CENTER OF MASS AND KÄHLER STRUCTURES

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Abstract. There is a sequence of positive numbers δ_{2n}, such that for any connected $2n$-dimensional Riemannian manifold M, there are two mutually exclusive possibilities:

(1) There is a complex structure on M making it into a Kähler manifold.
(2) For any almost complex structure J compatible with the metric, at every point $p \in M$, there is a smooth loop γ at p such that

$$dist(J_p, hol_\gamma^{-1} J_p hol_\gamma) > \delta_{2n}.$$

1. Introduction

Kähler manifolds possess a tremendous amount of interesting structure, and therefore have several equivalent characterizations. It has been a focus of much research to determine conditions under which manifolds do (or do not) admit a Kähler structure. This short note shows that the holonomy action on the space of almost hermitian structures determines two mutually exclusive cases, according to whether there is a structure that is *nearly preserved* at some point, by proving that any manifold with a nearly preserved almost hermitian structure at some point in fact admits a Kähler structure. The novel idea is to use a center of mass argument, averaging a given almost complex structure at a point over the holonomy action, and parallel transporting the result to obtain a global Kähler structure.

Due to the plethora of topological implications for having a Kähler structure, one may deduce several interesting consequences that do not mention the Kähler condition at all. For example, we deduce by [DGMS] that any almost hermitian manifold which has a non-trivial Massey product (or more generally is not formal) also has a holonomy action on the space almost hermitian structures which is bounded away from the identity in a way that is precisely quantifiable.

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2. Main result

For a $2n$-dimensional real vector space V with an inner product g, let

$$\mathbb{J}(V,g) = \{J : V \to V |\, J^2 = -id,\, g\,(Ju, Jv) = g\,(u,v)\}$$

denote the space of metric almost complex structures on V.

Lemma 1. *$\mathbb{J}(V,g)$ is a compact smooth manifold. The tangent space at a point J is*

$$T_J\mathbb{J}(V,g) = \{\phi : V \to V \mid \phi J = -J\phi \text{ and } \phi^\dagger = -\phi\}$$

and has a bilinear form $\tilde{g}(\phi,\psi) = tr(\phi\psi^\dagger)$, for all $\phi,\psi \in T_J\mathbb{J}(V,g)$, where $\psi^\dagger$ denotes the adjoint of ψ. This makes $\mathbb{J}(V,g)$ into a Riemannian manifold on which $O(V)$ acts transitively by isometries.

Proof. $\mathbb{J}(V,g)$ can be identified as the $O(V)$-homogenous space $O(V)/U(V)$ where $U(V)$ is defined using any fixed J on V, and the action of $O(V)$ on $\mathbb{J}(V,g)$ is given by conjugation. Therefore $\mathbb{J}(V,g)$ is a smooth manifold. The induced action of $O(V)$ on $T_J\mathbb{J}(V,g)$ is also given by conjugation, so that trace is invariant. □

Lemma 2. *For any two metric vector spaces (V,g) and (W,h) of the same dimension, the Riemannian manifolds $\mathbb{J}(V,g)$ and $\mathbb{J}(W,h)$ are isometric. In particular, for a vector space V and two metrics g and g', the Riemannian manifolds $\mathbb{J}(V,g)$ and $\mathbb{J}(V,g')$ are isometric.*

Proof. The Gramm-Schmidt process ensures that there is a linear isometry $f : (V,g) \to (W,h)$. Conjugation by this isometry gives the desired Riemannian isometry. □

For an argument below, we will require convex balls in $\mathbb{J}(V,g)$ for which is there is a well defined notion of center of mass. For any Riemannian manifold M, balls of radius r are convex if

$$r = \min\left\{\frac{\operatorname{inj} M}{2}, \frac{\pi}{2\sqrt{\epsilon}}\right\},$$

where $inj\, M$ denotes the injectivity radius of M, and ϵ is a finite positive upper bound on the sectional curvature (c.f. [Ch] Prop IX.6.1). Also, for any such ϵ, we have the following theorem.

Theorem 3 (Karcher, [K])**.** *Let $f : X \to M$ be a measurable map from a probability space (X,m) to a Riemannian manifold M. If $f(X)$ is contained in a convex subset B of M, with diameter less than or equal to $\pi/2\sqrt{\epsilon}$, then there is a unique center of mass in B, defined by the minimum of*

$$E(y) = \frac{1}{2}\int_X d^2\,(f(x), y)\; m(x).$$

Additionally, the center of mass is natural with respect to isometries ([K], 1.4.1). We refer the reader to [Ch] Prop IX.7.1 for an exposition of center of mass.

