ ):

Main Theorem 6. Given a pattern $P$ of length $m$, a text $T$ of length $n \leq \frac{3}{2} m$, and a positive integer threshold $k \leq m$, at least one of the following holds:

- The starting positions of all k-error occurrences of $P$ in $T$ lie in $\mathcal{O}(k)$ intervals of length $\mathcal{O}(k)$ each.
- There is a primitive string $Q$ of length $\mathcal{O}(m / k)$ and integers $i, j$ such that $\delta_{E}\left(P, Q^{\infty}[i \ldots j]\right)<2 k$. .

Again, we treat the (approximately) periodic case separately, thereby obtaining a result similar to Main Theorem 5


Figure 2 Consider a text $T:=\mathrm{a}^{n / 2} \cdot\left(\mathrm{c} \cdot \mathrm{a}^{k-1}\right)^{n / 2 k}$ and a pattern $P:=\mathrm{a}^{m / 2} \cdot\left(\mathrm{c} \cdot \mathrm{a}^{k-1}\right)^{m / 2 k}$ for $n:=m+2 k^{2}$. Now, for every $i \in[-k \ldots k]$, an $|i|$-mismatch occurrence of $P$ starts at position $n / 2-m / 2+i \cdot k$ in $T$. The remaining budget on the number of errors can be spent on shifting the starting positions, so for every $j \in[|i|-k \ldots k-|i|]$, there is a $k$-error occurrence starting at position $n / 2-m / 2+i \cdot k+j$ in $T$. Overall, the number of $k$-error occurrences of $P$ in $T$ is $\Omega\left(k^{2}\right)$, but neither $P$ nor $T$ is approximately periodic.

- Main Theorem 7. Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n$, and let $0 \leq k \leq m$ denote an integer threshold such that $n<3 / 2 m+k$. Suppose that the $k$-error occurrences of $P$ in $T$ include a prefix of $T$ and a suffix of $T$. If there is a positive integer $d \geq 2 k$ and a primitive string $Q$ satisfying $|Q| \leq m / 8 d$ and $\delta_{E}\left(P, Q^{\infty}[i \ldots j]\right) \leq d$ for some integers $i, j$, then each of following holds:
(a) For every $p \in \operatorname{Occ}_{k}^{E}(P, T)$, we have $p \bmod |Q| \leq 3 d$ or $p \bmod |Q| \geq|Q|-3 d$.
(b) The string $T$ satisfies $\delta_{E}\left(T, Q^{\infty}\left[i^{\prime} . . j^{\prime}\right]\right) \leq 3 d$ for some integers $i^{\prime}$ and $j^{\prime}$.
(c) The set $\operatorname{Occ}_{k}^{E}(P, T)$ can be decomposed into $\mathcal{O}\left(d^{3}\right)$ arithmetic progressions with difference $|Q|$.


## Technical Overview

## Gaining Structural Insights

To highlight the novelty of our approach, let us first outline the proof Theorem 1.2 by Bringmann et al. [11]. Consider a pattern $P$ of length $m$ and a text $T$ of length $n \leq \frac{3}{2} m$. Split the pattern into $\Theta(k)$ blocks of length $\Theta(m / k)$ each and process each such block $P_{i}$ as follows: Compute the shortest string period $Q_{i}$ of $P_{i}$ and align $P$ with a substring of $Q_{i}^{\infty}$, starting from $P_{i}=Q_{i}^{\infty}\left[0 . .\left|P_{i}\right|\right)$ and extending to both directions, with mismatches allowed. If there are $\mathcal{O}(k)$ mismatches for any block $P_{i}$, then $P$ is approximately periodic; otherwise, there are many mismatches for every block $P_{i}$. In particular, in every $k$-mismatch occurrence where a block $P_{i}$ is matched exactly, all but at most $k$ of these mismatches between $P$ and $Q_{i}^{\infty}$ must be aligned to the corresponding mismatches between $T$ and $Q_{i}^{\infty}$. Observing that, in any $k$-mismatch occurrence, all but at most $k$ of the blocks must be matched exactly, this yields an $\mathcal{O}\left(k^{2}\right)$ bound on the number of $k$-mismatch occurrences of $P$ in $T$.

The main shortcoming of this approach is the initial treatment of the pattern: Since the pattern $P$ is independently aligned with $Q_{i}^{\infty}$ for every block $P_{i}$, the same position in $P$ may be accounted for as a mismatch for multiple blocks $P_{i}$. In particular, this happens if several adjacent blocks share the same period. This leads to an overcounting of the $k$-mismatch occurrences that is hard to control.

What we do instead is a more careful analysis of the pattern. Instead of creating all blocks $P_{i}$ at once, we process $P$ from left to right, as described below. Suppose that $P[j \ldots m)$ is the unprocessed suffix of $P$. We first consider the length $-m / 8 k$ prefix $P^{\prime}$ of $P[j \ldots m)$ and compute its shortest string period $Q$. If $|Q|$ exceeds a certain constant fraction of $\left|P^{\prime}\right|$, we set $P^{\prime}$ aside as a break and continue processing $P\left[j+\left|P^{\prime}\right| \ldots m\right)$. Now, if $P^{\prime}$ is the $2 k$-th break that we set aside, our process stops, and we continue to work only with the breaks. If $P^{\prime}$ does not form a break, we try extending $P^{\prime}$ to a prefix $R$ of $P[j \ldots m)$ that satisfies $\delta_{H}\left(R, Q^{\infty}[0 . .|R|)\right)=\Theta(k \cdot|R| / m)$. If such a prefix $R$ exists, we set it aside as a repetitive region and continue processing $P[j+|R| \ldots m)$. Now, if all the repetitive regions

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collected so far have a total length of at least $3 / 8 \cdot m$, we stop our process and continue to work only with the repetitive regions computed so far. A repetitive region $R$ does not exist only if $P[j \ldots m$ ) has too few mismatches with $Q^{\infty}$. In this case, we try extending $P[j \ldots m)$ to a suffix $R^{\prime}$ of $P$ that satisfies $\delta_{H}\left(R^{\prime}, \bar{Q}^{\infty}\left[0 \ldots\left|R^{\prime}\right|\right)\right)=\Theta\left(k \cdot\left|R^{\prime}\right| / m\right)$; where $\bar{Q}$ is a suitable rotation of $Q$. If we fail again, we report that $P$ is approximately periodic; otherwise, we continue to work with the single repetitive region $R^{\prime}$, disregarding the previously generated repetitive regions. For this, we note that $\left|R^{\prime}\right| \geq 3 / 8 \cdot m$ because all breaks and repetitive regions found beforehand have a total length of at most $5 / 8 \cdot m$.

Overall, for every pattern $P$, we obtain either $2 k$ disjoint breaks, or disjoint repetitive regions of total length at least $3 / 8 \cdot m$, or a string with period $\mathcal{O}(m / k)$ at Hamming distance $\mathcal{O}(k)$ from $P$ (see Lemma 3.6).

If the analysis results in breaks, we observe that at least $k$ breaks need to be matched exactly in every $k$-mismatch occurrence of $P$ in $T$. As both the length and the shortest period of each break are $\Theta(n / k)$, there are at most $\mathcal{O}(k)$ exact matches of each break in the text. Now, a simple marking argument shows that the number of $k$-mismatch occurrences of $P$ in $T$ is $\mathcal{O}(k)$ (see Lemma 3.8.

If the analysis results in repetitive regions, for each region $R_{i}$, we consider its $k_{i}$-mismatch occurrences in $T$ with $k_{i}:=\Theta\left(k \cdot\left|R_{i}\right| / m\right)$. Intuitively, this distributes the available budget of $k$ mismatches among the repetitive regions according to their lengths. Next, we try extending each $k_{i}$-mismatch occurrence of each $R_{i}$ to an approximate occurrence of $P$, and we assign $\left|R_{i}\right|$ marks to this extension. Using insights gained in the periodic case, we bound the total number of marks by $\mathcal{O}\left(k \cdot \sum_{i}\left|R_{i}\right|\right)$. Independently, we show that each $k$-mismatch occurrence of $P$ has at least $\sum_{i}\left|R_{i}\right|-m / 4$ marks. Using $\sum_{i}\left|R_{i}\right| \geq 3 / 8 \cdot m$, we finally obtain a bound of $\mathcal{O}(k)$ on the number of $k$-mismatch occurrences of $P$ in $T$ (see Lemma 3.11).

In total, this proves Main Theorem 4 For the characterization of the periodic case (Main Theorem 5), we use a reasoning similar to that in [11. As in the theorem, assume that $P$ has $k$-mismatch occurrences both as a prefix and as a suffix of $T$. Further, fix a threshold $d \geq 2 k$ and a primitive string $Q$ such that $\delta_{H}\left(P, Q^{\infty}[0 \ldots m)\right) \leq d$. First, we show that every $k$-mismatch occurrence of $P$ in $T$ starts at a multiple of $|Q|$. In particular, $|Q|$ divides $n-m$ and, using this observation, we bound $\delta_{H}\left(T, Q^{\infty}[0 \ldots n)\right)$. Finally, to decompose $\mathrm{Occ}_{k}^{H}(P, T)$ into $\mathcal{O}\left(k^{2}\right)$ arithmetic progressions, we analyze the sequence of Hamming distances between $P$ and the length- $m$ fragments of $T$ starting at the multiples of $|Q|$ : we observe that the number of changes in this sequence is bounded by $\mathcal{O}\left(d^{2}\right)$, which then yields the claim.

For pattern matching with edits, surprisingly few modifications in our arguments are necessary. In fact, the analysis of the pattern stays essentially the same. The main difference in the subsequent arguments is that we need to account for shifts of up to $\mathcal{O}(k)$ positions; this causes the increase in the bound on the number of occurrences. Unfortunately, for the periodic case of pattern matching with edits, the situation is messier. The key difficulty that we overcome is that an alignment corresponding to a specific edit distance may not be unique. In particular, due to insertions and deletions, combining (the arguments for) two disjoint substrings is not as easy as in the Hamming distance case. We address these issues by enclosing individual errors between a string and its approximate period with so-called locked fragments, which admit a unique canonical alignment. (A similar idea was used by Cole and Hariharan [17].) Combining this with a more involved marking scheme, we then obtain Main Theorem 7 .

## A Unified Approach to Approximate Pattern Matching

The proofs of our new structural insights are already essentially algorithmic. To obtain algorithms for all the considered settings at once, we proceed in two steps. In the first step, we devise meta-algorithms that only rely on a core set of abstract operations; in the second step, we implement these operations in various settings. Specifically, we introduce the PILLAR model-a novel abstract interface to handle strings represented in a setting-specific manner. For two strings $S$ and $T$, the following operations are supported:

- Extract $(S, \ell, r)$ : Retrieve a string $S[\ell . . r]$.
- $\operatorname{LCP}(S, T)$ : Compute the length of the longest common prefix of $S$ and $T$.
- $\operatorname{LCP}^{R}(S, T)$ : Compute the length of the longest common suffix of $S$ and $T$.
- $\operatorname{IPM}(S, T)$ : Assuming that $|T| \leq 2|S|$, compute the starting positions of all exact occurrences of $S$ in $T$.
- Access $(S, i)$ : Retrieve the character $S[i]$.
= Length $(S)$ : Compute the length $|S|$ of the string $S$.
Using the PILLAR-model operations, the meta-algorithms for both pattern matching with mismatches and with errors follow the same overall structure:
- First, we implement the analysis of the pattern. Here, the key difficulty is to detect repetitive regions. Our algorithm finds the shortest repetitive region: Starting from the prefix $P^{\prime}$ of the unprocessed suffix $P\left[j \ldots m\right.$ ), we enumerate the mismatches (or errors) between $P\left[j \ldots m\right.$ ) and $Q^{\infty}$. We stop when the number of mismatches (or errors) within the constructed region $R$ reaches $\Theta(k / m \cdot|R|)$. Intuitively, this is correct because the number of mismatches (or errors) increases at most as fast as the length of $|R|$. We treat the special case when we reach the end of the pattern symmetrically.
Note that computing the next mismatch between two strings is a prime application of the LCP operation. For finding a next edit, we adapt the Landau-Vishkin algorithm [35], based on LCP operations as well.
- Next, we deal with the periodic case. This turns out to be the main difficulty. For the Hamming distance, implementing the proof of Main Theorem 5 is rather straightforward. However, for the edit distance case, the more complicated proof of Main Theorem 7 gets complemented with even more sophisticated algorithms. Hence, we do not discuss them in this outline.
- Finding the occurrences in the presence of $2 k$ breaks is easy: We first use IPM operations to locate exact occurrences of the breaks in the text and then perform a straightforward marking step; for the Hamming distance, we lose an $\mathcal{O}(\log \log k)$ factor for sorting marks.
- Finding the occurrences in the presence of repetitive regions is implemented similarly; the key difference is that we use our algorithm for the periodic case to find approximate occurrences of repetitive regions. Overall, this approach then yields the main technical results of this work (stated below for strings of arbitrary lengths):
『 Main Theorem 8. Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a positive integer $k \leq m$, we can compute (a representation of) the set $\operatorname{Occ}_{k}^{H}(P, T)$ using $\mathcal{O}\left(n / m \cdot k^{2} \log \log k\right)$ time plus $\mathcal{O}\left(n / m \cdot k^{2}\right)$ PILLAR operations.

For pattern matching with edits, the number of PILLAR-model operations matches the time cost of non-PILLAR-model operations; hence the simplified theorem statement.

『 Main Theorem 9. Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a positive integer $k \leq m$, we can compute (a representation of) the set $\operatorname{Occ}_{k}^{E}(P, T)$ using $\mathcal{O}\left(n / m \cdot k^{4}\right)$ time in the PILLAR model.

Finally, we show how to implement the PILLAR model in the settings that we consider:

- As a toy example, we start with the standard setting. Here, implementing the PILLAR-model operations boils down to collecting known tools on strings.
- For the fully compressed setting, we heavily rely on the recompression technique by Jeż [27, 28] (especially for internal pattern matching queries) and on other works on straight-line programs [8, 26].
- Finally, for the dynamic setting, we use the data structure by Gawrychowski et al. 23] (for LCP and $\mathrm{LCP}^{R}$ operations). Furthermore, we reuse some tools from the fully compressed setting, because the data structure of [23] actually works with (a form of) straight-line programs.
As the primitive operations of the PILLAR model are rather simple, we believe that they can be efficiently implemented in further settings not considered here.


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## 2 Preliminaries

## Sets and Arithmetic Progressions

For $n \in \mathbb{Z}_{\geq 0}$, we write [ $n$ ] to denote the set $\{0, \ldots, n-1\}$. Further, for $i, j \in \mathbb{Z}$, we write $[i . . j$ ] to denote $\{i, \ldots, j\}$ and $[i . . j)$ to denote $\{i, \ldots, j-1\}$; the sets $(i . . j]$ and $(i . . j)$ are defined similarly.

For $a, d, \ell \in \mathbb{Z}$ with $\ell>0$, the set $\{a+j \cdot d \mid j \in[0 . . \ell)\}$ is an arithmetic progression with starting value $a$, difference $d$, and length $\ell$. Whenever we use arithmetic progressions in an algorithm, we store them as a triple ( $a, d, \ell$ ) consisting of the first value, the difference, and the length.

For a set $X \subseteq \mathbb{Z}$, we write $k X$ to denote the set containing all elements of $X$ multiplied by $k$, that is, $k X:=\{k \cdot x \mid x \in X\}$. Similarly, we define $\lfloor X / k\rfloor:=\{\lfloor x / k\rfloor \mid x \in X\}$ and $k\lfloor X / k\rfloor:=\{k \cdot\lfloor x / k\rfloor \mid x \in X\}$.

## Strings

We write $T=T[0] T[1] \cdots T[n-1]$ to denote a string of length $|T|=n$ over an alphabet $\Sigma$. The elements of $\Sigma$ are called characters. We write $\varepsilon$ to denote the empty string.

For a string $T$, we denote the reverse string of $T$ by $T^{R}$, that is, $T^{R}:=T[n-1] T[n-2] \cdots T[0]$. For two positions $i \leq j$ in $T$, we write $T[i . . j+1):=T[i . . j]:=T[i] \cdots T[j]$ for the fragment of $T$ that starts at position $i$ and ends at position $j$. We set $T[i . . j]:=\varepsilon$ whenever $j<i$.

A prefix of a string $T$ is a fragment that starts at position 0 (that is, a prefix is a fragment of the form $T[0 . . j$ ) for some $j \geq 0$ ). A suffix of a string $T$ is a fragment that ends at position $|T|-1$ (that is, a suffix is a fragment of the form $T[i \ldots|T|$ ) for some $i \leq|T|)$. We denote the length of the longest common prefix (longest common suffix) of two strings $U$ and $V$, defined as the longest string that occurs as a prefix (suffix) of both $U$ and $V$, by $\operatorname{lcp}(U, V)$ (respectively, $\operatorname{lcp}{ }^{R}(U, V)$ ).

A string $P$ of length $m \in[0 \ldots|T|]$ is a substring of a string $T$ (denoted $P \preccurlyeq T$ ) if there is a fragment $T[i \ldots i+m$ ) matching $P$. In this case, we say that there is an exact occurrence of $P$ at position $i$ in $T$, or, more simply, that $P$ exactly occurs in $T$.

For two strings $U$ and $V$, we write $U V$ or $U \cdot V$ to denote their concatenation. We also write $U^{k}:=U \cdots U$ to denote the concatenation of $k$ copies of the string $U$. Furthermore, $U^{\infty}$ denotes an infinite string obtained by concatenating infinitely many copies of $U$. A string $T$ is called primitive if it cannot be expressed as $T=U^{k}$ for a string $U$ and an integer $k>1$.

A positive integer $p$ is called a period of a string $T$ if $T[i]=T[i+p]$ for all $i \in[0 . .|T|-p)$. We refer to the smallest period as the period $\operatorname{per}(T)$ of the string. The string $T[0 \ldots \operatorname{per}(T))$ is called the string period of $T$. We call a string periodic if its period is at most half of its length.

For a string $T$, we define the following rotation operations: The operation $\operatorname{rot}(\cdot)$ takes as input a string, and moves its last character to the front; that is, $\operatorname{rot}(T):=T[n-1] T[0 \ldots n-2]$. The inverse operation $\operatorname{rot}^{-1}(\cdot)$ takes as input a string and moves its initial character to the end; that is, $\operatorname{rot}^{-1}(T):=T[1 \ldots n-1] T[0]$. Note that a primitive string $T$ does not match any of its non-trivial rotations, that is, we have $T=\operatorname{rot}^{j}(T)$ if and only if $j \equiv 0(\bmod |T|)$.

Finally, the run-length encoding (RLE) of a string $T$ is a decomposition of $T$ into maximal blocks such that each block is a power of a single character. (For instance, the RLE of the string aaabbabbbb is $\mathrm{a}^{3} \mathrm{~b}^{2} \mathrm{ab}^{4}$.) Note that each block of the RLE can be represented in $\mathcal{O}(1)$ space.

## Hamming Distance and Pattern Matching with Mismatches

For two strings $S$ and $T$ of the same length $n$, we define the set mismatches between $S$ and $T$ as $\operatorname{Mis}(S, T):=\{i \in[n] \mid S[i] \neq T[i]\}$. Now, the Hamming distance of $S$ and $T$ is defined as the number of mismatches between $S$ and $T$, that is, $\delta_{H}(S, T):=|\operatorname{Mis}(S, T)|$.

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It is easy to verify that the Hamming distance satisfies the triangle inequality:
${ }^{-} \quad$ Fact 2.1 (Triangle inequality for Hamming distance). Any strings $A, B$, and $C$ of the same length satisfy $\delta_{H}(A, C)+\delta_{H}(C, B) \geq \delta_{H}(A, B) \geq\left|\delta_{H}(A, C)-\delta_{H}(C, B)\right|$.

As we are often concerned with the Hamming distance of a string $S$ and a prefix of $T^{\infty}$ for a string $T$, we write $\operatorname{Mis}\left(S, T^{*}\right):=\operatorname{Mis}\left(S, T^{\infty}[0 \ldots|S|)\right)$ and $\delta_{H}\left(S, T^{*}\right):=\left|\operatorname{Mis}\left(S, T^{*}\right)\right|$.

Now, for a string $P$ (also called a pattern), a string $T$ (also called a text), and an integer $k \geq 0$ (also called a threshold), we say that there is a $k$-mismatch occurrence of $P$ in $T$ at position $i$ if $\delta_{H}(P, T[i \ldots i+|P|)) \leq k$. We write $\operatorname{Occ}_{k}^{H}(P, T)$ to denote the set of all positions of $k$-mismatch occurrences of $P$ in $T$, that is, $\operatorname{Occ}_{k}^{H}(P, T):=\left\{i \mid \delta_{H}(P, T[i \ldots i+|P|) \leq k)\right\}$. Lastly, we define the pattern matching with mismatches problem.
${ }^{-}$Problem 2.2 (Pattern matching with mismatches). Given a pattern $P$, a text $T$, and a threshold $k$, compute the set $\mathrm{Occ}_{k}^{H}(P, T)$.

Note that, depending on the use case (especially if the set $\mathrm{Occ}_{k}^{H}(P, T)$ is relatively large), we may only want to compute the size $\left|\mathrm{Occ}_{k}^{H}(P, T)\right|$ or a space-efficient representation of the set $\mathrm{Occ}_{k}^{H}(P, T)$, e.g., as the union of disjoint arithmetic progressions.

## Edit Distance and Pattern Matching with Edits

The edit distance (also known as Levenshtein distance) between two strings $S$ and $T$, denoted $\delta_{E}(S, T)$, is the minimum number of character insertions, deletions, and substitutions required to transform $S$ into $T$.

Again, it is easy to verify that the edit distance satisfies the triangle inequality:
${ }^{-}$Fact 2.3 (Triangle inequality for edit distance). Any strings $A, B$, and $C$ of the same length satisfy $\delta_{E}(A, C)+\delta_{E}(C, B) \geq \delta_{E}(A, B) \geq\left|\delta_{E}(A, C)-\delta_{E}(C, B)\right|$.

Further, similarly to the Hamming distance, we write $\delta_{E}\left(S, T^{*}\right):=\min \left\{\delta_{E}\left(S, T^{\infty}[0 \ldots j)\right) \mid j \in \mathbb{Z}_{\geq 0}\right\}$ to denote the minimum edit distance between a string $S$ and any prefix of a string $T^{\infty}$. Further, we write $\delta_{E}\left(S,{ }^{*} T^{*}\right):=\min \left\{\delta_{E}\left(S, T^{\infty}[i \ldots j)\right) \mid i, j \in \mathbb{Z}_{\geq 0}, i \leq j\right\}$ to denote the minimum edit distance between $S$ and any substring of $T^{\infty}$, and we set $\delta_{E}\left(S,{ }^{*} T\right):=\min \left\{\delta_{E}\left(S, T^{\infty}[i \ldots j|T|)\right)\left|i, j \in \mathbb{Z}_{\geq 0}, i \leq j\right| T \mid\right\}$.

Now, for a string $P$ (also called a pattern), a string $T$ (also called a text), and an integer $k \geq 0$ (also called a threshold), we say that there is a $k$-error occurrence of $P$ in $T$ at position $i$ if $\delta_{E}(P, T[i \ldots j)) \leq k$ for some $j \geq i$. We write $\operatorname{Occ}_{k}^{E}(P, T)$ to denote the set of all positions of $k$-error occurrences of $P$ in $T$, that is, $\operatorname{Occ}_{k}^{E}(P, T):=\left\{i \mid \exists_{j \geq i} \delta_{E}(P, T[i \ldots j) \leq k)\right\}$. Lastly, we define the pattern matching with edits problem.

Problem 2.4 (Pattern matching with edits). Given a pattern $P$, a text $T$, and a threshold $k$, compute the set $\operatorname{Occ}_{k}^{E}(P, T)$.

Again, we may only want to compute the size $\left|\operatorname{Occ}_{k}^{E}(P, T)\right|$ or a space-efficient representation of $\mathrm{Occ}_{k}^{E}(P, T)$.

### 2.1 The PILLAR Model

In order to unify the implementations of our approach for approximate pattern matching in different settings, we introduce the PILLAR model. The PILLAR model captures certain primitive operations which can be implemented efficiently in all considered settings (see Section 7 for the actual implementations).

Thus，in the algorithmic sections of this work，Sections 4 and 6 we bound the running times in terms of the number of PILLAR operations；if this value is asymptotically smaller than the time complexity of the remaining computations，we also specify the extra running time．

In the PILLAR model，we are given a family of strings $\mathcal{X}$ for preprocessing．The elementary objects are fragments $X[\ell \ldots r)$ of strings $X \in \mathcal{X}$ ．Each such fragment $S$ is represented via a handle，which is how $S$ can be passed as input to PILLAR operations．Initially，the model provides a handle to each $X \in \mathcal{X}$ ， interpreted as $X[0 \ldots|X|)$ ．Handles to other fragments can be obtained through an Extract operation：
－Extract $(S, \ell, r)$ ：Given a fragment $S$ and positions $0 \leq \ell \leq r \leq|S|$ ，extract the（sub）fragment $S[\ell \ldots r)$ ．If $S=X\left[\ell^{\prime} \ldots r^{\prime}\right)$ for $X \in \mathcal{X}$ ，then $S[\ell \ldots r)$ is defined as $X\left[\ell^{\prime}+\ell \ldots \ell^{\prime}+r\right)$ ．
Furthermore，the following primitive operations are supported in the PILLAR model：
－ $\operatorname{LCP}(S, T):$ Compute the length of the longest common prefix of $S$ and $T$ ．
－ $\operatorname{LCP}^{R}(S, T):$ Compute the length of the longest common suffix of $S$ and $T$ ．
－ $\operatorname{IPM}(P, T)$ ：Assuming that $|T| \leq 2|P|$ ，compute $\operatorname{Occ}(P, T)$（represented as an arithmetic progression with difference $\operatorname{per}(P)$ ）．
－Access $(S, i)$ ：Assuming $i \in[|S|]$ ，retrieve the character $S[i]$ ．
－Length $(S)$ ：Retrieve the length $|S|$ of the string $S$ ．
We now provide a small toolbox that is to be used in Sections 4 and 6 First，we note that LCP $(S, T)$ ， $\operatorname{LCP}^{R}(S, T)$ ，or $\operatorname{IPM}(S, T)$ can be used to check whether two strings $S$ and $T$ are equal．

Lemma 2.5 （Equality，［31，Fact 2．5．2］）．Given strings $S$ and $T$ ，we can check whether $S$ and $T$ are equal in $\mathcal{O}(1)$ time in the PILLAR model．

A more involved operation allows checking if a given string is periodic and，if so，computing the period．
『 Lemma $2.6(\operatorname{Period}(S),[32$ 31］）．Given a string $S$ ，we can compute $\operatorname{per}(S)$ or declare that $\operatorname{per}(S)>$ $|S| / 2$ in $\mathcal{O}(1)$ time in the PILLAR model．

Next，we introduce an operation that checks cyclic equivalence and retrieves the witness shifts．
『 Lemma 2.7 （Rotations $(S, T)$ ，［32 31］）．Given strings $S$ and $T$ ，we can find all integers $j$ such that $T=\operatorname{rot}^{j}(S)$ in $\mathcal{O}(1)$ time in the PILLAR model．The output is represented as an arithmetic progression．』

Our subsequent goal is to generalize the primitive LCP operation to also support fragments of infinite powers of strings．We start with a special case，and then we cover the general case．
『 Lemma $2.8\left(\operatorname{LCP}\left(S, Q^{\infty}\right)\right.$ ，［31 Fact 2．5．2］；see also［5］）．Given strings $S$ and $Q$ ，we can compute $\operatorname{LCP}\left(S, Q^{\infty}\right)$ in $\mathcal{O}(1)$ time in the PILLAR model．
${ }^{r}$ Corollary $2.9\left(\operatorname{LCP}\left(S, Q^{\infty}[\ell \ldots r)\right)\right.$ ）．Given strings $S$ and $Q$ and integers $0 \leq \ell \leq r$ ，we can compute $\operatorname{LCP}\left(S, Q^{\infty}[\ell \ldots r)\right)$ in $\mathcal{O}(1)$ time in the PILLAR model．

Proof．We first compute $\operatorname{LCP}(S, Q[\ell \bmod |Q| \ldots|Q|))$ using a primitive operation．If we reach the end of the string $Q$ ，we continue with an $\operatorname{LCP}\left(S[|Q|-\ell \bmod |Q| \ldots|S|), Q^{\infty}\right)$ query，implemented using Lemma 2.8 This yields $\operatorname{LCP}\left(S, Q^{\infty}[\ell \ldots)\right)$ ，so we cap the obtained value with $r-\ell$ to retrieve $\operatorname{LCP}\left(S, Q^{\infty}[\ell \ldots r)\right)$ ．

A similar procedure yields an analogous generalization of the primitive LCP ${ }^{R}$ operation．
『 Corollary $2.10\left(\operatorname{LCP}^{R}\left(S, Q^{\infty}[\ell \ldots r)\right)\right.$ ．Given strings $S$ and $Q$ and integers $0 \leq \ell \leq r$ ，we can compute $\operatorname{LCP}^{R}\left(S, Q^{\infty}[\ell \ldots r)\right)$ in $\mathcal{O}(1)$ time in the PILLAR model．

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Lastly，we discuss an operation finding all exact occurrences of a given string $P$ in a given string $T$ ．
『 Lemma 2.11 （ExactMatches $(P, T)$ ）．Let $T$ denote a string of length $n$ and let $P$ denote a string of length $m$ ．We can compute the set $\operatorname{Occ}(P, T)$ using $\mathcal{O}(n / \operatorname{per}(P))$ time and $\mathcal{O}(n / m)$ PILLAR operations．

Proof．We perform an $\operatorname{IPM}\left(P, T_{i}\right)$ query with $T_{i}:=T[i m \ldots \min \{n,(i+2) m-1\})$ for each $i \in[\lfloor n / m\rfloor]$ ； that is a total of $\mathcal{O}(n / m)$ PILLAR operations．Each occurrence $j \in \operatorname{Occ}(P, T)$ corresponds to a single occur－ rence of $P$ in a single $T_{i}$ ，namely，$j \bmod m \in \operatorname{Occ}\left(P, T_{\lfloor j / m\rfloor}\right)$ ，and vice versa．Furthermore，each $\operatorname{IPM}\left(P, T_{i}\right)$ query returns an arithmetic progression with difference $\operatorname{per}(P)$ ，which thus consists of $\mathcal{O}(\mathrm{m} / \operatorname{per}(P))$ elements．Hence，the total number of elements of all arithmetic progressions is $\mathcal{O}(n / \operatorname{per}(P))$ ．

We conclude this section with introducing the concept of a generator．
『 Definition 2.12 （Generator of a set）．For an（ordered）set $S$ ，an $(\mathcal{O}(P), \mathcal{O}(Q))$－time generator of $S$ is a data structure that after $\mathcal{O}(P)$－time initialization in the PILLAR model，supports the following operation： －Next：In the ith call of Next，return the ith smallest element of the set $S$（or $\perp$ if $i>|S|$ ），using $\mathcal{O}(Q(i))$ time in the PILLAR model．

Note that a generator is not specific to the PILLAR model．We use generators to obtain positions where two strings differ－either by a mismatch or by an edit．Consult Sections 4.1 and 6.1 for the details，as well as for other PILLAR operations specific to pattern matching with mismatches or edits，respectively．

## 3 Improved Structural Insights into Pattern Matching with Mismatches

In this section，we provide insight into the structure of $k$－mismatch occurrences of a pattern $P$ in a text $T$ ． In particular，we improve the result of［11］and show the following asymptotically tight characterization （which is Main Theorem 4 with explicit constants and without the restriction on the length of $T$ ）．

Theorem 3．1．Given a pattern $P$ of length $m$ ，a text $T$ of length $n$ ，and a threshold $k \in[1 \ldots m]$ ，at least one of the following holds：
－The number of $k$－mismatch occurrences of $P$ in $T$ is bounded by $\left|\mathrm{Occ}_{k}^{H}(P, T)\right| \leq 576 \cdot n / m \cdot k$ ．
－There is a primitive string $Q$ of length $|Q| \leq m / 128 k$ that satisfies $\delta_{H}\left(P, Q^{*}\right)<2 k$ ．

## 3．1 Characterization of the Periodic Case

In order to prove Theorem 3．1．we first discuss the（approximately）periodic case，that is，the case when we have $\delta_{H}\left(P, Q^{*}\right)<2 k$ ．In particular，we prove the following theorem，which strengthens［11，Claim 3．1］．
－Theorem 3.2 （Compare Main Theorem 5and［11］Claim 3．1］）．Let $P$ denote a pattern of length $m$ ，let $T$ denote a text of length $n \leq 3 / 2 m$ ，and let $k \in[0 \ldots m]$ denote a threshold．Suppose that both $T[0 \ldots m)$ and $T\left[n-m \ldots n\right.$ ）are $k$－mismatch occurrences of $P$（that is，$\{0, n-m\} \subseteq \operatorname{Occ}_{k}^{H}(P, T)$ ）．If there is a positive integer $d \geq 2 k$ and a primitive string $Q$ with $|Q| \leq m / 8 d$ and $\delta_{H}\left(P, Q^{*}\right) \leq d$ ，then each of following holds：
（a）Every position in $\operatorname{Occ}_{k}^{H}(P, T)$ is a multiple of $|Q|$ ．
（b）The string $T$ satisfies $\delta_{H}\left(T, Q^{*}\right) \leq 3 d$ ．
（c）The set $\mathrm{Occ}_{k}^{H}(P, T)$ can be decomposed into $3 d(d+1)$ arithmetic progressions with difference $|Q|$ ．
（d）If $\delta_{H}\left(P, Q^{*}\right)=d$ ，then $\left|\mathrm{Occ}_{k}^{H}(P, T)\right| \leq 6 d$ ．
Before proving Theorem 3.2 ，we characterize the values $\delta_{H}(T[j|Q| \ldots j|Q|+m), P)$ under an extra assumption that $\delta_{H}\left(T, Q^{*}\right)$ is small as well；this assumption is dropped in Theorem 3．2


Figure 3 In both strings, all blocks apart from the last ones are of length $q$. For each string $X \in\{P, T\}$, we show the characters at positions in $\operatorname{Mis}\left(X, Q^{*}\right)$ only. At position $j q$ in $T$, we place $\mu_{j}\left(\tau_{2}, \pi_{1}\right)+\mu_{j}\left(\tau_{4}, \pi_{2}\right)+\mu_{j}\left(\tau_{5}, \pi_{3}\right)=$ $1+2+1=4$ marks. We have $\delta_{H}\left(P, Q^{*}\right)=\left|\left\{\pi_{1}, \pi_{2}, \pi_{3}\right\}\right|=3$ and $\delta_{H}\left(T[j q \ldots j q+m), Q^{*}\right)=\left|\left\{\tau_{2}, \tau_{3}, \tau_{4}, \tau_{5}\right\}\right|=4$. Using Claim 3.4 we obtain that $h_{j}=3$; the three corresponding mismatches are indicated by asterisks.

Lemma 3.3. Let $P$ denote a pattern of length $m$ and let $T$ denote a text of length $n \leq 3 / 2 m$. Further, let $Q$ denote a string of length $q$ and set $d:=\delta_{H}\left(P, Q^{*}\right)$ and $d^{\prime}:=\delta_{H}\left(T, Q^{*}\right)$. Then, the sequence of values $h_{j}:=\delta_{H}(T[j q \cdots j q+m), P)$ for $0 \leq j \leq(n-m) / q$ contains at most $d^{\prime}(2 d+1)$ entries $h_{j}$ with $h_{j} \neq h_{j+1}$ and, unless $d=0$, at most $2 d^{\prime}$ entries $h_{j}$ with $h_{j} \leq d / 2$.

Proof. For every $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$ and $\pi \in \operatorname{Mis}\left(P, Q^{*}\right)$, let us put $\left(2-\delta_{H}(P[\pi], T[\tau])\right)$ marks at position $\tau-\pi$ in $T$, if it exists. For each $0 \leq j \leq(n-m) / q$, let $\mu_{j}(\tau, \pi)$ denote the number of marks placed at position $j q$ due to the mismatches $\tau$ in $T$ and $\pi$ in $P$, that is,

$$
\mu_{j}(\tau, \pi):=\left\{\begin{array}{cl}
2-\delta_{H}(P[\pi], T[\tau]) & \text { if } \pi \in \operatorname{Mis}\left(P, Q^{*}\right) \text { and } \tau=j q+\pi \in \operatorname{Mis}\left(T, Q^{*}\right), \\
0 & \text { otherwise. }
\end{array}\right.
$$

Further, define $\mu_{j}:=\sum_{\tau, \pi} \mu_{j}(\tau, \pi)$ as the total number of marks at position $j q$.
Next, for every $0 \leq j \leq(n-m) / q$, we relate the Hamming distance $h_{j}:=\delta_{H}(T[j q . . j q+m), P)$ to the number of marks $\mu_{j}$ at position $j q$ and the Hamming distances $\delta_{H}\left(T[j q \ldots j q+m), Q^{*}\right)$ and $\delta_{H}\left(P, Q^{*}\right)$; consult Figure 3 for an illustration.

『Claim 3.4. For each $0 \leq j \leq(n-m) / q$, we have $h_{j}=\delta_{H}\left(P, Q^{*}\right)+\delta_{H}\left(T[j q \ldots j q+m), Q^{*}\right)-\mu_{j}$.
Proof. We show the following equivalent statement:

$$
\begin{equation*}
|\operatorname{Mis}(T[j q \ldots j q+m), P)|=\left|\operatorname{Mis}\left(P, Q^{*}\right)\right|+\left|\operatorname{Mis}\left(T[j q \ldots j q+m), Q^{*}\right)\right|-\sum_{\tau, \pi} \mu_{j}(\tau, \pi) . \tag{1}
\end{equation*}
$$

By construction, $\mu_{j}(\tau, \pi)=0$ whenever $\tau \neq \pi+j q$. Hence, we can prove (1) by showing that for every position $\pi \in[0 \ldots m)$ in $P$ and every position $\tau:=j q+\pi$ in $T$, the following equation holds:

$$
\delta_{H}(T[\tau], P[\pi])=\delta_{H}\left(P[\pi], Q^{\infty}[\pi]\right)+\delta_{H}\left(T[\tau], Q^{\infty}[\tau]\right)-\mu_{j}(\tau, \pi) .
$$

We proceed by case distinction on whether $\pi \in \operatorname{Mis}\left(P, Q^{*}\right)$ and whether $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$.

- If $\pi \notin \operatorname{Mis}\left(P, Q^{*}\right)$ and $\tau \notin \operatorname{Mis}\left(T, Q^{*}\right)$, then we have $P[\pi]=Q^{\infty}[\pi]=Q^{\infty}[\tau]=T[\tau]$ and thus

$$
\delta_{H}(T[\tau], P[\pi])=0=0+0-0=\delta_{H}\left(P[\pi], Q^{\infty}[\pi]\right)+\delta_{H}\left(T[\tau], Q^{\infty}[\tau]\right)-\mu_{j}(\tau, \pi) .
$$

- If $\pi \in \operatorname{Mis}\left(P, Q^{*}\right)$ and $\tau \notin \operatorname{Mis}\left(T, Q^{*}\right)$, then we have $P[\pi] \neq Q^{\infty}[\pi]=Q^{\infty}[\tau]=T[\tau]$ and thus

$$
\delta_{H}(T[\tau], P[\pi])=1=1+0-0=\delta_{H}\left(P[\pi], Q^{\infty}[\pi]\right)+\delta_{H}\left(T[\tau], Q^{\infty}[\tau]\right)-\mu_{j}(\tau, \pi) .
$$

- If $\pi \notin \operatorname{Mis}\left(P, Q^{*}\right)$ and $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$, then we have $P[\pi]=Q^{\infty}[\pi]=Q^{\infty}[\tau] \neq T[\tau]$ and thus

$$
\delta_{H}(T[\tau], P[\pi])=1=0+1-0=\delta_{H}\left(P[\pi], Q^{\infty}[\pi]\right)+\delta_{H}\left(T[\tau], Q^{\infty}[\tau]\right)-\mu_{j}(\tau, \pi) .
$$

## Faster Approximate Pattern Matching: A Unified Approach

- If $\pi \in \operatorname{Mis}\left(P, Q^{*}\right)$ and $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$, then we have $P[\pi] \neq Q^{\infty}[\pi]=Q^{\infty}[\tau] \neq T[\tau]$ and thus

$$
\delta_{H}(T[\tau], P[\pi])=1+1-\left(2-\delta_{H}(T[\tau], P[\pi])\right)=\delta_{H}\left(P[\pi], Q^{\infty}[\pi]\right)+\delta_{H}\left(T[\tau], Q^{\infty}[\tau]\right)-\mu_{j}(\tau, \pi)
$$

Combining the equations obtained for every pair of positions $\pi$ and $\tau$, we derive (1).
In particular, Claim 3.4 yields

$$
h_{j+1}-h_{j}=\left|\operatorname{Mis}\left(T, Q^{*}\right) \cap[j q+m \ldots(j+1) q+m)\right|-\left|\operatorname{Mis}\left(T, Q^{*}\right) \cap[j q \ldots(j+1) q)\right|-\mu_{j+1}+\mu_{j} .
$$

Hence, in order for $h_{j+1}$ not to equal $h_{j}$, at least one of the four terms on the right-hand side of the equation above must be non-zero. Let us analyze when this is possible. To that end, we first observe that the set $\operatorname{Mis}\left(T, Q^{*}\right) \cap[j q+m \ldots(j+1) q+m)$ contains only elements $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$ with $\tau \geq m$, and that the set $\operatorname{Mis}\left(T, Q^{*}\right) \cap[j q \ldots(j+1) q)$ only contains elements $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$ with $\tau<n-m$. Using $n \leq \frac{3}{2} m$, we observe that $h_{j+1}$ can be different from $h_{j}$ due to the first or second term at most $d^{\prime}$ times. Further, each non-zero value in one of the terms $\mu_{j+1}$ and $\mu_{j}$ can be attributed to a marked position ( $j q$ or $(j+1) q$, respectively). The total number of marked positions is at most $d d^{\prime}$, so $h_{j+1}$ can be different from $h_{j}$ due one of the terms $\mu_{j+1}$ or $\mu_{j}$ at most $2 d d^{\prime}$ times. In total, we conclude that the number of entries $h_{j}$ with $h_{j} \neq h_{j+1}$ is at most $d^{\prime}(2 d+1)$.

Next, observe that $\mu_{j} \leq 2\left|\operatorname{Mis}\left(T, Q^{*}\right) \cap[j q \ldots j q+m)\right|=2 \delta_{H}\left(T[j q \ldots j q+m), Q^{*}\right)$, and therefore

$$
h_{j}=\delta_{H}\left(P, Q^{*}\right)+\delta_{H}\left(T[j q \ldots j q+m), Q^{*}\right)-\mu_{j} \geq d-\mu_{j} / 2
$$

Consequently, $h_{j} \leq d / 2$ yields $\mu_{j} \geq d$, that is, that there are at least $d$ marks at position $j q$. Given that the total number of marks is at most $2 d d^{\prime}$, the number of entries $h_{j}$ with $h_{j} \leq d / 2$ is at most $2 d^{\prime}$, assuming that $d>0$.

