Opuscula Math. 40, no. 6 (2020), 703–723
https://doi.org/10.7494/OpMath.2020.40.6.703 Opuscula Mathematica

ON THE TWISTED DORFMAN–COURANT LIKE BRACKETS

Włodzimierz M. Mikulski

Communicated by P.A. Cojuhari

Abstract. There are completely described all $\mathcal{VB}_{m,n}$-gauge-natural operators C which, like to the Dorfman–Courant bracket, send closed linear 3-forms $H \in \Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$ on a smooth ($\mathcal{C}^\infty$) vector bundle E into $\mathbf{R}$-bilinear operators

$$C_H : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

transforming pairs of linear sections of $TE \oplus T^*E \to E$ into linear sections of $TE \oplus T^*E \to E$. Then all such C which also, like to the twisted Dorfman–Courant bracket, satisfy both some "restricted" condition and the Jacobi identity in Leibniz form are extracted.

Keywords: natural operator, linear vector field, linear form, (twisted) Dorfman-Courant bracket, Jacobi identity in Leibniz form.

Mathematics Subject Classification: 53A55, 53A45, 53A99.

1. INTRODUCTION

All manifolds considered in the paper are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class $\mathcal{C}^\infty$). Maps between manifolds are assumed to be $\mathcal{C}^\infty$.

In [3], the authors described all bilinear operators on sections of the Whitney sum $TN \oplus T^*N \to N$ of the tangent and cotangent bundles (for N a smooth manifold), which are $\mathcal{M}f_m$-natural, i.e. invariant under the morphisms in the category $\mathcal{M}f_m$ of m-dimensional manifolds and their submersions. The Courant bracket is an example of such operators and it is of particular interest, because it involves in the concepts of Dirac and generalized complex structures on N, see [2, 4, 5].

In [9], we described all $\mathcal{M}f_m$-natural operators A which send closed 3-forms H on N into bilinear operators A_H on sections of $TN \oplus T^*N \to N$ (for N a smooth manifold). The twisted (or H-twisted) Courant bracket is an example of such operators and it is of particular interest, because its properties were used in [8, 11] to define the concept of exact Courant algebroid.

In [10], we described all bilinear operators on linear sections of $TE \oplus T^*E \to E$ (for $E \to M$ a smooth vector bundle), which are $\mathcal{VB}_{m,n}$-gauge-natural, i.e. invariant under the morphisms in the category $\mathcal{VB}_{m,n}$ of rank-n vector bundles over m-dimensional bases and their vector bundle isomorphisms onto images. The Dorfman–Courant bracket is an example of such operators and it is of particular interest, because $(TE \oplus T^*E; E, TM \oplus E^*; M)$ is the *standard VB-Courant algebroid* and the Dorfman–Courant bracket is the part of this structure. The Dorfman–Courant bracket is the restriction of the Courant bracket to linear sections of $TE \oplus T^*E \to E$, see [6]. The Dorfman–Courant bracket can be also interpreted as the bracket of the Omni–Lie algebroid $Der(E^*) \oplus J^1(E^*)$, studied in [1].

In the present article, we describe all $\mathcal{VB}_{m,n}$-gauge-natural (i.e. invariant under the morphisms in the category $\mathcal{VB}_{m,n}$) operators

$$C : \Gamma^{l-\mathrm{clos}}(\bigwedge^3 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

which, like the twisted Dorfman–Courant bracket, transform closed linear 3-forms $H \in \Gamma_E^{l-\mathrm{clos}}(\bigwedge^3 T^*E)$ on E into bilinear operators

$$C_H : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

(for E a smooth vector bundle), where $\Gamma^l_E(TE \oplus T^*E)$ is the space of linear sections of $TE \oplus T^*E \to E$ (i.e. couples $X \oplus \omega$ of a linear vector field X on E and a linear 1-form ω on E). Thus the main result of the paper is the following

Theorem 1.1. *Let $m \geq 3$ and $n \geq 1$ be fixed integers. Any $\mathcal{VB}_{m,n}$-gauge-natural operator*

$$C : \Gamma^{l-\mathrm{clos}}(\bigwedge^3 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

is of the form

$$\begin{aligned} C_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \{&b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 \\ &+ b_4 d i_{X^2}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1 + c_1 i_{X^1} i_{X^2} H \\ &+ c_2 i_L i_{X^2} d i_{X^1} H + c_3 i_L i_{X^1} d i_{X^2} H + c_4 i_L d i_{X^2} i_{X^1} H\} \end{aligned} \tag{1.1}$$

for arbitrary (uniquely determined by C) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4$, where $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus T^*E)$, $H \in \Gamma_E^{l-\mathrm{clos}}(\bigwedge^3 T^*E)$, and where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

The problem of extracting of all operators C of the form (1.1) which, like the twisted Dorfman Courant bracket, satisfy the Jacobi identity in Leibniz form is rather technically complicated. In the last section, we solve this problem in the case of all C of the form (1.1) which, like the twisted Dorfman Courant bracket, satisfy the "restricted" condition $c_2 = c_3 = c_4 = 0$. Namely, we prove the following result.

Theorem 1.2. *Let $m \geq 2$ and $n \geq 1$. Let C be a $\mathcal{VB}_{m,n}$-gauge natural operator of the form* (1.1) *with $c_2 = c_3 = c_4 = 0$. Then C satisfies the Jacobi identity in Leibniz form (i.e. the condition $C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3))$ for any $\rho^i \in \Gamma^l_E(TE \oplus T^*E)$ for $i = 1, 2, 3$ and any $H \in \Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$) if and only if $(a, b_1, b_2, b_3, b_4, b_5, b_6, c_1)$ is from the following list of 8-tuples:*

$$\begin{aligned}
&(c,0,0,0,0,c,0,0),\ (c,0,0,0,0,c,-c,0),\\
&(c,c,0,0,0,-c,0,0),\ (c,c,-c,0,0,-c,c,0),\\
&(c,0,0,0,0,0,0,0),\ (c,c,0,0,0,0,0,0),\\
&(c,c,0,0,0,0,-c,0),\ (c,c,-c,0,0,0,0,0),\\
&(c,c,-c,0,c-\lambda,0,\lambda,0),\ (0,0,0,\lambda,\mu,-\lambda,-\mu,0),\\
&(c,c,-c,0,c,0,0,\nu),\ (0,0,0,0,0,0,0,\nu),
\end{aligned} \tag{1.2}$$

where c, λ, μ, ν are arbitrary real numbers with $c \neq 0$ and $\nu \neq 0$.

The concept of (gauge) natural operators can be found in [7]. However, our operators from Theorem 1.1 are probably unusual, because we do not know whether their domain is Whitney's extendible.

From now on, let $\mathbf{R}^{m,n}$ be the trivial vector bundle over $\mathbf{R}^m$ with the standard fibre $\mathbf{R}^n$ and let $x^1, \ldots, x^m, y^1, \ldots, y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$.

2. THE DORFMAN–COURANT LIKE BRACKETS

Let $E = (E \to M)$ be a vector bundle.

A vector field X on E is called linear if it has expression

$$X = \sum_{i=1}^{m} X^i(x^1, \ldots, x^m)\frac{\partial}{\partial x^i} + \sum_{j,k=1}^{n} X^k_j(x^1, \ldots x^m) y^j \frac{\partial}{\partial y^k}$$

in any local vector bundle trivialization $x^1, \ldots, x^m, \ldots, y^n$ on E. The Euler vector field L on E is an example of a linear vector field. (The coordinate expression of L is $L = \sum_{j=1}^n y^j \frac{\partial}{\partial y^j}$.) Equivalently, a vector field X on E is linear iff $\mathcal{L}_L X = 0$, where $\mathcal{L}$ denotes the Lie derivative.

A 1-form ω on E is called linear if it has expression

$$\omega = \sum_{i=1}^{m}\sum_{j=1}^{n} \omega_{ij}(x^1, \ldots, x^m) y^j dx^i + \sum_{j=1}^{n} \omega_j(x^1, \ldots, x^m) dy^j$$

in any local vector bundle trivialization $x^1, \ldots, x^m, \ldots, y^n$ on E. Equivalently, a 1-form ω on E is linear iff $\mathcal{L}_L\omega = \omega$, where L is the Euler vector field on E.

