Cyclic metric Lie groups*

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Abstract

Cyclic metric Lie groups are Lie groups equipped with a left-invariant metric which is in some way far from being biinvariant, in a sense made explicit in terms of Tricerri and Vanhecke's homogeneous structures. The semisimple and solvable cases are studied. We extend to the general case, Kowalski-Tricerri's and Bieszk's classifications of connected and simply-connected unimodular cyclic metric Lie groups for dimensions less than or equal to five.

Key words: Cyclic left-invariant metric, cyclic metric Lie group, homogeneous Riemannian structure.

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1 Introduction

For any metric Lie group (G, g), let $\mathfrak{g}$ be its Lie algebra, let ∇ denote the Levi-Civita connection and consider the left-invariant homogeneous structure S on (G, g) defined by $S_X Y = \nabla_X Y$, $X, Y \in \mathfrak{g}$.

We say that (G, g) is *cyclic* if the g-torsion of the (-)-connection of Cartan-Schouten $\widetilde{\nabla} := \nabla - S$ is cyclic (see [13]). This means that $S \in \mathcal{T}_1 \oplus \mathcal{T}_2$ in Tricerri and Vanhecke's classification (see [14]) or, equivalently, the corresponding inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ satisfies

$$\mathop{\mathfrak{S}}_{XYZ} \langle [X, Y], Z \rangle = 0,$$

for all $X, Y, Z \in \mathfrak{g}$. So nonabelian cyclic metric Lie groups are in some way far from being biinvariant Lie groups, for which the similarly defined structure S belongs to the last basic Tricerri and Vanhecke's class $\mathcal{T}_3$, or equivalently, the torsion of $\widetilde{\nabla}$ is totally skew-symmetric.

The homogeneous Riemannian spaces G/H that generalize in a natural way the cyclic metric Lie groups are the *cyclic* homogeneous Riemannian manifolds, that we have studied in [4].

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Note that Pfäffle and Stephan [12, p. 267] (see also Friedrich [3]) proved that the Dirac operator on a Riemannian manifold (M, g) does not contain any information on the Cartan-type component of the torsion of a generic Riemannian connection. In the case of homogeneous Riemannian manifolds, this is the $\mathcal{T}_2$-component of a homogeneous structure on the manifold. Hence, the connected and simply-connected traceless cyclic metric spin Lie groups have the simplest Dirac operator (that is, like that on a Riemannian symmetric spin space) among connected and simply-connected metric spin Lie groups.

The paper is organized as follows. In Section 2 we give some preliminaries on homogeneous Riemannian structures.

In Section 3 we consider cyclic metric Lie groups and exhibit some differences with biinvariant Lie groups. After proving (Proposition 3.4) that a connected cyclic metric Lie group is flat if and only if it is abelian, we study (Proposition 3.5) the different properties of sectional curvatures and scalar curvature according to the group being solvable, unimodular or nonunimodular.

The semisimple case is studied in Section 4. We prove (Lemma 4.1, Theorem 4.2) that every semisimple cyclic metric Lie group (and, more generally, every nonabelian cyclic metric Lie group) is not compact. Then we show (Corollary 4.3) that no cyclic metric Lie group has strictly positive sectional curvature and (Theorem 4.4) that the universal covering group of $SL(2, \mathbb{R})$ is the only connected, simply-connected simple real cyclic metric Lie group.

In Section 5 we consider the solvable case. After obtaining (formula (5.1) and Proposition 5.1) some specific properties useful in the classification for low dimensions, we characterize (Lemma 5.3) the cyclic metric Lie groups among semidirect products of cyclic metric Lie groups. Furthermore, we show (Proposition 5.4) that any solvable cyclic metric Lie group can be expressed as an orthogonal semidirect product. After proving (Proposition 5.5) that nonabelian nilpotent Lie groups do not admit any cyclic left-invariant metric, we construct (Examples 5.6 and 5.7) two multi-parameter families of solvable cyclic metric Lie groups.

The classification of connected, simply-connected nonabelian Lie groups G admitting a (nontrivial) traceless cyclic homogeneous structure, that is, for unimodular G, was given for dimensions three and four by Kowalski and Tricerri [8, Theorems 2.1, 3.1] and for (Lie algebras of) dimension five by Bieszk [2]. We extend their results, adding the corresponding nonunimodular Lie groups, in Theorems 6.1, 6.2 and 6.4.

2 Preliminaries

A homogeneous structure on a Riemannian manifold (M, g) is a tensor field S of type $(1, 2)$ satisfying $\widetilde{\nabla} g = \widetilde{\nabla} R = \widetilde{\nabla} S = 0$, where $\widetilde{\nabla}$ is (see [14]) the connection $\widetilde{\nabla} = \nabla - S$, ∇ being the Levi-Civita connection of g. The condition $\widetilde{\nabla} g = 0$ is equivalent to $S_{XYZ} = -S_{XZY}$, where $S_{XYZ} = g(S_X Y, Z)$.

Ambrose and Singer [1] gave the following characterization for homogeneous Riemannian manifolds: *A connected, simply-connected and complete Rieman-*

nian manifold (M,g) is homogeneous if and only if it admits a homogeneous structure S. Furthermore, considering $M = G/H$ as a reductive homogeneous manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then $\widetilde{\nabla} = \nabla - S$ is the canonical connection on M with respect to the given reductive decomposition (see [14]).

Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional Euclidean vector space. Tricerri and Vanhecke considered in [14] the vector space $\mathcal{T}(V)$ of tensors of type $(1,2)$, or equivalently (using the inner product $\langle \cdot, \cdot \rangle$) of type $(0,3)$ satisfying the same algebraic symmetry that a homogeneous structure, that is,

$$\mathcal{T}(V) = \{ S \in \otimes^3 V^* \, : \, S_{XYZ} = -S_{XZY} \, , \; X, Y, Z \in V \}.$$

They studied the decomposition of $\mathcal{T}(V)$ into invariant and irreducible components $\mathcal{T}_i(V)$, $i = 1,2,3$, under the action of the orthogonal group $O(n)$. The tensors S in the class $\mathcal{T}_1$ are those for which there exists $\xi \in V$ such that $S_X Y = \langle X, Y \rangle \xi - \langle \xi, Y \rangle X$. The tensors S in the class $\mathcal{T}_1 \oplus \mathcal{T}_2$ are those satisfying that the cyclic sum $\mathfrak{S}_{XYZ} S_{XYZ}$ vanishes. Those of type $\mathcal{T}_2$ are the ones satisfying moreover $c_{12}(S)(X) = 0$, for all $X \in V$, where $c_{12}(S)(X) = \sum_i S_{e_i e_i X}$ for an arbitrary orthonormal basis $\{e_i\}$ of V. Those of type $\mathcal{T}_3$ are the ones in $\mathcal{T}(V)$ satisfying $S_{XYZ} = -S_{YXZ}$.

A homogeneous structure S on (M,g) is said to be of type $\mathcal{T}$, $\mathcal{T}(V)$ being one of the eight invariant subspaces of $\mathcal{T}(V)$, if $S_p \in \mathcal{T}_p(T_p M)$ for all $p \in M$.

The torsion $\widetilde{T}$ of the connection $\widetilde{\nabla}$ is completely determined by the homogeneous structure S as follows:

$$\widetilde{T}_{XYZ} = S_{YXZ} - S_{XYZ}, \tag{2.1}$$

where $\widetilde{T}_{XYZ} = g(\widetilde{T}_X Y, Z)$. When it be necessary to refer to the metric, we shall say that $\widetilde{T}$, as a tensor field of type $(0,3)$, is the g-torsion of $\widetilde{\nabla}$. Hence,

$$2 \underset{XYZ}{\mathfrak{S}} S_{XYZ} = - \underset{XYZ}{\mathfrak{S}} \widetilde{T}_{XYZ} \quad \text{and} \quad c_{12}(S)(X) = \operatorname{tr} \, \widetilde{T}_X.$$

Conversely, since $\widetilde{\nabla}$ is a metric connection, S can be expressed in terms of $\widetilde{T}$ as ([7, p. 83])

$$2 S_{XYZ} = \widetilde{T}_{YXZ} + \widetilde{T}_{YZX} + \widetilde{T}_{XZY}. \tag{2.2}$$

Then, from (2.1) and (2.2) and according with the terminology in [13, p. 222], the g-torsion $\widetilde{T}$ is said to be

- *vectorial* if there exists a vector field ξ such that

$$\widetilde{T}_{XYZ} = g(X,Z) g(\xi, Y) - g(Y,Z) g(X, \xi),$$

  or equivalently if $S \in \mathcal{T}_1$;

- *cyclic* if $\mathfrak{S}_{XYZ} \widetilde{T}_{XYZ} = 0$, or equivalently if $S \in \mathcal{T}_1 \oplus \mathcal{T}_2$;

- *traceless* if tr $\widetilde{T}_X = 0$, or equivalently if $S \in \mathcal{T}_2 \oplus \mathcal{T}_3$;
- *traceless cyclic* if $\widetilde{T}$ is traceless and cyclic, or equivalently if $S \in \mathcal{T}_2$;
- *totally skew-symmetric* if $\widetilde{T}_{XYZ} = -\widetilde{T}_{XZY}$, or equivalently if $S \in \mathcal{T}_3$.

For the vectorial case, taking $\varphi(X) = g(\xi, X)$, one gets that $\widetilde{T}$ is vectorial if and only if there exists a one-form φ on M such that

$$\widetilde{T}_X Y = \varphi(Y)X - \varphi(X)Y.$$

Note that the properties of being vectorial or traceless do not depend on the metric g.