Now, we drop the assumption that $\delta_{H}\left(T, Q^{*}\right)$ is small and prove Theorem 3.2.
「 Theorem 3.2 (Compare Main Theorem 5 and [11] Claim 3.1]). Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n \leq 3 / 2 m$, and let $k \in[0 \ldots m]$ denote a threshold. Suppose that both $T[0 \ldots m)$ and $T\left[n-m \ldots n\right.$ ) are $k$-mismatch occurrences of $P$ (that is, $\{0, n-m\} \subseteq \operatorname{Occ}_{k}^{H}(P, T)$ ). If there is a positive integer $d \geq 2 k$ and a primitive string $Q$ with $|Q| \leq m / 8 d$ and $\delta_{H}\left(P, Q^{*}\right) \leq d$, then each of following holds:
(a) Every position in $\operatorname{Occ}_{k}^{H}(P, T)$ is a multiple of $|Q|$.
(b) The string $T$ satisfies $\delta_{H}\left(T, Q^{*}\right) \leq 3 d$.
(c) The set $\mathrm{Occ}_{k}^{H}(P, T)$ can be decomposed into $3 d(d+1)$ arithmetic progressions with difference $|Q|$.
(d) If $\delta_{H}\left(P, Q^{*}\right)=d$, then $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 6 d$.

Proof. Consider any position $\ell \in \mathrm{Occ}_{k}^{H}(P, T)$. By the definition of a $k$-mismatch occurrence, we have $\delta_{H}(T[\ell \ldots \ell+m), P) \leq k \leq d / 2$. Combining this inequality with $\delta_{H}\left(P, Q^{*}\right) \leq d$ via the triangle inequality yields $\delta_{H}\left(T[\ell \ldots \ell+m), Q^{*}\right) \leq 3 / 2 d$. Similarly, for the position $0 \in \operatorname{Occ}_{k}^{H}(P, T)$, we obtain $\delta_{H}(T[0 \ldots m)$, $\left.Q^{*}\right) \leq 3 / 2 d$, which lets us compare the overlapping parts of $Q^{\infty}$. Replacing strings by superstrings and applying the triangle inequality yields

$$
\begin{aligned}
\delta_{H}\left(Q^{\infty}[\ell \ldots m), Q^{\infty}[0 \ldots m-\ell)\right) & \leq \delta_{H}\left(T[\ell \ldots m), Q^{\infty}[\ell \ldots m)\right)+\delta_{H}\left(T[\ell \ldots m), Q^{\infty}[0 \ldots m-\ell)\right) \\
& \leq \delta_{H}\left(T[0 \ldots m), Q^{\infty}[0 \ldots m)\right)+\delta_{H}\left(T[\ell \ldots \ell+m), Q^{\infty}[0 \ldots m)\right) \\
& =\delta_{H}\left(T[0 \ldots m), Q^{*}\right)+\delta_{H}\left(T[\ell \ldots \ell+m), Q^{*}\right) \\
& \leq 3 d .
\end{aligned}
$$

Towards a proof by contradiction, suppose that $\ell$ is not a multiple of $|Q|$. As $Q$ is primitive, we have

$$
3 d \geq \delta_{H}\left(Q^{\infty}[\ell \ldots m), Q^{\infty}[0 \ldots m-\ell)\right) \geq\left\lfloor\frac{m-\ell}{|Q|}\right\rfloor \geq\left\lfloor\frac{m / 2}{m / 8 d}\right\rfloor=4 d
$$

where the second bound follows from $\ell \leq m / 2$ and $|Q| \leq m / 8 d$. This contradiction yields Claim (a).
In order to prove Claim (b), we observe that $n-m \in \operatorname{Occ}_{k}^{H}(P, T)$ is a multiple of $|Q|$. Consequently,

$$
\delta_{H}\left(T, Q^{*}\right)=\delta_{H}\left(T[0 \ldots n-m), Q^{*}\right)+\delta_{H}\left(T[n-m \ldots n), Q^{*}\right) \leq \delta_{H}\left(T[0 \ldots m), Q^{*}\right)+\frac{3}{2} d \leq 3 d
$$

which concludes the proof of Claim (b).
For a proof of Claims (c) and (d), we apply Lemma 3.3. Due to Claim (a), each position in $\operatorname{Occ}_{k}^{H}(P, T)$ corresponds to an entry $h_{j}$ with $h_{j} \leq k$. In particular, each block of consecutive entries $h_{j}, \ldots, h_{j^{\prime}}$ not exceeding $k$ yields an arithmetic progression (with difference $|Q|$ ) in $\operatorname{Occ}_{k}^{H}(P, T)$. The number of entries $h_{j}$ with $h_{j} \leq k<h_{j+1}$ or $h_{j}>k \geq h_{j+1}$ is in total at most $3 d(2 d+1)$, so the number of arithmetic progressions is at most $1+1 / 2 \cdot 3 d(2 d+1) \leq 3 d(d+1)$, which proves Claim (c).

For Claim (d), we observe that if $d=\delta_{H}\left(P, Q^{*}\right)$, then each position in $\operatorname{Occ}_{k}^{H}(P, T)$ corresponds to an entry $h_{j}$ with $h_{j} \leq k \leq d / 2$; thus $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 2 \cdot 3 d=6 d$.

『 Corollary 3.5. Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n$, and let $k \in[0 \ldots m]$ denote a threshold. If there is a positive integer $d \geq 2 k$ and a primitive string $Q$ with $|Q| \leq m / 8 d$ and $\delta_{H}\left(P, Q^{*}\right) \leq d$, then the set $\operatorname{Occ}_{k}^{H}(P, T)$ can be decomposed into $6 \cdot n / m \cdot d(d+1)$ arithmetic progressions with difference $|Q|$. Moreover, if $\delta_{H}\left(P, Q^{*}\right)=d$, then $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 12 \cdot n / m \cdot d$.

Proof. Partition the string $T$ into $\lfloor 2 n / m\rfloor$ blocks $T_{0}, \ldots, T_{\lfloor 2 n / m\rfloor-1}$ of length less than $\frac{3}{2} m$ each, where the $i$ th block starts at position $\lfloor i \cdot m / 2\rfloor$; formally, we set $T_{i}:=T[\lfloor i \cdot m / 2\rfloor \ldots \min \{n,\lfloor(i+3) \cdot m / 2\rfloor-1\})$. If $\operatorname{Occ}_{k}^{H}\left(P, T_{i}\right) \neq \emptyset$, we define $T_{i}^{\prime}$ to be the shortest fragment of $T_{i}$ containing all $k$-mismatch occurrences of $P$ in $T_{i}$. As a result, $T_{i}^{\prime}$ satisfies the assumptions of Theorem 3.2 Hence, $\mathrm{Occ}_{k}^{H}\left(P, T_{i}^{\prime}\right)$ can be decomposed into $3 d(d+1)$ arithmetic progressions with difference $|Q|$, and $\left|\operatorname{Occ}\left(P, T_{i}^{\prime}\right)\right| \leq 6 d$ if $\delta_{H}\left(P, Q^{*}\right)=d$.

We conclude that $\operatorname{Occ}_{k}^{H}\left(P, T_{i}^{\prime}\right)$ decomposes into $6 \cdot n / m \cdot d(d+1)$ arithmetic progressions with difference $|Q|$; further, $\left|\operatorname{Occ}\left(P, T_{i}\right)\right| \leq 12 \cdot n / m \cdot d$ if $\delta_{H}\left(P, Q^{*}\right)=d$.

### 3.2 The Non-Periodic Case

Having dealt with the (approximately) periodic case, we now turn to the general case. In particular, we show that whenever the string $P$ is sufficiently far from being periodic, the number of $k$-mismatch occurrences of $P$ in any string $T$ of length $n \leq \frac{3}{2} m$ is $\mathcal{O}(k)$.

Intuitively, we proceed (and thereby prove Theorem 3.1) as follows: We first analyze the string $P$ for useful structure that can help in bounding the number of occurrences of $P$ in any string $T$. If we fail to find any special structure in $P$, then we conclude that the string $P$ is close to a periodic string with a small period (compared to $|P|$ )—a case that we already understand thanks to the previous subsection.
『 Lemma 3.6. Given a string $P$ of length $m$ and and a threshold $k \in[1 \ldots m]$, at least one of the following holds:
(a) The string $P$ contains $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k}$ each having period $\operatorname{per}\left(B_{i}\right)>m / 128 k$ and length $\left|B_{i}\right|=\lfloor m / 8 k\rfloor$.
(b) The string $P$ contains $r$ disjoint repetitive regions $R_{1}, \ldots, R_{r}$ of total length $\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and has a primitive approximate period $Q_{i}$ with $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{H}\left(R_{i}, Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$.
(c) The string $P$ has a primitive approximate period $Q$ with $|Q| \leq m / 128 k$ and $\delta_{H}\left(P, Q^{*}\right)<8 k$.

```
Algorithm 1 A constructive proof of Lemma 3.6 .
    \(\mathcal{B} \leftarrow\} ; \mathcal{R} \leftarrow\{ \} ;\)
    while true do
        Consider the fragment \(P^{\prime}=P[j \ldots j+\lfloor m / 8 k\rfloor)\) of the next \(\lfloor m / 8 k\rfloor\) unprocessed characters
        of \(P\);
        if \(\operatorname{per}\left(P^{\prime}\right)>m / 128 k\) then
            \(\mathcal{B} \leftarrow \mathcal{B} \cup\left\{P^{\prime}\right\} ;\)
            if \(|\mathcal{B}|=2 k\) then return breaks \(\mathcal{B}\);
        else
            \(Q \leftarrow P\left[j \ldots j+\operatorname{per}\left(P^{\prime}\right)\right) ;\)
            Search for a prefix \(R\) of \(P[j \ldots m)\) with \(|R|>\left|P^{\prime}\right|\) and \(\delta_{H}\left(R, Q^{*}\right)=\lceil 8 k / m \cdot|R|\rceil ;\)
            if such \(R\) exists then
                \(\mathcal{R} \leftarrow \mathcal{R} \cup\{(R, Q)\} ;\)
                if \(\sum_{(R, Q) \in \mathcal{R}}|R| \geq 3 / 8 \cdot m\) then
                    return repetitive regions (and their corresponding periods) \(\mathcal{R}\);
            else
                Search for a suffix \(R^{\prime}\) of \(P\) with \(\left|R^{\prime}\right| \geq m-j\) and \(\delta_{H}\left(R^{\prime}, \operatorname{rot}^{\left|R^{\prime}\right|-m+j}(Q)^{*}\right)=\left\lceil 8 k / m \cdot\left|R^{\prime}\right|\right\rceil ;\)
            if such \(R^{\prime}\) exists then return repetitive region \(\left(R^{\prime}, \operatorname{rot}^{\left|R^{\prime}\right|-m+j}(Q)\right)\);
            else return approximate period \(\operatorname{rot}^{j}(Q)\);
```

Proof. We prove the claim constructively, that is, we construct either a set $\mathcal{B}$ of $2 k$ breaks, or a set $\mathcal{R}$ of repetitive regions, or, if we fail to construct either, we derive an approximate string period $Q$ of the string $P$ with the desired properties.

We process the string $P$ from left to right as follows: If the fragment $P^{\prime}$ of the next $\lfloor m / 8 k\rfloor$ (unprocessed) characters of $P$ has a long period, we have found a new break and continue (or return the found set of $2 k$ breaks). Otherwise, if $P^{\prime}$ has a short string period $Q$, we try to extend the fragment $P^{\prime}$ (to the right) into a repetitive region. If we succeed, we have found a new repetitive region and continue (or return the found set of repetitive regions if the total length of all repetitive regions found so far is at least $3 / 8 \cdot m$ ). If we fail to construct a new repetitive region, then we conclude that the suffix of $P$ starting with $P^{\prime}$ has an approximate period $Q$. We try to construct a repetitive region by extending this suffix to the left, dropping all the repetitive regions computed beforehand. If we fail again, we declare that an appropriate rotation of $Q$ is an approximate period of the string $P$. Consider Algorithm 1 for a detailed description.

By construction, all breaks in the set $\mathcal{B}$ and repetitive regions in the set $\mathcal{R}$ returned by the algorithm are disjoint and satisfy the claimed properties. To prove that the algorithm is also correct when it fails to find a new repetitive region, we start by bounding from above the length of the processed prefix of $P$.

『
Claim 3.7. Whenever we consider a new fragment $P^{\prime}=P[j \ldots j+\lfloor m / 8 k\rfloor)$ of the next $\lfloor m / 8 k\rfloor$ unprocessed characters of $P$, such a fragment starts at a position $j<5 / 8 \cdot m$.

Proof. Observe that whenever we consider a new fragment $P[j \ldots j+\lfloor m / 8 k\rfloor)$, the string $P[0 \ldots j)$ has been partitioned into breaks and repetitive regions. The total length of breaks is less than $2 k\lfloor m / 8 k\rfloor \leq 2 / 8 \cdot m$, and the total length of repetitive regions is less than $3 / 8 \cdot m$. Hence, $j<5 / 8 \cdot m$, yielding the claim.

Note that Claim 3.7 also shows that whenever we consider a new fragment $P^{\prime}$ of $\lfloor m / 8 k\rfloor$ characters, there is indeed such a fragment, that is, $P^{\prime}$ is well-defined.

Now, consider the case when, for a fragment $P^{\prime}=P[j \ldots j+\lfloor m / 8 k\rfloor$ ) (that is not a break) and its string period $Q=P\left[j \ldots j+\operatorname{per}\left(P^{\prime}\right)\right)$, we fail to obtain a new repetitive region $R$. In this case, we search for a repetitive region $R^{\prime}$ of length $\left|R^{\prime}\right| \geq m-j$ that is a suffix of $P$ and has an approximate period $Q^{\prime}:=\operatorname{rot}^{\left|R^{\prime}\right|-m+j}(Q)$. If we indeed find such a region $R^{\prime}$, then $\left|R^{\prime}\right| \geq m-j \geq m-5 / 8 \cdot m=3 / 8 \cdot m$ by Claim 3.7. so $R^{\prime}$ is long enough to be reported on its own. However, if we fail to find such $R^{\prime}$, we need to show that $\operatorname{rot}^{j}(Q)$ can be reported as an approximate period of $P$, that is, $\delta_{H}\left(P, \operatorname{rot}^{j}(Q)^{*}\right)<8 k$.

We first derive $\delta_{H}\left(P[j \ldots m), Q^{*}\right)<\lceil 8 k / m \cdot(m-j)\rceil$. For this, we inductively prove that the values $\Delta_{\rho}:=\lceil 8 k / m \cdot \rho\rceil-\delta_{H}\left(P[j \ldots j+\rho), Q^{*}\right)$ for $\rho \in\left[\left|P^{\prime}\right| \ldots m-j\right]$ are all at least 1 . In the base case of $\rho=\left|P^{\prime}\right|$, we have $\Delta_{\rho}=1-0$ because $Q$ is the string period of $P^{\prime}$. To carry out an inductive step, suppose that $\Delta_{\rho-1} \geq 1$ for some $\rho \in\left[\left|P^{\prime}\right| \ldots m-j\right]$. Notice that $\Delta_{\rho} \geq \Delta_{\rho-1}-1 \geq 0$ : The first term in the definition of $\Delta_{\rho}$ has not decreased compared to $\Delta_{\rho-1}$, and the second term $\delta_{H}\left(P[j \ldots j+\rho), Q^{*}\right)$ may have increased by at most one. Moreover, $\Delta_{\rho} \neq 0$ because $R=P[j \ldots j+\rho)$ could not be reported as a repetitive region. Since $\Delta_{\rho} \in \mathbb{Z}$, we conclude that $\Delta_{\rho} \geq 1$. This inductive reasoning ultimately shows that $\Delta_{m-j}>0$, that is, $\delta_{H}\left(P[j \ldots m), Q^{*}\right)<\lceil 8 k / m \cdot(m-j)\rceil$.

A symmetric argument holds for the values $\Delta_{\rho}^{\prime}:=\lceil 8 k / m \cdot \rho\rceil-\delta_{H}\left(P[m-\rho \ldots m), \operatorname{rot}^{\rho-m+j}(Q)^{*}\right)$ for $\rho \in[m-j \ldots m]$ because no repetitive region $R^{\prime}$ was found as an extension of $P[j \ldots m)$ to the left. Thus, $\delta_{H}\left(P, \operatorname{rot}^{j}(Q)^{*}\right)<8 k$, that is, $\operatorname{rot}^{j}(Q)$ is an approximate period of $P$.

In the next steps, we discuss how to exploit the structure obtained by Lemma 3.6. First, we discuss the case that a string $P$ contains $2 k$ disjoint breaks.

Lemma 3.8. Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n$, and let $k \in[1 \ldots m]$ denote a threshold. Suppose that $P$ contains $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k}$ each satisfying $\operatorname{per}\left(B_{i}\right) \geq$ $m / 128 k$. Then, $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 256 \cdot n / m \cdot k$.

Proof. For every break $B_{i}=P\left[b_{i} \ldots b_{i}+\left|B_{i}\right|\right)$ we mark a position $j$ in $T$ if $j+b_{i} \in \operatorname{Occ}\left(B_{i}, T\right)$.
${ }^{『}$ Claim 3.9. We place at most $256 \cdot n / m \cdot k^{2}$ marks in total.
Proof. Fix a break $B_{i}$ and notice that the positions in $\operatorname{Occ}\left(B_{i}, T\right)$ are at distance at least per $\left(B_{i}\right)$ from each other. Hence, for the break $B_{i}$, we place at most $128 \cdot n / m \cdot k$ marks in $T$. In total, we therefore place at most $2 k \cdot 128 n / m \cdot k=256 \cdot n / m \cdot k^{2}$ marks in $T$.

Next, we show that every $k$-mismatch occurrence of $P$ in $T$ starts at a position with at least $k$ marks.
『 Claim 3.10. Each position $\ell \in \operatorname{Occ}_{k}^{H}(P, T)$ has at least $k$ marks in $T$.
Proof. Fix $\ell \in \operatorname{Occ}_{k}^{H}(P, T)$. Out of the $2 k$ breaks, at least $k$ breaks are matched exactly, as not matching a break exactly incurs at least one mismatch. If a break $B_{i}$ is matched exactly, then we have $\ell+b_{i} \in \operatorname{Occ}\left(B_{i}, T\right)$. Hence, we have placed a mark at position $\ell$. Thus, there is a mark at position $\ell$ for every break $B_{i}$ matched exactly in the corresponding occurrence of $P$ in $T$. In total, there are at least $k$ marks at position $\ell$ in $T$.

By Claims 3.9 and 3.10 we have $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq\left(256 \cdot n / m \cdot k^{2}\right) / k=256 \cdot n / m \cdot k$.
Secondly, we discuss how to use repetitive regions in the string $P$ to bound $\left|\operatorname{Occ}_{k}^{H}(P, T)\right|$.
${ }^{\boldsymbol{r}}$ Lemma 3.11. Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n$, and let $k \in[1 \ldots m]$ denote a threshold. If $P$ contains disjoint repetitive regions $R_{1}, \ldots, R_{r}$ of total length at least $\sum_{i=1}^{r}\left|R_{i}\right| \geq$ $3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and has a primitive approximate period $Q_{i}$ with $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{H}\left(R_{i}, Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$, then $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 576 \cdot n / m \cdot k$.

Proof. Set $m_{R}:=\sum_{i=1}^{r}\left|R_{i}\right|$. For each repetitive region $R_{i}=P\left[r_{i} \ldots r_{i}+\left|R_{i}\right|\right)$, set $k_{i}:=\left\lfloor 4 k / m \cdot\left|R_{i}\right|\right\rfloor$, and place $\left|R_{i}\right|$ marks at each position $j$ with $j+r_{i} \in \operatorname{Occ}_{k_{i}}^{H}\left(R_{i}, T\right)$.

『
Claim 3.12. The total number of marks placed is at most $192 \cdot n / m \cdot k \cdot m_{R}$.
Proof. We use Corollary 3.5 to bound $\left|\operatorname{Occ}_{k_{i}}^{H}\left(R_{i}, T\right)\right|$. For this, we set $d_{i}:=\delta_{H}\left(R_{i}, Q_{i}^{*}\right)$ and notice that $d_{i}=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil \leq 16 \cdot k / m \cdot\left|R_{i}\right|$ since $\left|R_{i}\right| \geq m / 8 k$. Moreover, $d_{i} \geq 2 k_{i}$ and $\left|Q_{i}\right| \leq m / 128 k \leq\left|R_{i}\right| / 8 d_{i}$ due to $d_{i} \leq 16 \cdot k / m \cdot\left|R_{i}\right|$. Hence, the assumptions of Corollary 3.5 are satisfied. Consequently, $\left|\operatorname{Occ}_{k_{i}}^{H}\left(R_{i}, T\right)\right| \leq 12 \cdot n /\left|R_{i}\right| \cdot d_{i} \leq 192 \cdot n / m \cdot k$; the last inequality holds as $d_{i} \leq 16 \cdot k / m \cdot\left|R_{i}\right|$.

The total number of marks placed due to $R_{i}$ is therefore bounded by $192 \cdot n / m \cdot k \cdot\left|R_{i}\right|$. Across all repetitive regions, this sums up to $192 \cdot n / m \cdot k \cdot m_{R}$, yielding the claim.

Next, we show that every $k$-mismatch occurrence of $P$ in $T$, starts at a position with many marks.
『 Claim 3.13. Each $\ell \in \operatorname{Occ}_{k}^{H}(P, T)$ has at least $m_{R}-m / 4$ marks.
Proof. Let us fix $\ell \in \operatorname{Occ}_{k}^{H}(P, T)$ and denote $k_{i}^{\prime}:=\delta_{H}\left(R_{i}, T\left[\ell+r_{i} \ldots \ell+r_{i}+\left|R_{i}\right|\right)\right)$ to be the number of mismatches incurred by repetitive region $R_{i}$. Further, let $I:=\left\{i \in[1 \ldots r] \mid k_{i}^{\prime} \leq k_{i}\right\}=\{i \in[1 \ldots r] \mid$ $\left.k_{i}^{\prime} \leq 4 k / m \cdot\left|R_{i}\right|\right\}$ denote the set of indices of all repetitive regions that have $k_{i}$-mismatch occurrences at the corresponding positions in $T$. By construction, for each $i \in I$, we have placed $\left|R_{i}\right|$ marks at position $\ell$. Hence, the total number of marks at position $\ell$ is at least $\sum_{i \in I}\left|R_{i}\right|=m_{R}-\sum_{i \notin I}\left|R_{i}\right|$. It remains to bound the term $\sum_{i \notin I}\left|R_{i}\right|$. Using the definition of $I$, we obtain

$$
\sum_{i \notin I}\left|R_{i}\right|=\sum_{i \notin I} \frac{4 m k}{4 m k} \cdot\left|R_{i}\right|=\frac{m}{4 k} \cdot \sum_{i \notin I}\left(4 \cdot\left|R_{i}\right| / m \cdot k\right)<\frac{m}{4 k} \cdot \sum_{i \notin I} k_{i}^{\prime} \leq \frac{m}{4 k} \cdot \sum_{i=1}^{r} k_{i}^{\prime} \leq \frac{m}{4}
$$

where the last bound holds as, in total, all repetitive regions incur at most $\sum_{i=1}^{r} k_{i}^{\prime} \leq k$ mismatches (since all repetitive regions are pairwise disjoint). Hence, the number of marks placed is at least $m_{R}-m / 4$, completing the proof of the claim.

In total, by Claims 3.12 and 3.13 the number of $k$-mismatch occurrences of $P$ in $T$ is at most

$$
\left|\mathrm{Occ}_{k}^{H}(P, T)\right| \leq \frac{192 \cdot n / m \cdot k \cdot m_{R}}{m_{R}-m / 4}=\frac{192 \cdot n / m \cdot k}{1-m /\left(4 m_{R}\right)}
$$

As this bound is a decreasing function in $m_{R}$, the assumption $m_{R} \geq 3 / 8 \cdot m$ yields the upper bound

$$
\left|\mathrm{Occ}_{k}^{H}(P, T)\right| \leq \frac{192 \cdot n / m \cdot k \cdot 3 / 8 \cdot m}{3 / 8 \cdot m-m / 4}=576 \cdot n / m \cdot k,
$$

completing the proof.
Finally, we consider the case that $P$ is approximately periodic, but not too close to the periodic string in scope.

Lemma 3.14. Let $P$ denote a string of length $m$, let $T$ denote a string of length $n$, and let $k \in[1 \ldots m]$ denote threshold. If there is a primitive string $Q$ of length at most $|Q| \leq m / 128 k$ that satisfies $2 k \leq$ $\delta_{H}\left(P, Q^{*}\right) \leq 8 k$, then $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 96 \cdot n / m \cdot k$.

Proof. We apply Corollary 3.5 with $d=\delta_{H}\left(P, Q^{*}\right)$. As $2 k \leq d \leq 8 k$ yields $|Q| \leq m / 128 k \leq m / 8 d$, the assumptions of Corollary 3.5 are met. Consequently, $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 12 \cdot n / m \cdot d \leq 96 \cdot n / m \cdot k$.

Algorithm 2 A generator for the set $\operatorname{Mis}\left(S, Q^{*}\right)$.

```
MismGenerator( }S,\mp@subsup{Q}{}{*}\mathrm{ )
    return G }\leftarrow{S\leftarrowS;Q\leftarrowQ;i\leftarrow0}
Next(G}={S;Q;i}
    if i\geq|S| then return }\perp\mathrm{ ;
    \pi\leftarrow LCP(S[i.. |S| ), Q [ [i.. ));
    i\leftarrowi+\pi+1;
    if i> |S| then return }\perp\mathrm{ ;
    else return i-1;
```

Gathering Lemmas $3.6,3.8,3.11$ and 3.14 , we are now ready to prove Theorem 3.1, which we repeat here for convenience.
r Theorem 3.1. Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a threshold $k \in[1 \ldots m]$, at least one of the following holds:

- The number of $k$-mismatch occurrences of $P$ in $T$ is bounded by $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 576 \cdot n / m \cdot k$.
- There is a primitive string $Q$ of length $|Q| \leq m / 128 k$ that satisfies $\delta_{H}\left(P, Q^{*}\right)<2 k$.

Proof. We apply Lemma 3.6 on the string $P$ and proceed depending on the structure found in $P$.
If the string $P$ contains $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k}$ (in the sense of Lemma 3.6), we apply Lemma 3.8 and obtain that $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 256 \cdot n / m \cdot k$.

If the string $P$ contains $r$ disjoint repetitive regions $R_{1}, \ldots, R_{r}$ (again, in the sense of Lemma 3.6, we apply Lemma 3.11 and obtain that $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 576 \cdot n / m \cdot k$.

Otherwise, Lemma 3.6 guarantees that there is a primitive string $Q$ of length at most $|Q| \leq m / 128 k$ that satisfies $\delta_{H}\left(P, Q^{*}\right)<8 k$. If $\delta_{H}\left(P, Q^{*}\right) \geq 2 k$, then Lemma 3.14 yields $\left|\operatorname{Occ}_{k}^{H}(P, T)\right| \leq 96 \cdot n / m \cdot k$. If, however, $\delta_{H}\left(P, Q^{*}\right)<2 k$, then we are in the second alternative of the theorem statement.

## 4 Algorithm: Pattern Matching with Mismatches in the PILLAR Model

In this section, we implement the improved structural result (Theorem 3.1) from the previous section to obtain the following result.

『 Main Theorem 8. Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a positive integer $k \leq m$, we can compute (a representation of) the set $\operatorname{Occ}_{k}^{H}(P, T)$ using $\mathcal{O}\left(n / m \cdot k^{2} \log \log k\right)$ time plus $\mathcal{O}\left(n / m \cdot k^{2}\right)$ PILLAR operations.

In general, the algorithm follows the outline given by the proof of Theorem 3.1. We first show how to implement Lemma 3.6 to preprocess the given pattern $P$. Then, depending on the structure of $P$, we (construct and) use algorithms implementing the insights from the corresponding lemmas from the previous section.

### 4.1 Auxiliary PILLAR Model Operations for Pattern Matching with Mismatches

We start by introducing some commonly used operations for Pattern Matching with Mismatches and show how to implement them efficiently in the PILLAR model.

『 Lemma 4.1 （MismGenerator $\left(S, Q^{*}\right)$ ，MismGenerator ${ }^{R}\left(S,{ }^{*} Q\right)$ ）．For every pair of strings $S$ and $Q$ ，the sets $\operatorname{Mis}\left(S, Q^{*}\right)$ and $\operatorname{Mis}\left(S^{R},\left(Q^{R}\right)^{*}\right)$ admit $(\mathcal{O}(1), \mathcal{O}(1))$－time generators．

Proof．We only develop the MismGenerator generator；MismGenerator ${ }^{R}$ can be obtained similarly．
Given strings $S$ and $Q$ ，the generator itself just stores $S, Q$ ，and an index $i$ of the position after the last returned value by Next；initially，we set $i$ to 0 ．

We implement the Next operation by using Corollary 2.9 to compute $\pi=\operatorname{LCP}\left(S[i \ldots|S|), Q^{\infty}[i \ldots)\right)$ ． If we observe that $i+\pi=|S|$ ，that is，if we reached the end of the string $S$ ，then we return $\perp$ ．Otherwise， we report $i+\pi$ and update the index $i$ to $i+\pi+1$ ．See Algorithm 2 for a pseudo－code．

For the correctness，we observe that due to storing the index $i$ ，we are able to retrieve the suffixes of $S$ and $Q$ to be compared，so the correctness follows．

For the running time，we observe that the creation of a generator is only bookkeeping，which takes constant time．Further，the Next operation uses one call to the primitive LCP operation and a single call to the LCP operation from Corollary 2.9 ，which uses $\mathcal{O}(1)$ PILLAR operations．Thus in total，the Next operation also uses $\mathcal{O}(1)$ PILLAR operations，completing the proof．

「 Corollary 4.2 （Mismatches $\left.\left(S, Q^{*}\right)\right)$ ．Given strings $S$ and $Q$ ，we can compute the set $\operatorname{Mis}\left(S, Q^{*}\right)$ ， using $\mathcal{O}\left(\delta_{H}\left(S, Q^{*}\right)+1\right)$ primitive operations in the PILLAR model．

Proof．We use a MismGenerator from Lemma 4.1 and call its Next operation until the Next operation returns $\perp$ ．The claim follows．

Femma 4.3 （Verify $(S, T, k)$ ）．Let $S$ and $T$ denote strings of length $m$ each，and let $k \leq m$ denote a positive integer．Using $\mathcal{O}(k)$ PILLAR operations，we can check whether $\delta_{H}(S, T) \leq k$ ．

Proof．We use a MismGenerator from Lemma 4.1 and call its Next operation until either the Next operation returns $\perp$（in which case we return true）or until we obtain the（ $k+1$ ）st mismatch between $S$ and $T$（in which case we return false）．The claim follows．

## 4．2 Computing Structure in the Pattern

In this section，we show how to implement Lemma 3．6．While the proof of Lemma 3.6 is already constructive，we still need to fill in some implementation details．

『 Lemma 4.4 （Analyze $(P, k)$ ：Implementation of Lemma 3．6）．Let $P$ denote a string of length $m$ and let $k \leq m$ denote a positive integer．Then，there is an algorithm that computes one of the following：
（a） $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k} \preccurlyeq P$ each having period $\operatorname{per}\left(B_{i}\right)>m / 128 k$ and length $\left|B_{i}\right|=\lfloor m / 8 k\rfloor$ ； （b）disjoint repetitive regions $R_{1}, \ldots, R_{r} \preccurlyeq P$ of total length $\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and is constructed along with a primitive approximate period $Q_{i}$ such that $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{H}\left(R_{i}, Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil ;$ or
（c）a primitive approximate period $Q$ of $P$ with $|Q| \leq m / 128 k$ and $\delta_{H}\left(P, Q^{*}\right)<8 k$ ．
The algorithm uses $\mathcal{O}(k)$ time plus $\mathcal{O}(k)$ PILLAR operations．

Proof．Our implementation follows Algorithm 1 from the proof of Lemma 3.6 Recall that $P$ is processed from left to right and split into breaks and repetitive regions．In each iteration，the algorithm first considers a fragment of length $\lfloor m / 8 k\rfloor$ ．This fragment either becomes the next break（if its shortest period is long enough）or is extended to the right to a repetitive region（otherwise）．Having constructed sufficiently many breaks or repetitive regions of sufficiently large total length，the algorithm stops．Processing the string $P$

```
Algorithm 3 A PILLAR model implementation of Algorithm 1
Analyze \((P, k)\)
    \(j \leftarrow 0 ; r \leftarrow 1 ; b \leftarrow 1 ;\)
    while true do
        \(j^{\prime} \leftarrow j+\lfloor m / 8 k\rfloor ;\)
        if \(\operatorname{Period}\left(P\left[j \ldots j^{\prime}\right)\right)>m / 128 k\) then
            \(B_{b} \leftarrow P\left[j \ldots j^{\prime}\right) ;\)
            if \(b=2 k\) then return breaks \(B_{1}, \ldots, B_{2 k}\);
            \(b \leftarrow b+1 ; j \leftarrow j^{\prime} ;\)
        else
            \(q \leftarrow \operatorname{Period}\left(P\left[j \ldots j^{\prime}\right)\right) ;\)
            \(Q_{r} \leftarrow P[j \ldots j+q) ; \delta \leftarrow 0 ;\)
            generator \(\mathbf{G} \leftarrow\) MismGenerator \(\left(P[j \ldots m), Q_{r}{ }^{*}\right)\);
            while \(\delta<8 k / m \cdot\left(j^{\prime}-j\right)\) and \((\pi \leftarrow \operatorname{Next}(\mathbf{G})) \neq \perp\) do
                \(j^{\prime} \leftarrow j+\pi+1 ; \delta \leftarrow \delta+1 ;\)
            if \(\delta \geq 8 k / m \cdot\left(j^{\prime}-j\right)\) then
                    \(R_{r} \leftarrow P\left[j \ldots j^{\prime}\right) ;\)
                    if \(\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m\) then
                        return repetitive regions \(R_{1}, \ldots, R_{r}\) with periods \(Q_{1}, \ldots, Q_{r}\);
                    \(r \leftarrow r+1 ; j \leftarrow j^{\prime} ;\)
        else
            \(Q \leftarrow Q_{r} ; j^{\prime \prime} \leftarrow j ;\)
            generator \(\mathbf{G}^{\prime} \leftarrow\) MismGenerator \({ }^{R}\left(P[0 \ldots j),{ }^{*} Q\right)\);
            while \(\delta<8 k / m \cdot\left(m-j^{\prime \prime}\right)\) and \(\left(\pi \leftarrow \operatorname{Next}\left(\mathbf{G}^{\prime}\right)\right) \neq \perp\) do
                \(j^{\prime \prime} \leftarrow \pi ; \delta \leftarrow \delta+1 ;\)
            \(Q \leftarrow P\left[j+\left(j^{\prime \prime}-j\right) \bmod q \ldots j+\left(j^{\prime \prime}-j\right) \bmod q+q\right) ; \quad / / Q \leftarrow \operatorname{rot}^{j-j^{\prime \prime}}(Q)\)
            if \(\delta \geq 8 k / m \cdot\left(m-j^{\prime \prime}\right)\) then
                return repetitive region \(P\left[j^{\prime \prime} \ldots m\right)\) with period \(Q\)
            else return approximate period \(Q\);
```

in this manner guarantees disjointness of breaks and repetitive regions. As in the proof of Lemma 3.6. a slightly different approach is needed if the algorithm encounters the end of $P$ while growing a repetitive region. If this happens, the region is also extended to the left. This way, the algorithm either obtains a single repetitive region (which is not necessarily disjoint with the previously created ones, so it is returned on its own) or learns that the whole string $P$ is close to being periodic.

Next, we fill in missing details of the implementation of the previous steps in the PILLAR model. To that end, first note that the PILLAR model includes a Period operation of checking if the period of a string $S$ satisfies $\operatorname{per}(S) \leq|S| / 2$ and computing $\operatorname{per}(S)$ in case of a positive answer. Since our threshold $m / 128 k$ satisfies $\lfloor m / 128 k\rfloor \leq\lfloor m / 8 k\rfloor / 2$, no specific work is required to obtain the period of an unprocessed fragment of $\lfloor m / 8 k\rfloor$ characters of $P$.

To compute a repetitive region starting from a fragment $P^{\prime}=P[j \ldots j+\lfloor m / 8 k\rfloor)$ with string period $Q=P^{\prime}\left[0 \ldots \operatorname{per}\left(P^{\prime}\right)\right)$, we use a MismGenerator $\left(P[j \ldots m), Q^{*}\right)$ generator from Lemma 4.1. We extend $P^{\prime}$ up to the next mismatch between $P^{\prime}$ and $Q^{\infty}$ until we either reach the end of $P$ or the number $\delta=\delta_{H}\left(P^{\prime}, Q^{*}\right)$ reaches the bound $8 k / m \cdot\left|P^{\prime}\right|$. If we reach the end of $P$, we similarly extend $P^{\prime}=P[j \ldots m)$ to the left using a MismGenerator ${ }^{R}\left(P[0 \ldots j),{ }^{*} Q\right)$ generator from Lemma 4.1. Again, we
always extend $P^{\prime}$ up to the next mismatch until we reach the start of $P$ or the number $\delta=\delta_{H}\left(P^{\prime}, \bar{Q}^{*}\right)$ reaches the bound $8 k / m \cdot\left|P^{\prime}\right|$ (where $\bar{Q}=\operatorname{rot}^{\left|P^{\prime}\right|-m+j}(Q)$ is the corresponding cyclic rotation of $Q$ ). If we reach the start of the string, we return a suitable cyclic rotation of $Q$; otherwise we found a long repetitive region, which we then return. Consider Algorithm 3 for a detailed pseudo-code of the implementation.

For the correctness, since our algorithm follows the proof of Lemma 3.6 we only need to show that our implementation of finding repetitive regions correctly implements the corresponding step in Algorithm 1 However, this is easy, as with each extension of $P^{\prime}$, the number $\delta$ may increase by at most 1 . As we start with $\delta=\delta_{H}\left(P[j \ldots j+\lfloor m / 8 k\rfloor), Q^{*}\right)=0$, we thus never skip over a repetitive region. Further, the fragment $P^{\prime}=P[j \ldots j+\lfloor m / 8 k\rfloor)$ by construction contains at least two repetitions of the period $Q$, so we can obtain each cyclic rotation of $Q$ as a fragment of $P$. In particular we indeed compute a cyclic rotation of $Q$ in Line 25 of Algorithm 3 Consequently, Algorithm 3 indeed correctly implements Algorithm 1 .

For the running time analysis, observe that each iteration of the outer while loop processes at least $\lfloor m / 8 k\rfloor$ characters of $P$, so there are at most $\mathcal{O}(k)$ iterations of the outer while loop. In each iteration, we perform one Period operation, a constant number of Access operations, and at most $8 k / m \cdot\left(j^{\prime}-j\right)$ calls to the generator MismGenerator. Each of these calls uses $\mathcal{O}(1)$ PILLAR operations, which is $\mathcal{O}(8 k / m \cdot m)=\mathcal{O}(k)$ in total across all iterations. Similarly, we bound the running time of the calls to the generator MismGenerator ${ }^{R}$ : As we find at most $8 k / m \cdot m=8 k$ mismatches, MismGenerator ${ }^{R}$ uses at most $\mathcal{O}(k)$ operations. Overall, Algorithm 3 thus uses $\mathcal{O}(k)$ PILLAR operations.

The remaining running time is bounded by $\mathcal{O}(k)$ in the same way, completing the proof.

### 4.3 Computing Occurrences in the Periodic Case

「 Lemma 4.5 (FindRotation $(k, Q, S)$ ). Let $k$ denote a positive integer, let $Q$ denote a primitive string, and let $S$ denote a string with $|S| \geq(2 k+1)|Q|$. Then, we can compute a unique integer $j \in[0 \ldots|Q|)$ such that $\delta_{H}\left(S, \operatorname{rot}^{j}(Q)^{*}\right) \leq k$, or report $\perp$ if no such integer exists, using $\mathcal{O}(k)$ time plus $\mathcal{O}(k)$ PILLAR operations.
Proof. For every $0 \leq i \leq 2 k$, define $S_{i}:=S[i|Q| \ldots(i+1)|Q|)$. We compute the majority of $S_{0}, \ldots, S_{2 k}$ (using Lemma 2.5 for checking equality of fragments). If no majority exists, then we return $\perp$. Otherwise, we set $\bar{Q}$ to be the majority string of $S_{0}, \ldots, S_{2 k}$ and check if $\delta_{H}\left(S, \bar{Q}^{*}\right) \leq k$ using a MismGenerator from Lemma 4.1. If this test succeeds, we use a Rotations operation to retrieve all $j \in[0 . .|Q|)$ with $\bar{Q}=\operatorname{rot}^{j}(Q)$ and return any such $j$. If the test fails or if no such $j$ is found, then we return $\perp$.

For the correctness, observe that if $\delta_{H}\left(S, \bar{Q}^{*}\right) \leq k$, then at least $k+1$ fragments $S_{i}$ match $\bar{Q}$ exactly, so $\bar{Q}$ must be the majority of $S_{0}, \ldots, S_{2 k}$. Moreover, since $Q$ is primitive, there is at most one $j \in[0 \ldots|Q|)$ with $\bar{Q}=\operatorname{rot}^{j}(Q)$.

For the running time, note that we can compute the majority of $\mathcal{O}(k)$ elements with a classic linear-time algorithm by Boyer and Moore [9] using $\mathcal{O}(k)$ equality tests; as (by Lemma 2.5) each equality test takes $\mathcal{O}(1)$ time in the PILLAR model, we obtain the claimed running time and hence the claim.

- Lemma 4.6 (FindRelevantFragment $(P, T, d, Q)$ ). Let $P$ denote a pattern of length $m$ and let $T$ denote a text of length $n \leq \frac{3}{2} m$. Further, let d denote a positive integer and let $Q$ denote a primitive string that satisfies $|Q| \leq m / 8 d$ and $\delta_{H}\left(P, Q^{*}\right) \leq d$.

Then, using $\mathcal{O}(d)$ time plus $\mathcal{O}(d)$ PILLAR operations, we can report a fragment $T^{\prime}=T[\ell \ldots r)$ such that $\delta_{H}\left(T^{\prime}, Q^{*}\right) \leq 3 d$ and, for every $k \leq d / 2$, the set $\operatorname{Occ}_{k}^{H}\left(P, T^{\prime}\right)=\left\{p-\ell \mid p \in \operatorname{Occ}_{k}^{H}(P, T)\right\}$ contains only multiples of $|Q|$.
Proof. We start by using a call to FindRotation from Lemma 4.5 to find the unique integer $j$ such that $\delta_{H}\left(T[n-m \ldots m), \operatorname{rot}^{j}(Q)^{*}\right) \leq 3 / 2 d$. If no such $j$ exists, then we return the empty string $\varepsilon$. Otherwise, we

Algorithm 4 A PILLAR algorithm computing a relevant fragment $T$ : a fragment $T^{\prime}$ such that all $k$-mismatch occurrences (for any $k \leq d / 2$ ) of $P$ in $T$ start at a position in $T^{\prime}$ which is a multiple of $|Q|$.

```
FindRelevantFragment \((P, T, d, Q)\)
    \(j \leftarrow\) FindRotation \(\left(\left\lfloor\frac{3}{2} d\right\rfloor, Q, T[n-m \ldots m)\right)\);
    if \(j=\perp\) then return \(\varepsilon\);
    \(\delta \leftarrow 0 ; r \leftarrow n-m+j ;\)
    generator \(\mathbf{G} \leftarrow\) MismGenerator \(\left(T[n-m+j \ldots n), Q^{*}\right)\);
    while \(\delta \leq \frac{3}{2} d\) and \((\pi \leftarrow \operatorname{Next}(\mathbf{G})) \neq \perp\) do
        \(r \leftarrow n-m+j+\pi ;\)
        \(\delta \leftarrow \delta+1 ;\)
    if \(\delta \leq \frac{3}{2} d\) then \(r \leftarrow m\);
    \(\delta^{\prime} \leftarrow 0 ; \ell \leftarrow n-m+j ; \ell^{\prime} \leftarrow(n-m+j) \bmod |Q|\);
    generator \(\mathbf{G}^{\prime} \leftarrow\) MismGenerator \({ }^{R}\left(T\left[\ell^{\prime} \ldots n-m+j\right),{ }^{*} Q\right)\);
    while \(\delta^{\prime} \leq \frac{3}{2} d\) and \(\left(\pi \leftarrow \operatorname{Next}\left(\mathbf{G}^{\prime}\right)\right) \neq \perp\) do
        \(\ell \leftarrow \ell^{\prime}+|Q| \cdot\lceil(\pi+1) /|Q|\rceil ;\)
        \(\delta^{\prime} \leftarrow \delta^{\prime}+1 ;\)
    if \(\delta^{\prime} \leq 3 / 2 d\) then \(\ell \leftarrow \ell^{\prime}\);
    return \(T[\ell \ldots r)\);
```

proceed by computing the rightmost position $r$ such that $\delta_{H}\left(T[n-m+j \ldots r), Q^{*}\right) \leq \frac{3}{2} d$ and the leftmost position $\ell($ with $\ell \equiv(n-m+j)(\bmod |Q|))$ such that $\delta_{H}\left(T[\ell \ldots n-m+j), Q^{*}\right) \leq 3 / 2 d$; afterwards, we return the fragment $T[\ell \ldots r)$. Consider Algorithm 4 for implementation details.