We have the following definition being modification of the general one from [7].

Definition 2.1. A $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$$

is a $\mathcal{VB}_{m,n}$-invariant family of $\mathbf{R}$-bilinear operators

$$A : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

for all $\mathcal{VB}_{m,n}$-objects E, where $\Gamma^l_E(TE \oplus T^*E)$ is the vector space of linear sections of $TE \oplus T^*E$ (couples $X \oplus \omega$ of linear vector fields X and linear 1-forms ω on E).

Remark 2.2. The $\mathcal{VB}_{m,n}$-invariance of A means that if

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E)$$

and

$$(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in \Gamma^l_{\tilde{E}}(T\tilde{E} \oplus T^*\tilde{E}) \times \Gamma^l_{\tilde{E}}(T\tilde{E} \oplus T^*\tilde{E}))$$

are φ-related by an $\mathcal{VB}_{m,n}$-map $\varphi : E \to \tilde{E}$ (i.e. $\tilde{X}^i \circ \varphi = T\varphi \circ X^i$ and $\tilde{\omega}^i \circ \varphi = T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$.

Remark 2.3. The Dorfman–Courant bracket

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]] := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - i_{X^2} d\omega^1)$$

is an example of a $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator

$$\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*).$$

Theorem 2.4 ([10])**.** *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator $A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$ is of the form*

$$\begin{aligned} A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{&b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 \\ &+ b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 \\ &+ b_5 \mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1\} \end{aligned} \tag{2.1}$$

for arbitrary (uniquely determined by A) real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6$, where $[-,-]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

Moreover, such A satisfies the Jacobi identity in Leibniz form (i.e. the condition $A(\nu^1, A(\nu^2, \nu^3)) = A(A(\nu^1, \nu^2), \nu^3) + A(\nu^2, A(\nu^1, \nu^3))$ for any $\nu^i \in \Gamma^l_E(TE \oplus T^*E)$ for $i = 1, 2, 3$) if and only if $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ is from the following list of 7-tuples:

$$\begin{aligned} &(c, 0, 0, 0, c, 0),\ (c, 0, 0, 0, 0, c, -c), \\ &(c, c, 0, 0, 0, -c, 0),\ (c, c, -c, 0, 0, -c, c), \\ &(c, 0, 0, 0, 0, 0, 0),\ (c, c, 0, 0, 0, 0, 0), \\ &(c, c, 0, 0, 0, 0, -c),\ (c, c, -c, 0, 0, 0, 0), \\ &(c, c, -c, 0, c - \lambda, 0, \lambda),\ (0, 0, 0, \lambda, \mu, -\lambda, -\mu), \end{aligned} \tag{2.2}$$

where c, λ, μ are arbitrary real numbers with $c \neq 0$. In particular, the Dorfman–Courant bracket satisfies the Jacobi identity in Leibniz form.

3. THE RESTRICTED TWISTED DORFMAN–COURANT LIKE BRACKETS

A p-form Ω on E is called linear if $\mathcal{L}_L\Omega = \Omega$, where L is the Euler vector field on E. Equivalently, a p-form Ω on E is linear iff it has expression

$$\begin{aligned}\Omega = &\sum \Omega_{i_1,\dots,i_p,j}(x) y^j dx^{i_1} \wedge \dots \wedge dx^{i_p} \\ &+ \sum \Omega_{i_1,\dots,i_{p-1},j}(x) dy^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}\end{aligned}$$

in any local vector bundle trivialization $x^1, \dots, x^m, \dots, y^n$ on E, where $x = (x^1, \dots, x^m)$.

Definition 3.1. A $\mathcal{VB}_{m,n}$-gauge-natural operator

$$B : \Gamma^l(\bigwedge^2 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

sending linear 2-forms $F \in \Gamma^l_E(\bigwedge^2 T^*E)$ on $\mathcal{VB}_{m,n}$-objects E into $\mathbf{R}$-bilinear operators

$$B_F : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \to \Gamma^l_E(TE \oplus T^*E)$$

is a $\mathcal{VB}_{m,n}$-invariant family of regular operators (functions)

$$B : \Gamma^l_E(\bigwedge^2 T^*E) \to Lin_2(\Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E), \Gamma^l_E(TE \oplus T^*E))$$

for all $\mathcal{VB}_{m,n}$-objects E, where $Lin_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbf{R}$) functions $U \times V \to W$ for any real vector spaces U, V, W.

Remark 3.2. The invariance of B means that if $F \in \Gamma^l_E(\bigwedge^2 T^*E)$ and $\tilde{F} \in \Gamma^l_{\tilde{E}}(\bigwedge^2 T^*\tilde{E})$ are φ-related by a $\mathcal{VB}_{m,n}$-map $\varphi : E \to \tilde{E}$ and

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E)$$

and

$$(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in \Gamma^l_{\tilde{E}}(T\tilde{E} \oplus T^*\tilde{E}) \times \Gamma^l_{\tilde{E}}(T\tilde{E} \oplus T^*\tilde{E})$$

are also φ-related, then so are $B_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $B_{\tilde{F}}(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$. The regularity of B means that it transforms smoothly parametrized families $(F_t, X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$ into smoothly ones $B_{F_t}(X^1_t \oplus \omega^1_t, X^2_t \oplus \omega^2_t)$.

Definition 3.3. A $\mathcal{VB}_{m,n}$-gauge-natural operator B in the sense of Definition 3.1 is of order s if the following implication

$$(j^s_x F = j^s_x \tilde{F}, j^s_x \rho^1 = j^s_x \tilde{\rho}^1, j^s_x \rho^2 = j^s_x \tilde{\rho}^2) \Rightarrow B_F(\rho^1, \rho^2)_{|E_x} = B_{\tilde{F}}(\tilde{\rho}^1, \tilde{\rho}^2)_{|E_x}$$

holds for any $F, \tilde{F} \in \Gamma^l_E(\bigwedge^2 T^*E)$ and any $\rho^1, \rho^2, \tilde{\rho}^1, \tilde{\rho}^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object $E \to M$ and any $x \in M$.

Definition 3.4. A $\mathcal{VB}_{m,n}$-gauge-natural operator B in the sense of Definition 3.1 is admissible if

$$B_{F+dF'} = B_F \tag{3.1}$$

for any linear 2-form $F \in \Gamma^l_E(\bigwedge^2 T^*E)$ and any linear 1-form $F' \in \Gamma^l_E(T^*E)$ and any $\mathcal{VB}_{m,n}$-object E.

Remark 3.5. The restricted twisted Dorfman–Courant bracket

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_{dF} := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1 + i_{X_1}i_{X_2}dF)$$

is an example of an admissible $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 3.1

We are going to prove the following theorem.

Theorem 3.6. *Let B be an admissible $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definitions 3.1 and 3.4. Assume that $m \geq 3$ and $n \geq 1$. Then there exist uniquely determined real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4$ such that*

$$\begin{aligned} B_F(\rho^1, \rho^2) = a[X^1, X^2] \oplus \{&b_1\mathcal{L}_{X^1}\omega^2 + b_2\mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 \\ &+ b_4 d i_{X^2}\omega^1 + b_5\mathcal{L}_{X^1} d i_L\omega^2 + b_6\mathcal{L}_{X^2} d i_L\omega^1 \\ &+ c_1 i_{X^1} i_{X^2} dF + c_2 i_L i_{X^2} d i_{X^1} dF \\ &+ c_3 i_L i_{X^1} d i_{X^2} dF + c_4 i_L d i_{X^2} i_{X^1} dF\} \end{aligned} \tag{3.2}$$

for any $F \in \Gamma^l_E(\bigwedge^2 T^*E)$ and any $\rho^1, \rho^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

Proof. Operator B_0, where 0 is the zero linear 2-form, can be treated as the $\mathcal{VB}_{m,n}$-gauge-natural bilinear operator in the sense of Definition 2.1. Then B_0 is described in Theorem 2.4. So, replacing B by $B - B_0$, we can assume $B_0 = 0$.