3 Cyclic metric Lie groups

Let G be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. It is well known (see [10]) that there exists a one-to-one correspondence between the set of left-invariant affine connections on G and the set of bilinear functions α on $\mathfrak{g}\times\mathfrak{g}$ with values in $\mathfrak{g}$. The correspondence is given by $\alpha(X,Y) = \nabla_X Y$, for all $X, Y \in \mathfrak{g}$. For $\alpha = 0$ one gets the canonical connection $\widetilde{\nabla}$, called the $(-)$-connection of Cartan-Schouten, i.e., $\widetilde{\nabla}_X Y = 0$, for all $X, Y \in \mathfrak{g}$. Its torsion tensor $\widetilde{T}$ is given by $\widetilde{T}_X Y = -[X,Y]$ and its curvature tensor $\widetilde{R}$ is identically zero. Let ∇ be the Levi-Civita connection of a left-invariant metric g on G. Then the homogeneous structure $S = \nabla - \widetilde{\nabla}$ is left-invariant and it is determined by $S_X Y = \nabla_X Y$, for $X, Y \in \mathfrak{g}$. Using the Koszul formula, or directly from (2.2), we have

$$2\langle S_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle, \tag{3.1}$$

for all $X, Y, Z \in \mathfrak{g}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathfrak{g}$ corresponding with g. Let $U\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ be the symmetric bilinear mapping defined by

$$2\langle U(X,Y), Z\rangle = \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle.$$

Then S is characterized by

$$S_X Y = \frac{1}{2}[X,Y] + U(X,Y). \tag{3.2}$$

Note that $\langle S_X Y, Z\rangle + \langle Y, S_X Z\rangle = 0$ and $S = 0$ if and only if $\mathfrak{g}$ is abelian, or equivalently, because $\widetilde{T} = 0$, if $\widetilde{\nabla}$ is a flat connection. The curvature tensor R of the Levi-Civita connection is given by

$$R(X,Y) = S_{[X,Y]} - [S_X, S_Y], \qquad X, Y \in \mathfrak{g}. \tag{3.3}$$

A metric g on G which is both left- and right-invariant is called biinvariant. This is equivalent to $\langle\cdot,\cdot\rangle$ being $\mathrm{Ad}(G)$-invariant or also, under our hypothesis of connectedness, to the g-torsion $\widetilde{T}$ being totally skew-symmetric. Hence, $S \in \mathcal{T}_3$, or equivalently, $U = 0$. Then, for any biinvariant metric, $S_X = \frac{1}{2}\mathrm{ad}_X$ and one gets

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z], \quad \kappa(X,Y) = \frac{1}{4}\|[X,Y]\|^2,$$

where κ is the curvature function $\kappa\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $\kappa(X,Y) = \langle R(X,Y)X, Y\rangle$. So, the sectional curvature K is always nonnegative and there exists a section $\pi = \mathbb{R}\{X,Y\}$ such that $K(\pi) = 0$ if and only if $[X,Y] = 0$. The Ricci tensor Ric on $\mathfrak{g}$ is given by

$$\mathrm{Ric} = -\frac{1}{4}B, \tag{3.4}$$

where B is the Killing form of G.

Definition 3.1. A left-invariant metric g on a Lie group G (or a metric Lie group (G,g)) is said to be *vectorial, cyclic* or *traceless cyclic* if the g-torsion of $\widetilde{\nabla}$ so is.

Then a metric Lie group (G,g) is vectorial if the bracket product on $\mathfrak{g}$ satisfies

$$[X,Y] = \varphi(X)Y - \varphi(Y)X,$$

where $\varphi \in \mathfrak{g}^*$, $\mathfrak{g}^*$ being the dual space of $\mathfrak{g}$. Because $\mathrm{tr}\ \widetilde{T}_X = -\mathrm{tr}\,\mathrm{ad}_X$, for all $X \in \mathfrak{g}$, it follows that $\widetilde{T}$ is traceless if and only if the Lie group G is unimodular. Hence, (G,g) is cyclic if and only if

$$\mathop{\mathfrak{S}}_{XYZ} \langle [X,Y],Z\rangle = 0$$

and it is traceless cyclic if moreover G is unimodular.

It has been proved in [14, Theorem 5.2] that a connected, simply-connected and complete Riemannian manifold admits a non-trivial homogeneous structure $S \in \mathcal{T}_1$ if and only it is isometric to the real hyperbolic space. Moreover, for any given dimension n, this homogeneous structure corresponds to the representation of the real hyperbolic space as the solvable Lie group

$$H^n(c) = \left\{ \begin{pmatrix} \mathrm{e}^{c\,u} I_{n-1} & x \\ 0 & 1 \end{pmatrix} \in GL(n,\mathbb{R}) : u \in \mathbb{R},\ x \in \mathbb{R}^{n-1} \right\},$$

equipped with a suitable left-invariant metric with constant sectional curvature $-c^2$. See Example 5.6 for more details. Hence, arguing as in the proof of Theorem 5.2 in [14], we have the following result.

Proposition 3.2. *Any simply-connected, nonabelian vectorial metric Lie group is isometrically isomorphic to $H^n(c)$, for some $c \neq 0$.*

For cyclic left-invariant metrics, using (3.1), one has that

$$S_{XYZ} = \langle \nabla_X Y, Z\rangle = \langle [X,Y],Z\rangle. \tag{3.5}$$

Then it follows that

$$\kappa(X,Y) = -\|[X,Y]\|^2 + \langle S_X Y, S_Y X\rangle - \langle S_X X, S_Y Y\rangle. \tag{3.6}$$

Proposition 3.3. *If X belongs to the center of the Lie algebra $\mathfrak{g}$ of a cyclic metric Lie group then $\kappa(X, Y) = 0$ for all $Y \in \mathfrak{g}$.*

Proof. By (3.5), $S_X Y = 0$, and the result then follows from (3.6). □

From (3.3), any left-invariant metric on an abelian Lie group is flat, i.e., its Riemannian sectional curvature vanish. Next, we prove that the converse holds for the cyclic left-invariant case.

Proposition 3.4. *A connected cyclic metric Lie group is flat if and only if it is abelian.*

Proof. The Lie algebra $\mathfrak{g}$ of a Lie group equipped with a flat left-invariant metric splits as an orthogonal direct sum $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$ such that $\mathfrak{b}$ is an abelian subalgebra, $\mathfrak{u}$ is an abelian ideal and the linear transformation ad_B is skew-symmetric for every $B \in \mathfrak{b}$ (see [9, Theorem 1.5]). But using that the metric is cyclic left-invariant, it follows that ad_B is also selfadjoint. Hence, one has $\mathrm{ad}_B(X) = 0$, for all $X \in \mathfrak{g}$, and so the Lie group is abelian. □

Proposition 3.5. *Let G be a nonabelian cyclic metric Lie group. We have:*

(i) *If G is solvable then it has strictly negative scalar curvature.*

(ii) *If G is unimodular then there exist positive sectional curvatures. If moreover it is solvable, it has both positive and negative sectional curvatures.*

(iii) *If G is not unimodular there exist negative sectional curvatures.*

Proof. Properties (i) and (ii) follow from Proposition 3.4 together with Theorem 1.6, Theorem 3.1 and Corollary 3.2 in [9]. If G is not unimodular, its unimodular kernel $\mathfrak{u}$, that is,

$$\mathfrak{u} = \{X \in \mathfrak{g} \,:\, \mathrm{tr}\,\mathrm{ad}_X = 0\},$$

is an ideal of codimension one. Let W be a unit vector orthogonal to $\mathfrak{u}$. Then

$$\nabla_W W = 0, \quad \nabla_W X = \frac{1}{2}(\mathrm{ad}_W - \mathrm{ad}_W^t)X,$$

for all $X \in \mathfrak{u}$ (see [9] for the details). Hence, if the inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of G is cyclic left-invariant, it follows from (3.5) that $\mathrm{ad}_{W|\mathfrak{u}}$ is a selfadjoint operator and $S_W = 0$. So, from (3.6), $K(W, X) = -\|[W, X]\|^2$, for all $X \in \mathfrak{u}$. Since there exists $X \in \mathfrak{u}$ such that $\mathrm{ad}_W X \neq 0$, one gets $K(W, X) < 0$. This proves (iii). □

4 Semisimple cyclic metric Lie groups

The Killing form B of a semisimple Lie group G provides a biinvariant scalar product making G a biinvariant pseudo-Riemannian Lie group, which is, by (3.4), an Einstein manifold. When G is moreover compact, B is negative definite and $\langle \cdot, \cdot \rangle = -B(\cdot, \cdot)$ determines a biinvariant (Riemannian) metric.

For a general Lie group G with a biinvariant Riemannian metric determined by an inner product $\langle\cdot,\cdot\rangle$ on its Lie algebra $\mathfrak{g}$, the orthogonal complement of any ideal in $\mathfrak{g}$ is itself an ideal. So $\mathfrak{g}$ can be expressed as an orthogonal direct sum

$$\mathfrak{g} = \mathcal{Z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell,$$

where its center $\mathcal{Z}(\mathfrak{g})$ is isomorphic to $\mathbb{R}^k$ for some k, and $\mathfrak{g}_1, \dots, \mathfrak{g}_\ell$ are compact simple ideals (see [9] for details). Then $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is of the form

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0 + \beta_1 B_1 + \cdots + \beta_\ell B_\ell,$$

where $\langle\cdot,\cdot\rangle_0$ is the standard inner product on $\mathbb{R}^k$, B_i, $i = 1, \dots, \ell$, is the restriction of the Killing form B to $\mathfrak{g}_i \times \mathfrak{g}_i$ and $\beta_i < 0$. Hence, taking into account that $\mathcal{Z}(\mathfrak{g}) = 0$ for semisimple Lie groups, it follows that a semisimple Lie group is biinvariant if and only if it is compact.