For the correctness, first observe that $\delta_{H}\left(T[n-m \ldots m), \operatorname{rot}^{p-n+m}(Q)\right) \leq 3 / 2 d$ for each $p \in \operatorname{Occ}_{k}^{H}(P, T)$ : By triangle inequality (Fact 2.1), we have

$$
\delta_{H}\left(T[p \ldots p+m), Q^{*}\right) \leq k+\delta_{H}\left(P, Q^{*}\right) \leq 3 / 2 d
$$

Since $p \leq n-m$ and $p+m \geq m$, this yields $\delta_{H}\left(T[n-m \ldots m)\right.$, $\left.\operatorname{rot}^{p-n+m}(Q)\right) \leq 3 / 2 d$. Moreover, $|T[n-m \ldots m)|=2 m-n \geq m / 2 \geq 4 d|Q| \geq\left(2 \cdot\left\lfloor\frac{3}{2} d\right\rfloor+1\right)|Q|$, so the call to FindRotation is valid.

Hence, if the call to FindRotation returns $\perp$, then $\operatorname{Occ}_{k}^{H}(P, T)=\emptyset$ (for each $k \leq d / 2$ ). Otherwise, each position $p \in \operatorname{Occ}_{k}^{H}(P, T)$ satisfies $p \equiv n-m+j \equiv \ell(\bmod |Q|)$. Moreover, we have

$$
\begin{aligned}
\delta_{H}\left(T[n-m+j \ldots p+m), Q^{*}\right) & \leq \delta_{H}\left(T[p \ldots p+m), Q^{*}\right) \leq \frac{3}{2} d, \quad \text { and } \\
\delta_{H}\left(T[p \ldots n-m+j), Q^{*}\right) & \leq \delta_{H}\left(T[p \ldots p+m), Q^{*}\right) \leq \frac{3}{2} d .
\end{aligned}
$$

Hence, the fragment $T^{\prime}=T[\ell \ldots r$ ) contains all $k$-mismatch occurrences of $P$ in $T$ (for any $k \leq d / 2$ ), and all these occurrences start at multiples of $|Q|$ in $T^{\prime}$. Moreover, $\delta_{H}\left(T^{\prime}, Q^{*}\right)=\delta_{H}\left(T[\ell \ldots n-m+j), Q^{*}\right)+$ $\delta_{H}\left(T[n-m+j \ldots r), Q^{*}\right) \leq 3 d$.

For the running time (and the number of PILLAR operations used), the call to FindRotation uses $\mathcal{O}(d)$ time plus $\mathcal{O}(d)$ PILLAR operations; the same is true for the usage of MismGenerator and MismGenerator ${ }^{R}$. Thus, the algorithm uses $\mathcal{O}(d)$ time plus $\mathcal{O}(d)$ PILLAR operations in total, completing the proof.

『 Lemma 4.7 (DistancesRLE $(P, T, Q)$ : Implementation of Lemma 3.3). Let $P$ denote a pattern of length $m$ and let $T$ denote a text of length $n \leq 3 / 2 m$. Further, let d denote a positive integer and let $Q$ denote a string that satisfies $\delta_{H}\left(P, Q^{*}\right)=\mathcal{O}(d)$ and $\delta_{H}\left(T, Q^{*}\right)=\mathcal{O}(d)$.

Then, using $\mathcal{O}\left(d^{2} \log \log d\right)$ time plus $\mathcal{O}(d)$ PILLAR operations, we can compute a run-length encoded sequence of $h_{j}:=\delta_{H}(T[j|Q| \ldots j|Q|+m), P)$ for $0 \leq j \leq(n-m) /|Q|$.

Algorithm 5 A PILLAR algorithm for Lemma 3.3

```
DistancesRLE ( }P,T,Q
    // Marking phase
    M\leftarrow{};
    foreach }\tau\in\operatorname{Mismatches(T, Q*) do
        M\leftarrowM\cup{(\tau-m,1),(\tau,-1)};
        foreach }\pi\in\mathrm{ Mismatches( }P,\mp@subsup{Q}{}{*})\mathrm{ do
            M\leftarrowM\cup{(\tau-\pi-1, \delta
    // Sliding-window phase
    sort M;
    h\leftarrow |Mismatches( }P,\mp@subsup{Q}{}{*})|
    foreach }(\mp@subsup{i}{}{\prime},w)\inM with i'<0 do h\leftarrowh+w
    i\leftarrow0;
    foreach (i',w)\inM with 0\leq i'< n-m sorted by i' do
        Output a block of }\lceil(\mp@subsup{i}{}{\prime}+1)/q\rceil-\lceili/q\rceil\mathrm{ values }h\mathrm{ ;
        i\leftarrow\mp@subsup{i}{}{\prime}+1;
        h\leftarrowh+w;
        Output a block of }\lceil(n-m+1)/q\rceil-\lceili/q\rceil\mathrm{ values }h\mathrm{ ;
```

Proof. Observe that Claim 3.4 already gives rise to an algorithm: Starting with $h_{0}$, we can obtain the value $h_{j+1}$ from $h_{j}$ by adding the value

$$
h_{j+1}-h_{j}=\left|\operatorname{Mis}\left(T, Q^{*}\right) \cap[j q+m \ldots(j+1) q+m)\right|-\left|\operatorname{Mis}\left(T, Q^{*}\right) \cap[j q \ldots(j+1) q)\right|-\mu_{j+1}+\mu_{j},
$$

where $\mu_{j+1}$ and $\mu_{j}$ are defined as in Lemma 3.3 .
We implement this idea in two steps: In the first step, we compute the values $\mu_{j}$ (using marking) and the positions of mismatches in $\operatorname{Mis}\left(T, Q^{*}\right)$ (using Mismatches from Corollary 4.2. In the second step, we use a sliding-window approach (with the positions computed in the first step interpreted as events) to output the sequence of values of $h_{j}$. Consider the pseudo-code (Algorithm 5) for implementation details.

For the correctness, in the marking phase, the algorithm constructs a multiset $M$ of pairs $(i, w)$ (where $i$ can be interpreted as a position in $T$ and $w$ as the weight) so that $h_{j}-\delta_{H}\left(P, Q^{*}\right)=w(M, j|Q|)$, where $w(M, i)=\sum_{\left\{\left(i^{\prime}, w\right) \in M \mid i^{\prime}<i\right\}} w$ denotes the sum of weights of pairs $\left(i^{\prime}, w\right)$ with $i^{\prime}<i$.

Specifically, for each $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$, the algorithm first inserts to $M$ pairs $(\tau-m, 1)$ and $(\tau,-1)$. As a result, for each position $i$ with $0 \leq i \leq n-m$, we have $w(M, i)=\left|\operatorname{Mis}\left(T, Q^{*}\right) \cap[i \ldots i+m)\right|$. In particular, if $i=j|Q|$, then $w(M, i)=\delta_{H}\left(T[j|Q| \ldots j|Q|+m), Q^{*}\right)$. Next, for each $\tau \in \operatorname{Mis}\left(T, Q^{*}\right)$ and each $\pi \in \operatorname{Mis}\left(P, Q^{*}\right)$, the algorithm inserts to $M$ pairs $\left(\tau-\pi-1, \delta_{H}(P[\pi], T[\tau])-2\right)$ and $\left(\tau-\pi, 2-\delta_{H}(P[\pi], T[\tau])\right)$. As a result, the values $w(M, i)$ with $i \neq \tau-\pi$ are not altered, whereas $w(M, i)$ for $i=\tau-\pi$ is decreased by the number of marks placed in the proof of Lemma 3.3 at position $i=\tau-\pi$ of $T$ due to positions $\tau$ in $T$ and $\pi$ in $P$. Consequently, we have $w(M, j|Q|)=\delta_{H}\left(T[j|Q| \ldots j|Q|+m), Q^{*}\right)-\mu_{j}$, which yields $h_{j}=\delta_{H}\left(P, Q^{*}\right)+w(M, j|Q|)$ due to Claim 3.4.

Hence, in order to construct the sequence $h_{j}$, the algorithm sorts the pairs in $M$ and determines the partial sums $w(M, i)$. In each block of $\left[i \ldots i^{\prime}\right]$ of equal partial sums, the algorithm reports a block with all $\left\lceil\left(i^{\prime}+1\right) / q\right\rceil-\lceil i / q\rceil$ entries $h_{j}$ for $j|Q| \in\left[i \ldots i^{\prime}\right)$, which is indeed correct.

The running time is $\mathcal{O}\left(d^{2} \log \log d\right)$ (dominated by sorting $M$, which consists of $\mathcal{O}\left(d^{2}\right)$ integer pairs) plus $\mathcal{O}(d)$ PILLAR operations (for the calls to Mismatches $\left(P, Q^{*}\right)$ and Mismatches $\left(T, Q^{*}\right)$ and for accessing the mismatching positions of $P$ and $T$ ), thus completing the proof.

Algorithm 6 A PILLAR model algorithm for Lemma 3.8

```
BreakMatches \(\left(P, T,\left\{B_{1}=P\left[b_{1} \ldots b_{1}+\left|B_{1}\right|\right), \ldots, B_{2 k}=P\left[b_{2 k} \ldots b_{2 k}+\left|B_{2 k}\right|\right)\right\}, k\right)\)
    multi-set \(M \leftarrow\left\} ; \operatorname{Occ}_{k}^{H}(P, T) \leftarrow\{ \} ;\right.\)
    for \(i \leftarrow 1\) to \(2 k\) do
        foreach \(\tau \in \operatorname{ExactMatches}\left(B_{i}, T\right)\) do
            \(M \leftarrow M \cup\left\{\tau-b_{i}\right\} ; \quad / /\) Mark position \(\tau-b_{i}\) in \(T\)
    sort \(M\);
    foreach \(\pi \in[0 \ldots n-m]\) that appears at least \(k\) times in \(M\) do
        if Verify \((P, T[\pi \ldots \pi+m), k)\) then \(\operatorname{Occ}_{k}^{H}(P, T) \leftarrow \operatorname{Occ}_{k}^{H}(P, T) \cup\{\pi\} ;\)
    return \(\operatorname{Occ}_{k}^{H}(P, T)\);
```

- Lemma 4.8 (PeriodicMatches ( $P, T, k, d, Q$ ): Implementation of Corollary 3.5). Let $P$ denote $a$ pattern of length $m$ and let $T$ denote a text of length $n$. Further, let $k \leq m$ denote a non-negative integer, let $d \geq 2 k$ denote a positive integer, and let $Q$ denote a primitive string $Q$ that satisfies $|Q| \leq n / 8 d$ and $\delta_{H}\left(P, Q^{*}\right) \leq d$.

There is an algorithm that computes the set $\operatorname{Occ}_{k}^{H}(P, T)$, represented as $\mathcal{O}\left(n / m \cdot d^{2}\right)$ arithmetic progressions with difference $|Q|$ (or as $\mathcal{O}\left(n / m \cdot d\right.$ ) individual positions if $\delta_{H}\left(P, Q^{*}\right)=d$ ). The algorithm uses $\mathcal{O}\left(n / m \cdot d^{2} \log \log d\right)$ time plus $\mathcal{O}(n / m \cdot d)$ PILLAR operations.

Proof. First, we split the string $T$ into $\lfloor 2 n / m\rfloor$ blocks $T_{i}:=T[\lfloor i \cdot m / 2\rfloor \ldots \min \{n,\lfloor(i+3) \cdot m / 2\rfloor-1\})$ for $0 \leq i<\lfloor 2 n / m\rfloor$. For each block $T_{i}$, we call FindRelevantFragment ( $P, T_{i}, d, Q$ ) from Lemma 4.6 to obtain a fragment $T_{i}^{\prime}=T\left[\ell_{i} \ldots r_{i}\right.$ ) containing all $k$-mismatch occurrences of $P$ in $T_{i}$. Next, we call DistancesRLE $\left(P, T_{i}^{\prime}, Q\right)$ from Lemma 4.7. yielding a run-length encoded sequence of values $h_{t}:=\delta_{H}\left(T_{i}^{\prime}[t|Q| \ldots t|Q|+m), P\right)$ for $0 \leq t \leq\left(\left|T_{i}^{\prime}\right|-m\right) /|Q|$. For each run $h_{t}=\cdots=h_{t^{\prime}} \leq k$, we add the arithmetic progression $\left\{\ell_{i}+j \cdot|Q|: j \in\left[t \ldots t^{\prime}\right]\right\}$ to $\operatorname{Occ}_{k}^{H}(P, T)$. In the end, we return the set $\operatorname{Occ}_{k}^{H}(P, T)$.

For the correctness, note that we essentially follow the proof of Corollary 3.5 In particular, each $k$-mismatch occurrence of $P$ in $T$ is contained in exactly one of the fragments $T_{i}$. By Lemma 4.6 we see that $T_{i}^{\prime}$ contains all the $k$-mismatch occurrences of $P$ in $T_{i}$. Moreover, as $\operatorname{Occ}_{k}^{H}\left(P, T_{i}^{\prime}\right)$ only contains multiples of $|Q|$, each $p \in \operatorname{Occ}_{k}^{H}\left(P, T_{i}^{\prime}\right)$ corresponds to an entry $h_{j} \leq k$. Consequently, all the $k$-mismatch occurrences of $P$ in $T$ are found. Furthermore, since $h_{j}=\delta_{H}\left(T\left[\ell_{i}+j|Q| \ldots \ell_{i}+j|Q|+m\right), P\right) \leq k$ holds whenever $\ell_{i}+j|Q|$ is reported, there are no false positives.

If $\delta_{H}\left(P, Q^{*}\right)=d$, then for each $i$, the number of entries $h_{j}$ with $h_{j} \leq k$ is $\mathcal{O}(d)$ by Lemma 3.3. so the corresponding positions $\ell_{i}+j|Q|$ can be added to $\operatorname{Occ}_{k}^{H}(P, T)$ individually.

The bounds on the overall running time follow from Lemmas 4.6 and 4.7 due to $\delta_{H}\left(P, Q^{*}\right) \leq d$ and since $\delta_{H}\left(T_{i}, Q^{*}\right) \leq 3 d$ holds for each $i$ by Lemma 4.7

### 4.4 Computing Occurrences in the Non-Periodic Case

「 Lemma 4.9 (BreakMatches $\left(P, T,\left\{B_{1}, \ldots, B_{2 k}\right\}, k\right.$ ): Implementation of Lemma 3.8). Let $P$ denote a string of length $m$ having $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k} \preccurlyeq P$ each satisfying $\operatorname{per}\left(B_{i}\right) \geq m / 128 k$. Further, let $T$ denote a string of length $n \leq 3 / 2 m$.

Then, we can compute the set $\operatorname{Occ}_{k}^{H}(P, T)$ using $\mathcal{O}\left(k^{2} \log \log k\right)$ time plus $\mathcal{O}\left(k^{2}\right)$ PILLAR operations.
Proof. The implementation of (the marking in the proof of) Lemma 3.8 is straightforward: For each break $B_{i}=P\left[b_{i} \ldots b_{i}+\left|B_{i}\right|\right)$, we use a call to ExactMatches $\left(B_{i}, T\right)$ from Lemma 2.11 to find all exact occurrences $\operatorname{Occ}\left(B_{i}, T\right)$. For each occurrence $\pi \in \operatorname{Occ}\left(B_{i}, T\right)$, we mark position $\pi-b_{i}$ in $T$. Having

Algorithm 7 A PILLAR model algorithm for Lemma 3.11

```
RepetitiveMatches(P,T,{(R1}=P[\mp@subsup{r}{1}{}\ldots\mp@subsup{r}{1}{}+|\mp@subsup{R}{1}{}|),\mp@subsup{Q}{1}{})\ldots,(\mp@subsup{R}{r}{}=P[\mp@subsup{r}{r}{}\ldots\mp@subsup{r}{r}{}+|\mp@subsup{R}{r}{}|),\mp@subsup{Q}{r}{})},k
    multi-set }M\leftarrow{};\mp@subsup{\operatorname{Occ}}{k}{H}(P,T)\leftarrow{}
    for }i\leftarrow1\mathrm{ to }r\mathrm{ do
        foreach }\tau\in\mathrm{ PeriodicMatches ( }\mp@subsup{R}{i}{},T,\lfloor4\cdotk/m\cdot|\mp@subsup{R}{i}{}|\rfloor,\lceil8\cdotk/m\cdot|\mp@subsup{R}{i}{}|\rceil,\mp@subsup{Q}{i}{})\mathrm{ do
            M\leftarrowM\cup{(\tau-\mp@subsup{r}{i}{},|\mp@subsup{R}{i}{}|)};\quad // Place |R}|\mathrm{ marks at position }\tau-\mp@subsup{r}{i}{}\mathrm{ in }
```

    sort \(M\) by positions;
    foreach \(\pi \in[0 \ldots n-m]\) appearing at least \(\sum_{(\pi, v) \in M} v \geq \sum_{i=1}^{r}\left|R_{i}\right|-m / 4\) times in \(M\) do
        if \(\operatorname{Verify}\left(P, T[\pi \ldots \pi+m), k\right.\) ) then \(\operatorname{Occ}_{k}^{H}(P, T) \leftarrow \operatorname{Occ}_{k}^{H}(P, T) \cup\{\pi\} ;\)
    return \(\operatorname{Occ}_{k}^{H}(P, T)\);
    placed all marks, we run Verify from Lemma 4.3 for every position $\pi \in[0 . . n-m]$ in $T$ that has at least $k$ marks. In the end, we return all positions where Verify confirmed a $k$-mismatch occurrence. See Algorithm 6 for a pseudo-code.

For the correctness, note that we placed the marks as in the proof of Lemma 3.8 in particular, by Claim 3.10 any $\pi \in \operatorname{Occ}_{k}^{H}(P, T)$ has at least $k$ marks. As we verify every possible candidate using Verify, we report no false positives, and thus the algorithm is correct.

We continue with analyzing the number of PILLAR operations used. As every break $B_{i}$ has period $\operatorname{per}\left(B_{i}\right)>m / 128 k$, every call to ExactMatches uses $\mathcal{O}(k)$ basic PILLAR operations; thus, all calls to ExactMatches use $\mathcal{O}\left(k^{2}\right)$ basic operations in total. As there are at most $\mathcal{O}\left(k^{2} / k\right)=O(k)$ positions that we verify, and every call to Verify uses $\mathcal{O}(k)$ PILLAR operations, the verifications use $\mathcal{O}\left(k^{2}\right)$ PILLAR operations in total.

Finally, for the running time, by Claim 3.9, we place at most $\mathcal{O}\left(k^{2}\right)$ marks in $T$, so the marking step uses $\mathcal{O}\left(k^{2}\right)$ operations in total. Further, finding all positions in $T$ with at least $k$ marks can be done via a linear scan over the multiset $M$ of all marks after sorting $M$, which can be done in time $\mathcal{O}\left(k^{2} \log \log k\right)$. Overall, Algorithm 6 runs in time $\mathcal{O}\left(k^{2} \log \log k\right)$ plus $\mathcal{O}\left(k^{2}\right)$ PILLAR operations.

Lemma 4.10 (RepetitiveMatches $\left(P, T,\left\{\left(R_{1}, Q_{1}\right) \ldots,\left(R_{r}, Q_{r}\right)\right\}, k\right)$ : Implementation of Lemma 3.11). Let $P$ denote a string of length $m$, let $T$ denote a string of length $n \leq 3 / 2 m$, and let $k \leq m$ denote $a$ positive integer. Suppose that $P$ contains disjoint repetitive regions $R_{1}, \ldots, R_{r}$ of total length at least $\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and has a primitive approximate period $Q_{i}$ with $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{H}\left(R_{i}, Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$.

Then, we can compute the set $\operatorname{Occ}_{k}^{H}(P, T)$ using $\mathcal{O}\left(k^{2} \log \log k\right)$ time plus $\mathcal{O}\left(k^{2}\right)$ PILLAR operations.
Proof. As in the proof of Lemma 3.11 set $m_{R}:=\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ and define for every $1 \leq i \leq r$ the values $k_{i}:=\left\lfloor 4 \cdot k / m \cdot\left|R_{i}\right|\right\rfloor$ and $d_{i}:=\left\lceil 8 \cdot k / m \cdot\left|R_{i}\right|\right\rceil=\left|\operatorname{Mis}\left(R_{i}, Q_{i}^{*}\right)\right|$. Further, write $R_{i}=P\left[r_{i} . . r_{i}+\left|R_{i}\right|\right)$.

We implement the marking of the proof of Lemma 3.11 for every repetitive region $R_{i}$, we call PeriodicMatches ( $R_{i}, T, k_{i}, d_{i}, Q_{i}$ ) from Lemma 4.8 to obtain the set $\operatorname{Occ}_{k_{i}}^{H}\left(R_{i}, T\right)$. Next, for each position $\tau \in \operatorname{Occ}_{k_{i}}^{H}\left(R_{i}, T\right)$, we place $\left|R_{i}\right|$ marks at position $\tau-r_{i}$. Note that for performance reasons, instead of placing $\left|R_{i}\right|$ unweighted marks, we place a single mark of weight $\left|R_{i}\right|$ at position $\tau-r_{i}$.

Having placed all marks, we run Verify from Lemma 4.3 for every position $\pi \in[0 . . n-m]$ in $T$ that has marks of total weight at least $m_{r}-m / 4$. In the end, we return all positions where Verify confirmed a $k$-mismatch occurrence. See Algorithm 7 for a pseudo-code.

For the correctness, first note that in every call to PeriodicMatches from Lemma 4.8, we have $16 k / m \cdot\left|R_{i}\right| \geq d_{i}=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil=\delta_{H}\left(R_{i}, Q_{i}^{*}\right) \geq 2 k_{i}$, so $\left|Q_{i}\right| \leq m / 128 k \leq\left|R_{i}\right| / 8 d_{i}$; hence, we can indeed

```
Algorithm 8 A PILLAR model algorithm for Theorem 3.1
Mismatch0ccurrences \((P, T, k)\)
    \(\left(B_{1}, \ldots, B_{2 k}\right.\) or \(\left(R_{1}, Q_{1}\right), \ldots,\left(R_{r}, Q_{r}\right)\) or \(\left.Q\right) \leftarrow \operatorname{Analyze}(P, k)\);
    \(\operatorname{Occ}_{k}^{H}(P, T) \leftarrow\{ \} ;\)
    for \(i \leftarrow 0\) to \(\lfloor 2 n / m\rfloor\) do
        \(T_{i} \leftarrow T[\lfloor i \cdot m / 2\rfloor \ldots \min \{n,\lfloor(i+3) \cdot m / 2\rfloor-1\}) ;\)
        if breaks \(B_{1}, \ldots, B_{2 k}\) exist then
                \(\operatorname{Occ}_{k}^{H}\left(P, T_{i}\right) \leftarrow \operatorname{BreakMatches}\left(P, T_{i},\left\{B_{1}, \ldots, B_{2 k}\right\}, k\right) ;\)
        else if repetitive regions \(\left(R_{1}, Q_{1}\right), \ldots,\left(R_{r}, Q_{r}\right)\) exist then
                \(\operatorname{Occ}_{k}^{H}\left(P, T_{i}\right) \leftarrow\) RepetitiveMatches \(\left(P, T_{i},\left\{\left(R_{1}, Q_{1}\right), \ldots,\left(R_{r}, Q_{r}\right)\right\}, k\right) ;\)
        else \(\operatorname{Occ}_{k}^{H}\left(P, T_{i}\right) \leftarrow \operatorname{PeriodicMatches}\left(P, T_{i}, k, 8 k, Q\right)\);
        \(\operatorname{Occ}_{k}^{H}(P, T) \leftarrow \operatorname{Occ}_{k}^{H}(P, T) \cup\left\{\ell+i m / 2: \ell \in \operatorname{Occ}_{k}^{H}\left(P, T_{i}\right)\right\} ;\)
    return \(\operatorname{Occ}_{k}^{H}(P, T)\);
```

call PeriodicMatches in this case. Further, note that we placed the marks as in the proof of Lemma 3.11 in particular, by Claim 3.13 , any $\pi \in \operatorname{Occ}_{k}^{H}(P, T)$ has at least $m_{R}-m / 4$ marks. As we verify every possible candidate using Verify, we report no false positives, and thus the algorithm is correct.

For the number of PILLAR operations, observe that during the marking step, for every repetitive region $R_{i}$, we call PeriodicMatches once, and the call uses $\mathcal{O}\left(n /\left|R_{i}\right| \cdot d_{i}\right)=\mathcal{O}\left(m /\left|R_{i}\right| \cdot k / m \cdot\left|R_{i}\right|\right)=\mathcal{O}(k)$ PILLAR operations. Hence, the marking step uses $\mathcal{O}(r \cdot k)=\mathcal{O}\left(k^{2}\right)$ PILLAR operations in total. Next, during the verification step, by Claims 3.12 and 3.13 we call Verify at most $\mathcal{O}(k)$ times. As each call to Verify uses $\mathcal{O}(k)$ PILLAR operations, the verification step in total uses $\mathcal{O}\left(k^{2}\right)$ PILLAR operations. Overall, Algorithm 7 uses $\mathcal{O}\left(k^{2}\right)$ PILLAR operations.

Finally, for the running time, with similar calculations as for the number of PILLAR operations, we see that the marking step, including calls to PeriodicMatches, takes time $\sum_{i} \mathcal{O}\left(n /\left|R_{i}\right| \cdot d_{i}^{2} \log \log d_{i}\right)=$ $\sum_{i} \mathcal{O}\left(\left|R_{i}\right| / m \cdot k^{2} \log \log k\right)=\mathcal{O}\left(k^{2} \log \log k\right)$. Further, for every $R_{i}$, we place at most $\left|\operatorname{Occ}_{k_{i}}^{H}\left(R_{i}, T\right)\right|$ (weighted) marks, which can be bounded using Corollary 3.5 by $\left|\operatorname{Occ}_{k_{i}}^{H}\left(R_{i}, T\right)\right|=\mathcal{O}\left(n /\left|R_{i}\right| \cdot d_{i}\right)=\mathcal{O}(k)$. Thus, we place $|M|=\mathcal{O}\left(k^{2}\right)$ (weighted) marks in total. Therefore, we can sort $M$ (by positions) in time $\mathcal{O}\left(k^{2} \log \log k\right)$; afterwards, we can find the elements with total weight at least $m_{R}-m / 4$ via a linear scan over $M$ in time $\mathcal{O}\left(k^{2}\right)$. Hence, Algorithm 7 runs in $\mathcal{O}\left(k^{2} \log \log k\right)$ overall time, completing the proof. .

### 4.5 A PILLAR Model Algorithm for Pattern Matches with Mismatches

Main Theorem 8. Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a positive integer $k \leq m$, we can compute (a representation of) the set $\operatorname{Occ}_{k}^{H}(P, T)$ using $\mathcal{O}\left(n / m \cdot k^{2} \log \log k\right)$ time plus $\mathcal{O}\left(n / m \cdot k^{2}\right)$ PILLAR operations.

Proof. First, we split $T$ into overlapping parts $T_{1}, \ldots, T_{\lfloor 2 n / m\rfloor}$ of length less than $3 / 2 m$ each. In order to compute $\operatorname{Occ}_{k}^{H}\left(P, T_{i}\right)$ for each $i$, we follow the structure of the proof of Theorem 3.1. We first call Analyze ( $P, k$ ) from Lemma 4.4. If the call to Analyze $(P, k)$ yields $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k}$ in $P$, then we call BreakMatches $\left(P, T_{i},\left\{B_{1}, \ldots, B_{2 k}\right\}, k\right.$ ) from Lemma 4.9. If the call to Analyze $(P, k)$ yields disjoint repetitive regions $R_{1}, \ldots, R_{r}$ (and corresponding approximate periods $Q_{1}, \ldots, Q_{r}$ ), then we call RepetitiveMatches $\left(P, T_{i},\left\{\left(R_{1}, Q_{1}\right), \ldots,\left(R_{r}, Q_{r}\right)\right\}, k\right)$ from Lemma 4.10 Finally, if the call to Analyze $(P, k)$ yields an approximate period $Q$, then we call PeriodicMatches ( $P, T_{i}, k, 8 k, Q$ ) from Lemma 4.8. The resulting set $\operatorname{Occ}_{k}^{H}(P, T)$ is obtained by combining the sets $\mathrm{Occ}_{k}^{H}\left(P, T_{i}\right)$. Consider Algorithm 8 for a visualization as pseudo-code.

For the correctness，first observe that we do not lose any occurrences by splitting the string $T$ ，since every length－$m$ fragment of $T$ is contained in one of the fragments $T_{i}$ ．Second，by Lemma 4.4 and due to $\left|T_{i}\right| \leq \frac{3}{2} m$ ，the parameters in the calls to BreakMatches and RepetitiveMatches each satisfy the requirements．Lastly，if we use PeriodicMatches，notice that again by Lemma 4.4 the string $Q$ satisfies $\delta_{H}\left(P, Q^{*}\right) \leq 8 k$ and $|Q| \leq m / 128 k \leq m /(8 \cdot 8 k)$ ；hence，we can indeed call PeriodicMatches in this case．

For the number of PILLAR operations used，the call to Analyze uses $\mathcal{O}(k)$ PILLAR operations，each call to BreakMatches and RepetitiveMatches uses $\mathcal{O}\left(k^{2}\right)$ PILLAR operations，and each call to PeriodicMatches uses $\mathcal{O}(k)$ PILLAR operations．As there are at most $\mathcal{O}(n / m)$ calls to BreakMatches，RepetitiveMatches， and PeriodicMatches，we can bound the total number of PILLAR operations used by $\mathcal{O}\left(n / m \cdot k^{2}\right)$ ．

Similarly，for the running time，the call to Analyze takes $\mathcal{O}(k)$ time，whereas each call to BreakMatches， RepetitiveMatches，and PeriodicMatches takes $\mathcal{O}\left(k^{2} \log \log k\right)$ time．Again，since there are at most $\mathcal{O}(n / m)$ calls to BreakMatches，RepetitiveMatches，and PeriodicMatches each，and combining the sets $\mathrm{Occ}_{k}^{H}\left(P, T_{i}\right)$ to $\operatorname{Occ}_{k}^{H}(P, T)$ can be implemented in total time $\mathcal{O}\left(n / m \cdot k^{2}\right)$ ，we can bound the total running time by $\mathcal{O}\left(n / m \cdot k^{2} \log \log k\right)$ ，thus completing the proof．

## 5 Structural Insights into Pattern Matching with Edits

In this section，we develop insight into the structure of $k$－error occurrences of a pattern $P$ in a text $T$ ． We prove the following result，which is analogous to Theorem 3．1 and is Main Theorem 6 with explicit constants．

「 Theorem 5.1 （Compare Theorem 3．1）．Given a pattern $P$ of length $m$ ，a text $T$ of length $n$ ，and $a$ positive integer $k \leq m$ ，then at least one of the following holds．
－The $k$－error occurrences of $P$ in $T$ satisfy $\left.\| \operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor \mid \leq 642045 \cdot n / m \cdot k$ ．
－There is a primitive string $Q$ of length $|Q| \leq m / 128 k$ that satisfies $\delta_{E}\left(P,{ }^{*} Q^{*}\right)<2 k$ ．
Similarly to Section 3，we start with an analysis of the（approximately）periodic case．

## 5．1 Characterization of the Periodic Case

Theorem 5.2 （Compare Theorem 3．2）．Let $P$ denote a pattern of length $m$ ，let $T$ denote a text of length $n$ ，and let $k \leq m$ denote a non－negative integer such that $n<3 / 2 m+k$ ．Suppose that the $k$－error occurrences of $P$ in $T$ include a prefix of $T$ and a suffix of $T$ ．If there are a positive integer $d \geq 2 k$ and a primitive string $Q$ with $|Q| \leq m / 8 d$ and $\delta_{E}\left(P, Q^{*}\right)=\delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq d$ ，then each of following holds：
（a）For every $p \in \operatorname{Occ}_{k}^{E}(P, T)$ ，we have $p \bmod |Q| \leq 3 d$ or $p \bmod |Q| \geq|Q|-3 d$ ．
（b）The string $T$ satisfies $\delta_{E}\left(T,{ }^{*} Q^{*}\right) \leq 3 d$ ．
（c）If $\delta_{E}\left(P,{ }^{*} Q^{*}\right)=d$ ，then $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / d\right\rfloor\right| \leq 304 d$ ．
（d）The set $\operatorname{Occ}_{k}^{E}(P, T)$ can be decomposed into $617 d^{3}$ arithmetic progressions with difference $|Q|$ ．』
${ }^{-} \quad$ Lemma 5．3．Let $k$ denote a positive integer，let $Q$ denote a primitive string，and let $S$ denote a string of length $|S| \geq(2 k+1)|Q|$ ．If there are integers $\ell \leq r$ and $\ell^{\prime} \leq r^{\prime}$ such that $\delta_{E}\left(S, Q^{\infty}[\ell \ldots r)\right) \leq k$ and $\delta_{E}\left(S, Q^{\infty}\left[\ell^{\prime} \ldots r^{\prime}\right)\right) \leq k$ ，then there are integers $j, j^{\prime}$ and a decomposition $S=S_{L} \cdot S_{R}$ that satisfy

$$
\begin{aligned}
\delta_{E}\left(S, Q^{\infty}[\ell \ldots r)\right) & =\delta_{E}\left(S_{L}, Q^{\infty}[\ell \ldots j|Q|)\right)+\delta_{E}\left(S_{R}, Q^{\infty}[j|Q| \ldots r)\right) \quad \text { and } \\
\delta_{E}\left(S, Q^{\infty}\left[\ell^{\prime} \ldots r^{\prime}\right)\right) & =\delta_{E}\left(S_{L}, Q^{\infty}\left[\ell^{\prime} \ldots j^{\prime}|Q|\right)\right)+\delta_{E}\left(S_{R}, Q^{\infty}\left[j^{\prime}|Q| \ldots r^{\prime}\right)\right) .
\end{aligned}
$$

Furthermore，if $|Q|=1$ ，then the assumption $|S| \geq(2 k+1)|Q|$ is not required．


Figure 4 The setting in Lemma 5.3 Asterisks denote edit operations on the respective strings in their optimal alignment with $S . S_{i}$ is denoted by a shaded rectangle, $U=Q[|Q|-p \ldots|Q|)$ and $V=Q[0 \ldots|Q|-p)$.

Proof. If $|Q|=1$, we can set $S_{L}:=S, S_{R}:=\varepsilon, j:=r$, and $j^{\prime}:=r^{\prime}$.
Now assume that we have $|S| \geq(2 k+1)|Q|$ and define $S_{i}:=S[i|Q| \ldots(i+1)|Q|)$ for $0 \leq i \leq 2 k$. Further, fix optimal alignments between $S$ and $Q^{\infty}[\ell \ldots r)$ and between $S$ and $Q^{\infty}\left[\ell^{\prime} \ldots r^{\prime}\right)$.

Observe that at least one of the fragments $S_{i}$ is aligned without errors in both alignments. Let us fix such a fragment $S_{i}$ and observe that $S_{i}$ is a length- $|Q|$ substring of $Q^{\infty}$, so $S_{i}=\operatorname{rot}^{p}(Q)$ for some $p \in[0 \ldots|Q|)$. An illustration is provided in Figure 4. Based on this value, we set $S_{L}:=S[0 \ldots i|Q|+p)$ and $S_{R}:=S[i|Q|+p \ldots|S|)$.

Next, consider the fragment $Q^{\prime}$ of $Q^{\infty}[\ell \ldots r)$ that is aligned to $S_{i}$ in the alignment fixed earlier. The fragment $Q^{\prime}$ matches $\operatorname{rot}^{p}(Q)$. As $Q$ is primitive, $Q^{\prime}$ is thus of the form $Q^{\prime}=Q^{\infty}[j|Q|-p \ldots(j+1)|Q|-p)$ for some integer $j$. Consequently,

$$
\delta_{E}\left(S, Q^{\infty}[\ell \ldots r)\right)=\delta_{E}\left(S_{L}, Q^{\infty}[\ell \ldots j|Q|)\right)+\delta_{E}\left(S_{R}, Q^{\infty}[j|Q| \ldots r)\right)
$$

A similar argument shows that for some integer $j^{\prime}$, we also have

$$
\delta_{E}\left(S, Q^{\infty}\left[\ell^{\prime} \ldots r^{\prime}\right)\right)=\delta_{E}\left(S_{L}, Q^{\infty}\left[\ell^{\prime} \ldots j^{\prime}|Q|\right)\right)+\delta_{E}\left(S_{R}, Q^{\infty}\left[j^{\prime}|Q| \ldots r^{\prime}\right)\right)
$$

This completes the proof.

Lemma 5.4. Let $T$ denote a text of length $n$, let $k$ denote a positive integer, and let $Q$ denote a primitive string. Suppose that $\delta_{E}\left(T[0 \ldots q), Q^{\infty}[x \ldots y)\right) \leq k$ and $\delta_{E}\left(T[p \ldots n), Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right) \leq k$ holds for some integers $p \leq q, x \leq y$, and $x^{\prime} \leq y^{\prime}$. If $|Q|=1$ or $q-p \geq(2 k+1)|Q|$, then $\delta_{E}\left(T, Q^{\infty}\left[x^{\prime \prime} \ldots y\right)\right)=$ $\delta_{E}\left(T, Q^{\infty}\left[x \ldots y^{\prime \prime}\right)\right) \leq 2 k$ for some $x^{\prime \prime} \equiv x^{\prime}(\bmod |Q|)$ and $y^{\prime \prime} \equiv y^{\prime}(\bmod |Q|)$, and $\left(p+x-x^{\prime}+\right.$ $2 k) \bmod |Q| \leq 4 k$.

Proof. Observe that, for some integer $z \in[x \ldots y]$, we have

$$
\delta_{E}\left(T[0 \ldots q), Q^{\infty}[x \ldots y)\right)=\delta_{E}\left(T[0 \ldots p), Q^{\infty}[x \ldots z)\right)+\delta_{E}\left(T[p \ldots q), Q^{\infty}[z \ldots y)\right)
$$

Similarly, for some integer $z^{\prime} \in\left[x^{\prime} \ldots y^{\prime}\right]$, we have

$$
\delta_{E}\left(T[p \ldots n), Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(T[p \ldots q), Q^{\infty}\left[x^{\prime} \ldots z^{\prime}\right)\right)+\delta_{E}\left([q \ldots n), Q^{\infty}\left[z^{\prime} \ldots y^{\prime}\right)\right)
$$

Now, Lemma 5.3 applied to $S:=T\left[p \ldots q\right.$ ) yields an integer $r \in[p \ldots q]$ and integers $j, j^{\prime}$ such that (see also Figure 5

$$
\begin{aligned}
\delta_{E}\left(T[p \ldots q), Q^{\infty}[z \ldots y)\right) & =\delta_{E}\left(T[p \ldots r), Q^{\infty}[z \ldots j|Q|)\right)+\delta_{E}\left(T[r \ldots q), Q^{\infty}[j|Q| \ldots y)\right), \text { and } \\
\delta_{E}\left(T[p \ldots q), Q^{\infty}\left[x^{\prime} \ldots z^{\prime}\right)\right) & =\delta_{E}\left(T[p \ldots r), Q^{\infty}\left[x^{\prime} \ldots j^{\prime}|Q|\right)\right)+\delta_{E}\left(T[r \ldots q), Q^{\infty}\left[j^{\prime}|Q| \ldots z^{\prime}\right)\right)
\end{aligned}
$$



Figure 5 The setting in Lemma 5.4. $T$ is at edit distance at most $2 k$ from $Q^{\infty}[x \ldots j|Q|) Q^{\infty}\left[j^{\prime}|Q| \ldots y^{\prime}\right)=$ $Q^{\infty}\left[x+\left(j^{\prime}-j\right)|Q| \ldots y^{\prime}\right)=Q^{\infty}\left[x \ldots y^{\prime}+\left(j-j^{\prime}\right)|Q|\right)$.

This implies that

$$
\begin{aligned}
\delta_{E}\left(T[0 \ldots r), Q^{\infty}[x \ldots j|Q|)\right) & =\delta_{E}\left(T[0 \ldots p), Q^{\infty}[x \ldots z)\right)+\delta_{E}\left(T[p \ldots r), Q^{\infty}[z \ldots j|Q|)\right) \leq k, \text { and } \\
\delta_{E}\left(T[r \ldots n), Q^{\infty}\left[j^{\prime}|Q| \ldots y^{\prime}\right)\right) & =\delta_{E}\left(T[r \ldots q), Q^{\infty}\left[j^{\prime}|Q| \ldots z^{\prime}\right)\right)+\delta_{E}\left([q \ldots n), Q^{\infty}\left[z^{\prime} \ldots y^{\prime}\right)\right) \leq k .
\end{aligned}
$$

Combining the equations yields

$$
\begin{aligned}
& \delta_{E}\left(T, Q^{\infty}\left[x+\left(j^{\prime}-j\right)|Q| \ldots y^{\prime}\right)\right)=\delta_{E}\left(T, Q^{\infty}\left[x \ldots y^{\prime}+\left(j-j^{\prime}\right)|Q|\right)\right) \\
& \quad \leq \delta_{E}\left(T[0 \ldots r), Q^{\infty}[x \ldots j|Q|)\right)+\delta_{E}\left(T[r \ldots n), Q^{\infty}\left[j^{\prime}|Q| \ldots y^{\prime}\right)\right) \leq 2 k
\end{aligned}
$$

Moreover, we deduce $|j| Q|-x-r| \leq k$ and $\left|j^{\prime}\right| Q\left|-x^{\prime}-r+p\right| \leq k$, which yields $\left|p+x-x^{\prime}-\left(j-j^{\prime}\right)\right| Q|\mid \leq 2 k$, and therefore $\left(p+x-x^{\prime}+2 k\right) \bmod |Q| \leq 4 k$.
${ }^{r}$ Definition 5.5. Let $S$ denote a string and let $Q$ denote a primitive string. We say that a fragment $L$ of $S$ is locked (with respect to $Q$ ) if at least one of the following holds:

- For some integer $\alpha$, we have $\delta_{E}\left(L,{ }^{*} Q^{*}\right)=\delta_{E}\left(L, Q^{\alpha}\right)$.
- The fragment $L$ is a suffix of $S$ and $\delta_{E}\left(L,{ }^{*} Q^{*}\right)=\delta_{E}\left(L, Q^{*}\right)$.
- The fragment $L$ is a prefix of $S$ and $\delta_{E}\left(L,{ }^{*} Q^{*}\right)=\delta_{E}\left(L,{ }^{*} Q\right)$.
- We have $L=S$.