By the $\mathcal{VB}_{m,n}$-invariance of B, such B is determined by the values

$$B_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e \in T_e\mathbf{R}^{m,n} \oplus T^*_e\mathbf{R}^{m,n} \tag{3.3}$$

for all $F \in \Gamma^l_{\mathbf{R}^{m,n}}(\bigwedge^2 T^*\mathbf{R}^{m,n})$ and all $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_{\mathbf{R}^{m,n}}(T\mathbf{R}^{m,n} \oplus T^*\mathbf{R}^{m,n})$ and all $e = (e^1, \ldots, e^n) \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}^{m,n}_0$.

By Corollary 19.9 of the non-linear Petree theorem in [7], we may assume

$$F, X^1, X^2, \omega^1, \omega^2 \text{ are polynomial of degree not more than } r \in \mathbf{N}. \tag{3.4}$$

The proof of our Theorem 3.6 will be continued after proving several lemmas.

Lemma 3.7. *The operator B is of order 2. The values $B_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ are linear in F and independent of ω^1 and ω^2. Moreover, the vector field part of $B_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ is zero.*

Proof. Given $\rho^1 = X^1 \oplus \omega^1$, $\rho^2 = X^2 \oplus \omega^2$, $e = (e^1, \ldots, e^n) \in \mathbf{R}_0^{m,n}$ and F in question, we can write

$$B_F(\rho^1, \rho^2)_e = \Big(\sum a_i \frac{\partial}{\partial x^i}_e + \sum b_k^{k_1} e^k \frac{\partial}{\partial y^{k_1}}_e\Big) \oplus \sum (c_{ik} e^k d_e x^i + \sum f_k d_e y^k), \tag{3.5}$$

where $a_i = a_i(F, X^1, X^2, \omega^1, \omega^2)$ and $b_k^{k_1} = b_k^{k_1}(F, X^1, X^2, \omega^1, \omega^2)$ and $c_{ik} = c_{ik}(F, X^1, X^2, \omega^1, \omega^2)$ and $f_k = f_k(F, X^1, X^2, \omega^1, \omega^2)$ are the real numbers depending smoothly on $(F, X^1, X^2, \omega^1, \omega^2)$ and independent of e. Because of the invariance of B with respect to $(x^1, \ldots, x^m, \frac{1}{t}y^1, \ldots, \frac{1}{t}y^n)$ (preserving X^1 and X^2 (as X^1 and X^2 are linear) and sending F into tF (as F is linear) and sending ω^1 and ω^2 into $t\omega^1$ and $t\omega^2$ (as ω^1 and ω^2 are linear)), we get the homogeneity conditions

$$\begin{aligned} a_i(tF, X^1, X^2, t\omega^1, t\omega^2) &= a_i(F, X^1, X^2, \omega^1, \omega^2), \\ b_k^{k_1}(tF, X^1, X^2, t\omega^1, t\omega^2) &= b_k^{k_1}(F, X^1, X^2, \omega^1, \omega^2), \\ c_{ik}(tF, X^1, X^2, t\omega^1, t\omega^2) &= tc_{ik}(F, X^1, X^2, \omega^1, \omega^2), \\ f_k(tF, X^1, X^2, t\omega^1, t\omega^2) &= tf_k(F, X^1, X^2, \omega^1, \omega^2). \end{aligned} \tag{3.6}$$

Then, by the homogeneous function theorem and (3.4), c_{ik} and f_k are linear in F and independent of ω^1 and ω^2 because of the assumption $C_0 = 0$. Moreover, a_i and $b_k^{k_1}$ are independent of F, and they are zero because of the assumption $C_0 = 0$. So, the last two sentences of the lemma are complete.

It remains to prove the order part of the lemma. Let

$$h_t = \Big(\frac{1}{t}x^1, \ldots, \frac{1}{t}x^m, y^1, \ldots, y^n\Big).$$

Then

$$\begin{aligned} (h_t)_* F &= a_1(F)t + \ldots + a_{r+2}(F)t^{r+2}, \\ t(h_t)_* X^1 &= b_0(X^1) + \ldots + b_{r+2}(X^1)t^{r+2}, \\ t(h_t)_* X^2 &= b_0(X^2) + \ldots + b_{r+1}(X^2)t^{r+2}. \end{aligned} \tag{3.7}$$

The first above expression holds because of F is a linear 2-form. By the invariance of B with respect to h_t we have the homogeneous conditions

$$\begin{aligned} c_{ik}((h_t)_* F, t(h_t)_* X^1, t(h_t)_* X^2) &= t^3 c_{ik}(F, X^1, X^2)\ , \\ f_k((h_t)_* F, t(h_t)_* X^1, t(h_t)_* X^2) &= t^2 f_k(F, X^1, X^2)\ . \end{aligned} \tag{3.8}$$

Then the homogeneous function theorem and the assumption $B_0 = 0$ complete the order part of the lemma. □

Given $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}_0^{m,n}$, let $T_e^*(\mathbf{R}^m \times \mathbf{R}^n) = \mathbf{R}^{m*} \times \mathbf{R}^{n*}$ be the usual identification. Let

$$\begin{aligned} B_F^{\langle 1\rangle}(X^1, X^2)_e &= \text{the } \mathbf{R}^{m*}\text{-component of } B_F(X^1 \oplus 0, X^2 \oplus 0)_e, \\ B_F^{\langle 2\rangle}(X^1, X^2)_e &= \text{the } \mathbf{R}^{n*}\text{-component of } B_F(X^1 \oplus 0, X^2 \oplus 0)_e. \end{aligned} \tag{3.9}$$

Lemma 3.8. *If $m \geq 3$, B is determined by the collection*

$$\begin{aligned} &B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}, y^k\frac{\partial}{\partial y^{k_1}}\Big)_{e_1},\\ &B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(y^k\frac{\partial}{\partial y^{k_1}}, \frac{\partial}{\partial x^i}\Big)_{e_1},\\ &B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}, x^3\frac{\partial}{\partial x^{i_1}}\Big)_{e_1},\\ &B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^i}\Big)_{e_1},\\ &B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{i_1}}\Big)_{e_1} \end{aligned} \tag{3.10}$$

for all $i, i_1 = 1, \dots, m$ and $k, k_1 = 1, \dots, n$, where $e_1 = (1, 0, \dots, 0) \in \mathbf{R}^n = \mathbf{R}^{m,n}_0$.

Proof. By Lemma 3.7 and the assumption $B_{dF'} = B_0 = 0$ (a consequence of the admissibility of B), we derive that B is determined by the collection

$$\begin{aligned} &B_{f^1(x)d\varphi(y)\wedge df^2(x)}(X^1\oplus 0, X^2\oplus 0)_e,\\ &B_{f^3(x)\varphi(y)df^1(x)\wedge df^2(x)}(X^1\oplus 0, X^2\oplus 0)_e,\\ &B_{\varphi(y)df^1(x)\wedge df^2(x)}(X^1\oplus 0, X^2\oplus 0)_e \end{aligned} \tag{3.11}$$

for all $X^1, X^2 \in \Gamma^l_{\mathbf{R}^{m,n}}(T\mathbf{R}^{m,n})$ and all $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{m,n}$ and all maps $f^1, f^2, f^3 \colon \mathbf{R}^m \to \mathbf{R}$ with $f^1(0) = f^2(0) = f^3(0) = 0$ and all linear maps $\varphi : \mathbf{R}^m \to \mathbf{R}$. Of course, we can assume $\varphi(e) = 1$ and the rank of (d_0f^1, d_0f^2, d_0f^3) is maximal. Then, using the $\mathcal{VB}_{m,n}$-invariance of B, we can assume $e = e_1$, $\varphi = y^1$, $f^1 = x^1$, $f^2 = x^2$, $f^3 = x^3$ (we use $m \geq 3$). Further, using the invariance of B with respect to $(x^1, \dots, x^m, y^1 + x^3y^1, \dots, y^n)^{-1}$, we can see that the values

$$B_{y^1dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$$

determine the values

$$B_{(y^1+x^3y^1)\wedge dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1},$$

and then they determine the values

$$B_{x^3y^1dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}.$$

So, the values $B_{x^3y^1dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$ may be omitted. Moreover, since

$$B_{d(x^1y^1)\wedge dx^2} = -B_{d(x^2d(x^1y^1))} = -B_0 = 0$$

(because of the admissibility of B), then

$$B_{x^1dy^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1} = -B_{y^1dx^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}.$$

So, the values $B_{x^1dy^1\wedge dx^2}(X^1\oplus 0, X^2\oplus 0)_{e_1}$ may be omitted. So, B is determined by the values

$$\begin{aligned} &B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1},\\ &B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1} \end{aligned} \tag{3.12}$$

for all $\alpha,\beta,\gamma,\delta\in(\mathbf{N}\cup\{0\})^m$ and $i,i_1=1,\dots,m$ and $j,k,j_1,k_1=1,\dots,n$, where $(X^1=x^\alpha\frac{\partial}{\partial x^i}$ or $X^1=x^\beta y^j\frac{\partial}{\partial y^k})$ and $(X^2=x^\gamma\frac{\partial}{\partial x^{i_1}}$ or $X^2=x^\delta y^{j_1}\frac{\partial}{\partial y^{k_1}})$, where (of course) $x^\alpha:=(x^1)^{\alpha_1}\dots(x^m)^{\alpha_m}$. We are going to study this collection (3.12).