By contrast, we have in the cyclic left-invariant case the next result.

Lemma 4.1. *Every semisimple cyclic metric Lie group is not compact. Moreover, each cyclic left-invariant metric is determined by an inner product $\langle\cdot,\cdot\rangle$ on its Lie algebra making orthogonal an arbitrary B-orthonormal basis $\{e_i\}_{i=1}^n$ and $\langle e_i, e_i\rangle = \varepsilon_i\lambda_i$, $i = 1, \dots, n$, where $\varepsilon_i = B(e_i, e_i)$ and the λ_i's satisfy*

$$c_{ij}^k(\lambda_i + \lambda_j + \lambda_k) = 0, \qquad 1 \le i < j < k \le n, \tag{4.1}$$

c_{ij}^k being the structure constants given by $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$.

Proof. Let G be a semisimple Lie group and let $\langle\cdot,\cdot\rangle$ be a cyclic left-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. Then $\langle\cdot,\cdot\rangle$ can be expressed in terms of its Killing form B as

$$\langle X, Y\rangle = B(QX, Y), \qquad X, Y \in \mathfrak{g},$$

where Q is some selfadjoint operator on $\mathfrak{g}$. Let $\{e_1, \dots, e_n\}$ be a B-orthonormal basis of eigenvectors of Q and let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. Since $\langle\cdot,\cdot\rangle$ is nondegenerate, $\lambda_i \neq 0$ for all $i \in \{1, \dots, n\}$. Then $\{e_1, \dots, e_n\}$ is an orthogonal basis with respect to $\langle\cdot,\cdot\rangle$ satisfying $\langle e_i, e_i\rangle = \varepsilon_i\lambda_i$, $i = 1, \dots, n$, where $\varepsilon_i = B(e_i, e_i)$; with $\varepsilon_i = -1$ if $\lambda_i < 0$ and $\varepsilon_i = 1$ if $\lambda_i > 0$. Hence, the structure constants c_{ij}^k are given by $c_{ij}^k = \varepsilon_k B([e_i, e_j], e_k)$. Putting $\bar{c}_{ij}^k = \varepsilon_k c_{ij}^k$, one gets

$$\bar{c}_{ij}^k = -\bar{c}_{ji}^k, \quad \bar{c}_{ij}^k = -\bar{c}_{ik}^j. \tag{4.2}$$

The condition of cyclic left-invariance for $\langle\cdot,\cdot\rangle$ is equivalent to

$$\lambda_k\bar{c}_{ij}^k + \lambda_j\bar{c}_{ki}^j + \lambda_i\bar{c}_{jk}^i = 0.$$

Hence, using (4.2), we have

$$\bar{c}_{ij}^k(\lambda_i + \lambda_j + \lambda_k) = 0.$$

As $\mathcal{Z}(\mathfrak{g}) = 0$, it follows that for each $i \in \{1, \ldots, n\}$, $\bar{c}_{ij}^k$ is different from zero for some j and k, and so, $\lambda_i = -(\lambda_j + \lambda_k)$. This implies that not all the λ_i's have the same sign and so, B is not definite. Hence, G cannot be compact.

Every cyclic inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is then obtained taking as Q the operator given by $Qe_i = \lambda_i e_i$, where $\{e_i\}_{i=1}^n$ is an arbitrary B-orthonormal basis and the λ_i's satisfy our hypothesis. This proves the result. □

It is well known that every compact Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathcal{Z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, where the ideal $[\mathfrak{g}, \mathfrak{g}]$ is compact and semisimple. Since the restriction of a cyclic-invariant inner product to $[\mathfrak{g}, \mathfrak{g}]$ must be again cyclic-invariant, from Lemma 4.1 the following result follows.

Theorem 4.2. *Every nonabelian cyclic metric Lie group is not compact.*

Since $SU(2)$ is the only connected, simply-connected Lie group which admits a left-invariant metric of strictly positive sectional curvature (see [15]), one directly obtains

Corollary 4.3. *There is no cyclic metric Lie group with strictly positive sectional curvature.*

For the simple case we have the following result.

Theorem 4.4. *The universal covering group $\widetilde{SL(2, \mathbb{R})}$ of $SL(2, \mathbb{R})$ is the only connected, simply-connected simple real cyclic metric Lie group.*

Proof. Let $\mathfrak{g}_{\mathbb{C}}$ be a simple Lie algebra over $\mathbb{C}$ and let σ be an involutive automorphism of a compact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$, where $\mathfrak{u}$ and $\mathfrak{p}$ denote the eigenspaces of σ with eigenvalues $+1$ and -1, respectively. Let $(\mathfrak{g}^*, \sigma^*)$ be the dual orthogonal symmetric Lie algebra of $(\mathfrak{g}, \sigma)$, $\mathfrak{g}^*$ being the subspace of $\mathfrak{g}_{\mathbb{C}}$ defined by the Cartan decomposition $\mathfrak{g}^* = \mathfrak{u} \oplus i\mathfrak{p}$.

If $\mathfrak{g}^*$ admits a cyclic left-invariant inner product, since the Killing form B is strictly negative definite on the maximal compactly embedded subalgebra $\mathfrak{u}$, it follows from (4.1) that $\mathfrak{u}$ must be abelian. Now, according for instance to [6, pp. 695–718], the only simple real Lie algebras whose maximal compact Lie subalgebra is abelian are $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{su}(1, 1)$, $\mathfrak{so}(2, 1)$ and $\mathfrak{sp}(1, \mathbb{R})$, which are mutually isomorphic and define the connected, simply-connected simple Lie group $\widetilde{SL(2, \mathbb{R})}$. □

5 Solvable cyclic metric Lie groups

Let G be an n-dimensional solvable Lie group. Then its Lie algebra $\mathfrak{g}$ satisfies the chain condition or, equivalently, there exists a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_{n-1} \supset \mathfrak{g}_n = \{0\},$$

where $\mathfrak{g}_r$ is an ideal in $\mathfrak{g}_{r-1}$ of codimension 1, $1 \leq r \leq n$. Given an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ we construct an orthonormal basis $\{e_1, \ldots, e_n\}$, called *adapted* *to* $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, *such that* $\mathfrak{g}_{n-i} = \mathbb{R}\{e_1, \ldots, e_i\}$, $i = 1, \ldots, n$. Then the structure constants $c_{ij}^k = \langle [e_i, e_j], e_k \rangle$ of G with respect to this basis satisfy

$$c_{ij}^k = 0 \quad \text{for } k \geq \max\{i, j\}. \tag{5.1}$$

Hence, we have the next result.

Proposition 5.1. *The following conditions are equivalent for an n-dimensional solvable Lie group G.*

(i) *G is a cyclic metric Lie group.*

(ii) $c_{ik}^j = c_{jk}^i$, *for all* $1 \leq i < j < k \leq n$.

(iii) ad_{e_i} *is selfadjoint on* $\mathfrak{g}_{n-i}$, *for all* $i = 1 \ldots, n$.

Remark 5.2. The property (5.1) together with Proposition 5.1 (ii) determine completely the structure constants of any solvable cyclic metric Lie group. Then they can be useful to give examples and in fact to obtain classifications for low dimensions.

Let G_1 and G_2 be Lie groups equipped with left-invariant metrics determined by inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on their corresponding Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. For each homomorphism π of G_1 into the group $\mathrm{Aut}(G_2)$ of automorphisms of G_2, consider the semidirect product $G_1 \ltimes_\pi G_2$ equipped with the Riemannian product on $G_1 \times G_2$. Its Lie algebra is the semidirect sum $\mathfrak{g}_1 +_{\pi_*} \mathfrak{g}_2$ of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. The differential π_* of π is a Lie algebra homomorphism $\mathfrak{g}_1 \to \mathrm{Der}(\mathfrak{g}_2)$ and the bracket product on $\mathfrak{g}_1 +_{\pi_*} \mathfrak{g}_2$ is given by

$$[(X_1, X_2), (Y_1, Y_2)] = \big([X_1, Y_1], [X_2, Y_2] + \pi_*(X_1)(Y_2) - \pi_*(Y_1)(X_2)\big),$$

for all $X_i, Y_i \in \mathfrak{g}_i$, $i = 1, 2$. Since the inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ on $\mathfrak{g}_1 +_{\pi_*} \mathfrak{g}_2$ satisfies

$$\begin{aligned} \langle [(X_1, X_2), (Y_1, Y_2)], (Z_1, Z_2) \rangle \quad &= \quad \langle [X_1, Y_1], Z_1 \rangle_1 + \langle [X_2, Y_2], Z_2 \rangle_2 \\ &\quad + \langle \pi_*(X_1)(Y_2) - \pi_*(Y_1)(X_2), Z_2 \rangle_2. \end{aligned}$$

one gets the following result.