Definition 4. *For $n > 1$ let*

$$\delta_{2n} = min\Big\{\frac{inj\ \mathbb{J}(V,g)}{2}, \frac{\pi}{4\sqrt{\epsilon}}\Big\}$$

where V is a real vector space of dimension $2n$ with any metric g. Here we choose, once and for all, a finite positive upper bound ϵ on the sectional curvature of $\mathbb{J}(V,g)$ at one point, which by homogeneity works for all points. By Lemma 2, δ_{2n} depends only on the dimension of V.

An interesting question (that we will not address here) is whether there is a positive lower bound for the set of all least such δ_{2n}, independent of n.

Definition 5. *A complex structure $J \in \mathbb{J}(V,g)$ is said to be* nearly preserved *by a subgroup $H \subset O(V)$ if the orbit $HJ = \{\phi^{-1}J\phi \mid \phi \in H\}$ lies inside the ball $B(J, \delta_{2n}) \subset \mathbb{J}(V,g)$ with center J and radius δ_{2n}.*

Recall that any closed subgroup H of $O(V)$ is a compact Lie group, admitting a bi-invariant Haar measure, which is unique up to a constant. Therefore any such H has a unique probability measure (of total mass equal to one).

Proposition 6. *Let $J \in \mathbb{J}(V,g)$ be nearly preserved by a closed subgroup H of $O(V)$. Then there is a $J' \in \mathbb{J}(V,g)$ such that $HJ' = \{J'\}$.*

Proof. Consider the orbit $HJ \subset B(J,\delta_{2n}) \subset \mathbb{J}(V,g)$. By assumption, $B(J,\delta_{2n})$ is a convex ball about J. Consider the mapping $H \to HJ \subset \mathbb{J}(V,g)$, from the probability space H onto its orbit. By Definition 4 and Theorem 3, the set HJ has a unique center of mass J' in S. Since the orbit of the action of H on the set HJ is itself, and the center of mass is unique and natural with respect to isometries given by the H-action, H fixes J'. □

Given a point $p \in M$, let $H_p = Im(\rho_p)$ be the image subgroup of the holonomy homomorphism $\rho_p : \Omega_p(M) \to O(T_pV)$, which is known to be closed by the Ambrose-Singer theorem. The following proposition is a standard result which we include for completeness.

Proposition 7. *Let (M,g) be a connected Riemannian manifold and assume $J_p \in \mathbb{J}(T_pM,g)$ is invariant under the action of $H_p = Im(\rho_p)$. Then M admits a unique almost complex structure $J : TM \to TM$ agreeing with J_p and making (M,g,J) into a Kähler manifold.*

Proof. For any $q \in M$ define $J_q : T_qM \to T_qM$ by $P_\lambda \circ J_p \circ P_\lambda^{-1}$, where $P_\lambda : T_pM \to T_qM$ is the Riemannian parallel transport along any smooth path λ in M from p to q. Since M is connected, such a path λ always exists and J_q is independent of the choice of the path because $H_pJ_p = \{J_p\}$. Thus, we have defined a smooth complex structure $J : TM \to TM$, and by way of construction, J is compatible with all parallel transports, and therefore $\nabla J = 0$. Since the Levi-Cevita connection ∇ is torsion free, and J is still Hermitian, J is integrable; see Corollary 3.5 of [KN]. Therefore, (M,g,J) is a Kähler manifold. □

Corollary 8. *With (M, g) as above, if there is a $p \in M$ and a $J_p \in \mathbb{J}(T_pM, g)$ that is nearly preserved by $H_p = Im(\rho_p)$, then there is a $J' : TM \to TM$ making (M, g, J') into a Kähler manifold.*

Proof. This follows from Proposition 6 and Proposition 7. □

Theorem 9. *There is a sequence of positive numbers $\delta_{2n} \in \mathbb{R}$ such that for any connected $2n$-dimensional Riemannian manifold M, one of the two following mutually exclusive properties hold:*

(1) *There is a complex structure on M making it into a Kähler manifold.*
(2) *For any almost complex structure J compatible with the metric, at every point $p \in M$, there is a smooth loop γ at p such that*

$$dist(J_p, hol_\gamma^{-1} J_p hol_\gamma) > \delta_{2n}.$$

Proof. If 1) is true then 2) is clearly false, and the converse follows from the previous corollary. □

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