(i) At first we study the case of $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}$. We can see that if $X^1=x^\alpha\frac{\partial}{\partial x^i}$ and $X^2=x^\gamma\frac{\partial}{\partial x^{i_1}}$ then by the invariance of B with respect to $h_t:=(\frac{1}{t}x^1,\dots,\frac{1}{t}x^m,y^1,\dots,y^n)$, we get

$$t^{2+|\alpha|+|\gamma|-2}B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=tB^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1},$$

and then $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=0$ if $|\alpha|+|\gamma|\neq 1$. Similarly, if $X^1=x^\alpha\frac{\partial}{\partial x^i}$ and $X^2=x^\delta y^{j_1}\frac{\partial}{\partial y^{k_1}}$ then $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=0$ if $|\alpha|+|\delta|\neq 0$. Similarly, if $X^1=x^\beta y^j\frac{\partial}{\partial y^k}$ and $X^2=x^\gamma\frac{\partial}{\partial x^{i_1}}$, then $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=0$ if $|\beta|+|\gamma|\neq 0$. Similarly, $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=0$ in the rest sub-case.

Further, we can see that the values

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i},x^{i_2}\frac{\partial}{\partial x^{i_1}}\Big)_{e_1}$$

are determined by the values

$$B^{\langle 1\rangle}_{y^1df(x)\wedge dg(x))}(X^1(x),h(x)X^2(x))_{e_1}$$

for all “constant” vector fields X^1 and X^2 on $\mathbf{R}^m$ and all linear maps $f,g,h:\mathbf{R}^m\to\mathbf{R}$. Then (of course) we can assume that f, g, h are linearly independent (we use $m\geq 3$). Then, using the invariance of B with respect to $(\varphi(x^1,\dots,x^m),y^1,\dots,y^n)$ for a linear isomorphism $\varphi:\mathbf{R}^m\to\mathbf{R}^m$, we can assume $f=x^1$, $g=x^2$ and $h=x^3$. Because of the bi-linearity of B_F, we can else assume that $X^1=\frac{\partial}{\partial x^i}$ and $X^2=\frac{\partial}{\partial x^{i_1}}$. Quite similarly, one can proceed with

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^{i_2}\frac{\partial}{\partial x^{i_1}},\frac{\partial}{\partial x^i}\Big)_{e_1}$$

instead of

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i},x^{i_2}\frac{\partial}{\partial x^{i_1}}\Big)_{e_1}.$$

(ii) Now, we pass to $B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}$. If $X^1=x^\alpha\frac{\partial}{\partial x^i}$ and $X^2=x^\gamma\frac{\partial}{\partial x^{i_1}}$ then by the invariance of B with respect to h_t we get

$$t^{2+|\alpha|+|\gamma|-2}B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1},$$

and then $B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=0$ if $|\alpha|+|\gamma|\neq 0$. Quite similarly, we get that $B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}(X^1,X^2)_{e_1}=0$ in the rest three sub-cases. □

Lemma 3.9. *All values* $B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}})_{e_1}$ *are zero except (eventually) of the two ones. The exceptional values satisfy*

$$
\begin{aligned}
B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\Big)_{e_1} &= \tilde{a}d_{e_1}y^1,\\
B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^1}\Big)_{e_1} &= -\tilde{a}d_{e_1}y^1,
\end{aligned}
\tag{3.13}
$$

where $\tilde{a}$ *is the real number (determined by* B*).*

Proof. Let

$$
B^{\langle 2\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{i_1}}\Big)_{e_1} = \sum_{k=1}^{n} a_{ii_1k}d_{e_1}y^k,
$$

where $a_{ii_1k}\in\mathbf{R}$ are the real numbers. Then by the invariance of B with respect to

$$
\Big(\frac{1}{\tau^1}x^1,\dots,\frac{1}{\tau^m}x^m,y^1,\dots,y^n\Big)
$$

we get $\tau^1\tau^2\frac{1}{\tau^i}\frac{1}{\tau^{i_1}}a_{ii_1k}=a_{ii_1k}$. So, $a_{ii_1k}=0$ if $\{i,i_1\}\neq\{1,2\}$. Further, by the invariance of B with respect to $(x^1,\dots,x^m,y^1,\frac{1}{t}y^2,\dots,\frac{1}{t}y^n)$ we get $a_{12k}=ta_{12k}$ for $k=2,\dots,n$. Then $a_{12k}=0$ for $k=2,\dots,n$. Further, by the invariance of B with respect to the replacing x^1 by x^2 (and vice-versa) we get $a_{12k}=-a_{21k}$ for $k=1,\dots,n$. The lemma is complete. □

Lemma 3.10. *All values* $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(\frac{\partial}{\partial x^i},y^k\frac{\partial}{\partial y^{k_1}})_{e_1}$ *are zero except (eventually) of the two ones. The exceptional values satisfy*

$$
\begin{aligned}
B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1},y^1\frac{\partial}{\partial y^1}\Big)_{e_1} &= \tilde{c}d_{e_1}x^2\\
B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^2},y^1\frac{\partial}{\partial y^1}\Big)_{e_1} &= -\tilde{c}d_{e_1}x^1,
\end{aligned}
\tag{3.14}
$$

where $\tilde{c}$ *is the real number (determined by* B*).*

Proof. Let

$$
B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^i},y^k\frac{\partial}{\partial y^{k_1}}\Big)_{e_1} = \sum_{j=1}^{m} c_{ikk_1j}d_{e_1}x^j,
$$

where c_{ikk_1j} are the real numbers. By the invariance of B with respect to $(\frac{1}{\tau^1}x^1,\dots,\frac{1}{\tau^m}x^m,y^1,\dots,y^n)$ we get $\tau^1\tau^2\frac{1}{\tau^i}c_{ikk_1j}=\tau^jc_{ikk_1j}$. Then $c_{ikk_1j}=0$ if $\{i,j\}\neq\{1,2\}$. Further, by the invariance of B with respect to replacing x^1 by x^2 (and vice-versa) we get $c_{1kk_12}=-c_{2kk_11}$. Further, by invariance of B with respect to $(x^1,\dots,x^m,\frac{1}{\tau^1}y^1,\frac{1}{\tau^2}y^2,\dots,\frac{1}{\tau^n}y^n)$ with $\tau^1=1$, we get $\tau^k\frac{1}{\tau^{k_1}}c_{1kk_12}=c_{1kk_12}$. Then $c_{1kk_12}=0$ if $k\neq k_1$. Further, if $k\in\{2,\dots,n\}$, there exists a $\mathcal{VB}_m$-map

$$
\psi=(x^1,\dots,x^m,y^1,\tilde{\psi}(x^2,\dots,x^m,y^2\dots,y^n))
$$

sending $\frac{\partial}{\partial x^2}$ into $\frac{\partial}{\partial x^2} + y^k \frac{\partial}{\partial y^k}$. Then, using the invariance of B with respect to ψ, from

$$B_{y^1 dx^1 \wedge dx^2}\Big(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\Big)_{e_1} = 0 \oplus \tilde{a} d_{e_1} y^1,$$

where $\tilde{a}$ is from Lemma 3.9, we get

$$B_{y^1 dx^1 \wedge dx^2}\bigg(\frac{\partial}{\partial x^1} \oplus 0, \Big(\frac{\partial}{\partial x^2} + y^k \frac{\partial}{\partial y^k}\Big) \oplus 0\bigg)_{e_1} = 0 \oplus \tilde{a} d_{e_1} y^1.$$