Lemma 5.3. *Let G_1 and G_2 be Lie groups, each of them equipped with a left-invariant metric. The Riemannian product metric on $G_1 \times G_2$ defines a cyclic left-invariant metric on the semidirect product $G_1 \ltimes_\pi G_2$ if and only if the left-invariant metrics on G_1 and G_2 are cyclic and the derivation $\pi_*(X_1)$ on $\mathfrak{g}_2$, for each $X_1 \in \mathfrak{g}_1$, is selfadjoint with respect to $\langle \cdot, \cdot \rangle_2$.*

Next, we show that any solvable cyclic metric Lie group can be expressed as an orthogonal semidirect product. Concretely, we have the next result.

Proposition 5.4. *Any nontrivial connected, simply-connected solvable cyclic metric Lie group decomposes into an orthogonal semidirect product $\mathbb{R} \ltimes_\pi N$, where N is a unimodular normal Lie subgroup of codimension one and $\pi_*(\mathrm{d/dt})$ $= \mathrm{ad}_{\mathrm{d/dt}}$ is a selfadjoint derivation on the Lie algebra of N.*

Proof. Any (real) nonunimodular metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, not necessarily solvable, can be expressed as an orthogonal semidirect sum $\mathfrak{g} = \mathbb{R}W +_{\mathrm{ad}_W} \mathfrak{u}$, where $\mathfrak{u}$ is its unimodular kernel. For the unimodular case, we remark that if $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a nontrivial solvable Lie algebra equipped with a cyclic left-invariant inner product and $\{e_1, \ldots, e_n\}$ is an adapted orthonormal basis, then $\mathfrak{g}$ splits into the orthogonal semidirect sum of $\mathbb{R}e_n$ and the ideal $\mathfrak{g}_1 = \mathbb{R}\{e_1, \ldots, e_{n-1}\}$, the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{g}_1$ is also cyclic left-invariant and ad_{e_n} is selfadjoint on $\mathfrak{g}_1$. Hence, the connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$ satisfies the conditions in the statement. ☐

Proposition 5.5. *Nonabelian nilpotent Lie groups do not admit cyclic left-invariant metrics.*

Proof. Consider the central descending series of the Lie algebra $\mathfrak{g}$ of a Lie group G,

$$\mathfrak{g} = \mathcal{C}^0\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \cdots \supset \mathcal{C}^{m-1}\mathfrak{g} \supset \mathcal{C}^m\mathfrak{g} = \{0\},$$

defined by $\mathcal{C}^{p+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^p\mathfrak{g}]$, $p = 0, 1, \ldots, m-1$. Then $\mathcal{C}^{m-1}\mathfrak{g} \neq \{0\}$, it is in the center $\mathcal{Z}(\mathfrak{g})$ of $\mathfrak{g}$ and, if $m \geq 2$, then $\mathcal{C}^{m-1}\mathfrak{g} \subset [\mathfrak{g}, \mathfrak{g}]$. Because $\mathcal{Z}(\mathfrak{g})$ and the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ are orthogonal with respect to any cyclic left-invariant metric, m must be 1 and this implies that $\mathfrak{g}$ is abelian. ☐

Example 5.6. The metric solvable Lie group $G^n(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$.

For $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \in \mathbb{R}^{n-1} \setminus \{(0, \ldots, 0)\}$, let $\mathfrak{g} = \mathfrak{g}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ be the n-dimensional metric Lie algebra generated by the basis $\{e_1, \ldots, e_n\}$ with Lie brackets

$$[e_n, e_i] = \alpha_i e_i, \quad 1 \leq i \leq n-1\,; \quad [e_i, e_j] = 0, \quad 1 \leq i < j \leq n-1,$$

and equipped with the inner product $\langle\cdot,\cdot\rangle$ for which $\{e_1, \ldots, e_n\}$ is orthonormal. Then $\mathfrak{g}$ can be identified with the orthonormal semidirect sum $\mathbb{R}\{e_n\} +_{\mathrm{ad}_{e_n}} \mathbb{R}\{e_1, \ldots, e_{n-1}\}$ under the adjoint representation. It is unimodular if and only if $\sum_{i=1}^{n-1} \alpha_i = 0$.

The connected and simply-connected Lie group G generated by $\mathfrak{g}$ must be isomorphic to the orthogonal semidirect product $\mathbb{R} \ltimes_\pi \mathbb{R}^{n-1}$ under the action $\pi\colon \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^{n-1})$ given by $\pi(t) = \mathrm{diag}\,(\mathrm{e}^{\alpha_1 t}, \mathrm{e}^{\alpha_2 t}, \ldots, \mathrm{e}^{\alpha_{n-1} t})$. Because the structure constants c_{ij}^k with respect to $\{e_i\}_{i=1}^n$ satisfy (5.1), G is a solvable metric Lie group with $\{e_i\}_{i=1}^n$ as an adapted orthonormal basis. Moreover, from Proposition 5.1 or using Lemma 5.3, $\langle\cdot,\cdot\rangle$ determines a cyclic left-invariant metric. The group G can be described as the group $G^n(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ of matrices of the form

$$\begin{pmatrix} \mathrm{e}^{\alpha_1 u} & 0 & \cdots & 0 & x_1 \\ 0 & \mathrm{e}^{\alpha_2 u} & \cdots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathrm{e}^{\alpha_{n-1} u} & x_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We can consider the global coordinate system $(u, x_1, \ldots, x_{n-1})$ of this matrix group, and for each $A \in G^n(\alpha_1, \ldots, \alpha_{n-1})$, we have $u \circ L_A = u(A) + u$, $x_i \circ L_A = x_i(A) + \mathrm{e}^{\alpha_i u(A)} x_i$, $1 \leq i \leq n-1$. Hence the left-invariant Riemannian metric g on $G^n(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ defined by the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is

$$g = \mathrm{d}u^2 + \sum_{i=1}^{n-1} \mathrm{e}^{-2\alpha_i u} \mathrm{d}x_i^2.$$

The Levi-Civita connection ∇ of g is given by $\nabla_{e_n} e_n = \nabla_{e_n} e_i = 0$, $\nabla_{e_i} e_n = -\alpha_i e_i$, $\nabla_{e_i} e_j = \delta_{ij} \alpha_i e_n$, $1 \leq i, j \leq n-1$. This implies that $V = e_n$ is a geodesic vector and, from [5, Proposition 6.1], it is a harmonic vector field. Moreover, it defines a harmonic immersion into the unit tangent sphere of $G^n(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ if and only if $\sum_{i=1}^{n-1} \alpha_i^3 = 0$. The curvature tensor field satisfies $R_{e_i e_n} e_i = -\alpha_i^2 e_n$, $R_{e_i e_j} e_i = -\alpha_i \alpha_j e_j$, $1 \leq i \neq j \leq n-1$. Then $\{e_1, \ldots, e_n\}$ is a basis of eigenvectors for the Ricci tensor and the principal Ricci curvatures are

$$r(e_n) = -\sum_{j=1}^{n-1} \alpha_j^2, \quad r(e_i) = -\alpha_i \sum_{j=1}^{n-1} \alpha_j, \quad 1 \leq i \leq n-1. \tag{5.2}$$

The scalar curvature is

$$s = -2\Big(\sum_{i=1}^{n-1} \alpha_i^2 + \sum_{i<j}^{n-1} \alpha_i \alpha_j \Big).$$

For the sectional curvatures of basic sections we have $K(e_i, e_n) = -\alpha_i^2$, $K(e_i, e_j) = -\alpha_i \alpha_j$, $1 \leq i \neq j \leq n-1$.

If $G^n(\alpha_1, \ldots, \alpha_{n-1})$ is unimodular, i.e., $\sum_{i=1}^{n-1} \alpha_i = 0$, there exist $i, j \in \{1, \ldots, n-1\}$ such that $\alpha_i \alpha_j < 0$. Then, according with Proposition 3.5, one finds both positive and negative sectional curvatures. If $\alpha_1 = \cdots = \alpha_{n-1} = \alpha \neq 0$, the metric Lie group $G^n(\alpha_1, \ldots, \alpha_{n-1})$ gives the solvable description $H^n(\alpha)$ of the n-dimensional hyperbolic space with constant sectional curvature $-\alpha^2$.

Example 5.7. The metric solvable Lie group $H^{n+1}(\rho_1, \ldots, \rho_{n-1}; \lambda_1, \ldots, \lambda_{n-2})$.

Following Proposition 5.4, we determine the connected, simply-connected, solvable cyclic metric Lie groups, which are nontrivial one-dimensional orthogonal extensions of the unimodular Lie group $G^n(\alpha_1, \ldots, \alpha_{n-1})$, $(\alpha_1, \ldots, \alpha_{n-1}) \neq (0, \ldots, 0)$, $\alpha_1 + \cdots + \alpha_{n-1} = 0$.