The notion of locked fragments was also used in [17]. In order to develop some intuition, let us consider the following example: A string $U=Q^{k+1} S Q^{k+1}$ such that $\delta_{E}\left(U,{ }^{*} Q^{*}\right) \leq k$ and $Q$ is primitive. Then, in any optimal alignment of $U$ with a substring of $Q^{\infty}$ at least one of the leading (or trailing) $k+1$ occurrences of $Q$ in $U$ is matched exactly and hence also all occurrences preceding it (or succeeding it). Thus $U$ is locked with respect to $Q$.

Next, we show that we can identify short locked fragments covering all errors with respect to * $Q^{*}$. Intuitively, our strategy is to start with at most $k|Q|$-length fragments of $S$ that contain all the errors and extend or/and merge them (in a sense similar to that of the intuitive example provided above), so that the resulting fragments contain sufficiently many copies of $Q$

- Lemma 5.6. Let $S$ denote a string and let $Q$ denote a primitive string. There are disjoint locked fragments $L_{1}, \ldots, L_{\ell} \preceq S$ with $\delta_{E}\left(L_{i},{ }^{*} Q^{*}\right)>0$ such that

$$
\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\sum_{i=1}^{\ell} \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right) \quad \text { and } \quad \sum_{i=1}^{\ell}\left|L_{i}\right| \leq(5|Q|+1) \delta_{E}\left(S,{ }^{*} Q^{*}\right) .
$$



Figure 6 A partition $S_{0}^{(0)} \cdots S_{4}^{(0)}$ of a string $S$ is shown $\left(s^{(0)}=4\right)$, in which all fragments apart from $S_{0}^{(3)}$ are interesting. A merge of the fragments $S_{0}^{(0)}$ and $S_{1}^{(0)}$ yields the shown partition $S_{0}^{(1)} \cdots S_{3}^{(1)}$ of $S$.

Proof. Let us choose integers $x \leq y$ so that $\delta_{E}\left(S, Q^{*}\right)=\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right)$. Without loss of generality, we may assume that $x \in\left[0 \ldots|Q|\right.$ ). If $y \leq|Q|$, then $|S| \leq|Q|+\delta_{E}\left(S,{ }^{*} Q^{*}\right)$; thus setting the whole string $S$ as the only locked fragment satisfies the claimed conditions. Hence, we may assume that $y>|Q|$.

An arbitrary optimum alignment of $S$ and $Q^{\infty}[x \ldots y)$ yields a partition $S=S_{0}^{(0)} \cdots S_{S^{(0)}}^{(0)}$ with $s^{(0)}=\lfloor(y-1) /|Q|\rfloor$ such that $\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right)=\sum_{i=0}^{s^{(0)}} \Delta_{i}^{(0)}$, where

$$
\Delta_{i}^{(0)}= \begin{cases}\delta_{E}\left(S_{0}^{(0)}, Q[x \ldots|Q|)\right) & \text { if } i=0 \\ \delta_{E}\left(S_{i}^{(0)}, Q\right) & \text { if } 0<i<s^{(0)} \\ \delta_{E}\left(S_{s^{(0)}}^{(0)}, Q\left[0 \ldots y-s^{(0)}|Q|\right)\right) & \text { if } i=s^{(0)}\end{cases}
$$

See Figure 6 for an illustration.
We start with this partition and then coarsen it by exhaustively applying the merging rules specified below, where each rule is applied only if the previous rules cannot be applied. In each case, we re-index the unchanged fragments $S_{i}^{(t)}$ to obtain a new partition $S=S_{0}^{(t+1)} \cdots S_{s^{(t+1)}}^{(t+1)}$ and re-index the corresponding values $\Delta_{i}^{(t)}$ accordingly. We say that a fragment $S_{i}^{(t)}$ is interesting if $i=0, i=s^{(t)}, S_{i}^{(t)} \neq Q$, or $\Delta_{i}^{(t)}>0$. 1. If subsequent fragments $S_{i}^{(t)}$ and $S_{i+1}^{(t)}$ are both interesting, then merge $S_{i}^{(t)}$ and $S_{i+1}^{(t)}$, obtaining $S_{i}^{(t+1)}:=S_{i}^{(t)} S_{i+1}^{(t)}$ and $\Delta_{i}^{(t+1)}:=\Delta_{i}^{(t)}+\Delta_{i+1}^{(t)}$.
2. If $0<i<s^{(t)}$ and $\Delta_{i}^{(t)}>0$, then merge the subsequent fragments $S_{i-1}^{(t)}=Q, S_{i}^{(t)}$, and $S_{i+1}^{(t)}=Q$, obtaining $S_{i-1}^{(t+1)}:=S_{i-1}^{(t)} S_{i}^{(t)} S_{i+1}^{(t)}$, and set $\Delta_{i-1}^{(t+1)}:=\Delta_{i}^{(t)}-1$.
3. If $0<i=s^{(t)}$ and $\Delta_{i}^{(t)}>0$, then merge the subsequent fragments $S_{i-1}^{(t)}=Q$ and $S_{i}^{(t)}$, obtaining $S_{i-1}^{(t+1)}:=S_{i-1}^{(t)} S_{i}^{(t)}$, and set $\Delta_{i-1}^{(t+1)}:=\Delta_{i}^{(t)}-1$.
4. If $0=i<s^{(t)}$ and $\Delta_{i}^{(t)}>0$, then merge the subsequent fragments $S_{i}^{(t)}$ and $S_{i+1}^{(t)}=Q$, obtaining $S_{i}^{(t+1)}:=S_{i}^{(t)} S_{i+1}^{(t)}$, and set $\Delta_{i}^{(t+1)}:=\Delta_{i}^{(t)}-1$.
Let $S=S_{0} \cdots S_{s}$ denote the obtained final partition. We select as locked fragments all the fragments $S_{i}$ with $\delta_{E}\left(S_{i},{ }^{*} Q^{*}\right)>0$. Below, we show that this selection satisfies the desired properties. We start by proving that we indeed picked locked fragments.
${ }^{『}$ Claim 5.7. Each fragment $S_{i}^{(t)}$ of each partition $S=S_{0}^{(t)} \cdots S_{s^{(t)}}^{(t)}$ satisfies at least one of the following:

- $\delta_{E}\left(S_{i}^{(t)}, Q^{\alpha}\right) \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}$ for some integer $\alpha$;
- $\quad i=s^{(t)}$ and $\delta_{E}\left(S_{i}^{(t)}, Q^{*}\right) \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}$;
- $\quad i=0$ and $\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q\right) \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}$;
- $i=0=s^{(t)}$.

Proof. We proceed by induction on $t$. The base case follows from the definition of the values $\Delta_{i}^{(t)}$.
As for the inductive step, we assume that the claim holds for all fragments $S_{i}^{(t)}$ and we prove that it holds for all fragments $S_{i}^{(t+1)}$. We consider several cases based on the merge rule applied.

1. For a type 1 merge of interesting fragments $S_{i}^{(t)}$ and $S_{i+1}^{(t)}$ into $S_{i}^{(t+1)}$, it suffices to prove that $S_{i}^{(t+1)}$ satisfies the claim.

- If $0<i<s^{(t+1)}$, then $\delta_{E}\left(S_{i}^{(t)}, Q^{\alpha}\right) \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}$ and $\delta_{E}\left(S_{i+1}^{(t)}, Q^{\alpha^{\prime}}\right) \leq \delta_{E}\left(S_{i+1}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i+1}^{(t)}$ hold by the inductive assumption for some integers $\alpha, \alpha^{\prime}$. Consequently,

$$
\begin{aligned}
\delta_{E}\left(S_{i}^{(t+1)}, Q^{\alpha+\alpha^{\prime}}\right) & =\delta_{E}\left(S_{i}^{(t)} S_{i+1}^{(t)}, Q^{\alpha} Q^{\alpha^{\prime}}\right) \leq \delta_{E}\left(S_{i}^{(t)}, Q^{\alpha}\right)+\delta_{E}\left(S_{i+1}^{(t)}, Q^{\alpha^{\prime}}\right) \\
& \leq \delta_{E}\left(S_{i}^{(t)}, Q^{*}\right)+\Delta_{i}^{(t)}+\delta_{E}\left(S_{i+1}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i+1}^{(t)} \leq \delta_{E}\left(S_{i}^{(t+1)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t+1)} .
\end{aligned}
$$

- If $0<i=s^{(t+1)}$, then $\delta_{E}\left(S_{i}^{(t)}, Q^{\alpha}\right) \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}$ and $\delta_{E}\left(S_{i+1}^{(t)}, Q^{*}\right) \leq \delta_{E}\left(S_{i+1}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i+1}^{(t)}$ hold by the inductive assumption for some integer $\alpha$. Consequently,

$$
\begin{aligned}
\delta_{E}\left(S_{i}^{(t+1)}, Q^{*}\right)= & \delta_{E}\left(S_{i}^{(t)} S_{i+1}^{(t)}, Q^{*}\right) \leq \delta_{E}\left(S_{i}^{(t)}, Q^{\alpha}\right)+\delta_{E}\left(S_{i+1}^{(t)}, Q^{*}\right) \\
& \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}+\delta_{E}\left(S_{i+1}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i+1}^{(t)} \leq \delta_{E}\left(S_{i}^{(t+1)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t+1)} .
\end{aligned}
$$

- The analysis of the case that $0=i<s^{(t+1)}$ is symmetric to that of the above case - this can be seen by reversing all strings in scope.
- If $0=i=s^{(t+1)}$, then the claim holds trivially.

2. For a type- 2 merge of $S_{i-1}^{(t)}, S_{i}^{(t)}$, and $S_{i+1}^{(t)}$ into $S_{i-1}^{(t+1)}$, it suffices to prove that $S_{i-1}^{(t+1)}$ satisfies the claim.

- If $\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)>\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)$, we observe that $\delta_{E}\left(S_{i}^{(t)}, Q^{\alpha}\right) \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}$ holds by the inductive assumption for some integer $\alpha$. Consequently,

$$
\begin{aligned}
& \delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\alpha+2}\right)=\delta_{E}\left(Q S_{i}^{(t)} Q, Q Q^{\alpha} Q\right) \leq \delta_{E}\left(S_{i}^{(t)}, Q^{\alpha}\right) \\
& \quad \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)} \leq \delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)-1+\Delta_{i}^{(t)}=\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)+\Delta_{i-1}^{(t+1)} .
\end{aligned}
$$

- If $\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)=\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)$, then let $x^{\prime} \leq y^{\prime}$ denote integers that satisfy $\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)=$ $\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)$. This also yields integers $x^{\prime \prime}, y^{\prime \prime}$ with $x^{\prime} \leq x^{\prime \prime} \leq y^{\prime \prime} \leq y^{\prime}$ such that

$$
\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(Q, Q^{\infty}\left[x^{\prime} \ldots x^{\prime \prime}\right)\right)+\delta_{E}\left(S_{i}^{(t)}, Q^{\infty}\left[x^{\prime \prime} \ldots y^{\prime \prime}\right)\right)+\delta_{E}\left(Q, Q^{\infty}\left[y^{\prime \prime} \ldots y^{\prime}\right)\right) .
$$

Due to
$\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right) \leq \delta_{E}\left(S_{i}^{(t)}, Q^{\infty}\left[x^{\prime \prime} \ldots y^{\prime \prime}\right)\right) \leq \delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)=\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)$,
we have $\delta_{E}\left(Q, Q^{\infty}\left[x^{\prime} \ldots x^{\prime \prime}\right)\right)=0=\delta_{E}\left(Q, Q^{\infty}\left[y^{\prime \prime} \ldots y^{\prime}\right)\right)$. As the string $Q$ is primitive, this means that $x^{\prime}, x^{\prime \prime}, y^{\prime \prime}, y^{\prime}$ are all multiples of $|Q|$. Consequently,

$$
\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\left(y^{\prime}-x^{\prime}\right) /|Q|}\right)=\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right) \leq \delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)+\Delta_{i-1}^{(t-1)} .
$$

3. For a type -3 merge of $S_{i-1}^{(t)}$ and $S_{i}^{(t)}$ into $S_{i-1}^{(t+1)}$, it suffices to prove that $S_{i-1}^{(t+1)}$ satisfies the claim.

- If $\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)>\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)$, we observe that $\delta_{E}\left(S_{i}^{(t)}, Q^{*}\right) \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)}$ holds by the inductive assumption. Consequently,

$$
\begin{aligned}
\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{*}\right) & =\delta_{E}\left(Q S_{i}^{(t)}, Q^{*}\right) \leq \delta_{E}\left(S_{i}^{(t)}, Q^{*}\right) \\
\leq & \leq \delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{i}^{(t)} \leq \delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)-1+\Delta_{i}^{(t)}=\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)+\Delta_{i-1}^{(t+1)} .
\end{aligned}
$$

- If $\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)=\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)$, then let $x^{\prime} \leq y^{\prime}$ denote integers that satisfy $\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)=$ $\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)$. This also yields an integer $x^{\prime \prime}$ with $x^{\prime} \leq x^{\prime \prime} \leq y^{\prime}$ such that

$$
\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(Q, Q^{\infty}\left[x^{\prime} \ldots x^{\prime \prime}\right)\right)+\delta_{E}\left(S_{i}^{(t)}, Q^{\infty}\left[x^{\prime \prime} \ldots y^{\prime}\right)\right) .
$$

Due to
$\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right) \leq \delta_{E}\left(S_{i}^{(t)}, Q^{\infty}\left[x^{\prime \prime} \ldots y^{\prime}\right)\right) \leq \delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)=\delta_{E}\left(S_{i}^{(t)},{ }^{*} Q^{*}\right)$, we have $\delta_{E}\left(Q, Q^{\infty}\left[x^{\prime} \ldots x^{\prime \prime}\right)\right)=0$. As the string $Q$ is primitive, this means that $x^{\prime}, x^{\prime \prime}$ are both multiples of $|Q|$. Consequently,

$$
\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{*}\right)=\delta_{E}\left(S_{i-1}^{(t+1)}, Q^{\infty}\left[x^{\prime} . . y^{\prime}\right)\right)=\delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right) \leq \delta_{E}\left(S_{i-1}^{(t+1)},{ }^{*} Q^{*}\right)+\Delta_{i-1}^{(t-1)} .
$$

4. The analysis of type 4 merges is symmetrical to that of 3 merges - this can be seen by reversing all strings in scope.

This completes the proof of the inductive step.
Observe that if no merge rule can be applied to a partition $S=S_{0}^{(t)} \cdots S_{s^{(t)}}^{(t)}$, then $s^{(t)}=0$ or $\Delta_{0}^{(t)}=\cdots=\Delta_{s^{(t)}}^{(t)}=0$. Consequently, Claim 5.7 implies that all fragments $S_{i}$ in the final partition $S=S_{0} \cdots S_{s}$ are locked.
${ }^{『}$ Claim 5.8. For each partition $S=S_{0}^{(t)} \cdots S_{s^{(t)}}^{(t)}$, the total length $\lambda^{(t)}$ of interesting fragments satisfies

$$
\lambda^{(t)}+2|Q| \sum_{i=0}^{s^{(t)}} \Delta_{i}^{(t)} \leq(5|Q|+1) \delta_{E}\left(S,{ }^{*} Q^{*}\right) .
$$

Proof. We proceed by induction on $t$. In the base case of $t=0$, each interesting fragment other than $S_{0}^{(0)}$ and $S_{s^{(0)}}^{(0)}$ satisfies $\Delta_{i}^{(0)}>0$. Hence, the number of interesting fragments is at most $2+\sum_{i=0}^{s^{(0)}} \Delta_{i}^{(0)}=$ $2+\delta_{E}\left(S,{ }^{*} Q^{*}\right)$. Moreover, the length of each fragment $S_{i}^{(0)}$ does not exceed $|Q|+\Delta_{i}^{(0)}$. Consequently,

$$
\lambda^{(0)}+2|Q| \sum_{i=0}^{s^{(0)}} \Delta_{i}^{(0)} \leq\left(2+\delta_{E}\left(S,{ }^{*} Q^{*}\right)\right)|Q|+(2|Q|+1) \sum_{i=0}^{s^{(0)}} \Delta_{i}^{(0)} \leq(5|Q|+1) \delta_{E}\left(S,{ }^{*} Q^{*}\right) .
$$

This completes the proof in the base case.
As for the inductive step, it suffices to prove that $\lambda^{(t+1)}+2|Q| \sum_{i=0}^{s^{(t+1)}} \Delta_{i}^{(t+1)} \leq \lambda^{(t)}+2|Q| \sum_{i=0}^{s^{(t)}} \Delta_{i}^{(t)}$ :

- For a type -1 merge (where we merge two interesting fragments), we have

$$
\lambda^{(t+1)}+2|Q| \sum_{i=0}^{s^{(t+1)}} \Delta_{i}^{(t+1)}=\lambda^{(t)}+2|Q| \sum_{i=0}^{s^{(t)}} \Delta_{i}^{(t)} .
$$

- For a type- 2 type -3 or type- 4 merge (where we merge a fragment with its one or two non-interesting neighbors), we have

$$
\lambda^{(t+1)}+2|Q| \sum_{i=0}^{s^{(t+1)}} \Delta_{i}^{(t+1)} \leq \lambda^{(t)}+2|Q|+2|Q| \sum_{i=0}^{s^{(t+1)}} \Delta_{i}^{(t+1)}=\lambda^{(t)}+2|Q| \sum_{i=0}^{s^{(t)}} \Delta_{i}^{(t)} .
$$

Overall, we obtain the claimed bound.

We conclude that the total length of interesting fragments $S_{i}$ does not exceed $(5|Q|+1) \delta_{E}\left(S,{ }^{*} Q^{*}\right)$ ．
${ }^{『}$ Claim 5．9．We have $\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\sum_{i=0}^{s} \delta_{E}\left(S_{i},{ }^{*} Q^{*}\right)$ ．
Proof．The claim is immediate if $s=0$ ；hence，assume that $s \geq 1$ ．Observe that the inequality $\sum_{i=0}^{s} \delta_{E}\left(S_{i},{ }^{*} Q^{*}\right) \leq \delta_{E}\left(S,{ }^{*} Q^{*}\right)$ easily follows from disjointness of fragments $S_{i}$ ；thus，we focus on proving $\delta_{E}\left(S,{ }^{*} Q^{*}\right) \leq \sum_{i=0}^{s} \delta_{E}\left(S_{i},{ }^{*} Q^{*}\right)$ ．

For $0 \leq i \leq s$ ，let $Q_{i}$ denote a substring of $Q^{\infty}$ that satisfies $\delta_{E}\left(S_{i},{ }^{*} Q^{*}\right)=\delta_{E}\left(S_{i}, Q_{i}\right)$ ．Since each $S_{i}$ is locked（by Claim 5．7），we may assume that for $0<i<s$ the substring $Q_{i}$ is a power of $Q$ ，the substring $Q_{s}$ is a prefix of a power of $Q$ ，and the substring $Q_{0}$ is a suffix of a power of $Q$ ．Consequently，$Q_{0} \cdots Q_{s}$ is a substring of $Q^{\infty}$ ，and we have

$$
\delta_{E}\left(S,{ }^{*} Q^{*}\right) \leq \delta_{E}\left(S_{0} \cdots S_{s}, Q_{0} \cdots Q_{s}\right) \leq \sum_{i=0}^{s} \delta_{E}\left(S_{i}, Q_{i}\right)=\sum_{i=0}^{s} \delta_{E}\left(S_{i},{ }^{*} Q^{*}\right),
$$

thus completing the proof．
The locked fragments created satisfy $\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\sum_{i=1}^{\ell} \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right)$ due to Claim 5．9．Moreover，since $\delta_{E}\left(S_{i},{ }^{*} Q^{*}\right)>0$ holds only for interesting fragments，Claim 5.8 yields $\sum_{i=1}^{\ell}\left|L_{i}\right| \leq(5|Q|+1) \delta_{E}\left(S,{ }^{*} Q^{*}\right)$ ， completing the proof．

The definition and lemma that follow，as well as Theorem 5．2（d），are not needed for our proof of the main result of this section，Theorem 5．1－a reader interested only in that result can safely skip them． They，however，provide additional structural insights that we exploit in Section 6

『 Definition 5．10．Let $S$ denote a string，let $Q$ denote a primitive string，and let $k \geq 0$ denote an integer． We say that a prefix $L$ of $S$ is $k$－locked（with respect to $Q$ ）if at least one of the following holds：
－For every $p \in[0 . .|Q|)$ ，if $\delta_{E}\left(L, \operatorname{rot}^{p}(Q)^{*}\right) \leq k$ ，then $\delta_{E}\left(L, \operatorname{rot}^{p}(Q)^{*}\right)=\delta_{E}\left(L, Q^{\infty}[|Q|-p \ldots j|Q|)\right)$ holds for some integer $j$ ．
－We have $L=S$ ．
－Lemma 5．11．Let $S$ denote a string，let $Q$ denote a primitive string，and let $k \geq 0$ be an integer． There are disjoint locked fragments $L_{1}, \ldots, L_{\ell} \preceq S$ ，such that $L_{1}$ is a $k$－locked prefix of $S$ ，$L_{\ell}$ is a suffix of $S, \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right)>0$ for $1<i<\ell$ ，

$$
\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\sum_{i=1}^{\ell} \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right), \quad \text { and } \quad \sum_{i=1}^{\ell}\left|L_{i}\right| \leq(5|Q|+1) \delta_{E}\left(S,{ }^{*} Q^{*}\right)+2(k+1)|Q| .
$$

Proof．We proceed as in the proof of Lemma 5.6 except that $\Delta_{0}^{(0)}$ is artificially increased by $k+1$ ，the prefix $S_{0}$ in the final partition is included as $L_{1}$ among the locked fragments even if $\delta_{E}\left(S_{0},{ }^{*} Q^{*}\right)=0$ ，and the suffix $S_{s}$ is included as $L_{\ell}$ among the locked fragments even if $\delta_{E}\left(S_{s},{ }^{*} Q^{*}\right)=0$ ．

It is easy to see that Claims 5.7 and 5.9 remain satisfied，whereas the upper bound in Claim 5.8 is increased by $2(k+1)|Q|$ ．We only need to prove that $S_{0}$ is a $k$－locked prefix of $S$ ．For this，we prove the following claim using induction．
『 Claim 5．12．For each partition $S=S_{0}^{(t)} \cdots S_{s^{(t)}}^{(t)}$ ，at least one of the following holds：
＝For every $p \in[0 \ldots|Q|)$ ，if $\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right) \leq k-\Delta_{0}^{(t)}$ ，then $\delta_{E}\left(S_{0}^{(t)}, Q^{\infty}[|Q|-p \ldots j|Q|)\right) \leq$ $\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t)}$ holds for some integer $j$ ．
－We have $S_{0}^{(t)}=S$ ．

Proof. We proceed by induction on $t$. In the base case of $t=0$, the claim holds trivially since $\delta_{E}\left(S_{0}^{(0)}, \operatorname{rot}^{p}(Q)^{*}\right) \geq 0>k-\Delta_{0}^{(0)}$ holds for every $p$ due to $\Delta_{0}^{(0)} \geq k+1$.

As for the induction step, we assume that the claim holds for $\bar{S}_{0}^{(t)}$ and we prove that it holds for $S_{0}^{(t+1)}$. The claim holds trivially if the merge rule applied did not affect $S_{0}^{(t)}$. Given that $S_{0}^{(t)}$ is interesting by definition, the merges that might affect $S_{0}^{(t)}$ are of type 1 (if $S_{1}^{(t)}$ is interesting) or 4 (otherwise).

1. Consider a type-1 merge of $S_{0}^{(t)}$ and $S_{1}^{(t)}$. If $s^{(t)}=1$, then $S_{0}^{(t+1)}=S$ satisfies the claim trivially. Hence, we may assume that $1<s^{(t)}$ so that Claim 5.7 yields $\delta_{E}\left(S_{1}^{(t)}, Q^{\alpha}\right) \leq \delta_{E}\left(S_{1}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{1}^{(t)}$ for some integer $\alpha$. Let us fix $p \in[0 \ldots|Q|)$ with $\delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right) \leq k-\Delta_{0}^{(t+1)}$. Due to $\Delta_{0}^{(t+1)} \geq \Delta_{0}^{(t)}$, this yields $\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right) \leq k-\Delta_{0}^{(t)}$, so the inductive assumption implies $\delta_{E}\left(S_{0}^{(t)}, Q^{\infty}[|Q|-p \ldots j|Q|)\right) \leq$ $\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t)}$ for some integer $j$. Consequently,

$$
\begin{aligned}
& \delta_{E}\left(S_{0}^{(t+1)}, Q^{\infty}[|Q|-p \ldots(j+\alpha)|Q|)\right)=\delta_{E}\left(S_{0}^{(t)} S_{1}^{(t)}, Q^{\infty}[|Q|-p \ldots j|Q|) Q^{\alpha}\right) \\
& \leq \delta_{E}\left(S_{0}^{(t)}, Q^{\infty}[|Q|-p \ldots j|Q|)\right)+\delta_{E}\left(S_{1}^{(t)}, Q^{\alpha}\right) \leq \delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t)}+\delta_{E}\left(S_{1}^{(t)},{ }^{*} Q^{*}\right)+\Delta_{1}^{(t)} \\
& \leq \delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t+1)}
\end{aligned}
$$

2. Consider a type -4 merge of $S_{0}^{(t)}$ and $S_{1}^{(t)}$. Let us fix $p \in[0 \ldots|Q|)$ with $\delta_{E}\left(S_{0}^{(t+1)}\right.$, $\left.\operatorname{rot}^{p}(Q)^{*}\right) \leq$ $k-\Delta_{0}^{(t+1)}$.

- If $\delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)>\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)$, then

$$
\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right) \leq \delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)-1 \leq k-\Delta_{0}^{(t+1)}-1=k-\Delta_{0}^{(t)}
$$

Hence, the inductive assumption implies $\delta_{E}\left(S_{0}^{(t)}, Q^{\infty}[|Q|-p \ldots j|Q|)\right) \leq \delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t)}$ for some integer $j$. Consequently,

$$
\begin{aligned}
& \delta_{E}\left(S_{0}^{(t+1)}, Q^{\infty}[|Q|-p \ldots(j+1)|Q|)\right)=\delta_{E}\left(S_{0}^{(t)} Q, Q^{\infty}[|Q|-p \ldots j|Q|) Q\right) \\
& \begin{array}{r}
\leq \delta_{E}\left(S_{0}^{(t)}, Q^{\infty}[|Q|-p \ldots j|Q|)\right) \leq \delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t)} \leq \delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)-1+\Delta_{0}^{(t)} \\
\\
=\delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t+1)}
\end{array}
\end{aligned}
$$

- If $\delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)=\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)$, then let $y^{\prime}$ denote an arbitrary integer that satisfies $\delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)=\delta_{E}\left(S_{0}^{(t+1)}, Q^{\infty}\left[|Q|-p \ldots y^{\prime}\right)\right)$. This also yields an integer $y^{\prime \prime}$ with $|Q|-p \leq$ $y^{\prime \prime} \leq y^{\prime}$ such that

$$
\delta_{E}\left(S_{0}^{(t+1)}, Q^{\infty}\left[|Q|-p \ldots y^{\prime}\right)\right)=\delta_{E}\left(S_{0}^{(t)}, Q^{\infty}\left[|Q|-p \ldots y^{\prime \prime}\right)\right)+\delta_{E}\left(Q, Q^{\infty}\left[y^{\prime \prime} \ldots y^{\prime}\right)\right)
$$

Due to

$$
\begin{aligned}
\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right) \leq \delta_{E}\left(S_{0}^{(t)}, Q^{\infty}\left[|Q|-p \ldots y^{\prime \prime}\right)\right) & \leq \delta_{E}\left(S_{0}^{(t+1)}, Q^{\infty}\left[|Q|-p \ldots y^{\prime}\right)\right) \\
& =\delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)=\delta_{E}\left(S_{0}^{(t)}, \operatorname{rot}^{p}(Q)^{*}\right)
\end{aligned}
$$

we have $\delta_{E}\left(Q, Q^{\infty}\left[y^{\prime \prime} \ldots y^{\prime \prime}\right)\right)$. As the string $Q$ is primitive, this means that $y^{\prime \prime}, y^{\prime}$ are both multiples of $|Q|$. Consequently,

$$
\delta_{E}\left(S_{0}^{(t+1)}, Q^{\infty}\left[|Q|-p \ldots y^{\prime}\right)\right)=\delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right) \leq \delta_{E}\left(S_{0}^{(t+1)}, \operatorname{rot}^{p}(Q)^{*}\right)+\Delta_{0}^{(t+1)}
$$

This completes the proof of the inductive step.
Given that the final partition $S=S_{0}^{(t)} \cdots S_{s^{(t)}}^{(t)}$ satisfies $s^{(t)}=0$ or $\Delta_{0}^{(t)}=0$, we conclude that $S_{0}$ is indeed $k$-locked.

We are now ready to prove Theorem 5.2 ，which we restate here for convenience．
『 Theorem 5.2 （Compare Theorem 3．2）．Let $P$ denote a pattern of length $m$ ，let $T$ denote a text of length $n$ ，and let $k \leq m$ denote a non－negative integer such that $n<3 / 2 m+k$ ．Suppose that the $k$－error occurrences of $P$ in $T$ include a prefix of $T$ and a suffix of $T$ ．If there are a positive integer $d \geq 2 k$ and a primitive string $Q$ with $|Q| \leq m / 8 d$ and $\delta_{E}\left(P, Q^{*}\right)=\delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq d$ ，then each of following holds：
（a）For every $p \in \operatorname{Occ}_{k}^{E}(P, T)$ ，we have $p \bmod |Q| \leq 3 d$ or $p \bmod |Q| \geq|Q|-3 d$ ．
（b）The string $T$ satisfies $\delta_{E}\left(T,{ }^{*} Q^{*}\right) \leq 3 d$ ．
（c）If $\delta_{E}\left(P,{ }^{*} Q^{*}\right)=d$ ，then $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / d\right\rfloor\right| \leq 304 d$ ．
（d）The set $\operatorname{Occ}_{k}^{E}(P, T)$ can be decomposed into $617 d^{3}$ arithmetic progressions with difference $|Q|$ ．
Proof．Consider any $k$－error occurrence $T[\ell \ldots r)$ of $P$ ．By definition，$\delta_{E}(T[\ell \ldots r), P) \leq k \leq d / 2$ ． Combining this inequality with $\delta_{E}\left(P, Q^{*}\right) \leq d$ via the triangle inequality（Fact 2．3），we obtain the bound $\delta_{E}\left(T[\ell \ldots r), Q^{*}\right) \leq 3 / 2 d$ ．In particular，this inequality is true for the $k$－error occurrence of $P$ as a prefix of $T$ ．Hence，for some integer $m^{\prime} \in[m-k \ldots m+k]$ ，we have $\delta_{E}\left(T\left[0 \ldots m^{\prime}\right), Q^{*}\right) \leq 3 / 2 d$ ，and thus also $\delta_{E}\left(T\left[0 \ldots \min \left(r, m^{\prime}\right)\right), Q^{*}\right) \leq 3 / 2 d$ ．

Next，we apply Lemma 5.4 on the fragment $T[0 \ldots r)$ ，whose prefix $T\left[0 \ldots \min \left(r, m^{\prime}\right)\right)$ satisfies $\delta_{E}\left(T\left[0 \ldots \min \left(r, m^{\prime}\right)\right), Q^{*}\right) \leq 3 / 2 d$ and whose suffix $T[\ell \ldots r)$ satisfies $\delta_{E}\left(T[\ell \ldots r), Q^{*}\right) \leq \frac{3}{2} d$ ．Further，if $|Q|>1$ ，we also have $\left|T\left[\ell \ldots \min \left(r, m^{\prime}\right)\right)\right| \geq(3 d+1)|Q|$ ：Due to $r-\ell \geq m-k$ and $m^{\prime} \geq m-k$ ，we have $\min \left(r, m^{\prime}\right)-\ell \geq 2(m-k)-n>m / 2-3 k \geq 4 d|Q|-3 k$ ．Hence，it suffices to prove that $(d-1)|Q| \geq 3 k-1$ ． This equality holds trivially if $k=0$ ．For $k \geq 1$ ，we have $(d-1)|Q| \geq(2 k-1) \cdot 2=4 k-2 \geq 3 k-1$ due to $d \geq 2 k$ and $|Q| \geq 2$ ．Thus，we can indeed use Lemma 5.4

In particular，Lemma 5.4 implies $(\ell+3 d) \bmod |Q| \leq 6 d$ ．Since $T[\ell \ldots r)$ was an arbitrary $k$－error occurrence of $P$ in $T$ ，we conclude that Claim（a）holds．

Moreover，Lemma 5.4 implies $\delta_{E}\left(T[0 \ldots r), Q^{*}\right) \leq 3 d$ ．If we choose $T[\ell \ldots r)$ to be a $k$－error occurrence of $P$ that is a suffix of $T$ ，we have $r=n$ and therefore $\delta_{E}\left(T, Q^{*}\right) \leq 3 d$ ，which proves Claim（b）．

We proceed to the proof of Claim（c）．Let $L_{1}, \ldots, L_{\ell}$ denote locked fragments of $P$ obtained from Lemma5．6 Note that we thus have $\sum_{i=1}^{\ell} \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right)=\delta_{E}\left(P,{ }^{*} Q^{*}\right)=d, \ell \leq d$ ，and $\sum_{i=1}^{\ell}\left|L_{i}\right| \leq(5|Q|+1) d$ ．

Moreover，let us fix an optimal alignment between $T$ and a substring $Q^{\prime}$ of $Q^{*}$ ，and define $d^{\prime}:=$ $\delta_{E}\left(T, Q^{*}\right)$ ．This yields partitions $T=T_{0} \cdots T_{2 d^{\prime}}$ and $Q^{\prime}=Q_{0}^{\prime} \cdots Q_{2 d^{\prime}}^{\prime}$ such that：
－$T_{i}=Q_{i}^{\prime}$ for even $i$ ，
－$T_{i} \neq Q_{i}^{\prime}$ and $\left|T_{i}\right|,\left|Q_{i}^{\prime}\right| \leq 1$ for odd $i$ ．
We create a multi－set $E=\left\{\sum_{i^{\prime}<i}\left|T_{i^{\prime}}\right|: i\right.$ is odd $\}$ of size $d^{\prime}$ ．Its elements can be interpreted as positions in $T$ which incur errors in an optimal alignment with $Q^{\prime}$ ．In particular，we show the following：
${ }^{『}$ Claim 5．13．For every fragment $T[x \ldots y)$ ，we have $\delta_{E}\left(T[x \ldots y),{ }^{*} Q^{*}\right) \leq|\{e \in E \mid x \leq e<y\}|$ ．
Proof．It suffices to observe that the alignment between $T$ and $Q^{\prime}$ yields an alignment between $T[x \ldots y$ ） and a fragment $Q^{\prime}\left[x^{\prime} \ldots y^{\prime}\right)$ with $\delta_{E}\left(T[x \ldots y), Q^{\prime}\left[x^{\prime} \ldots y^{\prime}\right)\right) \leq|\{e \in E \mid x \leq e<y\}|$ edits．

We split $\mathbb{Z}$ into disjoint blocks of the form $[j d \ldots(j+1) d)$ for $j \in \mathbb{Z}$ ．We say that a block $[j d \ldots(j+1) d)$ is synchronized if it contains a position $p$ such that $(p+3 d) \bmod |Q| \leq 6 d$ ．For every locked fragment $L_{i}=P\left[\ell_{i} \ldots r_{i}\right)$ and every $e \in E$ ，we mark a synchronized block $[j d \ldots(j+1) d)$ if

$$
e \in\left[j d+\ell_{i}-k \ldots(j+1) d-1+r_{i}+k\right)
$$

『 Claim 5．14．If $[j d \ldots(j+1) d) \cap \operatorname{Occ}_{k}^{E}(P, T) \neq \emptyset$ ，then $[j d \ldots(j+1) d)$ has at least $d-k$ marks．

Proof. Consider a $k$-error occurrence of $P$ in $T$ starting at position $p \in[j d \ldots(j+1) d)$ and fix its arbitrary optimal alignment with $P$. For each locked fragment $L_{i}$, let $L_{i}^{\prime}$ be the fragment of $T$ aligned with $L_{i}$ in this alignment. Moreover, $L_{i}^{\prime}=T\left[\ell_{i}^{\prime} \ldots r_{i}^{\prime}\right)$ for

$$
\left[\ell_{i}^{\prime} \ldots r_{i}^{\prime}\right) \subseteq\left[p+\ell_{i}-k \ldots p+r_{i}+k\right) \subseteq\left[j d+\ell_{i}-k \ldots(j+1) d-1+r_{i}+k\right)
$$

Also, by Claim 5.13, we have

$$
\delta_{E}\left(L_{i}^{\prime},{ }^{*} Q^{*}\right) \leq\left|\left\{e \in E \mid \ell_{i}^{\prime} \leq e<r_{i}^{\prime}\right\}\right| \leq\left|\left\{e \in E \mid j d+\ell_{i}-k \leq e<(j+1) d-1+r_{i}+k\right\}\right|
$$

Hence, the number $\mu$ of marks at $[j d . .(j+1) d)$ is at least $\mu \geq \sum_{i=1}^{\ell} \delta_{E}\left(L_{i}^{\prime},{ }^{*} Q^{*}\right)$. On the other hand, by disjointness of regions $L_{i}$, we have $\sum_{i=1}^{\ell} \delta_{E}\left(L_{i}^{\prime}, L_{i}\right) \leq k$. By the triangle inequality (Fact 2.3), this yields $\sum_{i=1}^{\ell} \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right) \leq k+\mu$. Since $\sum_{i=1}^{\ell} \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right)=\delta_{E}\left(P,{ }^{*} Q^{*}\right)=d$, we conclude that $\mu \geq d-k$.

『 Claim 5.15. The total number of marks placed is at most $152 d^{2}$.
Proof. Let us fix $e \in E$ and a locked fragment $L_{i}=P\left[\ell_{i} \ldots r_{i}\right)$. Recall that a mark is placed at a synchronized block $[j d \ldots(j+1) d)$ if $e \in\left[j d+\ell_{i}-k \ldots(j+1) d-1+r_{i}+k\right)$, or, in other words,

$$
[j d \ldots(j+1) d) \cap\left[e-r_{i}-k . . e-\ell_{i}+k\right) \neq \emptyset .
$$

The length of the interval $I_{i, e}:=\left[e-r_{i}-k . . e-\ell_{i}+k\right)$ satisfies $\left|I_{i, e}\right|=2 k+\left|L_{i}\right| \leq d+\left|L_{i}\right|$.
We now consider two cases depending on whether or not the inequality $|Q|<9 d$ is satisfied. If $|Q|<9 d$, it suffices to observe that any interval $I$ overlaps with at most $2+|I| / d$ blocks. Hence, $I_{i, e}$ overlaps with at most $3+\left|L_{i}\right| / d$ blocks, and thus the number of marks we have placed due to $e$ and $L_{i}$ is bounded by $3+\left|L_{i}\right| / d$. The overall number of marks is therefore at most

$$
|E| \cdot \sum_{i=1}^{\ell}\left(3+\left|L_{i}\right| / d\right) \leq 9 d^{2}+3 \sum_{i=1}^{\ell}\left|L_{i}\right| \leq 9 d^{2}+3(5|Q|+1) d \leq 9 d^{2}+135 d^{2}=144 d^{2}
$$

On the other hand, if $|Q| \geq 9 d$, we utilize the fact that only synchronized blocks are marked. For this, observe that any interval $I$ overlaps at most $2+|I| /|Q|$ intervals of the form $\left[\left(j^{\prime}-1\right)|Q|-4 d \ldots j^{\prime}|Q|+4 d\right)$ each of which overlaps with at most 7 synchronized blocks (covering $\left[j^{\prime}|Q|-3 d \ldots j^{\prime}|Q|+3 d\right]$, which is of length $6 d+1)$. Hence, the total number of blocks marked due to $e$ and $L_{i}$ is bounded by $7\left(2+\left(d+\left|L_{i}\right|\right) /|Q|\right)$. The overall number of marks is therefore at most

$$
\begin{aligned}
& |E| \cdot \sum_{i=1}^{\ell} 7\left(2+\left(d+\left|L_{i}\right|\right) /|Q|\right) \leq 42 d^{2}+21 d^{2} /|Q|+21 d /|Q| \cdot \sum_{i=1}^{\ell}\left|L_{i}\right| \\
& \quad \leq 42 d^{2}+21 d^{2} /|Q|+21 d^{2} \cdot(5|Q|+1) /|Q| \leq 42 d^{2}+21 d^{2} /|Q|+105 d^{2}+21 d^{2} /|Q|<152 d^{2}
\end{aligned}
$$

This completes the proof of the claim.
Hence, the number of blocks with at least $d / 2$ marks is at most $304 d$, completing the proof of Claim (c).
We proceed to the proof of Claim (d). Let $L_{1}^{P}, \ldots, L_{\ell^{P}}^{P}$ denote locked fragments of $P$ obtained from Lemma 5.11 (so that $L_{1}^{P}$ is a $k$-locked prefix of $P$ ), and let $L_{1}^{T}, \ldots, L_{\ell^{T}}^{T}$ denote locked fragments of $T$ obtained from Lemma 5.6 Denote $L_{i}^{P}=P\left[\ell_{i}^{P} \ldots r_{i}^{P}\right.$ ) for $i \in\left[1 \ldots \ell^{P}\right]$ and $L_{j}^{T}=T\left[\ell_{j}^{T} \ldots r_{j}^{T}\right)$ for $j \in\left[1 \ldots \ell^{T}\right]$. We say that a position $p \in[0 \ldots n)$ is marked if $p \in[n-m-k \ldots n-m+k]$ or $\left[p+\ell_{i}^{P}-k \ldots p+r_{i}^{P}+k\right) \cap\left[\ell_{j}^{T} \ldots r_{j}^{T}\right) \neq \emptyset$ holds for some $i \in\left[1 \ldots \ell^{P}\right]$ and $j \in\left[1 \ldots \ell^{T}\right]$. (The positions in
[ $n-m-k \ldots n-m+k$ ] are marked for a technical reason that will become clear in the proof of Claim 5.17]) Furthermore, we say that $p$ is synchronized if $p \bmod |Q| \leq 3 d$ or $p \bmod |Q| \geq|Q|-3 d$.

Let us provide some intuition. Informally, if there is a $k$-occurrence in an unmarked position $p$, then no locked region of $P$ can overlap a locked region of $T$ in any corresponding optimal alignment and we can exploit this structure. On the other hand, we now show that there are only a few synchronized marked positions.

『
Claim 5.16. Marked positions can be decomposed into at most $10 d^{2}$ integer intervals. Moreover, the number of synchronized marked positions is at most $547 d^{3}$.

Proof. First, we have the interval $[n-m-k \ldots n-m+k]$. Observe that each pair $i \in\left[1 \ldots \ell^{P}\right]$ and $j \in\left[1 \ldots \ell^{T}\right]$ yields to marking positions $p \in\left(\ell_{j}^{T}-r_{i}^{P}-k \ldots r_{j}^{T}-\ell_{i}^{P}+k\right)$. Consequently, marked positions can be decomposed into $1+\ell^{P} \cdot \ell^{T}$ integer intervals. Due to $\ell^{P} \leq \delta_{E}\left(P,{ }^{*} Q^{*}\right)+2 \leq d+2 \leq 3 d$ and $\ell^{T} \leq \delta_{E}\left(T,{ }^{*} Q^{*}\right) \leq 3 d$, the number of intervals is at most $10 d^{2}$.