(That $B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})_{e_1} = 0$, see the proof of Lemma 3.8.) Then

$$B_{y^1 dx^1 \wedge dx^2}\Big(\frac{\partial}{\partial x^1} \oplus 0, y^k \frac{\partial}{\partial y^k} \oplus 0\Big)_{e_1} = 0 \oplus 0$$

for $k = 2, \dots, n$. The lemma is complete. □

Lemma 3.11. *All values $B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}(y^k \frac{\partial}{\partial y^{k_1}}, \frac{\partial}{\partial x^i})_{e_1}$ are zero except (eventually) of the two ones. The exceptional values satisfy*

$$\begin{aligned} B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}\Big(y^1 \frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^1}\Big)_{e_1} &= \tilde{e} d_{e_1} x^2 \\ B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}\Big(y^1 \frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2}\Big)_{e_1} &= -\tilde{e} d_{e_1} x^1, \end{aligned} \tag{3.15}$$

where $\tilde{e}$ is the real number (determined by B).

Proof. In fact, this lemma is Lemma 3.10 for B^{op} instead of B, where

$$B^{\mathrm{op}}_F(X^1 \oplus \omega^1, X^2 \oplus \omega^2) := B_F(X^2 \oplus \omega^2, X^1 \oplus \omega^1).$$ □

Lemma 3.12. *Let $m \geq 3$. All values $B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}(\frac{\partial}{\partial x^i}, x^3 \frac{\partial}{\partial x^{i_1}})_{e_1}$ are equal to zero except (eventually) of the four ones. The exceptional values satisfy*

$$\begin{aligned} B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}\Big(\frac{\partial}{\partial x^1}, x^3 \frac{\partial}{\partial x^2}\Big)_{e_1} &= \tilde{f} d_{e_1} x^3 \ , \\ B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}\Big(\frac{\partial}{\partial x^2}, x^3 \frac{\partial}{\partial x^1}\Big)_{e_1} &= -\tilde{f} d_{e_1} x^3 \ , \\ B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}\Big(\frac{\partial}{\partial x^3}, x^3 \frac{\partial}{\partial x^2}\Big)_{e_1} &= \tilde{g} d_{e_1} x^1 \ , \\ B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}\Big(\frac{\partial}{\partial x^3}, x^3 \frac{\partial}{\partial x^1}\Big)_{e_1} &= -\tilde{g} d_{e_1} x^2 \ , \end{aligned} \tag{3.16}$$

where $\tilde{f}$ and $\tilde{g}$ are the real numbers (determined by B).

Proof. Let

$$B^{\langle 1\rangle}_{y^1 dx^1 \wedge dx^2}\Big(\frac{\partial}{\partial x^i}, x^3 \frac{\partial}{\partial x^{i_1}}\Big)_{e_1} = \sum_{j=1}^{m} q_{i i_1 j} d_{e_1} x^j,$$

where $q_{ii_1j} \in \mathbf{R}$ are the numbers. Then by the invariance of B with respect to $(\frac{1}{\tau^1}x^1, \dots, \frac{1}{\tau^m}x^m, y^1, \dots, y^n)$ we get $\tau^1\tau^2\tau^3\frac{1}{\tau^i}\frac{1}{\tau^{i_1}}q_{ii_1j} = \tau^j q_{ii_1j}$. Then $q_{ii_1j} = 0$ if $\{i, i_1, j\} \neq \{1,2,3\}$. Further, there exists a 0-preserving embedding $\varphi : \mathbf{R} \to \mathbf{R}$ sending (the germ at 0 of) $\frac{\partial}{\partial x}$ into $\frac{\partial}{\partial x} + x\frac{\partial}{\partial x}$. Then, by the invariance of B with respect to $(x^1, x^2, \varphi(x^3), \dots, x^m, \dots, y^n)$, from $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3})_{e_1} = 0$ we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^3}\Big)_{e_1} = 0,$$

and then $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(\frac{\partial}{\partial x^1}, x^3\frac{\partial}{\partial x^3})_{e_1} = 0$, i.e. $q_{132} = 0$. Then using the invariance of B with respect to changing x^1 by x^2 (and vice-versa) we get that $q_{231} = -q_{132} = 0$ and $q_{321} = -q_{312}$ and $q_{123} = -q_{213}$. We put $\tilde f := q_{123}$ and $\tilde g := q_{321}$. □

Lemma 3.13. *Let $m \geq 3$. All values $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(x^3\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^i})_{e_1}$ are zero except (eventually) of the four ones. The exceptional values satisfy*

$$\begin{aligned}
B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}\Big)_{e_1} &= \tilde h d_{e_1}x^3,\\
B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\Big)_{e_1} &= -\tilde h d_{e_1}x^3,\\
B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\Big)_{e_1} &= \tilde k d_{e_1}x^1,\\
B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^3\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}\Big)_{e_1} &= -\tilde k d_{e_1}x^2,
\end{aligned} \tag{3.17}$$

where $\tilde h$ and $\tilde k$ are the real numbers (determined by B).

Proof. In fact, this lemma is Lemma 3.12 for B^{op} instead of B. □

Lemma 3.14. *Let $m \geq 3$. We have*

$$\tilde f = \tilde a + \tilde c \tag{3.18}$$

where $\tilde a$ is the real number from Lemma 3.9 and $\tilde c$ is the real number from Lemma 3.10 and $\tilde f$ is the number from Lemma 3.12.

Proof. Given $\tau > 0$,

$$\psi_\tau := \Big(x^1, \frac{x^2}{1+\tau x^3}, x^3, \dots, x^m, \dots, y^n\Big)$$

preserves e_1 and $\frac{\partial}{\partial x^1}$ and sends $y^1dx^1 \wedge dx^2$ into $y^1dx^1 \wedge d(x^2 + \tau x^2x^3)$ and $\frac{\partial}{\partial x^2}$ into $\frac{1}{1+\tau x^3}\frac{\partial}{\partial x^2}$. On the other hand, by invariance of B with respect to $(\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, y^1, \dots, y^n)$, we can easily see that $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1} = 0$. Then, by the invariance of B with respect to ψ_τ, we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge d(x^2+\tau x^2x^3)}\Big(\frac{\partial}{\partial x^1}, \frac{1}{1+\tau x^3}\frac{\partial}{\partial x^2}\Big)_{e_1} = 0.$$

Then by the order argument, we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge d(x^2+\tau x^2x^3)}\Big(\frac{\partial}{\partial x^1},(1-\tau x^3+\tau^2(x^3)^2)\frac{\partial}{\partial x^2}\Big)_{e_1}=0.$$

Then, comparing the coefficients on τ of both sides of this equality, we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1},x^3\frac{\partial}{\partial x^2}\Big)_{e_1}=B^{\langle 1\rangle}_{y^1dx^1\wedge d(x^2x^3)}\Big(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\Big)_{e_1}. \tag{3.19}$$

Further, by the invariance of B with respect to $(x^1,\dots,x^m,\frac{1}{1+x^3}y^1,y^2,\dots,y^n)$, from

$$B_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=0\oplus\tilde{a}d_{e_1}y^1,$$

we get

$$B_{(y^1+x^3y^1)dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=0\oplus(\tilde{a}d_{e_1}y^1+\tilde{a}d_{e_1}x^3),$$

and then

$$B^{\langle 1\rangle}_{x^3y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\Big)_{e_1}=\tilde{a}d_{e_1}x^3, \tag{3.20}$$

where $\tilde{a}$ is the number from Lemma 3.9.