Let (H, g) be one such extension. The corresponding metric Lie algebra $\mathfrak{h}$ is then an orthogonal semidirect sum $\mathfrak{h} = \mathbb{R}\{e_0\} +_{\mathrm{ad}_{e_0}} \mathbb{R}\{e_1, \ldots, e_n\}$, where $\{e_0, e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathfrak{h}$ such that $[e_n, e_i] = \alpha_i e_i$, $1 \leq i \leq n-1$, $[e_i, e_j] = 0$, $1 \leq i < j \leq n-1$, and ad_{e_0} must act on $\mathbb{R}\{e_1, \ldots, e_n\}$ as a selfadjoint operator, because g must be cyclic left-invariant. In terms of $\{e_1, \ldots, e_n\}$ we can write $\mathrm{ad}_{e_0} = (a_i^j)_{1 \leq i,j \leq n}$, $a_i^j = a_j^i$. Since ad_{e_0} must also act as a derivation on $\mathbb{R}\{e_1, \ldots, e_n\}$, applying ad_{e_0} to $[e_i, e_j]$ one has $a_i^n \alpha_j = 0$,

$1 \le i \neq j \le n-1$, and applying ad_{e_0} to $[e_n, e_j]$, we get $a_i^n \alpha_i = 0$, $a_n^n \alpha_i = 0$, $1 \le i \le n-1$, then $a_n^k = a_k^n = 0$ for $1 \le k \le n$, and hence $[e_0, e_n] = 0$. Thus ad_{e_0} and ad_{e_n} are commuting operators of $\mathbb{R}\{e_1, \dots, e_{n-1}\}$, hence there exist an orthonormal basis $\{v_1, \dots, v_{n-1}\}$ of $\mathbb{R}\{e_1, \dots, e_{n-1}\}$ which consists of eigenvectors for both ad_{e_0} and ad_{e_n}. We put $u_0 = e_0$, $v_0 = e_n$, and then $\{u_0, v_0, v_1, \dots, v_{n-1}\}$ is an orthonormal basis of $\mathfrak{h}$ such that

$$[u_0, v_i] = \rho_i v_i, \quad [v_0, v_i] = \lambda_i v_i, \qquad 1 \le i \le n-1,$$

for some $(\rho_1, \dots, \rho_{n-1}) \neq (0, \dots, 0)$ (because ad_{e_0} must act as a nontrivial operator on $\mathbb{R}\{e_1, \dots, e_{n-1}\}$), and $(\lambda_1, \dots, \lambda_{n-1}) \neq (0, \dots, 0)$, with $\lambda_1 + \cdots + \lambda_{n-1} = 0$ (because $\mathrm{tr}\,\mathrm{ad}_{e_n} = \alpha_1 + \cdots + \alpha_{n-1} = 0$). Therefore, $\mathfrak{h}$ is an orthogonal semidirect sum of its abelian subalgebras $\mathbb{R}\{u_0, v_0\}$ and $\mathbb{R}\{v_1, \dots, v_{n-1}\}$, and the connected simply-connected Lie group $H = H^{n+1}(\rho_1, \dots, \rho_{n-1}; \lambda_1, \dots, \lambda_{n-2})$ generated by $\mathfrak{h}$ is the semidirect product $\mathbb{R}^2 \ltimes_\pi \mathbb{R}^{n-1}$ under the action $\pi \colon \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{R}^n)$, given by $\pi(s,t) = \mathrm{diag}(\mathrm{e}^{\rho_1 s + \lambda_1 t}, \dots, \mathrm{e}^{\rho_{n-1} s + \lambda_{n-1} t})$, where $\lambda_{n-1} = -(\lambda_1 + \cdots + \lambda_{n-2})$. Then H can be described as $\mathbb{R}^{n+1}$ with the group operation

$$\begin{aligned}(u, v, x_1, \dots, x_{n-1}) &\cdot (u', v', x_1', \dots, x_{n-1}') \\ &= (u + u',\, v + v',\, x_1 + \mathrm{e}^{\rho_1 u + \lambda_1 v} x_1', \dots, x_{n-1} + \mathrm{e}^{\rho_{n-1} u + \lambda_{n-1} v} x_{n-1}'),\end{aligned}$$

where $(\rho_1, \dots, \rho_{n-1})$ and $(\lambda_1, \dots, \lambda_{n-1})$ are different from $(0, \dots, 0)$. The Lie group H is unimodular if and only if $\sum_{i=1}^{n-1} \rho_i = 0$.

If $(\rho_1, \dots, \rho_{n-1})$ is not a multiple of $(\lambda_1, \dots, \lambda_{n-1})$ (in particular if H is not unimodular) then H is the group of matrices of the form

$$\begin{pmatrix} \mathrm{e}^{\rho_1 u + \lambda_1 v} & \cdots & 0 & x_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \mathrm{e}^{\rho_{n-1} u + \lambda_{n-1} v} & x_{n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5.3}$$

In any case, since $(\lambda_1, \dots, \lambda_{n-2}) \neq (0, \dots, 0)$, we can suppose (reordering the vectors $v_1, \dots, v_{n-1}$ if necessary) that $\lambda_1 \neq 0$, and define $\hat{u}_0 = (1/\sqrt{\lambda_1^2 + \rho_1^2})$ $(\lambda_1 u_0 - \rho_1 v_0)$, $\hat{v}_0 = (1/\sqrt{\lambda_1^2 + \rho_1^2})(\rho_1 u_0 + \lambda_1 v_0)$. Then

$$[\hat{u}_0, v_i] = \sigma_i v_i, \quad [\hat{v}_0, v_i] = \mu_i v_i, \qquad 1 \le i \le n-1,$$

with $\sigma_i = (\lambda_1 \rho_i - \rho_1 \lambda_i)/\sqrt{\lambda_1^2 + \rho_1^2}$, $\mu_i = (\rho_1 \rho_i + \lambda_1 \lambda_i)/\sqrt{\lambda_1^2 + \rho_1^2}$. We have $\sigma_1 = 0$, $\mu_1 \neq 0$, then H can be described as the Lie group $\hat{H}^{n+1}(\sigma_2, \dots, \sigma_{n-1}; \mu_1, \dots, \mu_{n-1})$ of matrices of the form

$$\begin{pmatrix} \mathrm{e}^{\mu_1 v} & 0 & \cdots & 0 & x_1 \\ 0 & \mathrm{e}^{\sigma_2 u + \mu_2 v} & \cdots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{\sigma_{n-1} u + \mu_{n-1} v} & x_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5.4}$$

In particular, if H is unimodular, one gets that $\sigma_{n-1} = -\sum_{i=2}^{n-2}\sigma_i$, $\mu_{n-1} = -\sum_{i=1}^{n-2}\mu_i$, and the family of such unimodular metric Lie groups depends on $2n-5$ parameters.

We consider the global coordinate system $(u, v, x_1, \ldots, x_{n-1})$ of the matrix Lie group $H^{n+1}(\rho_1, \ldots, \rho_{n-1}; \lambda_1, \ldots, \lambda_{n-2})$. We put again $\lambda_{n-1} = -(\lambda_1 + \cdots + \lambda_{n-2})$. The generators of $\mathfrak{h}$ correspond to the left-invariant vector fields $u_0 = \partial/\partial u$, $v_0 = \partial/\partial v$, $v_i = \mathrm{e}^{\rho_i u + \lambda_i v}\partial/\partial x_i$, $1 \le i \le n-1$. The left-invariant Riemannian metric g is given by

$$g = \mathrm{d}u^2 + \mathrm{d}v^2 + \sum_{i=1}^{n-1} \mathrm{e}^{-2(\rho_i u + \lambda_i v)}\mathrm{d}x_i^2.$$

In terms of the basis $\{u_0, v_0, v_1, \ldots, v_{n-1}\}$ of $\mathfrak{h}$, the Levi-Civita connection ∇ of g is given by $\nabla_{v_i}u_0 = -\rho_i v_i$, $\nabla_{v_i}v_0 = -\lambda_i v_i$, $\nabla_{v_i}v_i = \rho_i u_0 + \lambda_i v_0$, $1 \le i \le n-1$, the other components being zero. The curvature tensor field satisfies

$$\begin{aligned}
&R_{u_0 v_i}u_0 = -\rho_i^2 v_i, \quad R_{v_0 v_i}v_0 = -\lambda_i^2 v_i, \quad R_{u_0 v_i}v_0 = R_{v_0 v_i}u_0 = -\lambda_i\rho_i v_i,\\
&R_{u_0 v_i}v_i = \rho_i(\rho_i u_0 + \lambda_i v_0), \quad R_{v_0 v_i}v_i = \lambda_i(\rho_i u_0 + \lambda_i v_0),\\
&R_{v_i v_j}v_i = -(\lambda_i\lambda_j + \rho_i\rho_j)v_j, \quad R_{v_i v_j}v_j = (\lambda_i\lambda_j + \rho_i\rho_j)v_i,
\end{aligned}$$

for $1 \le i \ne j \le n$, the other components being zero. Then the sectional curvatures of the basic sections are given by

$$\begin{aligned}
&K(u_0, v_0) = 0, \quad K(u_0, v_i) = -\rho_i^2, \quad K(v_0, v_i) = -\lambda_i^2, && 1 \le i \le n-1,\\
&K(v_i, v_j) = -(\rho_i\rho_j + \lambda_i\lambda_j), && 1 \le i \ne j \le n-1,
\end{aligned}$$

and the nonvanishing components of the Ricci curvature by

$$\begin{aligned}
&\mathrm{Ric}(u_0, u_0) = -\textstyle\sum_{i=1}^{n-1}\rho_i^2, \quad \mathrm{Ric}(v_0, v_0) = -\textstyle\sum_{i=1}^{n-1}\lambda_i^2,\\
&\mathrm{Ric}(v_i, v_i) = -\rho_i\textstyle\sum_{j=1}^{n-1}\rho_j, \quad \mathrm{Ric}(u_0, v_0) = -\textstyle\sum_{i=1}^{n-1}\lambda_i\rho_i.
\end{aligned}$$

The scalar curvature is then

$$s = -\sum_{i=1}^{n-1}\lambda_i^2 - \sum_{i=1}^{n-1}\rho_i^2 - \Big(\sum_{i=1}^{n-1}\rho_i\Big)^2.$$

If $H^{n+1}(\rho_1, \ldots, \rho_{n-1}; \lambda_1, \ldots, \lambda_{n-2})$ is unimodular, that is $\sum_{i=1}^{n-1}\rho_i = 0$, then the signature of the Ricci form is $(0, \overset{n-1}{\ldots}, 0, 0, -)$ or $(0, \overset{n-1}{\ldots}, 0, -, -)$, depending on whether or not $(\lambda_1, \ldots, \lambda_{n-2})$ and $(\rho_1, \ldots, \rho_{n-2})$ are proportional, respectively.