The interval of positions marked due to $i$ and $j$ is of length $2 k+\left|L_{i}^{P}\right|+\left|L_{j}^{T}\right|-1 \leq d+\left|L_{i}^{P}\right|+\left|L_{j}^{T}\right|-1$. Out of any $|Q|$ consecutive positions at most $6 d+1 \leq 7 d$ are synchronized. Hence, the number of synchronized positions in any such interval $I$ is at most $7 d(|I|-7 d+|Q|) /|Q|$. Consequently, the total number of synchronized marked positions does not exceed $2 k+1 \leq d^{3}$ (for [ $n-m-k . . n-m+k$ ]) plus

$$
\begin{aligned}
\sum_{i=1}^{\ell^{P}} \sum_{j=1}^{\ell^{T}} 7 d \frac{\left|L_{i}^{P}\right|+\left|L_{j}^{T}\right|-7 d+|Q|}{|Q|} & \leq \frac{7 d \ell^{T}}{|Q|} \sum_{i=1}^{\ell^{P}}\left|L_{i}^{P}\right|+\frac{7 d \ell^{P}}{|Q|} \sum_{i=1}^{\ell^{T}}\left|L_{i}^{T}\right|+\frac{7 d \ell^{P} \ell^{T}(|Q|-7 d)}{|Q|} \\
& \leq \frac{21 d^{2}}{|Q|}\left(\sum_{i=1}^{\ell^{P}}\left|L_{i}^{P}\right|+\sum_{i=1}^{\ell^{T}}\left|L_{i}^{T}\right|+3 d(|Q|-7 d)\right) \\
& \leq \frac{21 d^{2}}{|Q|}((5|Q|+1) d+2(k+1)|Q|+(5|Q|+1) 3 d+3 d(|Q|-7 d)) \\
& \leq \frac{21 d^{2}}{|Q|}((5|Q|+1) d+3 d|Q|+(5|Q|+1) 3 d+3 d(|Q|-7 d)) \\
& \leq \frac{21 d^{2}}{|Q|}\left(26 d|Q|+4 d-21 d^{2}\right) \\
& \leq 21 \cdot 26 d^{3}=546 d^{3}
\end{aligned}
$$

This completes the proof.
Next, we characterize unmarked positions $p \in \operatorname{Occ}_{k}^{E}(P, T)$. They can be decomposed into at most $10 d^{2}$ integer intervals by Claim 5.16 and by the fact that $p<n-m-k$. Consider any such interval $I$.『 Claim 5.17. For any $p, p^{\prime} \in I$ such that $p \equiv p^{\prime}(\bmod |Q|)$ we have $p \in \operatorname{Occ}_{k}^{E}(P, T)$ if and only if $p^{\prime} \in \operatorname{Occ}_{k}^{E}(P, T)$. In particular, $I \cap \operatorname{Occ}_{k}^{E}(P, T)$ can be decomposed into at most $6 d+1$ arithmetic progressions with difference $|Q|$.

Proof. By our marking scheme, for any $i, j$, for any pair $x \in\left[\ell_{i}^{P} \ldots r_{i}^{P}\right)$ and $y \in\left[\ell_{j}^{P} \ldots r_{j}^{P}\right)$, we see that $|(p+x)-y|>k$. Consider an unmarked position $p \in \operatorname{Occ}_{k}^{E}(P, T) \cap I$ and fix any alignment of some prefix $T[p \ldots t)$ of $T[p \ldots n)$ with $P$ with at most $k$ errors. Then, for all $i$, the fragment $T_{i}$ of $T$ aligned with $L_{i}^{P}$ is disjoint from all locked fragments of $T$ and hence is a substring of $Q^{\infty}$. Now, recall that $L_{1}^{P}$ is a prefix of $P$ and $L_{\ell^{P}}^{P}$ is a suffix of $P$. Hence, the locked fragments of $T$ that are considered are exactly those $L_{j}^{T} \mathrm{~s}$ such that $p+k<\ell_{j}<r_{j}<p+m-k$; say that this holds for $j \in\left[j_{1} \ldots j_{2}\right]$. For all $j \in\left[j_{1} \ldots j_{2}\right]$, the
fragment $P_{j}$ of $P$ aligned with $L_{j}^{T}$ is disjoint from all locked fragments of $P$ and is a substring of $Q^{\infty}$. In addition, since $I=\left[i_{1} \ldots i_{2}\right]$ is an interval of unmarked positions, $T\left[i_{1} \ldots i_{2}+\left|L_{1}^{P}\right|+k\right)$ is disjoint from all locked fragments of $T$ and hence is equal to a substring of $Q^{\infty}$. Thus, for some $r$ we see that:

$$
\begin{align*}
\delta_{E}(P, T[p \ldots t)) & \geq \delta_{E}\left(L_{1}^{P}, \operatorname{rot}^{r}(Q)^{*}\right)+\sum_{i=2}^{\ell^{P}} \delta_{E}\left(L_{i}^{P}, Q_{i}\right)+\sum_{j=j_{1}}^{j_{2}} \delta_{E}\left(\ell_{j}^{T}, P_{j}\right) \\
& \geq \delta_{E}\left(L_{1}^{P}, \operatorname{rot}^{r}(Q)^{*}\right)+\sum_{i=2}^{\ell^{P}} \delta_{E}\left(L_{i}^{P},{ }^{*} Q^{*}\right)+\sum_{j=j_{1}}^{j_{2}} \delta_{E}\left(\ell_{j}^{T},{ }^{*} Q^{*}\right) . \tag{2}
\end{align*}
$$

Note that since $L_{1}^{P}$ is $k$-locked and $\delta_{E}\left(L_{1}^{P}, \operatorname{rot}^{r}(Q)^{*}\right) \leq k$ there exists a $j$ such that $\delta_{E}\left(L_{1}^{P}, \operatorname{rot}^{r}(Q)^{*}\right)=$ $\delta_{E}\left(L_{1}^{P}, Q^{\infty}[|Q|-r \ldots j|Q|)\right)$. Let $b=\left|Q^{\infty}[|Q|-r \ldots j|Q|)\right|$.

Consider any position $p^{\prime} \in I$ with $p^{\prime} \equiv p(\bmod |Q|) . T\left[p^{\prime} . . p^{\prime}+b\right)=T[p \ldots p+b)$ since both fragments lie in $T\left[i_{1} \ldots i_{2}+\left|L_{1}^{P}\right|+k\right]$ and start a multiple of $|Q|$ positions apart. In addition, we have $P\left[\left|L_{1}^{P}\right| \ldots m\right)=Q^{\alpha_{1}} L_{2}^{P} Q^{\alpha_{2}} \cdots Q^{\alpha_{\ell}{ }^{P}-1} L_{\ell^{P}}^{P}$ for some non-negative integers $\alpha_{i}$ and that $T\left[p^{\prime}+b \ldots p^{\prime}+m+\right.$ $k)=Q^{\beta_{j_{1}-1}} L_{j_{1}}^{T} Q^{j_{1}} \cdots L_{j_{2}}^{T} Q^{\beta j_{2}} Q^{\prime}$ for some non-negative integers $\beta_{j}$ and a prefix $Q^{\prime}$ of $Q$. (Note that $p^{\prime}+m+k<n$ since $p^{\prime}<n-m-k$.) We claim that there exists a $t^{\prime}$ such that:

$$
\begin{aligned}
\delta_{E}\left(P, T\left[p^{\prime} \ldots t^{\prime}\right)\right) & \leq \delta_{E}\left(L_{1}^{P}, T\left[p^{\prime} \ldots p^{\prime}+b\right)\right)+\delta_{E}\left(P\left[\left|L_{1}^{P}\right| \ldots m\right), T\left[p^{\prime}+b \ldots t^{\prime}\right)\right) \\
& =\delta_{E}\left(L_{1}^{P}, \operatorname{rot}^{r}(Q)^{*}\right)+\sum_{i=2}^{\ell^{P}} \delta_{E}\left(L_{i}^{P},{ }^{*} Q^{*}\right)+\sum_{j=j_{1}}^{j_{2}} \delta_{E}\left(\ell_{j}^{T},{ }^{*} Q^{*}\right) \leq \delta_{E}(P, T[p \ldots t))
\end{aligned}
$$

In order to prove this, let us consider the following greedy alignment of $P\left[\left|L_{1}^{P}\right| \ldots m\right)$ and $T\left[p^{\prime}+b \ldots t^{\prime}\right)$. We start at the leftmost position in both strings. We will maintain the invariant that the remainder of each string starts with either $Q$ or a locked fragment, except possibly for the case that the remainder of $P$ is $L_{\ell^{P}}^{P}$, in which case the remainder of $T$ can be $Q^{\prime}$. While we have not reached the end of $P\left[\left|L_{1}^{P}\right| \ldots m\right)$ we repeat the following procedure. If both strings have a prefix equal to $Q$, we align those prefixes exactly. Else, the prefix of one of the strings is a locked fragment $L$. Let us first assume that $L \neq L_{\ell^{P}}^{P}$. Then, since $p^{\prime}$ is unmarked, $Q^{\infty}\left[0 \ldots|L|+k-k^{\prime}\right)$ is a prefix of the other string, where $k^{\prime}$ is the number of edits already performed by our greedy alignment. Since $L$ is locked, $\delta_{E}\left(L,{ }^{*} Q^{*}\right)=\delta_{E}\left(L, Q^{\alpha}\right)$ for some integer $\alpha$. We have $|\alpha| Q|-|L|| \leq k-k^{\prime}$ due to the fact that otherwise (2) would imply that $\delta_{E}(P, T[p \ldots t))>k$, a contradiction. Hence, $Q^{\alpha}$ is a prefix of $Q^{\infty}\left[0 \ldots|L|+k-k^{\prime}\right)$; we optimally align those two fragments. If $L=L_{\ell^{P}}^{P}$, an analogous argument shows that there exists a prefix $Q^{\prime \prime}$ of the remainder of $T\left[p^{\prime}+b \ldots t^{\prime}\right.$ ) such that $\delta_{E}\left(L,{ }^{*} Q^{*}\right)=\delta_{E}\left(L, Q^{\prime \prime}\right)$. Upon termination of this greedy alignment, the equality in the above equation holds.

We have thus proved that if $p \in I \cap \operatorname{Occ}_{k}^{E}(P, T)$ then all $p^{\prime} \in I$ such that $p^{\prime} \equiv p(\bmod |Q|)$ are also in $\operatorname{Occ}_{k}^{E}(P, T)$. Thus, for any fixed $0 \leq j<|Q|$, for $U=\{i \cdot|Q|+j \in I \mid i \in \mathbb{Z}\}$ either $U \subseteq \operatorname{Occ}_{k}^{E}(P, T)$ or $U \cap \operatorname{Occ}_{k}^{E}(P, T)=\emptyset$. By Claim (a), we can restrict our attention to synchronized positions. We can thus decompose $I \cap \operatorname{Occ}_{k}^{E}(P, T)$ into at most $6 d+1$ arithmetic progressions.

Combining Claims 5.16 and 5.17 we conclude that $\operatorname{Occ}_{k}^{E}(P, T)$ can be decomposed into at most $547 d^{3}+10 d^{2}(6 d+1) \leq 617 d^{3}$ arithmetic progressions with difference $|Q|$, thus completing the proof. $\quad$

- Corollary 5.18 (Compare Corollary 3.5). Let $P$ denote a pattern of length $m$, let $T$ denote a string of length $n$, and let $k \leq m$ denote a non-negative integer. Suppose that there are a positive integer $d \geq 2 k$ and a primitive string $Q$ with $|Q| \leq m / 8 d$ and $\delta_{E}\left(P,{ }^{*} Q^{*}\right)=d$. Then $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / d\right\rfloor\right| \leq 1216 \cdot n / m \cdot d$.

```
Algorithm 9 A constructive proof of Lemma 5.19. Changes to Algorithm 1 are highlighted.
    \(\mathcal{B} \leftarrow\} ; \mathcal{R} \leftarrow\{ \} ;\)
    while true do
        Consider the fragment \(P^{\prime}=P[j \ldots j+\lfloor m / 8 k\rfloor)\) of the next \(\lfloor m / 8 k\rfloor\) unprocessed characters
        of \(P\);
        if \(\operatorname{per}\left(P^{\prime}\right)>m / 128 k\) then
            \(\mathcal{B} \leftarrow \mathcal{B} \cup\left\{P^{\prime}\right\} ;\)
            if \(|\mathcal{B}|=2 k\) then return breaks \(\mathcal{B}\);
        else
            \(Q \leftarrow P\left[j \ldots j+\operatorname{per}\left(P^{\prime}\right)\right) ;\)
            Search for prefix \(R\) of \(P[j \ldots m)\) with \(\delta_{E}\left(R,{ }^{*} Q^{*}\right)=\lceil 8 k / m \cdot|R|\rceil\) and \(|R|>\left|P^{\prime}\right|\);
            if such \(R\) exists then
                \(\mathcal{R} \leftarrow \mathcal{R} \cup\{(R, Q)\} ;\)
                if \(\sum_{(R, Q) \in \mathcal{R}}|R| \geq 3 / 8 \cdot m\) then
                    return repetitive regions (and their corresponding periods) \(\mathcal{R}\);
            else
                Search for suffix \(R^{\prime}\) of \(P\) with \(\delta_{E}\left(R^{\prime},{ }^{*} Q^{*}\right)=\left\lceil 8 k / m \cdot\left|R^{\prime}\right|\right\rceil\) and \(\left|R^{\prime}\right| \geq m-j\);
            if such \(R^{\prime}\) exists then return repetitive region \(\left(R^{\prime}, Q\right)\);
            else return approximate period \(Q\);
```

Proof. Partition the string $T$ into $\lfloor 2 n / m\rfloor$ blocks $T_{0}, \ldots, T_{\lfloor 2 n / m\rfloor-1}$ of length at most $3 / 2 m+k-1$ each, where the $i$ th block starts at position $i \cdot m / 2$, that is, $T_{i}:=T[\lfloor i \cdot m / 2\rfloor \ldots \min \{n,\lfloor(i+3) \cdot m / 2\rfloor+k-1\})$. Observe that each $k$-error occurrence of $P$ in $T$ is contained in at least one of the fragments $T_{i}$ : Specifically, $T_{i}$ covers all the occurrences starting in $[\lfloor i \cdot m / 2\rfloor \ldots\lfloor(i+1) \cdot m / 2\rfloor)$. If $\operatorname{Occ}_{k}^{E}\left(P, T_{i}\right) \neq \emptyset$, we define $T_{i}^{\prime}:=T\left[t_{i}^{\prime} . . t_{i}^{\prime}+\left|T_{i}^{\prime}\right|\right)$ to be the shortest fragment of $T_{i}$ containing all $k$-error occurrences of $P$ in $T_{i}$. As a result, $T_{i}^{\prime}$ satisfies the assumptions of Theorem 5.2, so $\left\lfloor\left\lfloor\operatorname{Occ}_{k}^{E}\left(P, T_{i}^{\prime}\right)\right\rfloor / k \mid \leq 304 d\right.$. Each block $\left[j^{\prime} k \ldots\left(j^{\prime}+1\right) k\right)$ of positions in $T_{i}$ corresponds to a block $\left[t_{i}^{\prime}+j^{\prime} k \ldots t_{i}^{\prime}+\left(j^{\prime}+1\right) k\right.$ ) of positions in $T$, which intersects at most two blocks of the form $[j k \ldots(j+1) k)$. In total, we conclude that $\left|\left\lfloor\operatorname{Occ}\left(P, T_{i}\right)\right\rfloor / k\right| \leq$ $\lfloor 2 n / m\rfloor \cdot 2 \cdot 304 d \leq 1216 \cdot n / m \cdot d$.

### 5.2 Bounding the Number of Occurrences in the Non-Periodic Case

Lemma 5.19 (Compare Lemma 3.6). Let $P$ denote a string of length $m$ and let $k \leq m$ denote a positive integer. Then, at least one of the following holds:
(a) The string $P$ contains $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k}$ each having period $\operatorname{per}\left(B_{i}\right)>m / 128 k$ and length $\left|B_{i}\right|=\lfloor m / 8 k\rfloor$.
(b) The string $P$ contains disjoint repetitive regions $R_{1}, \ldots, R_{r}$ of total length $\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and has a primitive approximate period $Q_{i}$ with $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{E}\left(R_{i},{ }^{*} Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$.
(c) The string $P$ has a primitive approximate period $Q$ with $|Q| \leq m / 128 k$ and $\delta_{E}\left(P,{ }^{*} Q^{*}\right)<8 k$.

Proof. We use essentially the same algorithm as in the proof of Lemma 3.6. We replace all checks for a specific Hamming distance with the corresponding counterpart for the edit distance. Further, as we are only interested in (approximate) periods under an arbitrary rotation, we do not need to explicitly
rotate the string $Q$ in the algorithm anymore．Consider Algorithm 9 for a visualization；the changes to Algorithm 1 are highlighted．

In particular，we directly get an analogue of Claim 3．7．
『 Claim 5.20 （See Claim 3．7）．Whenever we consider a new fragment $P[j \ldots j+\lfloor m / 8 k\rfloor$ ）of $\lfloor m / 8 k\rfloor$ unprocessed characters of $P$ ，such a fragment starts at a position $j<5 / 8 \cdot m$ ．

Again，note that Claim 5.20 also shows that whenever we consider a new fragment $P^{\prime}$ of $\lfloor m / 8 k\rfloor$ characters， there is indeed such a fragment，that is，$P^{\prime}$ is well－defined．

Now consider the case when，for a fragment $P^{\prime}=P[j \ldots j+\lfloor m / 8 k\rfloor$ ）（that is not a break）and its corresponding period $Q=\left[j \ldots j+\operatorname{per}\left(P^{\prime}\right)\right)$ ，we fail to obtain a new repetitive region $R$ ．Recall that in this case，we search for a repetitive region $R^{\prime}$ of length $\left|R^{\prime}\right| \geq m-j$ that is a suffix of $P$ and has an approximate period $Q$ ．If we indeed find such a region $R^{\prime}$ ，then $\left|R^{\prime}\right| \geq m-j \geq m-5 / 8 \cdot m=3 / 8 \cdot m$ by Claim 5．20，so $R^{\prime}$ is long enough to be reported on its own．However，if we fail to find such $R^{\prime}$ ，we need to show that $Q$ can be reported as an approximate period of $P$ ，that is，$\delta_{E}\left(P,{ }^{*} Q^{*}\right)<8 k$ ．

Similar to Lemma 3．6，we first show that $\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)<\lceil 8 k / m \cdot(m-j)\rceil$ ．For this，we inductively prove that the values $\Delta_{\rho}:=\lceil 8 k / m \cdot \rho\rceil-\delta_{E}\left(P[j \ldots j+\rho),{ }^{*} Q^{*}\right)$ for $\left|P^{\prime}\right| \leq \rho \leq m-j$ are all at least 1．In the base case of $\rho=\left|P^{\prime}\right|$ ，we have $\Delta_{\rho}=1-0$ because $Q$ is the string period of $P^{\prime}$ ．To carry out an inductive step，suppose that $\Delta_{\rho-1} \geq 1$ for some $\left|P^{\prime}\right|<\rho \leq m-j$ ．Notice that $\Delta_{\rho} \geq \Delta_{\rho-1}-1 \geq 0$ ： the first term in the definition of $\Delta_{\rho}$ has not decreased，and the second term $\delta_{E}\left(P[j \ldots j+\rho),{ }^{*} Q^{*}\right)$ may have increased by at most 1 compared to $\Delta_{\rho-1}$ ．Moreover，$\Delta_{\rho} \neq 0$ because $R=P[j \ldots j+\rho)$ could not be reported as a repetitive region．Since $\Delta_{\rho}$ is an integer，we conclude that $\Delta_{\rho} \geq 1$ ．This inductive reasoning ultimately shows that $\Delta_{m-j}>0$ ，that is，$\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)<\lceil 8 k / m \cdot(m-j)\rceil$ ．

A symmetric argument holds for values $\Delta_{\rho}^{\prime}:=\lceil 8 k / m \cdot \rho\rceil-\delta_{E}\left(P[m-\rho \ldots m),{ }^{*} Q^{*}\right)$ for $m-j \leq \rho \leq m$ because no repetitive region $R^{\prime}$ was found as an extension of $P[j \ldots m)$ to the left．Note that in contrast to the proof of Lemma 3．6，the rotation of $Q$ is implicit．This completes the proof that $\delta_{E}\left(P,{ }^{*} Q^{*}\right)<8 k$ ， that is，$Q$ is an approximate period of $P$ ．

『 Lemma 5.21 （Compare Lemma 3．8．Let $P$ denote a pattern of length $m$ ，let $T$ denote a text of length $n$ ， and let $k \leq m$ denote a positive integer．Suppose that $P$ that contains $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k} \preccurlyeq P$ each satisfying $\operatorname{per}\left(B_{i}\right) \geq m / 128 k$ ．Then，$\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor\right| \leq 1024 \cdot n / m \cdot k$ ．

Proof．The proof proceeds similarly to the proof of Lemma 3.8 ．The only major difference is that we obtain length－$k$ blocks of possible starting positions instead of single starting positions．This is because the edit distance allows for deletions and insertions of characters．

Hence，we split $\mathbb{Z}$ into disjoint blocks of the form $[j k . .(j+1) k)$ for $j \in \mathbb{Z}$ ．Now for every break $B_{i}=P\left[b_{i} \ldots b_{i}+\left|B_{i}\right|\right)$ ，we mark a block $[j k \ldots(j+1) k)$ if

$$
\left[(j-1) \cdot k+b_{i} \ldots(j+2) \cdot k+b_{i}\right) \cap \operatorname{Occ}\left(B_{i}, T\right) \neq \emptyset
$$

Similarly to the proof of Lemma 3.8 ，we proceed to show that we place at most $\mathcal{O}\left(n / m \cdot k^{2}\right)$ marks and that every $k$－error occurrence starts in a block with at least $k$ marks．
${ }^{『}$ Claim 5．22．The total number of marks placed at blocks is at most $1024 \cdot n / m \cdot k^{2}$ ．
Proof．Fix a break $B_{i}$ ．Notice that positions in $\operatorname{Occ}\left(B_{i}, T\right)$ are at distance at least per $\left(B_{i}\right)$ from each other．Furthermore，note that for every occurrence in $\operatorname{Occ}\left(B_{i}, T\right)$ we mark at most 4 blocks．Hence，for the break $B_{i}$ ，we place at most $512 \cdot n / m \cdot k$ marks．In total，the number of marks placed is thus at most $2 k \cdot 512 n / m \cdot k=1024 \cdot n / m \cdot k^{2}$ ．

Next, we show that every $k$-error occurrence of $P$ in $T$ starts in a block with at least $k$ marks.
『Claim 5.23. If $[j k \ldots(j+1) k) \cap \operatorname{Occ}_{k}^{E}(P, T) \neq \emptyset$, then $[j k \ldots(j+1) k)$ has at least $k$ marks.
Proof. Consider a $k$-error occurrence of $P$ in $T$ starting at position $\ell \in[j k \ldots(j+1) k)$ and fix an arbitrary optimal alignment of it with $P$. Out of the $2 k$ breaks, at least $k$ breaks are matched exactly, as not matching a break exactly incurs at least one error. If a break $B_{i}$ is matched exactly, then for at least one $s \in[-k \ldots k]$, we have $\ell+b_{i}+s \in \operatorname{Occ}\left(B_{i}, T\right)$. Since $j k \leq \ell<(j+1) k$, we conclude that $\left[(j-1) \cdot k+b_{i} \ldots(j+2) \cdot k+b_{i}\right) \cap \operatorname{Occ}\left(B_{i}, T\right) \neq \emptyset$, that is, that the block $[j k \ldots(j+1) k)$ has been marked for $B_{i}$. In total, there are at least $k$ marks for the at least $k$ breaks matched exactly.

By Claims 5.22 and 5.23 the number of blocks where $k$-error occurrences of $P$ in $T$ may start is $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor\right| \leq\left(1024 \cdot n / m \cdot k^{2}\right) / k=1024 \cdot n / m \cdot k$.

F Lemma 5.24 (Compare Lemma 3.11). Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n$, and let $k \leq m$ denote a positive integer. Suppose that $P$ contains disjoint repetitive regions $R_{1}, \ldots, R_{r}$ of total length at least $\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and has a primitive approximate period $Q_{i}$ with $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{E}\left(R_{i},{ }^{*} Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$. Then, $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor\right| \leq 642045 \cdot n / m \cdot k$.

Proof. Again, the proof is similar to its Hamming distance counterpart; as before, a major difference is that we only obtain length- $k$ blocks of possible starting positions instead of single starting positions.

As in the proof of Lemma 5.21 , we split $\mathbb{Z}$ into disjoint blocks of the form $[j k . .(j+1) k)$ for $j \in \mathbb{Z}$. Further, we set $m_{R}:=\sum_{r}\left|R_{i}\right|$ and define $k_{i}:=\left\lfloor 4 \cdot\left|R_{i}\right| / m \cdot k\right\rfloor$ for every $1 \leq i \leq r$.

For every repetitive region $R_{i}=P\left[r_{i} \ldots r_{i}+\left|R_{i}\right|\right.$ ), we place $\left|R_{i}\right|$ marks on block $[j k \ldots(j+1) k)$ if

$$
\left[(j-1) \cdot k+r_{i} \cdots(j+2) \cdot k+r_{i}\right) \cap \operatorname{Occ}_{k_{i}}^{E}\left(R_{i}, T\right) \neq \emptyset
$$

Similarly to the proof Lemma 3.11, we proceed to show that we placed at most $\mathcal{O}\left(n / m \cdot k \cdot m_{R}\right)$ marks and that every $k$-error occurrence of $P$ in $T$ starts in a block with at last $m_{R}-m / 4$ marks.
${ }^{『}$ Claim 5.25. The total number of marks placed is at most $214015 \cdot n / m \cdot k \cdot m_{R}$,

Proof. We use Corollary 5.18 to analyze $\operatorname{Occ}_{k_{i}}^{E}\left(R_{i}, T\right)$. For this, we set $d_{i}:=\delta_{E}\left(R_{i},{ }^{*} Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$ and notice that $d_{i} \leq 16 k / m \cdot\left|R_{i}\right|$ since $\left|R_{i}\right| \geq m / 8 k$. Moreover, $d_{i} \geq 2 k_{i}$ and $\left|Q_{i}\right| \leq m / 128 k \leq\left|R_{i}\right| / 8 d_{i}$. Hence, the assumptions of Corollary 5.18 are satisfied, so $\left\lfloor\operatorname{Occ}_{k_{i}}^{E}\left(R_{i}, T\right) / d_{i}\right\rfloor \leq 1216 \cdot n /\left|R_{i}\right| \cdot d_{i} \leq 19456 \cdot k \cdot n / m$.

For a block $\left[j^{\prime} d_{i} \ldots\left(j^{\prime}+1\right) d_{i}\right)$ intersecting $\operatorname{Occ}_{k_{i}}^{E}\left(R_{i}, T\right)$, we mark a block $[j k \ldots(j+1) k)$ only if

$$
\left[(j-1) \cdot k+r_{i} \cdots(j+2) \cdot k+r_{i}\right) \cap\left[j^{\prime} d_{i} \cdots\left(j^{\prime}+1\right) d_{i}\right) \neq \emptyset
$$

which holds only if $j k \in\left[j^{\prime} d_{i}-r_{i}-2 k \ldots\left(j^{\prime}+1\right) d_{i}-r_{i}+k\right)$. The length of the latter interval is $d_{i}+3 k=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil+3 k \leq 11 k$, so the interval contains at most 11 multiples of $k$. Hence, the total number of marks placed due to $R_{i}$ is bounded by $11 \cdot 19456 \cdot n / m \cdot k \cdot\left|R_{i}\right|$. Across all repetitive regions, this sums up to no more than $214015 \cdot n / m \cdot k \cdot m_{R}$, yielding the claim.

Next, we show that every $k$-error occurrence of $P$ in $T$ starts in a block with many marks.
Claim 5.26. If $[j k \ldots(j+1) k) \cap \operatorname{Occ}_{k}^{E}(P, T) \neq \emptyset$, then $[j k \ldots(j+1) k)$ has at least $m_{R}-m / 4$ marks.

Proof. Consider a $k$-error occurrence of $P$ in $T$ starting at position $\ell \in[j k \ldots(j+1) k)$ and fix an arbitrary optimal alignment of it with $P$. For each repetitive region $R_{i}$, let $R_{i}^{\prime}$ be the fragment of $T$ aligned with $R_{i}$ in this alignment. Define $k_{i}^{\prime}=\delta_{E}\left(R_{i}, R_{i}^{\prime}\right)$ and observe that $R_{i}^{\prime}=T\left[r_{i}^{\prime} \ldots r_{i}^{\prime}+\left|R_{i}^{\prime}\right|\right)$ for some $r_{i}^{\prime} \in\left[\ell+r_{i}-k . . \ell+r_{i}+k\right] \subseteq\left[(j-1) \cdot k+r_{i} \ldots(j+2) \cdot k+r_{i}\right)$. Consequently,

$$
\left[(j-1) \cdot k+r_{i} \cdots(j+2) \cdot k+r_{i}\right) \cap \mathrm{Occ}_{k_{i}^{\prime}}^{E}\left(R_{i}, T\right) \neq \emptyset
$$

Further, let $I:=\left\{i \mid k_{i}^{\prime} \leq k_{i}\right\}=\left\{i\left|k_{i}^{\prime} \leq 4 \cdot\right| R_{i} \mid / m \cdot k\right\}$ denote the set of indices $i$ for which $R_{i}^{\prime}$ is a $k_{i}$-error occurrence of $R_{i}$. By construction, for each $i \in I$, we have placed $\left|R_{i}\right|$ marks at the block [ $\left.j k \ldots(j+1) k\right)$.

Hence, the total number of marks at the block $[j k \ldots(j+1) k)$ is at least $\sum_{i \in I}\left|R_{i}\right|=m_{R}-\sum_{i \notin I}\left|R_{i}\right|$. It remains to bound the term $\sum_{i \notin I}\left|R_{i}\right|$. Using the definition of $I$, we obtain

$$
\sum_{i \notin I}\left|R_{i}\right|=\frac{m}{4 k} \cdot \sum_{i \notin I}\left(4 \cdot\left|R_{i}\right| / m \cdot k\right)<\frac{m}{4 k} \cdot \sum_{i \notin I} k_{i}^{\prime} \leq \frac{m}{4 k} \cdot \sum_{i=1}^{r} k_{i}^{\prime} \leq \frac{m}{4}
$$

where the last bound holds because, in total, all repetitive regions incur at most $\sum_{i=1}^{r} k_{i}^{\prime} \leq k$ errors (since the repetitive regions are disjoint). Hence, the number of marks placed is at least $m_{r}-m / 4 k$, completing the proof of the claim.

In total, by Claims 5.25 and 5.26 the number of $k$-error occurrences of $P$ in $T$ is at most

$$
\operatorname{Occ}_{k}^{E}(P, T) \leq \frac{214015 \cdot n / m \cdot k \cdot m_{R}}{m_{R}-m / 4}
$$

As this bound is a decreasing function in $m_{R}$, the assumption $m_{R} \geq 3 / 8 \cdot m$ yields

$$
\operatorname{Occ}_{k}^{E}(P, T) \leq \frac{214015 \cdot n / m \cdot k \cdot 3 / 8 \cdot m}{3 / 8 \cdot m-m / 4}=642045 \cdot n / m \cdot k
$$

completing the proof.
Lemma 5.27 (Compare Lemma 3.14). Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n$, and let $k \leq m$ denote a positive integer. If there is a primitive string $Q$ of length at most $|Q| \leq m / 128 k$ that satisfies $2 k \leq \delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq 8 k$, then $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor\right| \leq 87551 \cdot n / m \cdot k$.

Proof. We apply Corollary 5.18 with $d=\delta_{E}\left(P,{ }^{*} Q^{*}\right)$. Observe that $d \geq 2 k$ and that $|Q| \leq m / 128 k \leq m / 8 d$ due to $d \leq 8 k$. Hence, the assumptions of Corollary 5.18 are met.

Consequently, $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / d\right\rfloor\right| \leq 1216 \cdot n / m \cdot d \leq 9728 \cdot n / m \cdot k$. Every block [ $\left.j^{\prime} d \ldots\left(j^{\prime}+1\right) d\right)$ is of length at most $8 k$, and thus may intersect at most 9 blocks of the form $[j k \ldots(j+1) k)$. Consequently, $\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor \leq 9 \cdot 9728 \cdot n / m \cdot k$, completing the proof.

『 Theorem 5.1 (Compare Theorem 3.1). Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a positive integer $k \leq m$, then at least one of the following holds.

- The $k$-error occurrences of $P$ in $T$ satisfy $\left.\| \operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor \leq 642045 \cdot n / m \cdot k$.
- There is a primitive string $Q$ of length $|Q| \leq m / 128 k$ that satisfies $\delta_{E}\left(P,{ }^{*} Q^{*}\right)<2 k$.

Proof. The proof proceeds similarly to the proof of Theorem 3.1 We apply Lemma 5.19 on the string $P$ and proceed depending on the structure found in $P$.

If the string $P$ contains $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k}$ (in the sense of Lemma 5.19), we apply Lemma 5.21 and obtain that $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor\right| \leq 1024 \cdot n / m \cdot k$.

## Faster Approximate Pattern Matching: A Unified Approach

If the string $P$ contains disjoint repetitive regions $R_{1}, \ldots, R_{r}$ (again, in the sense of Lemma 5.19), we apply Lemma 5.24 and obtain that $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor\right| \leq 642045 \cdot n / m \cdot k$.

Otherwise, Lemma 5.19 guarantees that there is a primitive string $Q$ of length at most $|Q| \leq m / 128 k$ that satisfies $\delta_{E}\left(P,{ }^{*} Q^{*}\right)<8 k$. If $\delta_{E}\left(P,{ }^{*} Q^{*}\right) \geq 2 k$, then Lemma 5.27 yields $\left|\left\lfloor\operatorname{Occ}_{k}^{E}(P, T) / k\right\rfloor\right| \leq 87551$. $n / m \cdot k$. If, however, $\delta_{E}\left(P,{ }^{*} Q^{*}\right)<2 k$, then we are in the second alternative of the theorem statement. 』

## 6 Algorithm: Pattern Matching with Edits in the PILLAR Model

In this section, we discuss how to solve pattern matching with edits in the PILLAR model. Specifically, we prove the following.

- Main Theorem 9. Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a positive integer $k \leq m$, we can compute (a representation of) the set $\operatorname{Occ}_{k}^{E}(P, T)$ using $\mathcal{O}\left(n / m \cdot k^{4}\right)$ time in the PILLAR model. 』

The overall structure of the algorithm is similar to the Hamming distance case: We first introduce useful tools for the algorithms later. Then, we implement Analyze, which is then followed by a discussion of the case when the pattern is periodic. Finally, we discuss the easier non-periodic case and conclude with combining the various auxiliary algorithms.

### 6.1 Auxiliary PILLAR Model Operations for Pattern Matching with Edits

As in the Hamming distance setting, we start the discussion of the algorithms with general tools that we use as auxiliary procedures in the remaining algorithms. Specifically, we discuss a generator that computes the "next" error between two strings. Further, we discuss a procedure to verify whether there is an occurrence of the pattern at a given (interval of) position(s) in the text.

Lemma 6.1 (EditGenerator $(S, Q)$, EditGenerator ${ }^{R}(S, Q)$. Let $S$ denote a string and let $Q$ denote a string (that is possibly given as a cyclic rotation $Q^{\prime}=\operatorname{rot}^{j}(Q)$ ). Then, there is an $(\mathcal{O}(1), \mathcal{O}(k))$ time generator that in the $k$-th call to Next returns the length of the longest prefix (suffix) $S^{\prime}$ of $S$ and the length of the corresponding prefix (suffix) $Q^{\prime}$ of $Q^{\infty}$ such that $\delta_{E}\left(S^{\prime}, Q^{\prime}\right) \leq k$. (Note that $k \geq 0$, that is, the initial call to Next is the zeroth call.)

Further, both generators support an additional operation Alignment, that outputs a witness for the result returned by $k$-th call to Next that is, Alignment outputs a sequence of edits $((i, j)$ for a replacement, $(i, \perp)$ for an insertion in $S$, and $(\perp, i)$ for an insertion in $Q)$. The operation Alignment takes $\mathcal{O}(k)$ time in the PILLAR model.

Proof. We focus on the generator EditGenerator $(S, Q)$; the generator EditGenerator ${ }^{R}(S, Q)$ can be obtained in a symmetric manner.

We construct the generator as follows: We run the dynamic programming algorithm by Landau and Vishkin [35] for one additional error (per call to Next) at a time, storing the dynamic programming table as a state in the generator. In particular, we maintain a sequence $\ell$ that after $k$ calls to Next stores at a position $i \in[-k \ldots k]$ the length of the longest prefix $S^{\prime}$ of $S$ that satisfies $\delta_{E}\left(S^{\prime}, Q^{\infty}\left[0 \ldots\left|S^{\prime}\right|+i\right]\right) \leq k$ (that is, intuitively, we store how far each of the "diagonals" of the dynamic programming table extends). In each call to Next, we update the values stored in the sequence $\ell$ by using three calls to LCP from Corollary 2.9 (one call for each of the insert, replace, delete cases of the edit distance) to compute each new entry. We then obtain the result as the maximum value of the newly computed sequence.

In order to support the Alignment operation, we additionally store every diagonal represented as list of pairs of insert, replace, and delete operations and the position(s) in the strings $S$ and $Q^{\infty}$ where

Algorithm 10 An analogue of Algorithm 2 for edit distance: We adapt Landau and Vishkin's algorithm to make its execution "resumable".

```
EditGenerator \(\left(S, Q^{\prime}=\operatorname{rot}^{j}(Q)\right)\)
    return \{
        \(S \leftarrow S ; Q^{\prime} \leftarrow Q^{\prime} ; k \leftarrow 0 ; j \leftarrow j ;\) end \(\leftarrow\) false;
        \(\left(\ell_{-1}^{(-1)}, \ldots, \ell_{1}^{(-1)}\right) \leftarrow(-\infty,-\infty,-\infty)\);
        \(A[-2 . .2] \leftarrow[(), \ldots,()]\);
    \};
    \(\operatorname{Next}\left(\mathbf{G}=\left\{S ; Q^{\prime} ; k ; j ;\left(\ell_{-k-1}^{(k-1)}, \ldots, \ell_{k+1}^{(k-1)}\right) ; A ;\right.\right.\) end \(\left.\}\right)\)
    if \(k=0\) then
        \(\left(\ell_{-2}^{(0)}, \ldots, \ell_{2}^{(0)}\right) \leftarrow\left(-\infty,-\infty, \operatorname{LCP}\left(S, Q^{\infty}\right),-\infty,-\infty\right) ;\)
        replace \(\ell^{(k-1)}\) with \(\ell^{(k)} ; k \leftarrow k+1\);
        return \(\ell_{0}^{(0)}\);
    \(\left(\ell_{-k-2}^{(k)}, \ldots, \ell_{k+2}^{(k)}\right) \leftarrow(-\infty, \ldots,-\infty) ;\)
    \(A^{\prime}[-k-2 \ldots k+2] \leftarrow[(), \ldots,()] ; r \leftarrow-\infty ; a_{r} \leftarrow-1 ;\)
    for \(i \leftarrow-k-1\) to \(k+1\) do
        // Compute new prefix lengths as long as we did not reach the end of \(S\)
        if not end then
            \(\ell_{\text {insert }} \leftarrow \ell_{i-1}^{(k-1)}+1+\operatorname{LCP}\left(S\left[\ell_{i-1}^{(k-1)}-i \ldots|S|\right), Q^{\prime \infty}\left[j+\ell_{i}^{(k-1)}+1 \ldots\right)\right)\);
            \(\ell_{\text {replace }} \leftarrow \ell_{i}^{(k-1)}+1+\operatorname{LCP}\left(S\left[\ell_{i}^{(k-1)}-i+1 \ldots|S|\right), Q^{\prime \infty}\left[j+\ell_{i}^{(k-1)}+1 \ldots\right)\right)\);
            \(\ell_{\text {delete }} \leftarrow \ell_{i+1}^{(k-1)}+\operatorname{LCP}\left(S\left[\ell_{i}^{(k-1)}-i+1 \ldots|S|\right), Q^{\prime \infty}\left[j+\ell_{i}^{(k-1)} \ldots\right)\right)\);
            \(\ell_{i}^{(k)} \leftarrow \max \left(\ell_{\text {insert }}, \ell_{\text {replace }}, \ell_{\text {delete }}\right) ;\)
        else \(\ell_{i}^{(k)} \leftarrow \ell_{i}^{(k-1)}\);
        \(r \leftarrow \max \left(r, \ell_{i}^{(k)}\right) ;\)
        if \(r=\ell_{i}^{(k)}-i\) then \(a_{r} \leftarrow \ell_{i}^{(k)}+i\);
        if end then continue;
        // Store witness for Alignment
        if \(\ell_{i}^{(k)}=\ell_{\text {insert }}\) then
        \(A^{\prime}[i] \leftarrow\left(A[i-1],\left(\perp, j+\ell_{i}^{(k-1)}\right)\right) ;\)
        if \(\ell_{i}^{(k)}=\ell_{\text {replace }}\) then
            \(A^{\prime}[i] \leftarrow\left(A[i],\left(\ell_{i}-i, j+\ell_{i}^{(k-1)}\right)\right) ;\)
        if \(\ell_{i}^{(k)}=\ell_{\text {delete }}\) then
            \(A^{\prime}[i] \leftarrow\left(A[i+1],\left(\ell_{i}^{(k-1)}-i, \perp\right)\right) ;\)
    replace \(\ell^{(k-1)}\) with \(\ell^{(k)} ; k \leftarrow k+1 ; A \leftarrow A^{\prime}\);
    if \(r \geq|S|\) then end \(\leftarrow\) true;
    return \(\left(r, a_{r}\right)\);
    \(\operatorname{Alignment}\left(\mathbf{G}=\left\{S ; Q^{\prime} ; k ; j ;\left(\ell_{-k-1}^{(k-1)}, \ldots, \ell_{k+1}^{(k-1)}\right) ; A ;\right.\right.\) end \(\left.\}\right)\)
    \(r \leftarrow-\infty ; a_{r} \leftarrow-1 ;\)
    for \(i \leftarrow-k-1\) to \(k+1\) do
        \(r \leftarrow \max \left(r, \ell_{i}^{(k-1)}\right) ;\)
        if \(r=\ell_{i}^{(k-1)}\) then \(a_{r} \leftarrow i\);
    return \(A\left[a_{r}\right]\);
```

the edits happened. In each call to Next, we also update the representations of the diagonals. (Note that, for performance reasons, we need to avoid copying whole diagonals; this can be done by storing all diagonals together as a (directed) graph.)

Consider Algorithm 10 for a visualization of the generator as pseudo-code; note that we simplified how we store the diagonals for the Alignment operation to improve readability.