Further, by invariance of B with respect to $(x^1,\frac{1}{t}x^2,x^3,\dots,x^m,\dots,y^n)$, we can easily see that

$$B_{y^1dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1}\oplus 0,\frac{\partial}{\partial x^2}\oplus 0\Big)_{e_1}=0\oplus 0.$$

Then, by the invariance of B with respect to $(x^1,\dots,x^m,\frac{1}{1+\tau x^2}y^1,y^2,\dots,y^n)$, we get

$$B_{(y^1+\tau x^2y^1)dx^1\wedge dx^3}\bigg(\frac{\partial}{\partial x^1}\oplus 0,\Big(\frac{\partial}{\partial x^2}-\frac{\tau}{1+\tau x^2}y^1\frac{\partial}{\partial y^1}\Big)\oplus 0\bigg)_{e_1}=0\oplus 0,$$

and then (by the order argument and comparing the coefficients on τ) we get

$$B^{\langle 1\rangle}_{x^2y^1dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\Big)_{e_1}=B^{\langle 1\rangle}_{dy^1\wedge dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1},y^1\frac{\partial}{\partial y^1}\Big)_{e_1},$$

Further, by the invariance of B with respect to $(x^1,x^2+x^3,x^3,\dots,x^m,\dots,y^n)$, from the first equality of (3.14), we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge(dx^2-dx^3)}\Big(\frac{\partial}{\partial x^1},y^1\frac{\partial}{\partial y^1}\Big)_{e_1}=\tilde{c}d_{e_1}(x^2-x^3),$$

and then $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^3}(\frac{\partial}{\partial x^1},y^1\frac{\partial}{\partial y^1})_{e_1}=\tilde{c}d_{e_1}x^3$, where $\tilde{c}$ is the number from Lemma 3.10. Then

$$B^{\langle 1\rangle}_{x^2y^1dx^1\wedge dx^3}\Big(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\Big)_{e_1}=\tilde{c}d_{e_1}x^3. \tag{3.21}$$

Now, from (3.16) and (3.19) and (3.20) and (3.21), since

$$x^3y^1dx^1 \wedge dx^2 + x^2y^1dx^1 \wedge dx^3 = y^1dx^1 \wedge d(x^2x^3),$$

we get

$$\tilde{f}d_{e_1}x^3 = B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}, x^3\frac{\partial}{\partial x^2}\Big)_{e_1} = (\tilde{a}+\tilde{c})d_{e_1}x^3,$$

as well. The lemma is complete. □

Lemma 3.15. *Let $m \geq 3$. We have*

$$\tilde{h} = -\tilde{a} + \tilde{e} \tag{3.22}$$

where $\tilde{a}$ is the real number from Lemma 3.9 and $\tilde{e}$ is the real number from Lemma 3.11 and $\tilde{h}$ is the number from Lemma 3.13.

Proof. In fact, this lemma is Lemma 3.14 for B^{op} instead of B. □

Lemma 3.16. *Let $m \geq 3$. We have*

$$\tilde{f} + \tilde{g} + \tilde{k} + \tilde{h} = 0 \tag{3.23}$$

where $\tilde{f}$ and $\tilde{g}$ are the numbers from Lemma 3.12 and $\tilde{h}$ and $\tilde{k}$ are the numbers from Lemma 3.13.

Proof. By the invariance of B with respect to $(x^1+\tau x^3, x^2, \dots, x^m, \dots, y^n)$, from the third equality of (3.16) we get

$$B^{\langle 1\rangle}_{y^1d(x^1-\tau x^3)\wedge dx^2}\Big(\frac{\partial}{\partial x^3}+\tau\frac{\partial}{\partial x^1}, x^3\frac{\partial}{\partial x^2}\Big)_{e_1} = \tilde{g}d_{e_1}(x^1-\tau x^3),$$

and then considering the coefficients on τ and using the first equation of (3.16) we obtain

$$-B^{\langle 1\rangle}_{y^1dx^3\wedge dx^2}\Big(\frac{\partial}{\partial x^3}, x^3\frac{\partial}{\partial x^2}\Big)_{e_1} + \tilde{f}d_{e_1}x^3 = -\tilde{g}d_{e_1}x^3.$$

Then using (in particular) the invariance of B with replacing x^3 by x^1 (and vice-versa) we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1}, x^1\frac{\partial}{\partial x^2}\Big)_{e_1} = (\tilde{g}+\tilde{f})d_{e_1}x^1. \tag{3.24}$$

Quite similarly, using (3.17) instead of (3.16) we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^1\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^1}\Big)_{e_1} = (\tilde{k}+\tilde{h})d_{e_1}x^1. \tag{3.25}$$

(In fact, the equality (3.25) is the equality (3.24) for B^{op} instead of B.)

Further, by invariance of B with respect to

$$(x^1, x^2+\tau(x^1)^2, x^3, \dots, x^m, \dots, y^n),$$

from $B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^1})_{e_1}=0$, we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge d(x^2-\tau(x^1)^2)}\Big(\frac{\partial}{\partial x^1}+2\tau x^1\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^1}+2\tau x^1\frac{\partial}{\partial x^2}\Big)_{e_1}=0,$$

and then considering the coefficients on τ we get

$$B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(\frac{\partial}{\partial x^1},x^1\frac{\partial}{\partial x^2}\Big)_{e_1}+B^{\langle 1\rangle}_{y^1dx^1\wedge dx^2}\Big(x^1\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^1}\Big)_{e_1}=0. \tag{3.26}$$

From (3.26) and (3.25) and (3.24) we obtain (3.23), as well. The lemma is complete. □

We are now in position to complete the proof of Theorem 3.6. By Lemmas 3.7–3.16, any admissible $\mathcal{VB}_{m,n}$-gauge-natural operator B with $B_0=0$ is uniquely determined by the corresponding 4-tuple $(\tilde a,\tilde c,\tilde g,\tilde k)$. Further, one can easily directly compute the corresponding 4-tuples of $\mathcal{VB}_{m,n}$-gauge natural operators $i_{X^1}i_{X^2}dF$ and $i_Li_{X^2}di_{X^1}dF$ and $i_Li_{X^1}di_{X^2}dF$ and $i_Ldi_{X^2}i_{X^1}dF$. They are $(-1,1,0,0)$ and $(0,0,0,1)$ and $(0,-1,1,0)$ and $(0,-1,0,0)$, respectively. The determinant of the matrix of the above vectors is -1. So, the dimension argument complete the proof of our Theorem 3.6. □

4. THE TWISTED DORFMAN–COURANT LIKE BRACKETS

Definition 4.1. A $\mathcal{VB}_{m,n}$-gauge-natural operator

$$C:\Gamma^{l-\mathrm{clos}}(\bigwedge^3 T^*)\rightsquigarrow Lin_2(\Gamma^l(T\oplus T^*)\times\Gamma^l(T\oplus T^*),\Gamma^l(T\oplus T^*))$$

sending closed linear 3-forms $H\in\Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$ on $\mathcal{VB}_{m,n}$-objects E into $\mathbf{R}$-bilinear operators

$$C_H:\Gamma^l_E(TE\oplus T^*E)\times\Gamma^l_E(TE\oplus T^*E)\to\Gamma^l_E(TE\oplus T^*E)$$

is a $\mathcal{VB}_{m,n}$-invariant family of regular operators (functions)

$$C:\Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)\to Lin_2(\Gamma^l_E(TE\oplus T^*E)\times\Gamma^l_E(TE\oplus T^*E),\Gamma^l_E(TE\oplus T^*E))$$

for all $\mathcal{VB}_{m,n}$-objects E.

Remark 4.2. The twisted Dorfman–Courant bracket

$$[[X^1\oplus\omega^1,X^2\oplus\omega^2]]_H:=[X^1,X^2]\oplus(\mathcal{L}_{X^1}\omega^2-i_{X^2}d\omega^1+i_{X_1}i_{X_2}H)$$

is an example of a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 4.1.