If $\rho_i = a\lambda_i$ for some $a \ne 0$, and for each $i = 1, \ldots, n-2$, the metric Lie group $H^{n+1}(\rho_1, \ldots, \rho_{n-1};\ \lambda_1, \ldots, \lambda_{n-2})$ is isometrically isomorphic to the direct product $G^n(\tilde\lambda_1, \ldots, \tilde\lambda_{n-2}, -\sum_{i=1}^{n-2}\tilde\lambda_i) \times \mathbb{R}$, with $\tilde\lambda_i = \sqrt{1+a^2}\lambda_i$, $i = 1, \ldots, n-2$.

6 Classification of connected, simply-connected cyclic metric Lie groups for dimensions $n \leq 5$

Because any homogeneous structure on a two-dimensional Riemannian manifold is of type $\mathcal{T}_1$ (see [14, Theorem 3.1]), it follows that every two-dimensional (nonabelian) metric Lie group is vectorial. Moreover, for the simply-connected case, Proposition 3.2 implies that it is isometrically isomorphic to the Poincaré half-plane $H^2(c)$, for some $c \neq 0$.

Next, using the same process than in [14, pp. 83-85], we shall give the classification for dimension three. First, suppose that (G, g) is a three-dimensional unimodular metric Lie group. Then there exists (cf. [9]) an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ of G such that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3, \tag{6.1}$$

where λ_1, λ_2, λ_3 are constants. The Ricci tensor Ric is given by

$$\mathrm{Ric} = 2\big(\mu_2\mu_3\,\theta^1 \otimes \theta^1 + \mu_1\mu_3\,\theta^2 \otimes \theta^2 + \mu_1\mu_2\,\theta^3 \otimes \theta^3\big),$$

where θ^i, $i = 1, 2, 3$, are the 1-forms dual to e_i, and $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$, $i = 1, 2, 3$. According with the signs of λ_i, $i = 1, 2, 3$, we have six kinds of Lie algebras.

If g is cyclic left-invariant, one gets from (6.1) that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This implies that $\mathfrak{g}$ is isomorphic to either $\mathfrak{sl}(2, \mathbb{R})$ with $\lambda_3 = -(\lambda_1 + \lambda_2)$, $\lambda_1, \lambda_2 > 0$, or to $\mathfrak{e}(1, 1)$ with $\lambda_1 = -\lambda_3 > 0$ and $\lambda_2 = 0$. In both cases, one gets $\mu_i = -\lambda_i$, $i = 1, 2, 3$. Then the signature of Ric is $(-, -, +)$ for $\mathfrak{sl}(2, \mathbb{R})$ and $(0, 0, -)$ for $\mathfrak{e}(1, 1)$. They give the simply-connected, traceless cyclic metric Lie groups $E(1, 1)$ and the universal covering group $\widetilde{SL(2, \mathbb{R})}$ of $SL(2, \mathbb{R})$ described by Kowalski and Tricerri [8, Theorem 2.1] in the cases (b) and (c), respectively.

The family of all cyclic left-invariant metrics on $\widetilde{SL}(2, \mathbb{R})$ can be explicitly determined as the inner products on $\mathfrak{so}(1, 2)$, under the natural identification with $\mathfrak{sl}(2, \mathbb{R})$, making orthonormal the bases $\{e_1, e_2, e_3\}$ given by

$$\begin{aligned} e_1 &= \sqrt{\lambda_2(\lambda_1 + \lambda_2)}(E_{12} + E_{21}), \\ e_2 &= \sqrt{\lambda_1(\lambda_1 + \lambda_2)}(E_{13} + E_{31}), \\ e_3 &= \sqrt{\lambda_1\lambda_2}(E_{32} - E_{23}), \end{aligned} \tag{6.2}$$

for all $\lambda_1, \lambda_2 > 0$, where E_{ij} denote the matrix on $\mathfrak{so}(1, 2)$ with entry 1 where the i th row and the j th column meet, all other entries being 0.

Next, let G be a nonunimodular Lie group. Because the unimodular kernel $\mathfrak{u}$ of $\mathfrak{g}$ is abelian, there exists an orthonormal basis $\{e_1, e_2, e_3\}$, with $\mathfrak{u} = \mathbb{R}\{e_2, e_3\}$, such that

$$[e_1, e_2] = ae_2 + be_3, \quad [e_1, e_3] = ce_2 + de_3, \quad [e_2, e_3] = 0,$$

where a, b, c, d are real constants such that $a + d \neq 0$. This implies that $\mathfrak{g}$ is solvable, even for the unimodular case $a + d = 0$. Moreover, if the metric g

on G is cyclic left-invariant it follows that ad_{e_1} must be selfadjoint on $\mathfrak{u}$. Then we can take a new orthonormal basis $\{u_1 = e_1, u_2, u_3\}$ such that u_2 and u_3 are eigenvectors of ad_{e_1}. Hence, the bracket operation is expressed as $[u_1, u_2] = \alpha u_2$, $[u_1, u_3] = \beta u_3$, $[u_2, u_3] = 0$, where α and β are the corresponding eigenvalues. Then G must be isomorphic to the orthogonal semidirect product $\mathbb{R} \ltimes_\pi \mathbb{R}^2$ such that $\pi(t) = \mathrm{e}^{t\,\mathrm{diag}(\alpha,\beta)} = \mathrm{diag}\big(\mathrm{e}^{\alpha\, t}, \mathrm{e}^{\beta t}\big)$.

According to Example 5.6, the group $\mathbb{R} \ltimes_\pi \mathbb{R}^2$ admits a description as the matrix group $G^3(\alpha, \beta)$, and the left-invariant metric making $\{u_1, u_2, u_3\}$ an orthonormal basis is that given in the statement of Theorem 6.1 (2) below.

For $\alpha = \beta$, $G^3(\alpha, \beta)$ is the three-dimensional hyperbolic space with constant sectional curvature $-\alpha^2$, and for the unimodular case $\alpha = -\beta$, putting $u_1' = u_1$, $u_2' = (1/\sqrt{2})(u_2 + u_3)$, $u_3' = (1/\sqrt{2})(u_2 - u_3)$, one gets

$$[u_1', u_2'] = \alpha u_3', \quad [u_2', u_3'] = 0, \quad [u_3', u_1'] = -\alpha u_2'.$$

Hence, $G^3(\alpha, -\alpha)$ is isometrically isomorphic to $E(1,1)$ equipped with a left-invariant Riemannian metric with principal Ricci curvatures $(0, 0, -2\alpha^2)$ (see (5.2)). Then, we have the following result.

Theorem 6.1. *A three-dimensional connected, simply-connected nonabelian cyclic metric Lie group is isometrically isomorphic to one of the following Lie groups with a suitable left-invariant metric:*

(1) *The group $\widetilde{SL(2,\mathbb{R})}$ with the family of metrics depending on two positive parameters λ_1, λ_2 making orthonormal the bases of Ricci eigenvectors described in* (6.2) *and with principal Ricci curvatures $(-2\lambda_2(\lambda_1 + \lambda_2), -2\lambda_1(\lambda_1 + \lambda_2), 2\lambda_1\lambda_2)$.*

(2) *The orthogonal semidirect product $\mathbb{R} \ltimes_\pi \mathbb{R}^2$, both factors with the additive group structure and where the action π is $\pi(t) = \mathrm{diag}\big(\mathrm{e}^{\alpha\, t}, \mathrm{e}^{\beta t}\big)$, $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$. This Lie group can be described as the matrix group $G^3(\alpha, \beta)$ with the left-invariant metric*

$$g = \mathrm{d}u^2 + \mathrm{e}^{-2\alpha u}\mathrm{d}x^2 + \mathrm{e}^{-2\beta u}\mathrm{d}y^2,$$

where (u, x, y) corresponds to the global coordinate system (u, x_1, x_2) in Example 5.6.

The case $\alpha = \beta$ corresponds to the hyperbolic space $H^3(\alpha)$ and the case $\alpha = -\beta$ to the group $E(1,1)$ of rigid motions of the Minkowski plane equipped with a one-parameter family of left-invariant metrics with signature of the Ricci form $(0, 0, -)$. Moreover, $G^3(\alpha, 0)$ (resp. $G^3(0, \beta)$) is isometrically isomorphic to the direct product of the 2-dimensional real hyperbolic space $H^2(\alpha)$ (resp. $H^2(\beta)$) and $\mathbb{R}$.

Note that for dimension three, according to [8, Corollary 2.2], a nonsymmetric manifold admitting a nontrivial structure of class $\mathcal{T}_3$ also admits a nontrivial structure of class $\mathcal{T}_2$.

The classification of connected, simply-connected cyclic metric Lie groups for dimension four was given by Kowalski and Tricerri [8, Theorem 3.1] for the unimodular case. We add the corresponding nonunimodular Lie groups in the following theorem.