For the correctness, we show by induction that the values computed in the array $\ell$ are indeed correct, that is, after $k$ calls to the Next operation, for all $i \in[-k \ldots k]$ we have

$$
\ell_{i}^{(k)}=\max _{r}\left\{r \mid \delta_{E}\left(S[0 \ldots r], Q^{\infty}[0 \ldots r+i]\right) \leq k\right\}
$$

or we reached the end of the string $S$ (in which case the output does not change anymore after calling Next). For the zeroth call to next, we explicitly return

$$
\ell_{0}^{(0)}=\operatorname{LCP}\left(S, Q^{\infty}\right)=\max _{r}\left\{r \mid \delta_{E}\left(S[0 \ldots r], Q^{\infty}[0 \ldots r+0]\right)=0\right\}
$$

which is thus correct. Now consider the $k$-th call to Next and assume that the values computed so far are correct. In particular, for all $i \in[-k+1 \ldots k-1]$, we have

$$
\ell_{i}^{(k-1)}=\max _{r}\left\{r \mid \delta_{E}\left(S[0 \ldots r], Q^{\infty}[0 \ldots r+i]\right) \leq k-1\right\}
$$

Now, fix a $j \in[k \ldots-k]$ and consider the longest prefix $S^{\prime}=S[0 \ldots r]$ with $\delta_{E}\left(S^{\prime}, Q^{\infty}[0 \ldots r+j]\right)=k^{\prime}$, for some $0<k^{\prime} \leq k$. By definition of the edit distance, there is an integer $r^{\prime}$ that satisfies $S^{\prime}\left(r^{\prime} \ldots r\right]=$ $Q^{\infty}\left(r^{\prime}+j \ldots r+j\right]$ and at least one of the following
$=k^{\prime}=\delta_{E}\left(S^{\prime}\left[0 \ldots r^{\prime}\right), Q^{\infty}\left[0 \ldots r^{\prime}+j\right)\right)+1$ and $S\left[r^{\prime}\right] \neq Q^{\infty}\left[r^{\prime}\right]$ (when changing the character $S\left[r^{\prime}\right]$ to the character $\left.Q^{\infty}\left[r^{\prime}+j\right]\right)$;

- $k^{\prime}=\delta_{E}\left(S^{\prime}\left[0 \ldots r^{\prime}\right), Q^{\infty}\left[0 \ldots r^{\prime}+j-1\right)\right)+1\left(\right.$ when inserting the character $\left.Q^{\infty}\left[r^{\prime}+j\right]\right)$; or
- $k^{\prime}=\delta_{E}\left(S^{\prime}\left[0 \ldots r^{\prime}\right], Q^{\infty}\left[0 \ldots r^{\prime}+j\right)\right)+1$ (when deleting the character $\left.Q^{\infty}\left[r^{\prime}+j\right]\right)$.

Note that (as $\left.S^{\prime}\left(r^{\prime} \ldots r\right]=Q^{\infty}\left(r^{\prime}+j \ldots r+j\right]\right)$ we may assume that $r^{\prime}$ is maximal (that is there is no larger integer with the same properties as $\left.r^{\prime}\right)$. In particular, we have

$$
r^{\prime}=\max \left(\ell_{j}^{(k-1)}+1, \ell_{j-1}^{(k-1)}+1, \ell_{j+1}^{(k-1)}\right)
$$

and hence Next computes $\ell_{j}^{(k)}$ indeed correctly.
Using the computed values $\ell^{(k)}$, we can easily compute the length $\left|S^{\prime}\right|$ of the longest prefix $S^{\prime}$ of $S$ and the length $\left|Q^{\prime}\right|$ of the corresponding prefix $Q^{\prime}$ of $Q^{\infty}$ such that $\delta_{E}\left(S^{\prime}, Q^{\prime}\right) \leq k$ : For $\left|S^{\prime}\right|$, we have

$$
\left|S^{\prime}\right|=\max _{j, r}\left\{r \mid \delta_{E}\left(S[0 \ldots r], Q^{\infty}[0 \ldots r+y]\right) \leq k\right\}=\max _{j} \ell_{j}^{(k)}
$$

as $k$ edits only allow for up to $k$ insertions or deletions (that is, operations that can change the shift between $Q^{\infty}$ and $\left.S\right)$. Hence, the computation of $\left|S^{\prime}\right|$ in Next is correct. For $\left|Q^{\prime}\right|$, observe that if $\left|S^{\prime}\right|=\ell_{j}^{(k)}$ for some $j \in[-k \ldots k]$, by construction, we have $\left|Q^{\prime}\right|=\ell_{j}^{(k)}+j$. Hence, also the computed value for $\left|Q^{\prime}\right|$ is indeed correct.

For the correctness of Alignment, observe that we store information computed in Next (which is correct); further we always synchronize the information stored, hence also Alignment is correct.

For the running time, observe that in the $k$-th call to Next we call LCP three times for each of the $2 k+3$ values $\ell_{i}^{(k)}$. Further, all other operations are essentially book-keeping that can be implemented in $\mathcal{O}(1)$ time. Hence in total, the $k$-th call to Next takes $\mathcal{O}(k)$ time in the PILLAR model.

For the Alignment operation, observe that we traverse the sequence $\ell^{(k)}$ exactly once, hence Alignment uses $\mathcal{O}(k)$ time in the PILLAR model, completing the proof.

『 Lemma 6.2 (Verify $(P, T, k, I)$, 17 Section 5]). Let $P$ denote a string of length $m$, let $T$ denote a string, and let $k \leq m$ and denote a positive integer. Further, let $I$ denote an interval of positive integers. Using $\mathcal{O}(k(k+|I|))$ PILLAR operations, we can compute $\left\{\left(\ell, \min _{r} \delta_{E}(P, T[\ell \ldots r))\right) \mid \ell \in \operatorname{Occ}_{k}^{E}(P, T) \cap I\right\}$.

Proof. Observe that the algorithm in [17, Section 5] only uses LCP operations, as it mainly implements 35]. In particular, the algorithm in [17, Section 5] uses $\mathcal{O}(k(k+|I|))$ LCP operations. The claim follows.

Note that we can also call Verify $(P, T, k, I)$ for strings $T=Q^{\infty}$ (for some primitive $Q$ ), as, by Corollary 2.9, we can also efficiently compute $\operatorname{LCP}\left(P, Q^{\infty}\right)$.

### 6.2 Computing Structure in the Pattern

We proceed to discuss the implementation of Lemma 5.19, that is, the analysation of the pattern. While the algorithm itself is similar to the Hamming distance case, the analysis requires more involved arguments. We start with an auxiliary combinatorial lemma:

Lemma 6.3. Let $S$ denote a string, let $k$ denote a positive integer, and let $Q$ denote a primitive string such that $|Q|=1$ or $|S| \geq(2 k+1)|Q|$. Suppose that $\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right) \leq k$ for integers $x \leq y$. Then, for any string $S^{\prime}$, if $\delta_{E}\left(S S^{\prime},{ }^{*} Q^{*}\right) \leq k$, then $\delta_{E}\left(S S^{\prime},{ }^{*} Q^{*}\right)=\delta_{E}\left(S S^{\prime}, \operatorname{rot}^{-x}(Q)^{*}\right)$, and if $\delta_{E}\left(S^{\prime} S,{ }^{*} Q^{*}\right) \leq k$, then $\delta_{E}\left(S^{\prime} S,{ }^{*} Q^{*}\right)=\delta_{E}\left(S^{\prime} S,{ }^{*} \operatorname{rot}^{-y}(Q)\right)$.

Proof. Suppose that $\delta_{E}\left(S S^{\prime},{ }^{*} Q^{*}\right)=\delta_{E}\left(S S^{\prime}, Q^{\infty}\left[x^{\prime} \ldots z^{\prime}\right)\right) \leq k$ for some integers $x^{\prime} \leq z^{\prime}$. Then, there is a position $y^{\prime} \in\left[x^{\prime} \ldots z^{\prime}\right]$ such that

$$
\delta_{E}\left(S S^{\prime},{ }^{*} Q^{*}\right)=\delta_{E}\left(S, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)+\delta_{E}\left(S^{\prime}, Q^{\infty}\left[y^{\prime} \ldots z^{\prime}\right)\right) \leq k
$$

Due to $\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right) \leq k$ and $\delta_{E}\left(S, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right) \leq k$, we may apply Lemma 5.3 , which yields a decomposition $S=S_{L} \cdot S_{R}$ and integers $j, j^{\prime}$ such that

$$
\begin{aligned}
& \delta_{E}\left(S, Q^{\infty}[x \ldots y)\right)=\delta_{E}\left(S_{L}, Q^{\infty}[x \ldots j|Q|)\right)+\delta_{E}\left(S_{R}, Q^{\infty}[j|Q| \ldots y)\right) \quad \text { and } \\
& \delta_{E}\left(S, Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(S_{L}, Q^{\infty}\left[x^{\prime} \ldots j^{\prime}|Q|\right)\right)+\delta_{E}\left(S_{R}, Q^{\infty}\left[j^{\prime}|Q| \ldots y^{\prime}\right)\right) .
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\delta_{E}\left(S_{L}, Q^{\infty}\right. & {[x \ldots j|Q|))+\delta_{E}\left(S_{R}, Q^{\infty}[j|Q| \ldots y)\right)=\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right)=\delta_{E}\left(S,{ }^{*} Q^{*}\right) } \\
& \leq \delta_{E}\left(S, Q^{\infty}\left[x^{\prime} \ldots y+\left(j^{\prime}-j\right)|Q|\right)\right) \leq \delta_{E}\left(S_{L}, Q^{\infty}\left[x^{\prime} \ldots j^{\prime}|Q|\right)\right)+\delta_{E}\left(S_{R}, Q^{\infty}[j|Q| \ldots y)\right),
\end{aligned}
$$

which implies $\delta_{E}\left(S_{L}, Q^{\infty}[x \ldots j|Q|)\right) \leq \delta_{E}\left(S_{L}, Q^{\infty}\left[x^{\prime} \ldots j^{\prime}|Q|\right)\right)$. Consequently,

$$
\begin{aligned}
\delta_{E}\left(S S^{\prime}, \operatorname{rot}^{-x}(Q)^{*}\right) & \leq \delta_{E}\left(S S^{\prime}, Q^{\infty}\left[x \ldots z^{\prime}+\left(j-j^{\prime}\right)|Q|\right)\right) \\
& \leq \delta_{E}\left(S_{L}, Q^{\infty}[x \ldots j|Q|)\right)+\delta_{E}\left(S_{R}, Q^{\infty}\left[j^{\prime}|Q| \ldots y^{\prime}\right)\right)+\delta_{E}\left(S^{\prime}, Q^{\infty}\left[y^{\prime} \ldots z^{\prime}\right)\right) \\
& \leq \delta_{E}\left(S_{L}, Q^{\infty}\left[x^{\prime} \ldots j^{\prime}|Q|\right)\right)+\delta_{E}\left(S_{R}, Q^{\infty}\left[j^{\prime}|Q| \ldots y^{\prime}\right)\right)+\delta_{E}\left(S^{\prime}, Q^{\infty}\left[y^{\prime} \ldots z^{\prime}\right)\right) \\
& =\delta_{E}\left(S S^{\prime},{ }^{*} Q^{*}\right),
\end{aligned}
$$

which implies $\delta_{E}\left(S S^{\prime},{ }^{*} Q^{*}\right)=\delta_{E}\left(S S^{\prime}, \operatorname{rot}^{-x}(Q)^{*}\right)$.
The claim regarding $\delta_{E}\left(S^{\prime} S,{ }^{*} Q^{*}\right)$ and $\delta_{E}\left(S^{\prime} S,{ }^{*} \operatorname{rot}^{-y}(Q)\right)$ is symmetric.

『 Lemma 6.4 (Analyze $(P, k)$ : Implementation of Lemma 5.19. . Let $P$ denote a string of length $m$ and let $k \leq m$ denote a positive integer. Then, there is an algorithm that computes one of the following:

Algorithm 11 Analyzing the pattern: A PILLAR model implementation of Algorithm 9

```
Analyze \((P, k)\)
    \(j \leftarrow 0 ; r \leftarrow 1 ; b \leftarrow 1 ;\)
    while true do
        \(j^{\prime} \leftarrow j+\lfloor m / 8 k\rfloor ;\)
        if \(\operatorname{Period}\left(P\left[j \ldots j^{\prime}\right)\right)>m / 128 k\) then
            \(B_{b} \leftarrow P\left[j \ldots j^{\prime}\right) ;\)
            if \(b=2 k\) then return breaks \(B_{1}, \ldots, B_{2 k}\);
            \(b \leftarrow b+1 ; j \leftarrow j^{\prime} ;\)
        else
            \(q \leftarrow \operatorname{Period}\left(P\left[j \ldots j^{\prime}\right)\right) ; Q_{r} \leftarrow P[j \ldots j+q) ;\)
            generator \(\mathbf{G} \leftarrow\) EditGenerator \(\left(P[j \ldots m), Q_{r}\right)\);
            \(\delta \leftarrow 0 ;\)
            while \(\delta<8 k / m \cdot\left(j^{\prime}-j\right)\) and \(j^{\prime} \leq m\) do
                \(\left(\pi, \pi^{\prime}\right) \leftarrow \operatorname{Next}(\mathbf{G}) ;\)
                \(j^{\prime} \leftarrow j+\pi+1 ; \delta \leftarrow \delta+1 ;\)
            if \(j^{\prime} \leq m\) then
                \(R_{r} \leftarrow P\left[j \ldots j^{\prime}\right) ;\)
                if \(\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m\) then
                    return repetitive regions \(R_{1}, \ldots, R_{r}\) with periods \(Q_{1}, \ldots, Q_{r}\);
            \(r \leftarrow r+1 ; j \leftarrow j^{\prime} ;\)
            else
                \(Q \leftarrow Q_{r} ;\)
            generator \(\mathbf{G}^{\prime} \leftarrow\) EditGenerator \({ }^{R}\left(P \operatorname{rot}^{-\pi^{\prime}}(Q)\right)\);
            \(j^{\prime \prime}=m ; \delta \leftarrow 0\);
            while \(\left(j^{\prime \prime} \geq j\right.\) or \(\left.\delta<8 k / m \cdot\left(m-j^{\prime \prime}\right)\right)\) and \(j^{\prime \prime} \geq 0\) do
                \(\left(\pi,{ }_{-}\right) \leftarrow \operatorname{Next}\left(\mathbf{G}^{\prime}\right) ;\)
                \(j^{\prime \prime} \leftarrow m-\pi-1 ; \delta \leftarrow \delta+1 ;\)
            if \(j^{\prime \prime} \geq 0\) then return repetitive region \(P\left[j^{\prime \prime} \ldots m\right)\) with period \(Q\);
            else return approximate period \(Q\);
```

(a) $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k} \preccurlyeq P$, each having period $\operatorname{per}\left(B_{i}\right)>m / 128 k$ and length $\left|B_{i}\right|=\lfloor m / 8 k\rfloor$.
(b) Disjoint repetitive regions $R_{1}, \ldots, R_{r} \preccurlyeq P$ of total length $\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and is constructed along with a primitive approximate period $Q_{i}$ such that $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{E}\left(R_{i},{ }^{*} Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$.
(c) A primitive approximate period $Q$ of $P$ with $|Q| \leq m / 128 k$ and $\delta_{E}\left(P,{ }^{*} Q^{*}\right)<8 k$.

The algorithm uses $\mathcal{O}\left(k^{2}\right)$ time plus $\mathcal{O}\left(k^{2}\right)$ PILLAR operations.
Proof. In a similar manner to the Hamming distance case, our implementation follows Algorithm 9 from the proof of Lemma 5.19

Recall that $P$ is processed from left to right and split into breaks and repetitive regions. In each iteration, the algorithm first considers a fragment of length $\lfloor m / 8 k\rfloor$. This fragment either becomes the next break (if its shortest period is long enough) or is extended to the right to a repetitive region (otherwise). Having constructed sufficiently many breaks or repetitive regions of sufficiently large total length, the algorithm stops. Processing the string $P$ in this manner guarantees disjointness of breaks and repetitive regions. As in the proof of Lemma 5.19, a slightly different approach is needed if the algorithm encounters
the end of $P$ while growing a repetitive region. If this happens, the region is also extended to the left. This way, the algorithm either obtains a single repetitive region (which is not necessarily disjoint with the previously created ones, so it is returned on its own) or learns that the whole string $P$ is approximately periodic.

Next, we fill in missing details of the implementation of the previous steps in the PILLAR model. To that end, first note that the PILLAR model includes a Period operation of checking if the period of a string $S$ satisfies $\operatorname{per}(S) \leq|S| / 2$; computing $\operatorname{per}(S)$ in case of a positive answer. Since our threshold $m / 128 k$ satisfies $\lfloor m / 128 k\rfloor \leq\lfloor m / 8 k\rfloor / 2$, no specific work is required to obtain the period of an unprocessed fragment of $\lfloor m / 8 k\rfloor$ characters of $P$.

To compute a repetitive region starting from a fragment $P\left[j \ldots j^{\prime}\right)$ with string period $Q$, we use a generator $\mathbf{G}=\operatorname{EditGenerator}(P[j \ldots m), Q)$ from Lemma 6.1 for subsequent values $\delta \geq 1$, we find the shortest prefix $P_{\delta}^{\prime}$ of $P[j \ldots m)$ such that $\delta_{E}\left(P_{\delta}^{\prime}, Q^{*}\right)=\delta$, until no such prefix exists or $\delta \geq$ $8 k / m \cdot\left|P_{\delta}^{\prime}\right|$. This is possible because the $(\delta-1)$-st call to $\operatorname{Next}(\mathbf{G})$ returns the length $\pi$ of the longest prefix of $P[j \ldots m)$ with $\delta_{E}\left(P[j \ldots j+\pi), Q^{*}\right)<\delta$. If $\delta \geq 8 k / m \cdot\left|P_{\delta}^{\prime}\right|$, then we have identified a repetitive region $P_{\delta}^{\prime}$. Otherwise, we reach $\pi=m-j$ and retrieve $\pi^{\prime}$ such that $\delta_{E}\left(P[j \ldots m), Q^{*}\right)=$ $\delta_{E}\left(P[j \ldots m), Q^{\infty}\left[0 \ldots \pi^{\prime}\right)\right)$ from the last call to $\operatorname{Next}(\mathbf{G})$. In this case, we similarly use a generator $\mathbf{G}^{\prime}=$ EditGenerator ${ }^{R}\left(P, \operatorname{rot}^{-\pi^{\prime}}(Q)\right.$ ) from Lemma 6.1. For subsequent values $\delta \geq 1$, we find the shortest suffix $P_{\delta}^{\prime \prime}$ of $P$ such that $\delta_{E}\left(P_{\delta}^{\prime \prime},{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)=\delta$, until no such suffix exists or $\left|P_{\delta}^{\prime \prime}\right| \geq|P[j \ldots m)|$ and $\delta \geq 8 k / m \cdot\left|P_{\delta}^{\prime \prime}\right|$. Again, this is possible because the ( $\delta-1$ )-st call to $\operatorname{Next}\left(\mathbf{G}^{\prime}\right)$ returns the length $\pi$ of the longest suffix of $P$ with $\delta_{E}\left(P[m-\pi \ldots m),{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)<\delta$. If we reach $\pi=m$, then we return $Q$ as an approximate period of $P$; otherwise, we return the final suffix $P_{\delta}^{\prime \prime}$ as a long repetitive region. Consider Algorithm 11 for implementation details.

For the correctness, since our algorithm follows the proof of Lemma 5.19 we only need to show that our implementation of finding repetitive regions correctly implements the corresponding steps in Algorithm 9

First, we inductively prove that each considered prefix $P_{\delta}^{\prime}$ of $P[j \ldots m)$ satisfies $\delta_{E}\left(P_{\delta}^{\prime},{ }^{*} Q^{*}\right)=\delta \leq$ $\left\lceil 8 k / m \cdot\left|P_{\delta}^{\prime}\right|\right\rceil$. The case of $\delta=1$ is easy since $\delta_{E}\left(P_{1}^{\prime},{ }^{*} Q^{*}\right) \leq \delta_{E}\left(P_{1}^{\prime}, Q^{*}\right)=1$, since $\delta_{E}\left(P_{1}^{\prime},{ }^{*} Q^{*}\right)=0$ would imply $\delta_{E}\left(P_{1}^{\prime}, Q^{*}\right)=0$ because $Q$ is a prefix of $P_{1}^{\prime}$, and since $8 k / m \cdot\left|P_{1}^{\prime}\right|>0$ due to $\left|P_{1}^{\prime}\right|>0$. Next, we prove that the claim holds for $\delta+1$ assuming that it holds for $\delta$. The inductive assumption guarantees $\delta_{E}\left(P_{\delta}^{\prime},{ }^{*} Q^{*}\right)=\delta_{E}\left(P_{\delta}^{\prime}, Q^{*}\right)=\delta$. Since the algorithm proceeded to the next step, we have $\delta<8 k / m \cdot\left|P_{\delta}^{\prime}\right|$. In particular, $\left|P_{\delta}^{\prime}\right| \geq(2 \delta+1) \cdot m / 128 k \geq(2 \delta+1)|Q|$, so we can apply Lemma 6.3 to $P_{\delta}^{\prime}$. If $\delta_{E}\left(P_{\delta+1}^{\prime},{ }^{*} Q^{*}\right) \leq \delta$, then Lemma 6.3 yields $\delta_{E}\left(P_{\delta+1}^{\prime}, Q^{*}\right)=\delta_{E}\left(P_{\delta+1}^{\prime},{ }^{*} Q^{*}\right) \leq \delta$, which contradicts the definition of $P_{\delta+1}^{\prime}$. Hence, $\delta_{E}\left(P_{\delta+1}^{\prime},{ }^{*} Q^{*}\right) \geq \delta+1$. Due to $\delta_{E}\left(P_{\delta+1}^{\prime}, Q^{*}\right)=\delta+1$, we have $\delta_{E}\left(P_{\delta+1}^{\prime},{ }^{*} Q^{*}\right)=\delta+1$. Moreover, $\left\lceil 8 k / m \cdot\left|P_{\delta+1}^{\prime}\right|\right\rceil \geq\left\lceil 8 k / m \cdot\left|P_{\delta}^{\prime}\right|\right\rceil>\delta$ guarantees $\left\lceil 8 k / m \cdot\left|P_{\delta+1}^{\prime}\right|\right\rceil \geq \delta+1$, which completes the inductive proof.

In particular, if we encounter a prefix $P_{\delta}^{\prime}$ that satisfies $\delta \geq 8 k / m \cdot\left|P_{\delta}^{\prime}\right|$, then $\delta_{E}\left(P_{\delta}^{\prime},{ }^{*} Q^{*}\right)=\left\lceil 8 k / m \cdot\left|P_{\delta}^{\prime}\right|\right\rceil$. However, if no such prefix $P_{\delta}^{\prime}$ exists, then $\delta_{E}\left(R,{ }^{*} Q^{*}\right)<8 k / m \cdot|R|$ holds for each non-empty prefix of $P\left[j \ldots m\right.$ ) (because $R=P_{\delta}^{\prime}$ is the shortest prefix $R$ of $P[j \ldots m)$ with $\delta_{E}\left(R,{ }^{*} Q^{*}\right)=\delta$ ). Thus, Line 9 of Algorithm 9 is implemented correctly.

In the following, we assume that no such prefix $R$ exists. In particular, we have $\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)<$ $8 k / m \cdot|P[j \ldots m)|$. Then, the last call to Next (G) resulted in $\left(m-j, \pi^{\prime}\right)$ such that $\delta_{E}\left(P[j \ldots m), Q^{*}\right)=$ $\delta_{E}\left(P[j \ldots m), Q^{\infty}\left[0 \ldots \pi^{\prime}\right)\right)$. Moreover, since $P_{\delta}^{\prime}$ with $\delta=\delta_{E}\left(P[j \ldots m), Q^{*}\right)$ satisfies $\delta_{E}\left(P_{\delta}^{\prime},{ }^{*} Q^{*}\right)=$ $\delta_{E}\left(P_{\delta}^{\prime}, Q^{*}\right)=\delta$, we have

$$
\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)=\delta_{E}\left(P[j \ldots m], Q^{*}\right)=\delta_{E}\left(P[j \ldots m),{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)=\delta .
$$

We inductively prove that each considered suffix of $P_{\delta}^{\prime \prime}$ of $P$ with $\left|P_{\delta}^{\prime \prime}\right|>j-m$ satisfies $\delta_{E}\left(P_{\delta}^{\prime \prime},{ }^{*} Q^{*}\right)=$ $\delta \leq\left\lceil 8 k / m \cdot\left|P_{\delta}^{\prime \prime}\right|\right\rceil$. Let us prove that this claim is true for $\delta+1$ assuming that is it true for $\delta$ (unless $\delta=0$, when there is no assumption). If $\delta<\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)$, then $\left|P_{\delta+1}^{\prime \prime}\right| \leq j-m$ and the claim is void, so
we only consider $\delta \geq \delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)$. If $\delta>\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)=\delta_{E}\left(P[j \ldots m),{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)$, then $\left|P_{\delta}^{\prime \prime}\right|>j-m$ and the inductive assumption guarantees $\delta_{E}\left(R,{ }^{*} Q^{*}\right)=\delta_{E}\left(R,{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)=\delta$ for $R=P_{\delta}^{\prime \prime}$. Otherwise, we have

$$
\delta=\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)=\delta_{E}\left(P[j \ldots m),{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right),
$$

in which case $\delta_{E}\left(R,{ }^{*} Q^{*}\right)=\delta_{E}\left(R,{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)=\delta$ holds for $R=P[j \ldots m)$. In either case, we also have $\delta<8 k / m \cdot|R|$, and hence $|R| \geq(2 \delta+1) \cdot m / 128 k \geq(2 \delta+1)|Q|$. Therefore, we can apply Lemma 6.3 to $R$. If $\delta_{E}\left(P_{\delta+1}^{\prime \prime},{ }^{*} Q^{*}\right) \leq \delta$, then Lemma 6.3 yields $\delta_{E}\left(P_{\delta+1}^{\prime \prime},{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)=\delta_{E}\left(P_{\delta+1}^{\prime \prime},{ }^{*} Q^{*}\right) \leq \delta$, which contradicts the definition of $P_{\delta+1}^{\prime \prime}$. Hence, $\delta_{E}\left(P_{\delta+1}^{\prime \prime},{ }^{*} Q^{*}\right) \geq \delta+1$. Due to $\delta_{E}\left(P_{\delta+1}^{\prime \prime},{ }^{*} \operatorname{rot}^{-\pi^{\prime}}(Q)\right)=\delta+1$, we have $\delta_{E}\left(P_{\delta+1}^{\prime \prime},{ }^{*} Q^{*}\right)=\delta+1$. Moreover, $\left\lceil 8 k / m \cdot\left|P_{\delta+1}^{\prime \prime}\right|\right\rceil \geq\lceil 8 k / m \cdot|R|\rceil>\delta$ guarantees $\left\lceil 8 k / m \cdot\left|P_{\delta+1}^{\prime \prime}\right|\right\rceil \geq \delta+1$, which completes the inductive proof.

In particular, if we encounter a suffix $P_{\delta}^{\prime \prime}$ that satisfies $\left|P_{\delta}^{\prime \prime}\right|>m-j$ and $\delta \geq 8 k / m \cdot\left|P_{\delta}^{\prime \prime}\right|$, then $\delta_{E}\left(P_{\delta}^{\prime \prime},{ }^{*} Q^{*}\right)=\left\lceil 8 k / m \cdot\left|P_{\delta}^{\prime \prime}\right|\right\rceil$. On the other hand, if no such suffix $P_{\delta}^{\prime \prime}$ exists, then $\delta_{E}\left(R,{ }^{*} Q^{*}\right)<8 k / m \cdot|R|$ holds for each suffix $R$ of $P$ of length $|R|>m-j$ (because $R=P_{\delta}^{\prime}$ is the shortest suffix $R$ of $P$ with $\delta_{E}\left(R,{ }^{*} Q^{*}\right)=\delta$ assuming $\left.\delta>\delta_{E}\left(P[j \ldots m),{ }^{*} Q^{*}\right)\right)$. Thus, Line 15 of Algorithm 9 is also implemented correctly.

For the running time analysis, observe that each iteration of the outer while loop processes at least $\lfloor m / 8 k\rfloor$ characters of $P$, so there are at most $\mathcal{O}(k)$ iterations of the outer while loop. In each iteration, we perform one Period operation, a constant number of Access operations, and at most $8 k / m \cdot\left(j^{\prime}-j\right)$ calls to the generator EditGenerator. These calls use $\mathcal{O}\left(\left(8 k / m \cdot\left(j^{\prime}-j\right)\right)^{2}\right)$ time in the PILLAR model, which is $\mathcal{O}\left(k^{2}\right)$ in total across all iterations (since the function $x \mapsto x^{2}$ is convex). Similarly, we bound the running time of the calls to the generator EditGenerator ${ }^{R}$ : As we find at most $8 k / m \cdot m=8 k$ errors, EditGenerator ${ }^{R}$ uses at most $\mathcal{O}\left(k^{2}\right)$ time. Overall, Algorithm 11 thus uses $\mathcal{O}\left(k^{2}\right)$ time in the PILLAR model.

### 6.3 Computing Occurrences in the Periodic Case

We start this subsection with a subroutine to compute a witness that a string $S$ has a small edit distance to a string $Q^{\infty}$.
r Lemma 6.5 (FindAWitness $(k, Q, S)$ ). Let $k$ denote a positive integer, let $S$ denote a string, and let $Q$ denote a primitive string that satisfies $|S| \geq(2 k+1)|Q|$ or $|Q| \leq 3 k+1$.

Then, we can be compute a witness $Q^{\infty}[x \ldots y)$ such that $\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right)=\delta_{E}\left(S,{ }^{*} Q^{*}\right) \leq k$, or report that $\delta_{E}\left(S, Q^{*}\right)>k$. The algorithm takes $\mathcal{O}\left(k^{2}\right)$ time in the PILLAR model.

Proof. For a set $A \subseteq \mathbb{Z}$ and an integer $p>0$, we define $A \bmod p:=\{a \bmod p \mid a \in A\}$. We first compute a (short) interval $J$ such that $\operatorname{Occ}_{k}^{E}\left(S, Q^{\infty}\right) \bmod |Q| \subseteq J \bmod |Q|$. If $|Q| \leq 3 k+1$, then we simply set $J=[0 \ldots|Q|)$. Otherwise, we proceed similarly as in the Hamming distance setting (Lemma 4.5), where we computed a majority string of the first $2 k+1$ length- $|Q|$ subsequent fragments $S_{1}, \ldots, S_{2 k}$ of $S$. However, now we need to accommodate for insertions and deletions of a $k$-error occurrence. Hence, we first compute an auxiliary set $I$ defined as the union of intervals $[p \ldots p+k]$ such that for at least $k+1$ fragments $S_{i}$ of $S$, we have $Q=\operatorname{rot}^{j}\left(S_{i}\right)$ for $j \in[p \ldots p+k]$. Finally, we set $J \subseteq[0 \ldots 2|Q|)$ to be a shortest interval satisfying $I \bmod |Q| \subseteq J \bmod |Q|$. Here, $J \bmod |Q|$ can be interpreted as a shortest cyclic interval (modulo $|Q|$ ) containing $I \bmod |Q|$.

Having computed the set $J$, we use Verify from Lemma 6.2 to determine at which starting position $x$ in $J$ we have an occurrence with the fewest number of errors (or to report that the number of errors is greater than $k$ everywhere). Finally, we use an EditGenerator from Lemma 6.1 to compute the ending

```
Algorithm 12 Finding a witness \(Q^{\infty}[x \ldots y)\) for \(\delta_{E}\left(S,{ }^{*} Q^{*}\right) \leq k\).
1 FindAWitness ( \(k, Q, S\) )
    // Compute "correct" rotation(s) of \(Q\)
    if \(|Q| \leq 3 k+1\) then \(J \leftarrow[0 \ldots|Q|)\);
    else
        multi-set \(R \leftarrow\}\);
        for \(i \leftarrow 0\) to \(2 k\) do
            \(R \leftarrow R \cup \operatorname{Rotations}(S[i|Q| \ldots(i+1)|Q|), Q) ;\)
        \(I \leftarrow \bigcup\{[p \ldots p+k]: p \in \mathbb{Z}\) and \([p \ldots p+k]\) contains at least \(k+1\) elements of \(R\}\);
        Let \(J \subseteq[0 \ldots 2|Q|)\) denote a shortest interval that satisfies \(I \bmod |Q| \subseteq J \bmod |Q|\);
    // Compute the start position of the witness
    \(O c c \leftarrow \operatorname{Verify}\left(S, Q^{\infty}, k, J\right)\);
    if \(O c c=\emptyset\) then return \(\perp\);
    Let \(\left(x, k^{\prime}\right) \in O c c\) be an arbitrary element minimizing \(k^{\prime}\);
    // Compute the end position of the witness
    generator \(\mathbf{G} \leftarrow\) EditGenerator \(\left(S\right.\), \(\left.\operatorname{rot}^{-x}(Q)\right)\);
    for \(i \leftarrow 1\) to \(k^{\prime}\) do \(\operatorname{Next}(\mathbf{G})\);
    \(\left(\lambda, \lambda^{\prime}\right) \leftarrow \operatorname{Next}(\mathbf{G}) ;\)
    return \(Q^{\infty}\left[x \ldots x+\lambda^{\prime}\right)\);
```

position $y$ of the occurrence of $S$ as a prefix of $Q^{\infty}[x \ldots)$. Consider Algorithm 12 for a pseudo-code implementation.

The correctness is based on the aforementioned characterization of $J$ :
${ }^{『}$ Claim 6.6. The interval $J$ satisfies $\operatorname{Occ}_{k}^{E}\left(S, Q^{\infty}\right) \bmod |Q| \subseteq J \bmod |Q|$.
Proof. The claim trivially holds if $|Q| \leq 3 k+1$, so we assume that $|Q|>3 k+1$. For every $i \in[0 \ldots 2 k]$, define $S_{i}:=S[i|Q| \ldots(i+1)|Q|)$. Consider an optimum alignment between $S$ and its $k$-error occurrence $Q^{\infty}[x \ldots y)$, and let $Q_{i}=Q^{\infty}\left[x_{i} \ldots x_{i+1}\right)$ denote the fragment aligned to $S_{i}$. Consider the multi-set $R:=\bigcup_{i}$ Rotations $\left(S_{i}, Q\right)$. Next, consider the values $\delta_{i}:=x_{i}-i|Q|$ for $i \in[0 \ldots 2 k]$. We have $\delta_{0}=x$, and $\delta_{i+1}=\delta_{i}+\left|Q_{i}\right|-\left|S_{i}\right|$ for $i>0$. Since

$$
\sum_{i=0}^{2 k}| | Q_{i}\left|-\left|S_{i}\right|\right| \leq \sum_{i=0}^{2 k} \delta_{E}\left(Q_{i}, S_{i}\right) \leq k
$$

all values $\delta_{i}$ belong to an interval of the form $[p \ldots p+k]$ for some integer $p$. Moreover, note that $Q_{i}=S_{i}$ holds for at least $k+1$ fragments $Q_{i}$; these fragments satisfy $Q=\operatorname{rot}^{\delta_{i}}\left(S_{i}\right)$ and thus contribute $\delta_{i}$ to $R$. We conclude that there is an interval $[p \ldots p+k]$ containing $x$ and at least $k+1$ elements of $R$. Consequently, we have $x \in I$. By definition of $J$, this yields $x \bmod |Q| \in J \bmod |Q|$.

Now, let $Q^{\infty}[x \ldots y)$ denote a witness that satisfies $\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right)$. By Claim 6.6 there is a matching fragment $Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)=Q^{\infty}[x \ldots y)$ starting at $x^{\prime} \in J$. Thus, we may assume without loss of generality that $x \in J$. As we verify all possible starting positions in $J$ using Verify from Lemma 6.2, we correctly compute the starting position $x$ of a witness occurrence of $S$ in $Q^{\infty}$. Further, as we use an EditGenerator from Lemma 6.1. we also compute the corresponding ending position correctly.

As for the running time, we prove the following characterization of $J$ :

『 Claim 6.7. The interval $J$ satisfies $|J| \leq 3 k+1$.
Proof. The claim trivially holds if $|Q| \leq 3 k+1$, so we assume that $|Q|>3 k+1$. Recall that the multiset $R$ in Algorithm 12 is the union of $2 k+1$ sets $\left\{j \in \mathbb{Z}: \operatorname{rot}^{j}\left(S_{i}\right)=Q\right\}$. As the string $Q$ is primitive, $R$ is the union of at most $2 k+1$ infinite arithmetic progressions with difference $|Q|$. In particular, if $[p \ldots p+k]$ and $\left[p^{\prime} \ldots p^{\prime}+k\right]$ contain at least $k+1$ elements of $R$ each, then $([p \ldots p+k] \bmod |Q|) \cap\left(\left[p^{\prime} \ldots p^{\prime}+k\right] \bmod |Q|\right) \neq \emptyset$, and thus $\left[p^{\prime} \ldots p^{\prime}+k\right] \bmod |Q| \subseteq[p-k \ldots p+2 k] \bmod |Q|$. Since $J$ is the union of such intervals $\left[p^{\prime} \ldots p^{\prime}+k\right]$, we have $J \bmod |Q| \subseteq[p-k \ldots p+2 k] \bmod |Q|$. By definition of $I$, we conclude that $|I| \leq|[p-k . . p+2 k]|=3 k+1$.

Now, observe that computing the multiset $R$ (represented as the union of infinite arithmetic progressions modulo $|Q|)$ takes $\mathcal{O}(k)$ time in the PILLAR model; computing the sets $I$ and $J$ can be done in $\mathcal{O}(k \log \log k)$ time by sorting $R($ restricted to $[0 . .|Q|))$ and a subsequent cyclic scan over $R$. Further, by Claim 6.7. we call Verify on an interval of length $\mathcal{O}(k)$; hence the call to Verify takes $\mathcal{O}\left(k^{2}\right)$ time in the PILLAR model. Finally, as we query the EditGenerator for up to $k$ errors, the last step of the algorithm takes $\mathcal{O}\left(k^{2}\right)$ time in the PILLAR model as well. Hence in total, FindAWitness runs in $\mathcal{O}\left(k^{2}\right)$ time in the PILLAR model, completing the proof.

「 Lemma 6.8 (FindRelevantFragment $(P, T, k, d, Q)$ ). Let $P$ denote a pattern of length $m$, let $T$ denote a text of length $n$, and let $0 \leq k \leq m$ denote a threshold such that $n<3 / 2 m+k$. Further, let $d \geq 2 k$ denote a positive integer and let $Q$ denote a primitive string that satisfies $|Q| \leq m / 8 d$ and $\delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq d$.

Then, there is an algorithm that computes a fragment $T^{\prime}=T[\ell \ldots r)$ and an integer range $I$ such that $\delta_{E}\left(T^{\prime},{ }^{*} Q^{*}\right) \leq 3 d$, $\left|\operatorname{Occ}_{k}^{E}(P, T)\right|=\left|\operatorname{Occ}_{k}^{E}\left(P, T^{\prime}\right)\right|,|I| \leq 6 d+1$, and $\operatorname{Occ}_{k}^{E}\left(P, T^{\prime}\right) \bmod |Q| \subseteq I \bmod |Q|$. The algorithm runs in $\mathcal{O}\left(d^{2}\right)$ time in the PILLAR model,

Proof. We start with two calls to FindAWitness from Lemma 4.5 in order to find a fragment $Q^{\infty}[x \ldots y)$ such that $\delta_{E}\left(P, Q^{\infty}[x \ldots y)\right)=\delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq d$ and a fragment $Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)$ such that

$$
\delta_{E}\left(T[n-m+k \ldots m-k), Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right)\right)=\delta_{E}\left(T[n-m+k \ldots m-k),{ }^{*} Q^{*}\right) \leq \frac{3}{2} d
$$

If the latter fragment does not exist, then we return the empty string $T^{\prime}=\varepsilon$ and the empty interval $I=\emptyset$. Otherwise, we proceed by computing the rightmost position $r$ such that $\delta_{E}\left(T[n-m+k \ldots r)\right.$, $\left.\operatorname{rot}^{-x^{\prime}}(Q)^{*}\right) \leq$ $3 / 2 d$ and the leftmost position $\ell$ such that $\delta_{E}\left(T[\ell \ldots m-k),{ }^{*} \operatorname{rot}^{-y^{\prime}}(Q)\right) \leq \frac{3}{2} d$. That is, we "extend" the fragment found in the text as much as possible. Afterwards, we return the fragment $T^{\prime}=T[\ell \ldots r$ ) and the interval $I=\left[n-m+k-\ell+x-x^{\prime}-3 d \ldots n-m+k-\ell+x-x^{\prime}+3 d\right]$. Consider Algorithm 13 for implementation details.

For the correctness, note that the due to the assumption on $Q$, the first call to FindAWitness is valid and returns a witness $Q^{\infty}[x \ldots y)$ (rather than $\perp$ ). Next, consider a $k$-error occurrence $T[p \ldots q$ ) of $P$. By triangle inequality (Fact 2.3), we have

$$
\left.\delta_{E}\left(T[p \ldots q), Q^{\infty}[x \ldots y)\right) \leq k+\delta_{E}\left(P, Q^{\infty}[x \ldots y)\right)\right) \leq \frac{3}{2} d
$$

Due to $|T[p \ldots q)| \geq m-k$, we have $q \geq m-k$ and $p \leq n-m+k$, which yields

$$
\delta_{E}\left(T[n-m+k \ldots m-k), Q^{\infty}\left[x^{\prime \prime} \ldots y^{\prime \prime}\right)\right) \leq 3 / 2 d
$$

for some integers $x^{\prime \prime}, y^{\prime \prime}$ with $x \leq x^{\prime \prime} \leq y^{\prime \prime} \leq y$. Moreover, as in the proof of Theorem 5.2, we have $|T[n-m+k \ldots m-k)|=2(m-k)-n \geq(3 d+1)|Q|$ or $|Q|=1$, so the second call to FindAWitness is valid. Thus, if the call returns $\perp$, then $\operatorname{Occ}_{k}^{E}(P, T)=\emptyset$.

We henceforth assume that the call returned a witness $Q^{\infty}\left[x^{\prime} . . y^{\prime}\right)$. Next, we apply Lemma 6.3 for $S=T[n-m+k \ldots m-k)$. This is indeed possible because $|S| \geq(3 d+1)|Q|$ or $|Q|=1$. Due to

$$
\delta_{E}\left(T[n-m+k \ldots q),{ }^{*} Q^{*}\right) \leq \delta_{E}\left(T[n-m+k \ldots q), Q^{\infty}\left[x^{\prime \prime} \ldots y\right)\right) \leq \frac{3}{2} d,
$$

Lemma 6.3 yields

$$
\delta_{E}\left(T[n-m+k \ldots q), \operatorname{rot}^{-x^{\prime}}(Q)^{*}\right)=\delta_{E}\left(T[n-m+k \ldots q),{ }^{*} Q^{*}\right) \leq \frac{3}{2} d .
$$

Consequently, we have $q \leq r$, because $r$ is computed correctly using EditGenerator from Lemma 6.1. Symmetrically, due to

$$
\delta_{E}\left(T[p \ldots m+k),{ }^{*} Q^{*}\right) \leq \delta_{E}\left(T[p \ldots m+k), Q^{\infty}\left[x \ldots y^{\prime \prime}\right)\right) \leq \frac{3}{2} d,
$$

Lemma 6.3 yields

$$
\delta_{E}\left(T[p \ldots m-k),{ }^{*} \operatorname{rot}^{-y^{\prime}}(Q)\right)=\delta_{E}\left(T[p \ldots m+k),{ }^{*} Q^{*}\right) \leq \frac{3}{2} d .
$$

Consequently, we have $p \geq \ell$, because $\ell$ is computed correctly using EditGenerator ${ }^{R}$ from Lemma 6.1. We conclude that $T[p \ldots q)$ is contained in $T^{\prime}=T[\ell \ldots r)$. Since $T[p \ldots q$ ) was an arbitrary $k$-error occurrence of $P$ in $T$, this implies $\left|\operatorname{Occ}_{k}^{E}\left(P, T^{\prime}\right)\right|=\left|\operatorname{Occ}_{k}^{E}(P, T)\right|$.