The main result of this paper is the following:

Theorem 4.3. *Let C be a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 4.1. Assume that $m \geq 3$ and $n \geq 1$. Then there exist uniquely determined real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4$ such that*

$$\begin{aligned} C_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \big\{ & b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 + b_3 d i_{X^1}\omega^2 \\ & + b_4 d i_{X^2}\omega^1 + b_5 \mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1 \\ & + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^2} d i_{X^1} H \\ & + c_3 i_L i_{X^1} d i_{X^2} H + c_4 i_L d i_{X^2} i_{X^1} H \big\} \end{aligned} \tag{4.1}$$

for any $H \in \Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$ and any $\rho^1, \rho^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

Proof. Using C, we define

$$B : \Gamma^l(\bigwedge^2 T^*) \rightsquigarrow L_2\big(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*)\big)$$

by $B_F := C_{dF}$. Clearly, B is admissible. So, by Theorem 3.6, there exist uniquely determined real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4$ such that we have (4.1) for any exact linear 3-form $H \in \Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$ and any $\rho^1, \rho^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$. Then, since C is (in particular) local, we can write "closed" instead of "exact" because of the Poincaré lemma. The theorem is complete. □

5. THE TWISTED DORFMAN–COURANT LIKE BRACKETS SATISFYING THE JACOBI IDENTITY IN LEIBNIZ FORM

Let C be a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 4.1.

Definition 5.1. We say that C satisfies the Jacobi identity in Leibniz form if

$$C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3)) \tag{5.1}$$

for all $H \in \Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$ and all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma^l_E(TE \oplus T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$-objects E.

Remark 5.2. It is well-known that the twisted Dorfman Courant bracket from Remark 4.2 satisfies the Jacobi identity in Leibniz form.

Lemma 5.3. *Let C be a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 4.1 of the form*

$$\begin{aligned} C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \{ & b_1 \mathcal{L}_{X^1}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^1 \\ & + b_3 d i_{X^1}\omega^2 + b_4 d i_{X^2}\omega^1 \\ & + b_5 \mathcal{L}_{X^1} d i_L \omega^2 + b_6 \mathcal{L}_{X^2} d i_L \omega^1 \\ & + c_1 i_{X^1} i_{X^2} H \} \end{aligned} \tag{5.2}$$

for any $H \in \Gamma_E^{l-\mathrm{clos}}(\bigwedge^3 T^*E)$ *and any* $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus T^*E)$ *and any* $\mathcal{VB}_{m,n}$*-object* E*, where* $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1$ *are real numbers.* (*It means, we assume that* $c_2 = c_3 = c_4 = 0$.) *If* C *satisfies the Jacobi identity in Leibniz form, then the following conditions* (a) *and* (b) *are satisfied:*

(a) *The* 7*-tuple* $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ *is from the following list of* 7*-tuples:*

$$\begin{aligned}
&(c,0,0,0,0,c,0),\ (c,0,0,0,0,c,-c),\\
&(c,c,0,0,0,-c,0),\ (c,c,-c,0,0,-c,c),\\
&(c,0,0,0,0,0,0),\ (c,c,0,0,0,0,0),\\
&(c,c,0,0,0,0,-c),\ (c,c,-c,0,0,0,0),\\
&(c,c,-c,0,c-\lambda,0,\lambda),\ (0,0,0,\lambda,\mu,-\lambda,-\mu),
\end{aligned} \tag{5.3}$$

where c, λ, μ *are arbitrary real numbers with* $c \neq 0$*;*

(b) *It holds*

$$\begin{aligned}
&b_1c_1\mathcal{L}_{X^1}i_{X^2}i_{X^3}H + b_3c_1di_{X^1}i_{X^2}i_{X^3}H\\
&+ b_5c_1\mathcal{L}_{X^1}di_Li_{X^2}i_{X^3}H + ac_1i_{X^1}i_{[X^2,X^3]}H\\
&= b_2c_1\mathcal{L}_{X^3}i_{X^1}i_{X^2}H + b_4c_1di_{X^3}i_{X^1}i_{X^2}H\\
&\quad + b_6c_1\mathcal{L}_{X^3}di_Li_{X^1}i_{X^2}H + ac_1i_{[X^1,X^2]}i_{X^3}H\\
&\quad + b_1c_1\mathcal{L}_{X^2}i_{X^1}i_{X^3}H + b_3c_1di_{X^2}i_{X^1}i_{X^3}H\\
&\quad + b_5c_1\mathcal{L}_{X^2}di_Li_{X^1}i_{X^3}H + ac_1i_{X^2}i_{[X^1,X^3]}H
\end{aligned} \tag{5.4}$$

for any linear vector fields X^1, X^2, X^3 *and any closed linear* 3*-form* H *on* $\mathbf{R}^{m,n}$*.*

Proof. If C satisfies the Jacobi identity in Leibniz form, then C_0 satisfies the Jacobi identity in Leibniz form. Then we have the condition (a) because of Theorem 2.4. Further, for any linear vector fields X^1, X^2, X^3 on $\mathbf{R}^{m,n}$ and any closed linear 3-form H on $\mathbf{R}^{m,n}$, we can write

$$\begin{aligned}
C_H(X^1 \oplus 0, C_H(X^2 \oplus 0, X^3 \oplus 0)) &= a^2[X^1,[X^2,X^3]] \oplus \Omega,\\
C_H(C_H(X^1 \oplus 0, X^2 \oplus 0), X^3 \oplus 0) &= a^2[[X^1,X^2],X^3] \oplus \Theta,\\
C_H(X^2 \oplus 0, C_H(X^1 \oplus 0, X^3 \oplus 0)) &= a^2[X^2,[X^1,X^3]] \oplus \mathcal{T},
\end{aligned}$$

where

$$\begin{aligned}
\Omega &= b_1\mathcal{L}_{X^1}\{c_1i_{X^2}i_{X^3}H\} + b_3di_{X^1}\{c_1i_{X^2}i_{X^3}H\}\\
&\quad + b_5\mathcal{L}_{X^1}di_L\{c_1i_{X^2}i_{X^3}H\} + c_1i_{X^1}i_{a[X^2,X^3]}H,\\
\Theta &= b_2\mathcal{L}_{X^3}\{c_1i_{X^1}i_{X^2}H\} + b_4di_{X^3}\{c_1i_{X^1}i_{X^2}H\}\\
&\quad + b_6\mathcal{L}_{X^3}di_L\{c_1i_{X^1}i_{X^2}H\} + c_1i_{a[X^1,X^2]}i_{X^3}H,\\
\mathcal{T} &= b_1\mathcal{L}_{X^2}\{c_1i_{X^1}i_{X^3}H\} + b_3di_{X^2}\{c_1i_{X^1}i_{X^3}H\}\\
&\quad + b_5\mathcal{L}_{X^2}di_L\{c_1i_{X^1}i_{X^3}H\} + c_1i_{X^2}i_{a[X^1,X^3]}H,
\end{aligned}$$

From the Jacobi identity in Leibniz form of C it follows $\Omega = \Theta + \mathcal{T}$, i.e. (5.4). □

Theorem 5.4. *Let $m \geq 2$ and $n \geq 1$. Let C be a $\mathcal{VB}_{m,n}$-gauge-natural operator in the sense of Definition 4.1 of the form* (5.2) *for any $H \in \Gamma^{l-\mathrm{clos}}_E(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma^l_E(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$-object E, where $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1$ are real numbers.* (*It means, we assume that $c_2 = c_3 = c_4 = 0$.*) *Then C satisfies the Jacobi identity in Leibniz form if and only if the* 8*-tuple $(a, b_1, b_2, b_3, b_4, b_5, b_6, c_1)$ is from the following list of* 8*-tuples:*

$$\begin{gathered}(c,0,0,0,0,c,0,0),\ (c,0,0,0,0,c,-c,0),\\ (c,c,0,0,0,-c,0,0),\ (c,c,-c,0,0,-c,c,0),\\ (c,0,0,0,0,0,0,0),\ (c,c,0,0,0,0,0,0),\\ (c,c,0,0,0,0,-c,0),\ (c,c,-c,0,0,0,0,0),\\ (c,c,-c,0,c-\lambda,0,\lambda,0),\ (0,0,0,\lambda,\mu,-\lambda,-\mu,0),\\ (c,c,-c,0,c,0,0,\nu),\ (0,0,0,0,0,0,0,\nu),\end{gathered} \tag{5.5}$$

where c, λ, μ, ν are arbitrary real numbers with $c \neq 0$ and $\nu \neq 0$.