Theorem 6.2. *A four-dimensional connected and simply-connected nonabelian cyclic metric Lie group G is isometrically isomorphic to one of the following metric Lie groups:*

(1) *The direct product $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$, where $\widetilde{SL(2,\mathbb{R})}$ is equipped with any of the cyclic left-invariant metrics described in* Theorem 6.1 (1).

(2) *The matrix Lie group $G^4(\alpha,\beta,\gamma)$, $(\alpha,\beta,\gamma) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, with the left-invariant metric*

$$g = \mathrm{d}u^2 + \mathrm{e}^{-2\alpha u}\mathrm{d}x^2 + \mathrm{e}^{-2\beta u}\mathrm{d}y^2 + \mathrm{e}^{-2\gamma u}\mathrm{d}z^2,$$

where (u,x,y,z) corresponds to the global coordinate system (u,x_1,x_2,x_3) in Example 5.6.

The Lie group $G^4(\alpha,-\alpha,0)$ corresponds to the metric direct product $E(1,1)\times \mathbb{R}$, where $E(1,1)$ is equipped with a left-invariant Riemannian metric with principal Ricci curvatures $(0,0,-2\alpha^2)$.

(3) *The orthogonal semidirect product $\mathbb{R}^2 \ltimes_\pi \mathbb{R}^2$ under $\pi\colon \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{R}^2)$ given by $\pi(s,t) = \mathrm{diag}(\mathrm{e}^{\rho s+\lambda t}, \mathrm{e}^{\sigma s-\lambda t})$, $\rho+\sigma \neq 0$, $\lambda > 0$. This Lie group can be described as the group $H^4(\rho,\sigma;\lambda)$ of matrices of the form*

$$\begin{pmatrix} \mathrm{e}^{\rho u+\lambda v} & 0 & x \\ 0 & \mathrm{e}^{\sigma u-\lambda v} & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho+\sigma\neq 0, \quad \lambda > 0,$$

with the left-invariant metric

$$g = \mathrm{d}u^2 + \mathrm{d}v^2 + \mathrm{e}^{-2(\rho u+\lambda v)}\mathrm{d}x^2 + \mathrm{e}^{-2(\sigma u-\lambda v)}\mathrm{d}y^2.$$

Proof. From [8, Proposition 3.6], one has that a four-dimensional connected, simply-connected, unimodular, nonabelian cyclic metric Lie group G is either $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ or $E(1,1)\times\mathbb{R}$ or $G^4(\alpha,\beta,\gamma)$ for $(\alpha,\beta,\gamma) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ with $\alpha+\beta+\gamma = 0$.

Suppose now that the Lie algebra $\mathfrak{g}$ of G is not unimodular, then its unimodular kernel $\mathfrak{u}$ has dimension three and $\mathfrak{g}$ is the semidirect sum $\mathfrak{u}^\perp \oplus \mathfrak{u}$ under the adjoint representation, orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle$ defining the left-invariant Riemannian metric g on G. The unimodular ideal $\mathfrak{u}$ of $\mathfrak{g}$, with its induced inner product, must be isometrically isomorphic to either $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{e}(1,1)$ or $\mathbb{R}^3$, and there exists an orthonormal basis $\{e_0,e_1,e_2,e_3\}$ of $\mathfrak{g}$ such that $\mathfrak{u}^\perp = \mathbb{R}\{e_0\}$ and $\mathfrak{u} = \mathbb{R}\{e_1,e_2,e_3\}$ such that

$$[e_2,e_3] = \lambda_1 e_1, \quad [e_3,e_1] = \lambda_2 e_2, \quad [e_1,e_2] = \lambda_3 e_3, \qquad \lambda_1+\lambda_2+\lambda_3 = 0. \quad (6.3)$$

Since g is cyclic left-invariant, ad_{e_0} is selfadjoint on $\mathfrak{u}$, and then

$$\mathrm{ad}_{e_0} = \begin{pmatrix} \mu_1 & a_3 & a_2 \\ a_3 & \mu_2 & a_1 \\ a_2 & a_1 & \mu_3 \end{pmatrix}, \quad (6.4)$$

in terms of the basis $\{e_1, e_2, e_3\}$ of $\mathfrak{u}$. Since ad_{e_0} acts as a derivation on the Lie algebra $\mathfrak{u}$, we get

$$\lambda_k(\mu_i + \mu_j - \mu_k) = 0, \quad (\lambda_i + \lambda_j)a_k = 0, \tag{6.5}$$

for each cyclic permutation (i, j, k) of $(1, 2, 3)$.

First case. If $\mathfrak{u} = \mathfrak{sl}(2, \mathbb{R})$ then $\lambda_1\lambda_2\lambda_3 \neq 0$, and for all distinct i, j, k, we have $\lambda_k = -(\lambda_i + \lambda_j)$, then $a_k = 0$ and $\mu_i = \mu_j = \mu_k = 0$. Hence $\mathrm{ad}_{e_0} = 0$ and $\mathfrak{g}$ would be the direct sum $\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$, which is unimodular.

Second case. If $\mathfrak{u}$ is the abelian Lie algebra $\mathbb{R}^3$, we can take an orthonormal basis of eigenvectors $\{u_1, u_2, u_3\}$ of the selfadjoint operator ad_{e_0} acting on $\mathfrak{u}$. The structure equations are given, in terms of the orthonormal basis $\{e_0, u_1, u_2, u_3\}$ of $\mathfrak{g}$, by

$$[e_0, u_1] = \alpha u_1, \quad [e_0, u_2] = \beta u_2, \quad [e_0, u_3] = \gamma u_3,$$

with $\alpha + \beta + \gamma \neq 0$, because $\mathfrak{g}$ is not unimodular. This gives (see Example 5.6) the nonunimodular version of (2) in the statement.

Third case. If $\mathfrak{u} = \mathfrak{e}(1, 1)$ we can suppose $\lambda_1 = -\lambda_2 > 0$, $\lambda_3 > 0$ in (6.3). By (6.5), we have $a_1 = a_2 = 0$, $\mu_3 = 0$ and $\mu_1 = \mu_2$ in (6.4), and put $\lambda = \lambda_1 = -\lambda_2$, $\mu = \mu_1 = \mu_2$, $a = a_3$. Then $[e_0, e_1] = \mu e_1 + a e_2$, $[e_0, e_2] = a e_1 + \mu e_2$, $[e_2, e_3] = \lambda e_1$, $[e_1, e_3] = \lambda e_2$, and $\mathbb{R}\{e_0, e_3\}$ and $\mathbb{R}\{e_1, e_2\}$ are mutually orthogonal abelian subalgebras of $\mathfrak{g}$. We now consider the orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathfrak{g}$ given by $u_1 = e_0$, $u_2 = -e_3$, $u_3 = (1/\sqrt{2})(e_1 + e_2)$, $u_4 = (1/\sqrt{2})(e_1 - e_2)$. Then

$$[u_1, u_3] = \rho u_3, \quad [u_1, u_4] = \sigma u_4, \quad [u_2, u_3] = \lambda u_3, \quad [u_2, u_4] = -\lambda u_4, \tag{6.6}$$

where we have put $\rho = \mu + a$, $\sigma = \mu - a$, so that, since $\mathfrak{g}$ is not unimodular, $\rho + \sigma = 2\mu \neq 0$. Thus, $\mathfrak{g}$ is the direct sum of its abelian subalgebra $\mathbb{R}\{u_1, u_2\}$ and its abelian ideal $\mathbb{R}\{u_3, u_4\}$. Hence, the Lie group G is isomorphic to the orthogonal semidirect product $\mathbb{R}^2 \ltimes_\pi \mathbb{R}^2$, where the action π is $\pi(s, t) = \mathrm{diag}(\mathrm{e}^{\rho s + \lambda t}, \mathrm{e}^{\sigma s - \lambda t})$, that is, G can be considered as $\mathbb{R}^4$ with the group operation

$$(u, v, x, y) \cdot (u', v', x', y') = (u + u',\ v + v',\ x + \mathrm{e}^{\rho u + \lambda v} x',\ y + \mathrm{e}^{\sigma u - \lambda v} y'),$$

and it admits a description as the matrix group $H^4(\rho, \sigma; \lambda)$. With respect to the global coordinate system (u, v, x, y) of $H^4(\rho, \sigma; \lambda)$, the generators u_i of $\mathfrak{g}$ correspond to the left-invariant vector fields $u_1 = \partial/\partial u$, $u_2 = \partial/\partial v$, $u_3 = \mathrm{e}^{\rho u + \lambda v}\partial/\partial x$, $u_4 = \mathrm{e}^{\sigma u - \lambda v}\partial/\partial y$. □

Remark 6.3. One can prove that, if $\rho\sigma = \lambda^2$, then $H^4(\rho, \sigma; \lambda)$ is isometrically isomorphic to the metric direct product Lie group $H^2(\alpha) \times H^2(\beta)$, with $\alpha = \sqrt{\lambda^2 + \rho^2}$ and $\beta = \lambda(\rho + \sigma)/\sqrt{\lambda^2 + \rho^2}$ and that $H^4(\rho, -\rho; \lambda)$ is isometrically isomorphic to the direct product $E(1, 1) \times \mathbb{R}$, where $E(1, 1)$ is equipped with a left-invariant Riemannian metric with principal Ricci curvatures $\big(0, 0, -2(\lambda^2 + \rho^2)\big)$.