Now consider the fragment $T[p \ldots m-k)$ whose prefix $T[p \ldots m-k)$ satisfies

$$
\delta_{E}\left(T[p \ldots m-k), Q^{\infty}\left[x \ldots y^{\prime \prime}\right)\right) \leq 3 / 2 d
$$

and whose suffix $T[n-m+k \ldots m-k)$ satisfies

$$
\delta_{E}\left(T[n-m+k \ldots m-k), Q^{\infty}\left[x^{\prime} . . y^{\prime}\right)\right) \leq \frac{3}{2} d .
$$

We apply Lemma 5.4 to $T[p \ldots m-k)$; this is indeed possible because $|T[n-m+k \ldots m-k)| \geq(3 d+1)|Q|$ or $|Q|=1$. Lemma 5.4 now implies $\left(n-m+k-p+x-x^{\prime}+3 d\right) \bmod |Q| \leq 6 d$. Consequently, we have $p \bmod |Q| \in\left[n-m+k+x-x^{\prime}-3 d . . n-m+k+x-x^{\prime}+3 d\right] \bmod |Q|$.

Since $T[p \ldots q$ ) was an arbitrary $k$-error occurrence of $P$ in $T$, this implies
$\operatorname{Occ}_{k}^{E}\left(P, T^{\prime}\right) \bmod |Q| \subseteq I \bmod |Q|$.
Moreover, $|I| \leq 6 d+1$ holds trivially by construction.
Now consider the fragment $T$, whose prefix $T[\ell . . m-k)$ satisfies

$$
\delta_{E}\left(T[\ell \ldots m-k),{ }^{*} Q^{*}\right) \leq \frac{3}{2} d
$$

and whose suffix $T[n-m+k \ldots r)$ satisfies

$$
\delta_{E}\left(T[n-m+k . . r),{ }^{*} Q^{*}\right) \leq 3 / 2 d .
$$

Again, we apply Lemma 5.4 to $T^{\prime}$; this is indeed possible because $|T[n-m+k \ldots m-k)| \geq(3 d+1)|Q|$ or $|Q|=1$. Lemma 5.4 now implies $\delta_{E}\left(T^{\prime},{ }^{*} Q^{*}\right) \leq 3 d$, as claimed.

As for the running time in the PILLAR model, the calls to FindAWitness use $\mathcal{O}\left(d^{2}\right)$ time; the same is true for the usage of EditGenerator and EditGenerator ${ }^{R}$. Thus, the algorithm takes $\mathcal{O}\left(d^{2}\right)$ time in the PILLAR model.

Algorithm 13 A PILLAR algorithm computing a relevant fragment $T^{\prime}$ of $T$ containing all $k$-error occurrences of $P$ in $T$, and an interval $I$ such that $\operatorname{Occ}_{k}^{E}\left(P, T^{\prime}\right) \bmod |Q| \subseteq I \bmod |Q|$.

```
FindRelevantFragment \((P, T, k, d, Q)\)
    \(Q^{\infty}[x \ldots y) \leftarrow\) FindAWitness \((d, Q, P) ;\)
    \(Q^{\prime} \leftarrow\) FindAWitness \(\left(\left\lfloor\frac{3}{2} d\right\rfloor, Q, T[n-m+k \ldots m-k)\right.\) );
    if \(Q^{\prime}=\perp\) then return \((\varepsilon, \emptyset)\);
    \(Q^{\infty}\left[x^{\prime} \ldots y^{\prime}\right) \leftarrow Q^{\prime} ;\)
    // Extend \(Q^{\prime}\) as much as possible to the right.
    generator \(\mathbf{G} \leftarrow\) EditGenerator \(\left(T[n-m+k \ldots n)\right.\), \(\left.\operatorname{rot}^{-x^{\prime}}(Q)\right)\);
    for \(i \leftarrow 0\) to \(\lfloor 3 / 2 d\rfloor\) do \((\lambda, \quad\) ) \(\leftarrow \operatorname{Next}(\mathbf{G})\);
    \(r \leftarrow n-m+k+\lambda ;\)
    // Extend \(Q^{\prime}\) as much as possible to the left.
    generator \(\mathbf{G}^{\prime} \leftarrow\) EditGenerator \({ }^{R}\left(T[0 \ldots m-k)\right.\), \(\left.\operatorname{rot}^{-y^{\prime}}(Q)\right)\);
    for \(i \leftarrow 0\) to \(\lfloor 3 / 2 d\rfloor\) do \(\left(\lambda^{\prime}, \ldots\right) \leftarrow \operatorname{Next}\left(\mathbf{G}^{\prime}\right)\);
    \(\ell \leftarrow m-k-\lambda^{\prime}\);
    return \(\left(T[\ell . . r),\left[n-m+k-\ell+x-x^{\prime}-3 d \ldots n-m+k-\ell+x-x^{\prime}+3 d\right]\right)\);
```

Lemma $6.9(\operatorname{Locked}(S, Q, d, k)$ : Implementation of Lemma 5.11). Let $S$ denote a string, let $Q$ denote a primitive string, let d denote a positive integer such that $\delta_{E}\left(S,{ }^{*} Q^{*}\right) \leq d$ and $|S| \geq(2 d+1)|Q|$, and let $k$ denote a non-negative integer.

Then, there is an algorithm that computes disjoint locked fragments $L_{1}, \ldots, L_{\ell} \preceq S$ such that $L_{1}$ is a $k$-locked prefix of $S, L_{\ell}$ is a suffix of $S$, and $\delta_{E}\left(L_{i},{ }^{*} Q^{*}\right)>0$ for $1<i<\ell$. Moreover, we have

$$
\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\sum_{i=1}^{\ell} \delta_{E}\left(L_{i},{ }^{*} Q^{*}\right) \quad \text { and } \quad \sum_{i=1}^{\ell}\left|L_{i}\right| \leq(5|Q|+1) d+2(k+1)|Q| .
$$

The algorithm takes $\mathcal{O}\left(d^{2}+k\right)$ time in the PILLAR model.

Proof. We implement the process from in the proof of Lemma 5.11. Our algorithm is described below; see also Algorithm 14 for implementation details.

First, we construct an optimal alignment between $S$ and a substring of $Q^{\infty}$. For this, we first use FindAWitness of Lemma 6.5 to obtain positions $x \leq y$ such that $\delta_{E}\left(S,{ }^{*} Q^{*}\right)=\delta_{E}\left(S, Q^{\infty}[x \ldots y)\right)$. Then, we apply a generator EditGenerator $\left(S, \operatorname{rot}^{-x}(Q)\right)$ of Lemma 6.1 to construct an optimal alignment $A$ between $S$ and $Q^{\infty}\left[x \ldots x+\pi^{\prime}\right.$ ) for some integer $\pi^{\prime} \geq 0$ (note that we cannot guarantee $y=x+\pi^{\prime}$ ).

Then, based on the alignment $A$, we construct a decomposition $S=S_{0}^{(0)} \cdots S_{s^{(0)}}^{(0)}$ such that $S_{i}^{(0)}$ is aligned against

$$
Q_{i}^{(0)}:=Q^{\infty}[\max (x,(|Q|-1)\lceil x /|Q|\rceil) \ldots \min (|Q|\lceil x /|Q|\rceil, x+\pi))
$$

in the decomposition $A$, and a sequence $\Delta_{i}^{(0)}$ such that we have $\Delta_{i}^{(0)}=\delta_{E}\left(S_{i}^{(0)}, Q_{i}^{(0)}\right)$ for $i>0$ and $\Delta_{0}^{(0)}=\delta_{E}\left(S_{0}^{(0)}, Q_{0}^{(0)}\right)+k+1$. Since this sequence might be long, we only generate interesting fragments $S_{i}^{(0)}$ and store them, along with the values $\Delta_{i}^{(0)}$, in a queue $F$ in left-to-right order. (Recall that $S_{i}^{(t)}$ is interesting if $i=0, i=s^{(t)}, S_{i}^{(t)} \neq Q$, or $\Delta_{i}^{(t)}>0$.)

The process of constructing the interesting fragments $S_{i}^{(0)}$ is somewhat tedious. We maintain a fragment $Q^{\infty}\left[\ell_{Q} \ldots r_{Q}\right)$, interpreted as $Q_{i}^{(0)}$ for increasing values of $i$, a fragment $S\left[\ell_{S} \ldots r_{S}\right)$, interpreted

Algorithm 14 Computing locked fragments in a string $S$.

```
\(\operatorname{Locked}(S, Q, d, k)\)
    \(Q^{\infty}[x \ldots y) \leftarrow\) FindAWitness \((d, Q, S)\);
    generator \(\mathbf{G} \leftarrow\) EditGenerator \(\left(S\right.\), \(\left.\operatorname{rot}^{-x}(Q)\right)\);
    do \(\left(\pi, \pi^{\prime}\right) \leftarrow \operatorname{Next}(\mathbf{G})\) while \(\pi<|S|\);
    \(A \leftarrow\) Alignment (G);
    \(\ell_{Q} \leftarrow x ; \ell_{S} \leftarrow 0 ;\)
    \(r_{Q} \leftarrow|Q|\lceil x /|Q|\rceil ; r_{S} \leftarrow r_{Q}-\ell_{Q} ;\)
    \(\Delta \leftarrow k+1 ;\)
    queue \(F\);
    foreach \((s, q) \in A \cup\left(\pi, \pi^{\prime}\right)\) do
        if \(s=\perp\) then \(s \leftarrow q+x+r_{S}-r_{Q}-1\);
        if \(q=\perp\) then \(q \leftarrow s-x+r_{Q}-r_{S}-1\);
        if \(x+q \geq r_{Q}\) then
            \(\operatorname{push}\left(F,\left(S\left[\ell_{S} \ldots r_{S}\right), \Delta\right)\right)\);
            \(\ell_{Q} \leftarrow|Q|\lfloor(x+q) /|Q|\rfloor ; \ell_{S} \leftarrow r_{S}+\ell_{Q}-r_{Q} ;\)
            \(r_{Q} \leftarrow \ell_{Q}+|Q| ; r_{S} \leftarrow r_{S}+|Q| ;\)
            \(\Delta \leftarrow 0 ;\)
        \(r_{S} \leftarrow r_{Q}-x+s-q ;\)
        if \((s, q) \neq\left(\pi, \pi^{\prime}\right)\) then \(\Delta \leftarrow \Delta+1\);
    \(\operatorname{push}\left(F,\left(S\left[\ell_{S} \ldots|S|\right), \Delta\right)\right)\);
    stack \(L\);
    while \(F\) is not empty do
        \((S[\ell \ldots r), \Delta) \leftarrow \operatorname{front}(F) ; \operatorname{pop}(F)\);
        while true do
            if \(\operatorname{top}(L)=S\left[\ell^{\prime} \ldots r^{\prime}\right)\) and \(r^{\prime}=\ell\) then
                \(\ell \leftarrow \ell^{\prime} ;\)
            pop ( \(L\) );
        else if \(\operatorname{front}(F)=\left(S\left[\ell^{\prime} \ldots r^{\prime}\right), \Delta^{\prime}\right)\) and \(\ell^{\prime}=r\) then
            \(r \leftarrow r^{\prime} ;\)
            \(\Delta \leftarrow \Delta+\Delta^{\prime} ;\)
            pop \((F)\);
            else if \(\Delta>0\) then
                \(\ell \leftarrow \max (0, \ell-|Q|) ;\)
                \(r \leftarrow \min (|S|, r+|Q|) ;\)
                \(\Delta \leftarrow \Delta-1 ;\)
            else
                push (L, \(S[\ell \ldots r)\) );
                break;
    return \(L\);
```

as a candidate for $S_{i}^{(0)}$, and an integer $\Delta$, interpreted as $\Delta_{i}^{(0)}$. They are initialized to $Q^{\infty}[x \ldots|Q|\lceil x /|Q|\rceil$ ), $S[0 \ldots Q \mid\lceil x /|Q|\rceil-x)$, and $k+1$, respectively.

Next, we process pairs $(s, q)$ corresponding to subsequent errors in the alignment $A$. The interpretation of the $j$-th pair $(s, q)$ is that $S[0 \ldots s)$ is aligned with $Q^{\infty}[x \ldots x+q)$ with $j$ errors so that the $j$-th error is a substitution of $S[s]$ into $Q^{\infty}[x+q]$, and insertion of $Q^{\infty}[x+q]$, or a deletion of $S[s]$.

The first step of processing $(s, q)$ is only performed if $Q^{\infty}[x \ldots x+q)$ is not (yet) contained in $Q^{\infty}\left[\ell_{Q} \ldots r_{Q}\right)$. If this is not the case, then we push $S\left[\ell_{S} \ldots r_{S}\right)$ with budget $\Delta$ to the queue $F$ of interesting fragments, and we update the maintained data: The fragment $Q^{\infty}\left[\ell_{Q} \ldots r_{Q}\right)$ is set to be the fragment of $Q^{\infty}$ matching $Q$ and containing $Q^{\infty}[x+q]$; between the previous and the current value of $Q^{\infty}\left[\ell_{Q} \ldots r_{Q}\right)$, there are zero or more copies of $Q$ aligned in $A$ without error. Hence, we skip the same number of copies of $Q$ in $S$ (these are the uninteresting fragments $S_{i}^{(0)}$ ) and set $S\left[\ell_{S} \ldots r_{S}\right.$ ) to be the subsequent fragment of length $|Q|$. Finally, the budget $\Delta$ is reset to 0 .

In the second step, we update $r_{S}$ according to the type of the currently processed error: We increment $r_{S}$ in case of deletion of $S[s]$ and we decrement $r_{S}$ in case of insertion of $Q^{\infty}[x+q]$. This way, we guarantee that $\left|S\left(s \ldots r_{S}\right)\right|=\left|Q^{\infty}\left(x+q \ldots r_{Q}\right)\right|$, and that $A$ aligns $S\left[\ell_{S} \ldots r_{S}\right)$ with $Q^{\infty}\left[\ell_{Q} \ldots r_{Q}\right)$ provided that we have already processed all errors involving $Q^{\infty}\left[\ell_{Q} \ldots r_{Q}\right)$. Additionally, we increase $\Delta$ to acknowledge the currently processed error between $S\left[\ell_{S} \ldots r_{S}\right)$ and $Q^{\infty}\left[\ell_{Q} \ldots r_{Q}\right)$.

In a similar way, we process $(s, q)=\left(|S|, \pi^{\prime}\right)$, interpreting it as extra substitution. This time, however, we do not increase $\Delta$ (because this is a not a real error). Finally, we push $S\left[\ell_{S} \ldots|S|\right)=S_{s^{(0)}}^{(0)}$ with budget $\Delta$ to the queue $F$.

In the second phase of the algorithm, we transform the decomposition $S=S_{0}^{(0)} \cdots S_{s^{(0)}}^{(0)}$ and the sequence $\Delta_{0}^{(0)} \cdots \Delta_{s^{(0)}}^{(0)}$ using the four types of merge operations described in the proof of Lemma 5.6 .

We maintain an invariant that a stack $L$ contains already processed interesting fragments, all with budget equal to 0 , in left-to-right order (so that $\operatorname{top}(L)$ represents the rightmost one), while $F$ contains fragments that have not been processed yet (and may have positive budgets) also in the left-to-right order (so that front $(F)$ represents the leftmost one). Additionally, the currently processed fragment $S[\ell \ldots r$ ) is guaranteed to be to the right of all fragments in $L$ and to the left of all fragments in $F$. The fragments in $L$, the fragment $S[\ell \ldots r)$, and the fragments in $F$ form the sequence of all interesting fragments in the current decomposition $S=S_{0}^{(t)} \cdots S_{s^{(t)}}^{(t)}$.

In each iteration of the main loop, we pop the front fragment $S[\ell \ldots r)$ with budget $\Delta$ from the queue $F$ and exhaustively perform merge operations involving it: We first try applying a type -1 merge with the fragment to the left (which must be $\operatorname{top}(L)$ ). If this is not possible, we type applying a type -1 merge with the fragment to the right (which must be front $(F)$ ). If also this is not possible, then $S[\ell \ldots r$ ) is surrounded by uninteresting fragments. In this case, we perform a type- 2 type- 3 or 4 merge provided that $\Delta>0$. Otherwise, we push $S[\ell \ldots r)$ to $L$ and proceed to the next iteration.

Finally, the algorithm returns the sequence of (locked) fragments represented in the stack $L$.
The correctness of the algorithm follows from Lemma 5.11 no deep insight is needed to prove that our implementation indeed follows the procedure described in the proof of Lemma 5.6 and extended in the proof of Lemma 5.11

For the running time, the initial call to FindAWitness and applying the generator $\mathbf{G}$ each take $\mathcal{O}\left(d^{2}\right)$ time in the PILLAR model. As the alignment $A$ is of size $|A| \leq d$, the for loop in Line 10 takes $\mathcal{O}(d)$ time and generates $\mathcal{O}(d)$ interesting locked fragments with total budget $\mathcal{O}(d+k)$. Each iteration of the while loop in Line 24 decreases the number of interesting locked fragments or their total budget, so there are at most $\mathcal{O}(d+k)$ iterations in total. Overall the algorithm runs in $\mathcal{O}\left(d^{2}+k\right)$ time in the PILLAR model. 』

[^1]$0 \leq k \leq m$ denote a threshold, and let $T$ denote a text of length $n \leq 3 / 2 m+k$. Further, let I denote an integer range and let $Q$ denote a primitive string that satisfies $\delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq d$ and $\delta_{E}\left(T,{ }^{*} Q^{*}\right) \leq d^{\prime}$.

There is an algorithm that computes the set $\operatorname{Occ}_{k}^{E}(P, T) \cap(I+|Q| \mathbb{Z})$ as $\mathcal{O}\left(|I| d^{\prime}(d+k)\right)$ arithmetic progressions. The algorithm takes $\mathcal{O}\left(k d^{\prime}(d+k)\left(k+|I|+d+d^{\prime}\right)\right)$ time in the PILLAR model.

Proof. The algorithm resembles the proof of Theorem 5.2 (d). Consult Algorithm 15 for a visualization of the algorithm as pseudo-code; in the interest of readability we use $\operatorname{Occ}_{k}^{E}(P, T)$ instead of $\operatorname{Occ}_{k}^{E}(P, T) \cap$ $(I+|Q| \mathbb{Z})$ in the pseudo-code. Note that if $|I|>|Q|$ we can replace $I$ with $[0 \ldots Q)$, since in this case $I+|Q| \mathbb{Z}=\mathbb{Z}=[0 . . Q)+|Q| \mathbb{Z}$. Hence, we can assume that $|I| \leq|Q|$.

We first compute $\mathcal{L}^{P}:=\operatorname{Locked}(P, Q, d, k)$ and $\mathcal{L}^{T}:=\operatorname{Locked}\left(T, Q, d^{\prime}, 0\right)$ using Algorithm 14 We have $\ell^{P}:=\left|\mathcal{L}^{P}\right| \leq \delta_{E}\left(P,{ }^{*} Q^{*}\right)+2 \leq d+2$ and $\ell^{T}=\left|\mathcal{L}^{T}\right| \leq \delta_{E}\left(T,{ }^{*} Q^{*}\right) \leq d^{\prime}+2$.

Then, for each of the $\mathcal{O}\left(d d^{\prime}\right)$ pairs of locked fragments $L_{i}^{P}=P[\ell \ldots r) \in \mathcal{L}^{P}$ and $L_{j}^{T}=T\left[\ell^{\prime} . . r^{\prime}\right) \in \mathcal{L}^{T}$ we (implicitly) mark the positions in the interval $\left[\ell^{\prime}-r-k . . r^{\prime}-\ell+k\right.$ ). We also mark all positions in [ $n-m-k \ldots n-m+k]$. We decompose the set of marked positions $M$ into $\mathcal{O}\left(d d^{\prime}\right)$ maximal ranges $J \subseteq M$. For each such maximal range $J$, for each maximal range $J^{\prime} \subseteq J \cap(I+|Q| \mathbb{Z})$, we call Verify ( $P, T, k, J^{\prime}$ ) and add its output to $\operatorname{Occ}_{k}^{E}(P, T) \cap(I+|Q| \mathbb{Z})$. This guarantees that we correctly compute all elements of the set $\operatorname{Occ}_{k}^{E}(P, T) \cap(I+|Q| \mathbb{Z}) \cap M$.

The decomposition of $M$ into maximal ranges yields a decomposition of $[0 \ldots n-m+k) \backslash M$ into $\mathcal{O}\left(d d^{\prime}\right)$ maximal ranges. For each such maximal range $J$, we rely on the characterization of Claim 5.17 in order to compute $\operatorname{Occ}_{k}^{E}(P, T) \cap(I+|Q| \mathbb{Z}) \cap J$. Recall that for $p, p^{\prime} \in J$ with $p \equiv p^{\prime}(\bmod |Q|)$ we have $p \in \operatorname{Occ}_{k}^{E}(P, T)$ if and only if $p^{\prime} \in \operatorname{Occ}_{k}^{E}(P, T)$. Hence, it suffices to restrict our attention to the intersection of the first (at most) $|Q|$ positions of $J$ with $I+|Q| \mathbb{Z}$. This intersection consists of at most two intervals of total size at most $|I|$. We call Verify for each of them, and for each position returned by these Verify queries, we add an arithmetic progression to $\operatorname{Occ}_{k}^{E}(P, T) \cap(I+|Q| \mathbb{Z})$.

We now proceed to analyze the time complexity of the algorithm in the PILLAR model. The two calls to Locked require $\mathcal{O}\left(d^{2}+k+d^{\prime 2}\right)$ time in total in the PILLAR model due to Lemma 6.9 We then decompose $M$ into maximal ranges, which can be implemented in $\mathcal{O}\left(d d^{\prime} \log \log \left(d d^{\prime}\right)\right)$ time. The interval $R$ of positions marked due to locked regions $L_{i}^{P}$ and $L_{j}^{T}$ is of size $\left|L_{i}^{P}\right|+\left|L_{j}^{T}\right|+2 k-1$; the number of maximal ranges $R^{\prime} \subseteq R \cap(I+|Q| \mathbb{Z})$ is at most $(|R|+2|Q|-2|I|) /|Q|$. Consequently, the total number of maximal ranges of size at most $|I|$ that we need to consider intervals does not exceed $2 k+1$ plus

$$
\begin{aligned}
\sum_{i=1}^{\ell^{P}} \sum_{j=1}^{\ell^{T}} \frac{\left|L_{i}^{P}\right|+\left|L_{j}^{T}\right|+2|Q|-2|I|}{|Q|} & \leq \frac{\ell^{T}}{|Q|} \sum_{i=1}^{\ell^{P}}\left|L_{i}^{P}\right|+\frac{\ell^{P}}{|Q|} \sum_{i=1}^{\ell^{T}}\left|L_{i}^{T}\right|+\frac{2 \ell^{P} \ell^{T}(|Q|-|I|)}{|Q|} \\
& =\mathcal{O}\left(\left(d^{\prime}(d|Q|+k|Q|)+d d^{\prime}|Q|+d d^{\prime}|Q|\right) /|Q|\right) \\
& =\mathcal{O}\left(d^{\prime}(d+k)\right) .
\end{aligned}
$$

Each call to Verify in Line 11 of Algorithm 15 requires time $\mathcal{O}\left(k\left(k+\left|J^{\prime}\right|\right)\right)$ by Lemma 6.2 By the above analysis, we make $\mathcal{O}\left(d^{\prime}(d+k)\right)$ calls to Verify, each time for an interval of size at most $|I|$. Hence, we can upper bound the overall running time for this step by $\mathcal{O}\left(d^{\prime}(d+k) k(k+|I|)\right)$. Finally, the total time required by Verify queries in Line 15 of Algorithm 15 is $\mathcal{O}\left(d d^{\prime} k(k+|I|)\right)$ as we call Verify $\mathcal{O}\left(d d^{\prime}\right)$ times, each time for an interval of size $\mathcal{O}(|I|)$. Thus, the overall running time is $\mathcal{O}\left(d^{\prime}(d+k) k(k+|I|)+d^{2}+d^{\prime 2}+d d^{\prime} \log \log \left(d d^{\prime}\right)\right)=$ $\mathcal{O}\left(k d^{\prime}(d+k)\left(k+|I|+d+d^{\prime}\right)\right)$.

The bounds obtained in the time complexity analysis also imply that our representation of $\operatorname{Occ}_{k}^{E}(P, T) \cap$ $(I+|Q| \mathbb{Z})$ consists of $\mathcal{O}\left(|I| d^{\prime}(d+k)\right)$ arithmetic progressions.

Algorithm 15 Computing $k$-error occurrences in the presence of locked regions in text and pattern.

```
SynchedMatches \(\left(P, T, I, k, d, d^{\prime}, Q\right)\)
    \(\mathcal{L}^{P} \leftarrow \operatorname{Locked}(P, Q, d, k) ;\)
    \(\mathcal{L}^{T} \leftarrow \operatorname{Locked}\left(T, Q, d^{\prime}, 0\right) ;\)
    \(M \leftarrow[n-m-k \ldots n-m+k] ;\)
    foreach \(P[\ell \ldots r) \in \mathcal{L}^{P}\) do
            foreach \(T\left[\ell^{\prime} \ldots r^{\prime}\right) \in \mathcal{L}^{T}\) do
                \(M \leftarrow M \cup\left(\ell^{\prime}-r-k \ldots r^{\prime}-\ell+k\right) ;\)
    \(M \leftarrow M \cap[0 \ldots n-m+k) ;\)
    foreach maximal range \(J \subseteq M\) do
            foreach maximal range \(J^{\prime} \subseteq J \cap(I+|Q| \mathbb{Z})\) do
            \(\operatorname{Occ}_{k}^{E}(P, T) \leftarrow \operatorname{Occ}_{k}^{E}(P, T) \cup\left\{\operatorname{pos} \mid\left(\right.\right.\) pos,\(\left.\left.k_{p o s}\right) \in \operatorname{Verify}\left(P, T, k, J^{\prime}\right)\right\} ;\)
    foreach maximal range \([\ell \ldots r) \subseteq[0 \ldots n-m+k) \backslash M\) do
        \(J \leftarrow[\ell \ldots \min (r, \ell+|Q|)) ;\)
            foreach maximal range \(J^{\prime} \subseteq J \cap(I+|Q| \mathbb{Z})\) do
            foreach \(\left(p o s, k_{p o s}\right) \in \operatorname{Verify}\left(P, T, k, J^{\prime}\right)\) do
                \(\operatorname{Occ}_{k}^{E}(P, T) \leftarrow \operatorname{Occ}_{k}^{E}(P, T) \cup((\operatorname{pos}+|Q| \mathbb{Z}) \cap[\ell \ldots r)) ;\)
    return \(\mathrm{Occ}_{k}^{E}(P, T)\);
```

Lemma 6.11 (PeriodicMatches $(P, T, k, d, Q)$ ). Let $P$ denote a pattern of length $m$ and let $T$ denote a text of length $n$. Further, let $0 \leq k \leq m$ denote a threshold, let $d \geq 2 k$ denote a positive integer, and let $Q$ denote a primitive string that satisfies $|Q| \leq m / 8 d$ and $\delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq d$.

There is an algorithm that computes the set $\operatorname{Occ}_{k}^{E}(P, T)$, using $\mathcal{O}\left(n / m \cdot d^{4}\right)$ time in the PILLAR model.

Proof. We consider $\lfloor 2 n / m\rfloor$ blocks $T_{0}, \ldots, T_{\lfloor 2 n / m\rfloor-1}$ of $T$, each of length at most $3 / 2 m+k-1$, where the $i$-th block starts at position $i \cdot m / 2$, that is,

$$
T_{i}:=T[\lfloor i \cdot m / 2\rfloor \ldots \min \{n,\lfloor(i+3) \cdot m / 2\rfloor+k-1\})
$$

Observe that each $k$-error occurrence of $P$ in $T$ is contained in at least one of the fragments $T_{i}$ : Specifically, $T_{i}$ covers all occurrences starting in $\lfloor\lfloor i \cdot m / 2\rfloor \ldots\lfloor(i+1) \cdot m / 2\rfloor)$. For each block $T_{i}$, we call FindRelevantFragment $\left(P, T_{i}, k, d, Q\right)$ from Lemma 6.8 and obtain a fragment $T_{i}^{\prime}=T\left[\ell_{i} \ldots r_{i}\right)$ containing all $k$-error occurrences of $P$ in $T_{i}$ and an integer range $I_{i}$. Lemma 6.8 guarantees that $\delta_{E}\left(T_{i}^{\prime}, Q\right) \leq 3 d$ and $\left|I_{i}\right| \leq 6 d+1$. Next, we call SynchedMatches $\left(P, T_{i}^{\prime}, I_{i}, d, 3 d, k, Q\right.$ ) from Lemma 6.10. The output of the call to SynchedMatches consists of $\mathcal{O}((6 d+1) 3 d(d+k))=\mathcal{O}\left(d^{3}\right)$ arithmetic progressions. For each obtained arithmetic progression, we first add $\lfloor i \cdot m / 2\rfloor$ to all of its elements, and, if $i<\lfloor 2 n / m\rfloor-1$, we intersect the resulting arithmetic progression with $[\lfloor i \cdot m / 2\rfloor \ldots\lfloor(i+1) \cdot m / 2\rfloor$ ); finally, we add the obtained set to $\operatorname{Occ}_{k}^{E}(P, T)$. The intersection step guarantees that each $k$-error occurrence is accounted for by exactly one block.

For the correctness, note that by Lemma 6.8. for each $i$, we have $\operatorname{Occ}_{k}^{E}\left(P, T_{i}^{\prime}\right) \bmod |Q| \subseteq I_{i} \bmod |Q|$. Hence, the call to SynchedMatches indeed computes all occurrences of $P$ in $T_{i}$.

Each call to FindRelevantFragment requires $\mathcal{O}\left(d^{2}\right)$ time, while each call to SynchedMatches requires time $\mathcal{O}(3 k d(d+k)(k+(6 d+1)+d+3 d))=\mathcal{O}\left(d^{4}\right)$. The claimed overall running time follows.

```
Algorithm 16 A PILLAR model algorithm for Lemma 5.21.
BreakMatches \(\left(P, T,\left\{B_{1}=P\left[b_{1} \ldots b_{1}+\left|B_{1}\right|\right), \ldots, B_{2 k}=P\left[b_{2 k} \ldots b_{2 k}+\left|B_{2 k}\right|\right)\right\}, k\right)\)
    multi-set \(M \leftarrow\left\} ; \operatorname{Occ}_{k}^{E}(P, T) \leftarrow\{ \} ;\right.\)
    for \(i \leftarrow 1\) to \(2 k\) do
        foreach \(\tau \in\) ExactMatches \(\left(B_{i}, T\right)\) do
            \(M \leftarrow M \cup\left\{\left\lfloor\left(\tau-b_{i}-k\right) / k\right\rfloor\right\} ; \quad / /\) Mark block \(\left\lfloor\left(\tau-b_{i}-k\right) / k\right\rfloor\) of \(T\)
            \(M \leftarrow M \cup\left\{\left\lfloor\left(\tau-b_{i}\right) / k\right\rfloor\right\} ; \quad / /\) Mark block \(\left\lfloor\left(\tau-b_{i}\right) / k\right\rfloor\) of \(T\)
            \(M \leftarrow M \cup\left\{\left\lfloor\left(\tau-b_{i}+k\right) / k\right\rfloor\right\} ; \quad / /\) Mark block \(\left\lfloor\left(\tau-b_{i}+k\right) / k\right\rfloor\) of \(T\)
            \(M \leftarrow M \cup\left\{\left\lfloor\left(\tau-b_{i}+2 k\right) / k\right\rfloor\right\} ; \quad / /\) Mark block \(\left\lfloor\left(\tau-b_{i}+2 k\right) / k\right\rfloor\) of \(T\)
    sort \(M\);
    foreach \(\pi \in[0 \ldots n-m]\) that appears at least \(k\) times in \(M\) do
        \(\operatorname{Occ}_{k}^{E}(P, T) \leftarrow \operatorname{Occ}_{k}^{E}(P, T) \cup\left\{\operatorname{pos} \mid\left(\operatorname{pos}, k_{p o s}\right) \in \operatorname{Verify}(P, T, k,[\pi \cdot k \ldots(\pi+1) \cdot k))\right\} ;\)
    return \(\operatorname{Occ}_{k}^{E}(P, T)\);
```


### 6.4 Computing Occurrences in the Non-Periodic Case

Lemma 6.12 (BreakMatches $\left(P, T,\left\{B_{1}, \ldots, B_{2 k}\right\}, k\right.$ ): Implem. of Lemma 5.21). Let $k$ denote $a$ threshold and let $P$ denote a pattern of length $m$ having $2 k$ disjoint breaks $B_{1}, \ldots, B_{2 k} \preccurlyeq P$ each satisfying $\operatorname{per}\left(B_{i}\right) \geq m / 128 k$. Further, let $T$ denote a text of length $n \leq \frac{3}{2} m+k$.

Then, we can compute the set $\operatorname{Occ}_{k}^{E}(P, T)$ using $\mathcal{O}\left(k^{3}\right)$ time in the PILLAR model.
Proof. We proceed similarly to Lemma 4.9 Instead of marking positions, we now mark blocks of length $k$; in the end, we verify complete blocks at once using Verify from Lemma 6.2 Consider Algorithm 16 for the complete algorithm visualized as pseudo-code.

For the correctness, note that we have placed the marks as in the proof of Lemma 5.21 in particular, by Claim 5.23 any block $[j k \ldots(j+1) k)$ that contains any position $\pi \in \operatorname{Occ}_{k}^{E}(P, T)$ has at least $k$ marks. As we verify each such block using Verify from Lemma 6.2 , we report no false positives, and thus the algorithm is correct.

We continue with analyzing the running time. As every break $B_{i}$ has period $\operatorname{per}\left(B_{i}\right)>m / 128 k$, every call to ExactMatches uses $\mathcal{O}(k)$ time in the PILLAR model by Lemma 2.11 thus, all calls to ExactMatches in total take $\mathcal{O}\left(k^{2}\right)$ time in total. Next, by Claim 5.22 , we place at most $\mathcal{O}\left(k^{2}\right)$ marks in $T$, so the marking step uses $\mathcal{O}\left(k^{2}\right)$ operations in total. Further, finding all positions in $T$ with at least $k$ marks can be done via a linear scan over the multi-set $M$ of all marks after sorting $M$, which can be done in time $\mathcal{O}\left(k^{2} \log \log k\right)$. Finally, as there are at most $\mathcal{O}\left(k^{2} / k\right)=O(k)$ blocks that we verify, and every call to Verify takes time $\mathcal{O}\left(k^{2}\right)$ in the PILLAR model, the verifications take $\mathcal{O}\left(k^{3}\right)$ time in the PILLAR in total. Overall, Algorithm 16 thus takes $\mathcal{O}\left(k^{3}\right)$ time in the PILLAR model.

Lemma 6.13 (RepetitiveMatches $\left(P, T,\left\{\left(R_{1}, Q_{1}\right) \ldots,\left(R_{r}, Q_{r}\right)\right\}, k\right)$ : Implementation of Lemma 5.24). Let $P$ denote a pattern of length $m$ and let $k \leq m$ denote a threshold. Further, let $T$ denote a string of length $n \leq 3 / 2 m+k$. Suppose that $P$ contains disjoint repetitive regions $R_{1}, \ldots, R_{r}$ of total length at least $\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ such that each region $R_{i}$ satisfies $\left|R_{i}\right| \geq m / 8 k$ and has a primitive approximate period $Q_{i}$ with $\left|Q_{i}\right| \leq m / 128 k$ and $\delta_{H}\left(R_{i}, Q_{i}^{*}\right)=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil$.

Then, we can compute the set $\mathrm{Occ}_{k}^{E}(P, T)$ using $\mathcal{O}\left(k^{4}\right)$ time in the PILLAR model.
Proof. As in the proof of Lemma 5.24 set $m_{R}:=\sum_{i=1}^{r}\left|R_{i}\right| \geq 3 / 8 \cdot m$ and define for every $1 \leq i \leq r$ the values $k_{i}:=\left\lfloor 4 \cdot k / m \cdot\left|R_{i}\right|\right\rfloor$ and $d_{i}:=\left\lceil 8 \cdot k / m \cdot\left|R_{i}\right|\right\rceil=\left|\operatorname{Mis}\left(R_{i}, Q_{i}^{*}\right)\right|$. Further, write $R_{i}=P\left[r_{i} . . r_{i}+\left|R_{i}\right|\right)$.

Algorithm 17 A PILLAR model algorithm for Lemma 5.24

```
RepetitiveMatches \(\left(P, T,\left\{\left(R_{1}=P\left[r_{1} \ldots r_{1}+\left|R_{1}\right|\right), Q_{1}\right) \ldots,\left(R_{r}=P\left[r_{r} \ldots r_{r}+\left|R_{r}\right|\right), Q_{r}\right)\right\}, k\right)\)
    multi-set \(M \leftarrow\left\} ; \operatorname{Occ}_{k}^{E}(P, T) \leftarrow\{ \} ;\right.\)
    for \(i \leftarrow 1\) to \(r\) do
        set \(M_{i} \leftarrow\{ \}\);
        foreach \(\tau \in\) PeriodicMatches ( \(R_{i}, T,\left\lfloor 4 \cdot k / m \cdot\left|R_{i}\right|\right\rfloor,\left\lceil 8 \cdot k / m \cdot\left|R_{i}\right|\right\rceil, Q_{i}\) ) do
            \(M_{i} \leftarrow M_{i} \cup\left\{\left(\left\lfloor\left(\tau-r_{i}-k\right) / k\right\rfloor,\left|R_{i}\right|\right)\right\} ; \quad / /\) Place \(\left|R_{i}\right|\) marks at bl. \(\left\lfloor\left(\tau-r_{i}-k\right) / k\right\rfloor\)
            \(M_{i} \leftarrow M_{i} \cup\left\{\left(\left\lfloor\left(\tau-r_{i}\right) / k\right\rfloor,\left|R_{i}\right|\right)\right\} ; \quad / /\) Place \(\left|R_{i}\right|\) marks at block \(\left\lfloor\left(\tau-r_{i}\right) / k\right\rfloor\)
            \(M_{i} \leftarrow M_{i} \cup\left\{\left(\left\lfloor\left(\tau-r_{i}+k\right) / k\right\rfloor,\left|R_{i}\right|\right)\right\} ; \quad / /\) Place \(\left|R_{i}\right|\) marks at bl. \(\left\lfloor\left(\tau-r_{i}+k\right) / k\right\rfloor\)
            \(M_{i} \leftarrow M_{i} \cup\left\{\left(\left\lfloor\left(\tau-r_{i}+2 k\right) / k\right\rfloor,\left|R_{i}\right|\right)\right\} ; \quad / /\) Place \(\left|R_{i}\right|\) marks at bl. \(\left\lfloor\left(\tau-r_{i}+2 k\right) / k\right\rfloor\)
        \(M \leftarrow M \cup M_{i} ;\)
    sort \(M\) by positions;
    foreach \(\pi \in[0 \ldots n-m]\) appearing at least \(\sum_{(\pi, v) \in M} v \geq \sum_{i=1}^{r}\left|R_{i}\right|-m / 4\) times in \(M\) do
        \(\operatorname{Occ}_{k}^{E}(P, T) \leftarrow \operatorname{Occ}_{k}^{E}(P, T) \cup\left\{\operatorname{pos} \mid\left(p o s, k_{p o s}\right) \in \operatorname{Verify}(P, T, k,[\pi \cdot k \ldots(\pi+1) \cdot k))\right\} ;\)
    return \(\operatorname{Occ}_{k}^{E}(P, T)\);
```

Again, we proceed similarly to the Hamming distance setting (Lemma 4.10). However, instead of marking positions, we now mark blocks of length $k$; in the end, we then verify complete blocks at once using Verify from Lemma 6.2. Note that we need to ensure that we mark a block of $T$ only at most once for each repetitive part $R_{i}$; we do so by first computing a set of all blocks to be marked due to $R_{i}$ (thereby removing duplicates) and then merging the sets computed for every $R_{i}$ into a multi-set. Consider Algorithm 17 for the complete algorithm visualized as pseudo-code.

For the correctness, first note that in every call to PeriodicMatches from Lemma 6.11 we have

$$
16 k / m \cdot\left|R_{i}\right| \geq d_{i}=\left\lceil 8 k / m \cdot\left|R_{i}\right|\right\rceil=\delta_{H}\left(R_{i}, Q_{i}^{*}\right) \geq 2 k_{i}
$$

hence $\left|Q_{i}\right| \leq m / 128 k \leq\left|R_{i}\right| / 8 d_{i}$; thus, we can indeed call PeriodicMatches in this case. Further, note that we have placed the marks as in the proof of Lemma 5.24, in particular, by Claim 5.26, any block $[j k \ldots(j+1) k]$ that contains any position $\pi \in \operatorname{Occ}_{k}^{E}(P, T)$ has at least $m_{R}-m / 4$ marks. As we verify every possible candidate using Verify from Lemma 6.2 we report no false positives, and thus the algorithm is correct.

For the running time in the PILLAR model, observe that during the marking step, for every repetitive region $R_{i}$ we call PeriodicMatches once. In total, all calls to PeriodicMatches take

$$
\sum_{i} \mathcal{O}\left(n /\left|R_{i}\right| \cdot d_{i}^{4}\right)=\sum_{i} \mathcal{O}\left(\left|R_{i}\right| / m \cdot k^{4}\right)=\mathcal{O}\left(k^{4}\right)
$$

time in the PILLAR model. Further, for every $R_{i}$, we place at most $\mathcal{O}\left(\left\|\operatorname{Occ}_{k_{i}}^{E}\left(R_{i}, T\right) / k\right\| \mid\right)$ (weighted) marks, which can be bounded by $\mathcal{O}\left(\left|\left\lfloor\operatorname{Occ}_{k_{i}}^{E}\left(R_{i}, T\right) / k\right\rfloor\right|\right)=\mathcal{O}\left(n /\left|R_{i}\right| \cdot d_{i}\right)=\mathcal{O}(k)$ using Corollary 5.18 Thus, we place $|M|=\mathcal{O}\left(k^{2}\right)$ (weighted) marks in total. Hence, the marking step in total takes $\mathcal{O}\left(k^{4}\right)$ time in the PILLAR model.

As the multi-set $M$ contains at most $\mathcal{O}\left(k^{2}\right)$ (weighted) marks, we can sort $M$ (by positions) in time $\mathcal{O}\left(k^{2} \log \log k\right)$; afterwards, we can find the elements with total weight at least $m_{R}-m / 4$ via a linear scan over $M$ in time $\mathcal{O}\left(k^{2}\right)$. As there are (by Claims 5.25 and 5.26) at most $O(k)$ blocks with at least $m_{R}-m / 4$ marks, we call Verify at most $\mathcal{O}(k)$ times. As we call verify always on a whole block of length $k$ at once, each call to Verify takes $\mathcal{O}\left(k^{2}\right)$ time in the PILLAR model. Hence, the verification step in total takes $\mathcal{O}\left(k^{3}\right)$ time in the PILLAR model.