Proof. At first we prove the implication $\Rightarrow$. For, assume that C in question satisfies the Jacobi identity in Leibniz form. We will study the 8-tuple $(a, b_1, b_2, b_3, b_4, b_5, b_6, c_1)$ of C. This 8-tuple satisfies Lemma 5.3. So, the 7-tuple $(a, b_1, b_2, b_3, b_4, b_5, b_6)$ obtained by restriction from our 8-tuple is from the list (5.3). More, we have (5.4).

Putting (linear vector fields) $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = y^1 \frac{\partial}{\partial y^1}$ and (closed linear 3-form) $H = x^1 dy^1 \wedge dx^1 \wedge dx^2$ into (5.4), we get

$$\begin{aligned}&- b_1c_1y^1dx^1 - b_3c_1d(x^1y^1) + 0 + 0\\ &= -b_2c_1x^1dy^1 - b_4c_1d(x^1y^1) - b_6c_1d(x^1y^1) + 0 + 0 + b_3c_1d(x^1y^1) + 0 + 0,\end{aligned}$$

i.e.

$$\begin{aligned}&- b_1c_1^1y^1dx^1 - b_3c_1d(x^1y^1)\\ &= -b_2c_1x^1dy^1 - b_4c_1d(x^1y^1) - b_6c_1d(x^1y^1) + b_3c_1d(x^1y^1).\end{aligned} \tag{5.6}$$

Let us consider several cases.

(I) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, c, 0, 0, 0, -c, 0)$, $c \neq 0$. Then from (5.6) we get

$$-cc_1y^1dx^1 - 0 = -0 - 0 - 0 - 0 + 0.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

(II) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, c, 0, 0, 0, 0, 0)$, $c \neq 0$. Then from (5.6) we get

$$-cc_1y^1dx^1 - 0 = -0 - 0 - 0 + 0.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

(III) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, c, -c, 0, 0, 0, 0)$, $c \neq 0$. Then from (5.6) we get

$$-cc_1y^1dx^1 - 0 = cc_1x^1dy^1 - 0 - 0 + 0.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

(IV) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, c, 0, 0, 0, 0, -c)$, $c \neq 0$. Then from (5.6) we get

$$-cc_1 y^1 dx^1 - 0 = -0 - 0 + cc_1 d(x^1 y^1) + 0.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

(V) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, 0, 0, 0, 0, c, -c)$, $c \neq 0$. Then from (5.6) we get

$$-0 - 0 = -0 - 0 + cc_1 d(x^1 y^1) + 0.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

Similarly, putting (linear vector fields) $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = x^2 \frac{\partial}{\partial x^2}$ and (closed linear 3-form) $H = dy^1 \wedge dx^1 \wedge dx^2$ into (5.4), we obtain

$$0 + 0 + 0 - ac_1 dy^1 = 0 + 0 + 0 + 0 - b_1 c_1 dy^1 + 0 - b_5 c_1 dy^1 + 0,$$

i.e.

$$-ac_1 dy^1 = -b_1 c_1 dy^1 - b_5 c_1 dy^1 \tag{5.7}$$

Let us consider next cases.

(VI) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, 0, 0, 0, 0, 0, 0)$, $c \neq 0$. Then from (5.7) we get

$$cc_1 dy^1 = -0 - 0.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

(VII) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, c, -c, 0, 0, -c, c)$, $c \neq 0$. Then from (5.7) we get

$$cc_1 dy^1 = -cc_1 dy^1 + cc_1 dy^1.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

Similarly, putting (linear vector fields) $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = L$ (the Euler vector field) and $X^3 = \frac{\partial}{\partial x^2}$ and (closed linear 3-form) $H = dx^2 \wedge dx^1 \wedge dy^1$ into (5.4), we obtain

$$0 - b_3 c_1 dy^1 + 0 + 0 = 0 - b_4 c_1 dy^1 + 0 + 0 + b_1 c_1 dy^1 + b_3 c_1 dy^1 + b_5 c_1 dy^1 + 0,$$

i.e.

$$-b_3 c_1 dy^1 = -b_4 c_1 dy^1 + b_1 c_1 dy^1 + b_3 c_1 dy^1 + b_5 c_1 dy^1. \tag{5.8}$$

Let us consider next cases.

(VIII) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (0, 0, 0, \lambda, \mu, -\lambda, -\mu)$. Assume $\lambda \neq 0$ or $\mu \neq 0$. From (5.7), we get

$$-0 = -0 + \lambda c_1 dy^1,$$

i.e. $\lambda c_1 = 0$. Moreover, from (5.8), we get

$$-\lambda c_1 dy^1 = -\mu c_1 dy^1 + 0 + \lambda c_1 dy^1 - \lambda c_1 dy^1,$$

i.e. $(\lambda - \mu) c_1 = 0$. Consequently, $c_1 = 0$.

(IX) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, 0, 0, 0, 0, c, 0)$, $c \neq 0$. Then from (5.8) we get

$$0 = 0 + 0 + 0 + cc_1 dy^1.$$

Then $cc_1 = 0$. Then $c_1 = 0$.

(X) Let $(a, b_1, b_2, b_3, b_4, b_5, b_6) = (c, c, -c, 0, c - \lambda, 0, \lambda)$, $c \neq 0$, $\lambda \neq 0$. Then from (5.8) we get

$$0 = -(c - \lambda)c_1 dy^1 + cc_1 dy^1 + 0 + 0.$$

Then $\lambda c_1 = 0$. Then $c_1 = 0$.

So, we have considered all possibilities for the 7-tuple $(a, b_1, b_2, b_3, b_4, b_5, b_6)$. Hence we have proved that the 8-tuple $(a, b_1, b_2, b_3, b_4, b_5, b_6, c_1)$ is from the list presented in the theorem, i.e. the implication $\Rightarrow$ is complete.

Conversely, if $(a, b_1, b_2, b_3, b_4, b_5, b_6, c_1)$ is from the list of the theorem and $c_1 = 0$, then the corresponding operator C is independent of H, and then C satisfies the Jacobi identity in Leibniz form because of Theorem 2.4. If $c_1 \neq 0$ and $a \neq 0$, then C is "proportional" to the twisted Dorfman–Courant bracket, i.e. satisfies (as it is well-known) the Jacobi identity in Leibniz form for closed linear 3-forms. Then putting $a \to 0$, we complete the case $(0, 0, 0, 0, 0, 0, 0, c_1)$, too. Theorem 5.4 is complete. □

REFERENCES

[1] Z. Chen, Z. Liu, *Omni-Lie algebroids*, J. Geom. Phys. **60** (2010) 5, 799–808.

[2] T. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), 631–661.

[3] M. Doupovec, J. Kurek, W.M. Mikulski, *The natural brackets on couples of vector fields and* 1*-forms*, Turk. Math. J. **42** (2018), 1853–1862.

[4] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. **174** (2011) 1, 75–123.

[5] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. **54** (2003), 281–308.

[6] M. Jotz Lean, C. Kirchhoff-Lukat, *Natural lifts of Dorfman brackets*, arXiv:1610.05986v2 [math.DG] 14 Jul 2017.

[7] I. Kolář, P.W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Berlin, Germany Springer-Verlag, 1993.

[8] Z.J. Liu, A. Weinstein, P. Xu, *Main triples for Lie bialgebroids*, J. Differ. Geom. **45** (1997), 547–574.

[9] W.M. Mikulski, *The natural operators similar to the twisted Courant bracket one*, Mediter. J. Math. **16** (2019), Article no. 101.

[10] W.M. Mikulski, *The gauge-natural bilinear operators similar to the Dorfman–Courant bracket*, Mediter. J. Math. **17** (2020), Article no. 40.

[11] P. Severa, A. Weinstein, *Poisson geometry with a* 3*-form background*, Prog. Theor. Phys., Suppl. **144** (2001), 145–154.

Włodzimierz Mikulski
Wlodzimierz.Mikulski@im.uj.edu.pl
https://orcid.org/0000-0002-2905-0461

Jagiellonian University
Department of Mathematics
S. Łojasiewicza 6, Cracow, Poland

Received: July 1, 2020.
Accepted: October 2, 2020.