In total, Algorithm 17 thus takes time $\mathcal{O}\left(k^{4}\right)$ in the PILLAR model.

```
Algorithm 18 Computing \(k\)-error occurrences in the PILLAR model.
EditOccurrences \((P, T, k)\)
    \(\left(B_{1}, \ldots, B_{2 k}\right.\) or \(\left(R_{1}, Q_{1}\right), \ldots,\left(R_{r}, Q_{r}\right)\) or \(\left.Q\right) \leftarrow \operatorname{Analyze}(P, k)\);
    \(\operatorname{Occ}_{k}^{E}(P, T) \leftarrow\{ \} ;\)
    if approximate period \(Q\) exists then
        return PeriodicMatches ( \(P, T, k, 8 k, Q\) );
    for \(i \leftarrow 0\) to \(\lfloor 2 n / m\rfloor-1\) do
        \(T_{i} \leftarrow T[\lfloor i \cdot m / 2\rfloor \ldots \min \{n,\lfloor(i+3) \cdot m / 2\rfloor-1+k\}) ;\)
        if breaks \(B_{1}, \ldots, B_{2 k}\) exist then
                \(\operatorname{Occ}_{k}^{E}\left(P, T_{i}\right) \leftarrow \operatorname{BreakMatches}\left(P, T_{i},\left\{B_{1}, \ldots, B_{2 k}\right\}, k\right) ;\)
            else if repetitive regions \(\left(R_{1}, Q_{1}\right), \ldots,\left(R_{r}, Q_{r}\right)\) exist then
                \(\operatorname{Occ}_{k}^{E}\left(P, T_{i}\right) \leftarrow \operatorname{RepetitiveMatches}\left(P, T_{i},\left\{\left(R_{1}, Q_{1}\right), \ldots,\left(R_{r}, Q_{r}\right)\right\}, k\right) ;\)
            if \(i<\lfloor 2 n / m\rfloor-1\) then
                \(V \leftarrow\left\{\ell+\lfloor i \cdot m / 2\rfloor \mid \ell \in \operatorname{Occ}_{k}^{E}\left(P, T_{i}\right)\right\} \cap[\lfloor i \cdot m / 2\rfloor . .\lfloor(i+1) \cdot m / 2\rfloor) ;\)
            \(\operatorname{Occ}_{k}^{E}(P, T) \leftarrow \operatorname{Occ}_{k}^{E}(P, T) \cup V ;\)
    return \(\operatorname{Occ}_{k}^{E}(P, T)\);
```


### 6.5 A PILLAR Model Algorithm for Pattern Matching with Edits

Finally, we are ready to prove Main Theorem 9
$\Gamma$
Main Theorem 9. Given a pattern $P$ of length $m$, a text $T$ of length $n$, and a positive integer $k \leq m$, we can compute (a representation of) the set $\operatorname{Occ}_{k}^{E}(P, T)$ using $\mathcal{O}\left(n / m \cdot k^{4}\right)$ time in the PILLAR model.

Proof. We proceed, as in Main Theorem 8 by separately considering each of the three possible outcomes of Analyze $(P, k)$. Consider Algorithm 18 for a visualization of the whole algorithm as pseudo-code.

If there is an approximate period $Q$ of $P$ we call PeriodicMatches (from Lemma 6.11). Else, for each of the $\lfloor 2 n / m\rfloor$ blocks $T_{0}, \ldots, T_{\lfloor 2 n / m\rfloor-1}$, where $T_{i}:=T[\lfloor i \cdot m / 2\rfloor \ldots \min \{n,\lfloor(i+3) \cdot m / 2\rfloor+k-1\}$ ), we call BreakMatches (from Lemma 6.12) or RepetitiveMatches (from Lemma 6.13), depending on the case we are in, and add the computed occurrences in $\operatorname{Occ}_{k}^{E}(P, T)$.

The correctness in the approximately periodic case follows from Lemma 6.11 and the fact that we can indeed call PeriodicMatches since, due to Lemma 6.4 string $Q$ satisfies $\delta_{E}\left(P,{ }^{*} Q^{*}\right) \leq 8 k$ and $|Q| \leq m / 128 k \leq m /(8 \cdot 8 k)$. In the other cases, first observe that each length- $(m+k)$ fragment of $T$ is contained in at least one of the fragments $T_{i}$ and hence we do not lose any occurrences. Second, by Lemma 6.4 and due to $\left|T_{i}\right| \leq \frac{3}{2} m+k$, the parameters in the calls to BreakMatches (from Lemma 6.12) and RepetitiveMatches (from Lemma 6.13) each satisfy the requirements. Finally, the intersection step in Line 13 of Algorithm 18 guarantees that we account for each $k$-error occurrence exactly once.

For the running time in the PILLAR model, we have that the call to Analyze takes $\mathcal{O}\left(k^{2}\right)$ time in the PILLAR model, the call to PeriodicMatches takes $\mathcal{O}\left(n / m \cdot k^{4}\right)$ time in the PILLAR model, each call to BreakMatches takes $\mathcal{O}\left(k^{3}\right)$ time in the PILLAR model, and each call to RepetitiveMatches takes $\mathcal{O}\left(k^{4}\right)$ time in the PILLAR model. Finally, as the output of our calls to BreakMatches and RepetitiveMatches is of size $\mathcal{O}\left(k^{2}\right)$ and is sorted, Lines 13 and 14 require $\mathcal{O}\left(k^{2}\right)$ time. As there are at most $\mathcal{O}(n / m)$ calls to BreakMatches and RepetitiveMatches, we can bound the total time in the PILLAR model by $\mathcal{O}\left(n / m \cdot k^{4}\right)$, completing the proof.

## Faster Approximate Pattern Matching：A Unified Approach

## 7 Implementing the PILLAR Model：Faster Approximate Pattern Matching

In this section we implement the PILLAR model in the static，fully compressed and dynamic settings， thereby lifting Main Theorems 8 and 9 to these settings．For each setting，we first show how to implement each of the following primitive PILLAR operations：
－Extract $(S, \ell, r)$ ：Retrieve a string $S[\ell . . r]$ ．
－ $\operatorname{LCP}(S, T):$ Compute the length of the longest common prefix of $S$ and $T$ ．
$=\operatorname{IPM}(P, T)$ ：Assuming that $|T| \leq 2|P|$ ，compute $\operatorname{Occ}(P, T)$（represented as an arithmetic progression with difference $\operatorname{per}(P)$ ）．
－ $\operatorname{Access}(S, i)$ ：Retrieve the character $S[i]$ ．
＝Length $(S)$ ：Compute the length $|S|$ of the string $S$ ．
Operation $\operatorname{LCP}^{R}(S, T)$ can be implemented analogously to $\operatorname{LCP}(S, T)$ by reversing all the strings in scope． We then apply our main algorithmic results（Main Theorems 8 and 9 ）in order to obtain efficient algorithms for approximate pattern matching．

## 7．1 Implementing the PILLAR Model in the Standard Setting

We start with the implementation of the PILLAR model in the standard setting．This turns out to be a straightforward application of known tools for strings．

Recall that in the PILLAR model，we are to maintain a collection $\mathcal{X}$ of strings．Let us denote the total length of all strings in $\mathcal{X}$ by $n$ ．In the standard setting，a handle to $S=X[\ell . . r)$ is implemented as a pointer to $X \in \mathcal{X}$（recall that $X$ is stored explicitly）along with the indices $\ell$ and $r$ ．Hence，the implementations of Extract，Access，and Length are trivial．

Next， $\operatorname{LCP}(S, T)$ queries can by implemented efficiently as follows．We construct the generalized suffix tree for the elements of $\mathcal{X}$ in $\mathcal{O}(n)$ time［18 and preprocess it within the same time complexity so that we can support $\mathcal{O}(1)$－time lowest common ancestor queries［7］．

As for efficiently answering $\operatorname{IPM}(P, T)$ queries，we build the data structure of Kociumaka et al．32，31， encapsulated in the following theorem，for the concatenation of the elements of $\mathcal{X}$ ．

## 『 Lemma 7.1 （［32［31］）．For every string $S$ of length n，there is a data structure of size $\mathcal{O}(n)$ ，which can

 be constructed in $\mathcal{O}(n)$ time and answers $\operatorname{IPM}(P, T)$ queries in $\mathcal{O}(1)$ time for fragments $P$ and $T$ of $S$ ．』We summarize the above discussion in the following theorem．
『 Theorem 7．2．After an $\mathcal{O}(n)$－time preprocessing of a collection of strings of total length $n$ ，each PILLAR operation can be performed in $\mathcal{O}(1)$ time．

This yields the following theorems，which are not new，but as a warm－up，they are perhaps somewhat instructive．

Combining Theorem 7.2 and Main Theorem 8 ，we obtain an algorithm for pattern matching with mismatches with the same running time as the algorithm of Clifford et al．［15，which is essentially optimal when $k=\mathcal{O}(\sqrt{m}){ }^{\text {c }}$

『 Theorem 7．3．Given a text $T$ of length $n$ ，a pattern $P$ of length $m$ and a threshold $k$ ，we can compute the set $\operatorname{Occ}_{k}^{H}(P, T)$ in time $\mathcal{O}\left(n+n / m \cdot k^{2} \log \log k\right)$ ．

[^2]Similarly, combining Theorem 7.2 and Main Theorem 9 we obtain an algorithm for pattern matching with edits that is, again, not slower than the known algorithm [17].

「 Theorem 7.4. Given a text $T$ of length $n$, a pattern $P$ of length $m$ and a threshold $k$, we can compute the set $\operatorname{Occ}_{k}^{E}(P, T)$ in time $\mathcal{O}\left(n+n / m \cdot k^{4}\right)$.

『 Remark 7.5. The discussion of this subsection implies that our algorithms also apply to the internal setting. That is, a string $S$ of length $n$ can be preprocessed in $\mathcal{O}(n)$ time, so that given fragments $P$ and $T$ of $S$, and a threshold $k$, we can compute $\operatorname{Occ}_{k}^{H}(P, T)$ in time $\mathcal{O}\left(|T| /|P| \cdot k^{2} \log \log k\right)$ and $\operatorname{Occ}_{k}^{E}(P, T)$ in time $\mathcal{O}\left(|T| /|P| \cdot k^{4}\right)$.

### 7.2 Implementing the PILLAR Model in the Fully Compressed Setting

Next, we focus on the fully compressed setting, where we want to solve approximate pattern matching when both the text and the pattern are given as a straight-line programs - that is, in this setting, we maintain a collection $\mathcal{X}$ of straight-line programs and show how to implement the primitive PILLAR operations on this collection. We start with a short exposition on straight-line programs and related concepts.

## Straight-Line Programs

We denote the set of non-terminals of a context-free grammar $\mathcal{G}$ by $N_{\mathcal{G}}$ and call the elements of $\mathcal{S}_{G}=N_{\mathcal{G}} \cup \Sigma$ symbols. Then, a straight line program $(S L P) \mathcal{G}$ is a context-free grammar that consists of a set $N_{\mathcal{G}}=\left\{A_{1}, \ldots, A_{n}\right\}$ of non-terminals, such that each $A \in N_{\mathcal{G}}$ is associated with a unique production rule $A \rightarrow f_{\mathcal{G}}(A)$, where $f_{\mathcal{G}}(A) \in \mathcal{S}_{\mathcal{G}}^{*}$. For SLPs given as input, we can assume without loss of generality that each production rule is of the form $A \rightarrow B C$ for some symbols $B$ and $C$ (that is, the given SLP is in Chomsky normal form).

Every symbol $A \in \mathcal{S}_{\mathcal{G}}$ generates a unique string, which we denote by gen $(A) \in \Sigma^{*}$. The string gen $(A)$ can be obtained from $A$ by repeatedly replacing each non-terminal by its production. In addition, $A$ is associated with its parse tree $\mathrm{PT}(A)$ consisting of a root labeled with $A$ to which zero or more subtrees are attached:

- If $A$ is a terminal, there are no subtrees.
- If $A$ is a non-terminal $A \rightarrow B_{1} \cdots B_{p}$, then PT $\left(B_{i}\right)$ are attached in increasing order of $i$.

Note that if we traverse the leaves of $\mathrm{PT}(A)$ from left to right, spelling out the corresponding non-terminals, then we obtain gen $(A)$.

The parse tree $\mathrm{PT}_{\mathcal{G}}$ of $\mathcal{G}$ is the parse tree of the starting symbol $A_{n} \in N_{\mathcal{G}}$; gen $\left(A_{n}\right)=S$, where $S$ is the unique string generated by $\mathcal{G}$. We write $\operatorname{gen}(\mathcal{G}):=S$. Finally, an SLP can be represented naturally as a directed acyclic graph $H_{\mathcal{G}}$ of size $\left|\mathcal{S}_{\mathcal{G}}\right|$. Consult Figure 7 for an example of an SLP, its parse tree, and the corresponding acyclic graph.

We define the value $\operatorname{val}(v)$ of a node $v$ in $\mathrm{PT}_{\mathcal{G}}$ to be the fragment $S[a \ldots b]$ corresponding to the leaves $S[a], \ldots, S[b]$ in the subtree of $v$. Note that $\operatorname{val}(v)$ is an occurrence of $\operatorname{gen}(A)$ in gen $(\mathcal{G})$, where $A$ is the label of $v$. A sequence of nodes in $\mathrm{PT}_{\mathcal{G}}$ is a chain if their values are consecutive fragments in $T$.

Given an SLP $\mathcal{G}$ of size $n$, we can compute $|\operatorname{gen}(\mathcal{G})|$ in $\mathcal{O}(n)$ time using dynamic programming. We compute the topological order of $H_{\mathcal{G}}$ and process the nodes in the reverse order: For each node corresponding to a non-terminal $A$ with production rule $A \rightarrow B C$, we just need to compute $|\operatorname{gen}(B)|+|\operatorname{gen}(C)|$.

Bille et al. 8] have shown that we can efficiently access any character in gen $(\mathcal{G})$.

$$
\begin{aligned}
A_{5} & \rightarrow A_{4} A_{4} \\
A_{4} & \rightarrow A_{1} A_{3} \\
A_{3} & \rightarrow A_{1} A_{2} \\
A_{2} & \rightarrow \mathrm{~b} \\
A_{1} & \rightarrow \mathrm{a}
\end{aligned}
$$

(b)


(c)

Figure 7 (a) An SLP $\mathcal{G}$ generating aabaab. (b) The corresponding parse tree $\mathrm{PT}_{\mathcal{G}}$. (c) The corresponding directed acyclic graph $H_{\mathcal{G}}$.

Theorem 7.6 ([8]). An SLP $\mathcal{G}$ of size $n$, generating a string $S$ of size $N$, can be preprocessed in time $\mathcal{O}(n \log (N / n))$ so that, for any $i \in[0 \ldots N)$, we can access $\operatorname{gen}(\mathcal{G})[i]$ in $\mathcal{O}(\log N)$ time.

I in [26] presented an efficient data structure for answering longest common prefix queries for suffixes of a string given by an SLP, which we encapsulate in the following theorem. This data structure is based on the recompression technique, which we discuss in the next subsection in more detail.

Theorem 7.7 ([26]). An $S L P \mathcal{G}$ of size n, generating a string $S$ of size $N$, can be preprocessed in time $\mathcal{O}(n \log (N / n))$ so that for any $i$ and $j$, we can compute $\operatorname{LCP}(S[i \ldots N), S[j \ldots N))$ in $\mathcal{O}(\log N)$ time. 』

Finally, we discuss how to "concatenate" two SLPs. Given two SLPs $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$, with gen $\left(\mathcal{G}_{1}\right)=S_{1}$ and $\operatorname{gen}\left(\mathcal{G}_{2}\right)=S_{2}$, we can construct an SLP generating $S_{1} S_{2}$ in $\mathcal{O}\left(\left|\mathcal{G}_{1}\right|+\left|\mathcal{G}_{2}\right|\right)$ time as follows. We first rename the non-terminals in $N_{\mathcal{G}_{2}}$ to make sure that they are disjoint from the non-terminals in $N_{\mathcal{G}_{1}}$. Next, let $R_{1}$ and $R_{2}$ denote the starting non-terminals of $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$, respectively. We construct a new SLP $\mathcal{G}$ with $N_{\mathcal{G}}:=N_{\mathcal{G}_{1}} \cup N_{\mathcal{G}_{2}} \cup\{R\}$, where $R$ has production rule $R \rightarrow R_{1} R_{2}$. Note that this procedure can be applied to more than two strings: We first apply a global renaming, and then repeatedly "concatenate" two strings in the collection.

Let us now denote the total size of all SLPs in $\mathcal{X}$ by $n$, and the total length of all strings generated by those SLPs by $N$. We implement the handle of a fragment $S=X[\ell \ldots r)$ so that it consists of a pointer to the SLP $\mathcal{G} \in \mathcal{X}$ generating $X$ along with the positions $\ell$ and $r$; this makes Extract and Length trivial.

The above discussion on computing the length of the string generated by an SLP implies that the handles to $\operatorname{gen}(\mathcal{G})[0 \ldots|\operatorname{gen}(\mathcal{G})|)$ for all $\mathcal{G} \in \mathcal{X}$ can be constructed in $\mathcal{O}(n)$ time. Moreover, Theorem 7.6 implies that we can preprocess $\mathcal{X}$ in $\mathcal{O}(n \log (N / n))$ time so that operation Access requires $\mathcal{O}(\log N)$ time.

For efficiently answering LCP queries, we rely on Theorem 7.7. We build I's data structure for an SLP that generates the concatenation of all elements in the multi-set $\{\operatorname{gen}(\mathcal{G}) \mid \mathcal{G} \in \mathcal{X}\}$. Thus, after an $\mathcal{O}(n \log (N / n))$-time preprocessing, each LCP operation takes $\mathcal{O}(\log N)$ time.

To implement the IPM operation efficiently, we rely on the recompression technique due to Jeż 27, 28], which we discuss next.

## Recompression of Straight-Line Programs

We start with some additional notation. A run-length straight line program ( $R L S L P$ ) is a straight-line program $\mathcal{G}$ that contains two kinds of non-terminals:
－Concatenations：Non－terminals with production rules of the form $A \rightarrow B C$（for symbols $B$ and $C$ ）．
－Powers：Non－terminals with production rules of the form $A \rightarrow B^{p}$（for a symbol $B$ and an integer $p \geq 2$ ）．

The key idea of the recompression technique by Jez［27，［28］is the construction of a particular RLSLP $\mathcal{R}$ that generates the input string $S$ ．The parse tree $\mathrm{PT}_{\mathcal{R}}$ is of depth $\mathcal{O}(\log N)$ and it can be traversed efficiently based on the underlying directed acyclic graph $H_{\mathcal{R}}$ ．In particular，the name of the technique stems from the fact that an SLP $\mathcal{G}$ of size $n$ generating a string $S$ of length $N$ can be efficiently recompressed to the $\operatorname{RLSLP} \mathcal{R}(\mathcal{G})$ in－place，that is，without first uncompressing $\mathcal{G}$ ；efficiently here means in $\mathcal{O}(n \log N)$ time．As observed by I［26］，the parse tree $\mathrm{PT}_{\mathcal{R}(\mathcal{G})}$ is locally consistent in a certain sense．To formalize this property，he introduced the popped sequence of every fragment $S[a . . b]$ ，which is a sequence of symbols labeling a certain chain of nodes of $\mathrm{PT}_{\mathcal{R}(\mathcal{G})}$ ，whose values constitute $S[a \ldots b]$ ．
「 Theorem 7.8 （［26］）．If two fragments match，then their popped sequences are equal．Moreover，each popped sequence consists of $\mathcal{O}(\log N)$ runs（maximal powers of a single symbol）and can be constructed in $\mathcal{O}(\log N)$ time．The nodes corresponding to symbols in a run share a single parent．Furthermore，the popped sequence consists of a single symbol only for fragments of length 1 ．

Let $F_{1}^{p_{1}} \cdots F_{t}^{p_{t}}$ denote the run－length encoding of the popped sequence of a substring $U$ of $S$ and set

$$
L(U):=\left\{\left|\operatorname{gen}\left(F_{1}\right)\right|,\left|\operatorname{gen}\left(F_{1}^{p_{1}}\right)\right|,\left|\operatorname{gen}\left(F_{1}^{p_{1}} F_{2}^{p_{2}}\right)\right|, \ldots,\left|\operatorname{gen}\left(F_{1}^{p_{1}} \cdots F_{t-1}^{p_{t-1}}\right)\right|,\left|\operatorname{gen}\left(F_{1}^{p_{1}} \cdots F_{t-1}^{p_{t-1}} F_{t}^{p_{t}-1}\right)\right|\right\} .
$$

By Theorem 7.8 the set $L(U)$ can be constructed in $\mathcal{O}(\log N)$ time given a fragment $S[a \ldots b]=U$ ．Now， the following lemma from［14］allows us to efficiently implement internal pattern matching queries．

『 Lemma 7.9 （［14］）．Let $v$ denote a non－leaf node of $\mathrm{PT}_{\mathcal{R}(\mathcal{G})}$ and let $S[a . . b]$ denote an occurrence of $S$ that is contained in val（ $v$ ），but not contained in val（ $u$ ）for any child $u$ of $v$ ．If $S[a \ldots c]$ is the longest prefix of $S[a \ldots b]$ contained in val $(u)$ for a child $u$ of $v$ ，then $|S[a \ldots c]| \in L(U)$ ．Symmetrically，if $S\left[c^{\prime}+1 . . . b\right]$ is the longest suffix of $S[a . . b]$ contained in val（ $u$ ）for a child $u$ of $v$ ，then $\left|S\left[a \ldots c^{\prime}\right]\right| \in L(U)$ ．

Finally，we discuss how to implement the IPM operation．
『 Lemma 7．10．Given $H_{\mathcal{R}(\mathcal{G})}$ ，a fragment $T=S[j \ldots j+\nu)$ ，and a fragment $P=S[i \ldots i+\mu)$ with $|T| \leq 2|P|$ ，we can compute $\operatorname{Occ}(P, T)$ in the time required by $\mathcal{O}\left(\log ^{2} N\right) \operatorname{LCP}$ and $\mathrm{LCP}^{R}$ operations on fragments of $S$ ．

In particular，the LCP and $\mathrm{LCP}^{R}$ operations are between pairs consisting in one of $\mathcal{O}(\log N)$ fragments of $P$ and one of $\mathcal{O}(\log N)$ fragments of $S$ ．
Proof．We can assume that $|T| \geq|P|>1$ ；otherwise it suffices to perform a constant number of letter comparisons which can be done in $\mathcal{O}(\log N)$ time．We first compute the popped sequence of $P$ and $L(P)$ using Theorem 7．8．Let $v \in \mathrm{PT}_{\mathcal{R}(\mathcal{G})}$ denote the lowest common ancestor of the leaves representing $S[j]$ and $S[j+\nu-1]$ ；the node $v$ can be naively computed in $\mathcal{O}(\log N)$ time by a forward search from the root of the parse tree．As $T$ is a fragment of $\operatorname{val}(v)$ ，all occurrences of $P$ in $T$ are also contained in val $(v)$ ． Our first aim is to compute all occurrences of $P$ in $T$ that are not contained in val $(u)$ for any child $u$ of $v$ ； we then appropriately recurse on the children of $v$ that may contain sought occurrences．

Let us first analyze the case that the label of $v$ is a concatenation symbol $A \rightarrow B C$ ．Write $v_{\ell}$ for the left child of $v$ and $v_{r}$ for the right child of $v$ ．Further，let $T_{\ell}=S[j \ldots t)$ denote the longest prefix of $T$ that is completely contained in val $\left(v_{\ell}\right)$ and，similarly，let $T_{r}=S[t \ldots j+\nu)$ denote the longest suffix of $T$ that is completely contained in val $\left(v_{r}\right)$ ．Suppose that there is a fragment $U=T[a . . b]$ of $T$ that equals $P$ and overlaps with both val $\left(v_{\ell}\right)$ and $\operatorname{val}\left(v_{r}\right)$ ．The fragment $U$ can then be naturally decomposed

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into a non-empty suffix $U_{\ell}$ of $T_{\ell}$ and a non-empty prefix $U_{r}$ of $T_{r}$. Lemma 7.9 implies that $\left|U_{\ell}\right| \in L(P)$. It thus suffices to check for each $q \in L(P)$ whether $P[0 \ldots q)$ is a suffix of $T_{\ell}$ and $P[q \ldots m)$ is a prefix of $T_{r}$. There are $|L(P)|=\mathcal{O}(\log N)$ choices for $q$, and for each of them we can perform the check using $\operatorname{LCP}^{R}\left(P[0 \ldots q), T_{\ell}\right)$ and $\operatorname{LCP}\left(P[q \ldots m), T_{r}\right)$ operations.

We now consider the case that the label of $v$ is a power symbol $A \rightarrow B^{p}$ and denote the children of $v$ in the left-to-right order by $v_{1}, \ldots, v_{p}$. Let $T[x \ldots y)$ denote the overlap of $T$ with val $(v)$.

If $T$ overlaps with $\operatorname{val}\left(v_{d}\right)$ for two children $v_{d}$ of $v$, then we can process $v$ as in the previous case. In the case that $T$ overlaps with exactly three children of $v$, some care is needed to avoid double-counting occurrences that overlap with all of them. In particular, let these three children be $v_{x}, v_{x+1}$, and $v_{x+2}$. We consider separately:

- occurrences that overlap with both $\operatorname{val}\left(v_{x}\right)$ and $\operatorname{val}\left(v_{x+1}\right)$ by setting $T_{\ell}$ to be the longest prefix of $T$ that is completely contained in $\operatorname{val}\left(v_{x}\right)$ and $T_{r}$ to be the longest suffix of $T$ that is completely contained in $\operatorname{val}\left(v_{x+1}\right) \operatorname{val}\left(v_{x+2}\right)$, and
- occurrences that overlap with both $\operatorname{val}\left(v_{x+1}\right)$ and $\operatorname{val}\left(v_{x+2}\right)$, but not with val $\left(v_{x}\right)$, by setting $T_{\ell}$ to be $\operatorname{val}\left(v_{x+1}\right)$ and $T_{r}=S[t \ldots j+\nu)$ to be the longest suffix of $T$ that is completely contained in $\operatorname{val}\left(v_{x+2}\right)$.

We can thus assume that $T$ overlaps with $\operatorname{val}\left(v_{d}\right)$ for more than three children $v_{d}$ of $v$. In that case, for all $d$, we have $\operatorname{val}\left(v_{d}\right)<\mu$ and hence no occurrence of $P$ in $T[x \ldots y)$ can be completely contained in $\operatorname{val}\left(v_{d}\right)$. We set $T_{\ell}:=\operatorname{val}\left(v_{1}\right)$ and $T_{r}:=\operatorname{val}\left(v_{2}\right) \cdots \operatorname{val}\left(v_{p}\right)$. Using $|L(P)|$ many LCP operations and $|L(P)|$ many LCP ${ }^{R}$ operations, we can compute the set $Y$ of occurrences of $P$ in $S$ (but not necessarily in $T$ ) which can be decomposed into a prefix $U_{\ell}$ that is a suffix of $T_{\ell}$ and a suffix $U_{r}$ that is a prefix of $T_{r}$.

Then, by the periodicity of $\operatorname{val}(v)=\operatorname{gen}(B)^{p}$, the desired set of occurrences is

$$
Z:=\{i+j \cdot|\operatorname{gen}(B)|: i \in Y, j \in[0 \ldots p-1]\} \cap[x \ldots y-\mu]
$$

Note that $Z$ trivially decomposes into $\mathcal{O}(\log N)$ arithmetic progressions; these arithmetic progressions can be replaced by a single arithmetic progression with difference $\operatorname{per}(P)$ in $\mathcal{O}(\log N)$ time.

If the overlap of the value of each child of $v$ with $T$ has length less than $\mu$, we terminate the algorithm. Otherwise, if $v$ has two children whose values have an overlap of length $\mu$ with $T$, we check whether either of them is equal to $P$ using a single LCP operation and terminate the algorithm. Finally, at most one of $v$ 's children has length greater than $\mu$. In that case, we repeat the above procedure for this child. As the depth of $\mathrm{PT}_{\mathcal{R}(\mathcal{G})}$ is $\mathcal{O}(\log N)$, the overall running time is upper bounded by the time required for $\mathcal{O}\left(\log ^{2} N\right)$ LCP and LCP ${ }^{R}$ operations.

In the end, we have $\operatorname{Occ}(P, T)$ represented by at most one arithmetic progression and $\mathcal{O}\left(\log ^{2} N\right)$ single occurrences. We postprocess this representation in $\mathcal{O}\left(\log ^{2} N\right)$ time, in order to represent $\operatorname{Occ}(P, T)$ with a single arithmetic progression with difference per $(P)$.

Note that, in each level, the LCP (resp. LCP ${ }^{R}$ ) queries we perform are between $\mathcal{O}(1)$ fragments of $S$ and each $P[0 \ldots q)($ resp. $P[q \ldots m)$ ) for $q \in L(P)$. This observation implies the last claim of the statement of this lemma, concluding its proof.

『 Remark 7.11. Essentially the same proof of the above lemma has recently appeared in [29].
We can get an $\mathcal{O}\left(\log ^{3} N\right)$-time implementation of IPM by employing Lemma 7.10 and answering each LCP or LCP ${ }^{R}$ query in $\mathcal{O}(\log N)$ time. However, the structure of the queries allows for a more efficient implementation.
「 Lemma 7.12. We can preprocess an $S L P \mathcal{G}$ of size $n$, generating a string $S$ of length $N$, in $\mathcal{O}(n \log N)$ time, so that $\operatorname{IPM}(P, T)$ queries for fragments $P$ and $T$ of $S$ can be answered in $\mathcal{O}\left(\log ^{2} N \log \log N\right)$ time.

Proof. We first build, in $\mathcal{O}(n \log (N / n))$ time, the data structures of Theorems 7.6 and 7.7 for performing Access, LCP, and LCP ${ }^{R}$ operations in $\mathcal{O}(\log N)$ time. Then, we recompress $\mathcal{G}$ to an $\operatorname{RLSLP} \mathcal{R}(\mathcal{G})$ in $\mathcal{O}(n \log N)$ time.

Lemma 7.10 then reduces the task at hand to answering $\mathcal{O}\left(\log ^{2} N\right)$ LCP and LCP ${ }^{R}$ queries between pairs consisting in one of $\mathcal{O}(\log N)$ fragments of $P$ and one of $\mathcal{O}(\log N)$ fragments of $S$. Let us focus on efficiently answering all such LCP queries; LCP ${ }^{R}$ queries can be answered analogously.

We sort all fragments in scope using $\mathcal{O}(\log N \log \log N)$ comparisons, implementing each comparison in $\mathcal{O}(\log N)$ time using an LCP operation, followed by two Access operations. This step thus takes $\mathcal{O}\left(\log ^{2} N \log \log N\right)$ time. Then, we construct an array $A$ of $\operatorname{size} \mathcal{O}(\log N)$ such that $A[i]$ stores the length of the longest common prefix of the $i$-th and the $(i+1)$-st elements in our sorted list. After preprocessing array $A$ in $\mathcal{O}(\log N)$ time so that range minimum queries over it can be answered in constant time [7], we answer each of the $\mathcal{O}\left(\log ^{2} N\right)$ LCP queries in $\mathcal{O}(1)$ time.

In total, we have thus proved the following result.
『 Theorem 7.13. Given a collection of SLPs of total size n, generating strings of total length N, each PILLAR operation can be performed in $\mathcal{O}\left(\log ^{2} N \log \log N\right)$ time after an $\mathcal{O}(n \log N)$-time preprocessing. 」

In the next subsection (cf. Remark 7.16), we discuss how to perform each PILLAR operation in $\mathcal{O}\left(\log ^{2} N\right)$ after an $\mathcal{O}(n \log N)$-time preprocessing at the cost of randomization.

## Approximate Pattern Matching in Fully Compressed Strings

We are now ready to present efficient algorithms for approximate pattern matching in the fully compressed setting. We choose to state our results using our deterministic implementation of the PILLAR model, that is Theorem 7.13

We are given an SLP $\mathcal{G}_{T}$ of size $n$ with $T=\operatorname{gen}\left(\mathcal{G}_{T}\right)$, an SLP $\mathcal{G}_{P}$ of size $m$ with $P=\operatorname{gen}\left(\mathcal{G}_{T}\right)$ and a threshold $k$ and are required to compute the $k$-mismatch or $k$-error occurrences of $P$ in $T$.

Set $N:=|T|+|P|$ and $\mathcal{X}:=\left\{\mathcal{G}_{T}, \mathcal{G}_{P}\right\}$. The overall structure of our algorithm is as follows: We first preprocess the collection $\mathcal{X}$ in $\mathcal{O}((n+m) \log N)$ time according to Theorem 7.13 Next, we traverse $\mathcal{G}_{T}$ and compute for every non-terminal $A$ of $\mathcal{G}_{T}$ the approximate occurrences of $P$ in $T$ that "cross" $A$. Depending on the setting, we combine Theorem 7.13 with Main Theorem 8 or Main Theorem 9 to compute the occurrences. Finally, we combine the computed occurrences using dynamic programming.

Formally, for each non-terminal $A \in N_{\mathcal{G}_{T}}$, with production rule $A \rightarrow B C$, we wish to compute all approximate occurrences of $P$ in the string

$$
\operatorname{gen}(B)[|\operatorname{gen}(B)|-|P|+1 . .|\operatorname{gen}(B)|) \operatorname{gen}(C)[0 . .|P|-1),
$$

which is indeed a fragment of $\operatorname{gen}\left(\mathcal{G}_{T}\right)$ and is of length $2|P|-2$. These approximate occurrences can be computed in time:
$=\mathcal{O}\left(k^{2} \log ^{2} N \log \log N\right)$ in the Hamming distance case by combining Theorem 7.13 and Main Theorem 8 - $\mathcal{O}\left(k^{4} \log ^{2} N \log \log N\right)$ in the edit distance case, by combining Theorem 7.13 and Main Theorem 9 Other approximate occurrences in gen $(A)$ lie entirely in gen $(B)$ or gen $(C)$; hence they are computed when considering $B$ and $C$. (Compare [11, Theorem 4.1] for a similar algorithm.)

Now, the number of approximate occurrences of $P$ in $T$ (that is $\left|\mathrm{Occ}_{k}^{H}(P, T)\right|$ or $\left.\left|\mathrm{Occ}_{k}^{E}(P, T)\right|\right)$ can be computed by dynamic programming that is analogous to the dynamic programming used to compute the length of a string generated by an SLP. Further, all approximate occurrences can be reported in time proportional to their number by performing a traversal of $\mathrm{PT}_{\mathcal{G}}$, avoiding to explore subtrees that correspond to fragments of $T$ that do not contain any approximate occurrences.

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We hence obtain the following algorithm for pattern matching with mismatches in the fully compressed setting.
${ }^{-}$Main Theorem 1. Let $\mathcal{G}_{T}$ denote an SLP of size $n$ generating a text $T$, let $\mathcal{G}_{P}$ denote an SLP of size $m$ generating a pattern $P$, let $k>0$ denote an integer threshold, and set $N:=|T|+|P|$.

Then, we can compute $\left|\operatorname{Occ}_{k}^{H}(P, T)\right|$ in time $\mathcal{O}\left(m \log N+n k^{2} \log ^{2} N \log \log N\right)$. The elements of $\operatorname{Occ}_{k}^{H}(P, T)$ can be reported within $\mathcal{O}\left(\left|\operatorname{Occ}_{k}^{H}(P, T)\right|\right)$ extra time.

Similarly, we obtain the following algorithm for pattern matching with edits in the fully compressed setting.

## - Main Theorem 2. Let $\mathcal{G}_{T}$ denote an SLP of size $n$ generating a string $T$, let $\mathcal{G}_{P}$ denote an $S L P$

 of size $m$ generating a string $P$, let $k>0$ denote an integer threshold, and set $N:=|T|+|P|$.Then, we can compute $\left|\operatorname{Occ}_{k}^{E}(P, T)\right|$ in time $\mathcal{O}\left(m \log N+n k^{4} \log ^{2} N \log \log N\right)$. The elements of $\mathrm{Occ}_{k}^{E}(P, T)$ can be reported within $\mathcal{O}\left(\left|\operatorname{Occ}_{k}^{E}(P, T)\right|\right)$ extra time.

### 7.3 Implementing the PILLAR Model in the Dynamic Setting

Lastly, we consider the dynamic setting. In particular, we consider the dynamic maintenance of a collection of non-empty persistent strings $\mathcal{X}$ that is initially empty and undergoes updates specified by the following operations:

- makestring $(U)$ : Insert a non-empty string $U$ to $\mathcal{X}$.
- concat $(U, V):$ Insert $U V$ to $\mathcal{X}$, for $U, V \in \mathcal{X}$.
$=\operatorname{split}(U, i): \operatorname{Insert} U[0 \ldots i)$ and $U[i \ldots|U|)$ in $\mathcal{X}$, for $U \in \mathcal{X}$ and $i \in[0 \ldots|U|)$.
Let $N$ denote an upper bound on the total length of all strings in $\mathcal{X}$ throughout the execution of the algorithm. Gawrychowski et al. [23] presented a data structure that efficiently maintains such a collection and allows for efficient longest common prefix queries.

『 Theorem 7.14 ([23]). A collection $\mathcal{X}$ of non-empty persistent strings of total length $N$ can be dynamically maintained with update operations makestring $(U)$, concat $(U, V), \operatorname{split}(U, i)$ requiring time $\mathcal{O}(\log N+|U|), \mathcal{O}(\log N)$, and $\mathcal{O}(\log N)]^{\text {d }}$ respectively, so that $\operatorname{LCP}(U, V)$ queries for $U, V \in \mathcal{X}$ can be answered in time $\mathcal{O}(1)$.

LCP and LCP ${ }^{R}$ operations for arbitrary fragments of elements of $\mathcal{X}$ can be answered in $\mathcal{O}(\log N)$ time (w.h.p.) by first performing a constant number of split operations to add the corresponding fragments to the collection and then asking an LCP query between them.

The lengths of the strings in $\mathcal{X}$ can be maintained explicitly. Upon a makestring operation we naively compute the length of $U$, while upon a concat or a split operation, we can compute the lengths of the strings that are inserted in $\mathcal{X}$ in constant time from the arguments of the operation.

For each string of the collection $\mathcal{X}$, the data structure of [23] maintains an RLSLP stemming from recompression that is of depth $\mathcal{O}(\log N)$ w.h.p. Given a string $X \in \mathcal{X}$, a pointer to the root of the parse tree of $X$ can be retrieved in $\mathcal{O}(1)$ time.

Each Extract $(X, \ell, r)$ operation can be performed using at most two split operations in $\mathcal{O}(\log N)$ time w.h.p.

[^3]We now show that $\operatorname{Access}(X, i)$ for $X \in \mathcal{X}$ can be performed efficiently. Although the parse trees of the strings in $\mathcal{X}$ are not maintained explicitly, given a pointer to some node $v$ in the parse tree of $X$, we can retrieve in $\mathcal{O}(1)$ time the endpoints $a, b$ of the fragment $\operatorname{val}(v)=X[a \ldots b]$, the degree of $v$, a pointer to the parent of $v$, and a pointer to the $j$-th child of $v$, provided that such a child exists. Thus, an Access $(X, i)$ operation can be implemented in time proportional to the height of the parse tree, that is, in $\mathcal{O}(\log N)$ time w.h.p.

We are left with showing that IPM operations can be performed efficiently. Let us remark that the RLSLPs of all $X \in \mathcal{X}$ maintained by the data structure underlying Theorem 7.14 are locally consistent with each other, that is Theorem 7.8 is also true for fragments of different strings $X_{1}, X_{2} \in \mathcal{X}$. Thus, Lemmas 7.9 and 7.10 also hold in this setting. Combining Lemma 7.10 and Theorem 7.14 we get that $\operatorname{IPM}(P, T)$ queries can be answered in $\mathcal{O}\left(\log ^{2} N\right)$ time (w.h.p.) by performing $\mathcal{O}(\log N)$ split operations and $\mathcal{O}\left(\log ^{2} N\right)$ LCP queries (cf. the last statement of Lemma 7.10. Note that the only other component of the proof of Lemma 7.10 is a forward search from the root of the relevant parse tree, which can be efficiently performed given the available pointers.

We summarize the above discussion in the following theorem.
${ }^{-}$Theorem 7.15. A collection $\mathcal{X}$ of non-empty persistent strings of total length $N$ can be dynamically maintained with operations makestring $(U)$, concat $(U, V), \operatorname{split}(U, i)$ requiring time $\mathcal{O}(\log N+|U|)$, $\mathcal{O}(\log N)$ and $\mathcal{O}(\log N)$, respectively, so that PILLAR operations can be performed in time $\mathcal{O}\left(\log ^{2} N\right), e^{e}$

『 Remark 7.16. Given an SLP $\mathcal{G}$ of size $n$, generating a string $S$ of size $N$, we can efficiently implement the PILLAR operations through dynamic strings. Let us start with an empty collection $\mathcal{X}$ of dynamic strings. Using $\mathcal{O}(n)$ makestring $(a)$ operations, for $a \in \Sigma$, and $\mathcal{O}(n)$ concat operations (one for each non-terminal of $\mathcal{G})$, we can insert $S$ to $\mathcal{X}$ in $\mathcal{O}(n \log N)$ time w.h.p. Then, we can perform each PILLAR operation in $\mathcal{O}\left(\log ^{2} N\right)$ time w.h.p., due to Theorem 7.15 thus outperforming Theorem 7.13 at the cost of randomization.

Combining Theorem 7.15 and Main Theorems 8 and 9 we obtain an algorithm for approximate pattern matching for dynamic strings.

- Main Theorem 3. A collection $\mathcal{X}$ of non-empty persistent strings of total length $N$ can be maintained subject to makestring $(U)$, concat $(U, V)$, and $\operatorname{split}(U, i)$ operations requiring $\mathcal{O}(\log N+|U|), \mathcal{O}(\log N)$, and $\mathcal{O}(\log N)$ time, respectively, so that given two strings $P, T \in \mathcal{X}$ with $|P|=m$ and $|T|=n$ and an integer threshold $k>0$, we can compute $\left|\mathrm{Occ}_{k}^{H}(P, T)\right|$ in time $\mathcal{O}\left(n / m \cdot k^{2} \log ^{2} N\right)$ and $\left|\operatorname{Occ}_{k}^{E}(P, T)\right|$ in time $\mathcal{O}\left(n / m \cdot k^{4} \log ^{2} N\right) ป^{f}$ The elements of $\operatorname{Occ}_{k}^{H}(P, T)$ and $\operatorname{Occ}_{k}^{E}(P, T)$ can be reported in $\mathcal{O}\left(\left|\mathrm{Occ}_{k}^{H}(P, T)\right|\right)$ and $\mathcal{O}\left(\left|\mathrm{Occ}_{k}^{E}(P, T)\right|\right)$ extra time, respectively.

[^4]
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[^0]:    ${ }^{\text {a }}$ Strictly speaking, makestring $(U)$ costs $\mathcal{O}(|U|+\log N)$ time.
    ${ }^{\mathrm{b}}$ All running time bounds hold with high probability (i.e., $1-N^{\Omega(1)}$ ).

[^1]:    Lemma 6.10 (SynchedMatches $\left(P, T, I, d, d^{\prime}, k, Q\right)$ ). Let $P$ denote a pattern of length $m$, let

[^2]:    ${ }^{c}$ Strictly speaking，the algorithm in［15］runs in time $\mathcal{O}\left(n\right.$ polylog $\left.(m)+n / m \cdot k^{2} \log k\right)$ ，so our algorithm is slightly faster． However，an even better improvement in the logarithmic factors was already obtained recently in［13．

[^3]:    ${ }^{\mathrm{d}}$ These running times hold w.h.p.

[^4]:    ${ }^{e}$ All running time bounds hold w.h.p.
    ${ }^{f}$ All running time bounds hold with high probability (i.e., $1-N^{\Omega(1)}$ ).