In some cases, the nonunimodular Lie group $G^4(\alpha, \beta, \gamma)$ is a decomposable metric Lie group. In fact, $G^4(\alpha, \beta, 0)$ is isometrically isomorphic to the direct product $G^3(\alpha,\beta)\times\mathbb{R}$, and $G^4(\alpha,\alpha,0)$ and $G^4(\alpha,0,0)$ are isometrically isomorphic to the direct products $H^3(\alpha)\times\mathbb{R}$ and $H^2(\alpha)\times\mathbb{R}^2$, respectively. On the other hand, the indecomposable metric Lie group $G^4(\alpha,\beta,\gamma)$ for $\alpha=\beta=\gamma$ is the four-dimensional hyperbolic space $H^4(\alpha)$.

The classification of unimodular Lie algebras of dimension five corresponding to traceless cyclic metric Lie groups was given by Bieszk [2]. We add the corresponding nonunimodular Lie groups in the next theorem.

Theorem 6.4. *A five-dimensional connected and simply-connected nonabelian cyclic metric Lie group G is isometrically isomorphic to one of the following metric Lie groups:*

(1) *The metric direct products $\widetilde{SL(2,\mathbb{R})}\times\mathbb{R}^2$ and $\widetilde{SL(2,\mathbb{R})}\times H^2(\alpha)$, where $\widetilde{SL(2,\mathbb{R})}$ is equipped with any of the cyclic left-invariant metrics described in* Theorem 6.1 (1).

(2) *The Lie group $G^5(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, for $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbb{R}^4$, with some $\alpha_i\neq 0$.*

(3) *The orthogonal semidirect product $\mathbb{R}^2\ltimes_\pi\mathbb{R}^3$ under $\pi\colon\mathbb{R}^2\to\mathrm{Aut}(\mathbb{R}^3)$ given by $\pi(s,t)=\mathrm{diag}(\mathrm{e}^{\rho s+\lambda t},\mathrm{e}^{\sigma s+\mu t},\mathrm{e}^{\tau s-(\lambda+\mu)t})$. This Lie group can be described as the group $H^5(\rho,\sigma,\tau;\lambda,\mu)$ of matrices of the form*

$$\begin{pmatrix} \mathrm{e}^{\rho u+\lambda v} & 0 & 0 & x\\ 0 & \mathrm{e}^{\sigma u+\mu v} & 0 & y\\ 0 & 0 & \mathrm{e}^{\tau u-(\lambda+\mu)v} & z\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{6.7}$$

with the left-invariant metric

$$g=\mathrm{d}u^2+\mathrm{d}v^2+\mathrm{e}^{-2(\rho u+\lambda v)}\mathrm{d}x^2+\mathrm{e}^{-2(\sigma u+\mu v)}\mathrm{d}y^2+\mathrm{e}^{2((\lambda+\mu)v-\tau u)}\mathrm{d}z^2.$$

In particular, if $\rho+\sigma\neq 0$ and $\lambda\neq 0$, $H^5(\rho,\sigma,0;\lambda,-\lambda)$ is isometrically isomorphic to the direct product $H^4(\rho,\sigma;\lambda)\times\mathbb{R}$.

Proof. Let g be the cyclic left-invariant metric on G, and let $\langle\cdot,\cdot\rangle$ be the corresponding inner product on the Lie algebra $\mathfrak{g}$ of G. First, suppose that $\mathfrak{g}$ is not unimodular and let $\mathfrak{u}$ be its unimodular kernel. Then $\mathfrak{g}$ is the semidirect sum $\mathfrak{u}^\perp\oplus\mathfrak{u}$ under the adjoint representation, orthogonal with respect to $\langle\cdot,\cdot\rangle$. With its induced inner product, $\mathfrak{u}$ must be isometrically isomorphic to either the direct sum $\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$ or $\mathbb{R}^4$ or the Lie algebra $\mathfrak{g}(\alpha,\beta,\gamma)$, with $\alpha+\beta+\gamma=0$. In each case, there exists a suitable orthonormal basis $\{e_0,e_1,e_2,e_3,e_4\}$ of $\mathfrak{g}$ such that $\mathfrak{u}^\perp=\mathbb{R}\{e_0\}$ and $\mathfrak{u}=\mathbb{R}\{e_1,e_2,e_3,e_4\}$. Since g is cyclic left-invariant, ad_{e_0} is selfadjoint on $\mathfrak{u}$, and in terms of the basis $\{e_1,e_2,e_3,e_4\}$ of $\mathfrak{u}$ we can write

$$\mathrm{ad}_{e_0}=\begin{pmatrix} \mu_1 & a_3 & a_2 & a_4\\ a_3 & \mu_2 & a_1 & a_5\\ a_2 & a_1 & \mu_3 & a_6\\ a_4 & a_5 & a_6 & \mu_4\end{pmatrix}. \tag{6.8}$$

We have to consider three cases.

First case. If $\mathfrak{u} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ we can suppose that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3, \quad [e_i, e_4] = 0, \qquad i = 1, 2, 3,$$

with all λ_i not zero and $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Since ad_{e_0} acts as a derivation on $\mathfrak{u}$, we get $\lambda_k(\mu_i + \mu_j - \mu_k) = 0$, $(\lambda_i + \lambda_j)a_k = 0$, $\lambda_1 a_4 = \lambda_2 a_5 = \lambda_3 a_6 = 0$, for each cyclic permutation (i, j, k) of $(1, 2, 3)$. Then $\mu_1 = \mu_2 = \mu_3 = 0$ and $a_i = 0$ in (6.8) for each $i = 1, \ldots, 6$. If $\mu_4 = 0$ in (6.8) then $\mathfrak{g}$ should be be the unimodular Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^2$. Then $\mu_4 \neq 0$ and $\mathfrak{g}$ must be the orthogonal direct sum of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and the Lie algebra $\mathbb{R}\{e_0, e_4\}$ with the structure equation $[e_0, e_4] = \mu_4 e_4$, and hence we have (1) in the statement, with $\alpha = \mu_4$.

Second case. If $\mathfrak{u} = \mathbb{R}^4$, we can take a basis formed by eigenvectors u_1, u_2, u_3, u_4 of the selfadjoint operator ad_{e_0} acting on $\mathfrak{u}$. In terms of the orthonormal basis $\{e_0, u_1, u_2, u_3, u_4\}$ of $\mathfrak{g}$, the structure equations of $\mathfrak{g}$ are given by

$$[e_0, u_i] = \alpha_i u_i, \qquad i = 1, \ldots, 4,$$

with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \neq 0$ because $\mathfrak{g}$ is not unimodular. In accordance with Example 5.6, we have the metric Lie groups in (2) in the statement with the condition $\sum_{i=1}^4 \alpha_i \neq 0$.

Third case. If $\mathfrak{u} = \mathfrak{g}(\alpha, \beta, \gamma)$, with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, $\alpha + \beta + \gamma = 0$, we can suppose that

$$[e_4, e_1] = \alpha e_1, \quad [e_4, e_2] = \beta e_2, \quad [e_4, e_3] = \gamma e_3.$$

Then, according to the discussion in Example 5.7, the nonunimodular Lie group G can be described as the group of matrices of the form (6.7), with $\rho + \sigma + \tau \neq 0$ and $(\lambda, \mu) \neq (0, 0)$.

Finally, from the classification of unimodular Lie algebras of dimension five with a cyclic left-invariant metric, given by Bieszk [2], we also have the metric direct product Lie group $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}^2$ in (1) in the statement, and the metric Lie group $G^5(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in (2) with $\sum_{i=1}^4 \alpha_i = 0$ (which includes the cases (c) and (e) in [2, Theorem 1]). The metric Lie group $H^5(\rho, \sigma, \tau; \lambda, \mu)$ in (3) with $\rho + \sigma + \tau = 0$ corresponds to the case (d) in [2, Theorem 1]; in fact, it can be described (as a particular case of the description (5.4) of the matrix Lie group in Example 5.7) as the group $\hat{H}^5(\sigma, -\sigma, \mu_1, \mu_2, -\mu_1 - \mu_2)$ of matrices of the form

$$\begin{pmatrix} \mathrm{e}^{\mu_1 v} & 0 & 0 & x \\ 0 & \mathrm{e}^{\sigma u + \mu_2 v} & 0 & y \\ 0 & 0 & \mathrm{e}^{-\sigma u - (\mu_1 + \mu_2) v} & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\mu_1, \sigma \neq 0$, which gives the Lie group in case (d) in [2, Theorem 1]. $\square$

Remark 6.5. Many decomposable nonunimodular metric Lie groups appear as particular cases of the metric Lie groups $H^5(\rho, \sigma, \tau; \lambda, \mu)$. For example, if $\rho + \sigma \neq 0$, $\tau = 0$, $\lambda \neq 0$, and $\mu = -\lambda$, we have the direct product $H^4(\rho, \sigma; \lambda) \times \mathbb{R}$.

Moreover, one can prove that $H^2(c) \times G^3(\alpha, \beta)$ is isometrically isomorphic to $H^5(\rho, \sigma, \tau; \lambda, \mu)$. In particular, if $\alpha = \beta$ this metric Lie group corresponds to $H^2(c) \times H^3(\alpha)$. If $\beta = 0$ we have $H^5(\rho, \sigma, 0; \lambda, -\lambda)$, which is isometrically isomorphic to $H^2(c) \times H^2(\alpha) \times \mathbb{R}$. If $\alpha = -\beta$, we have the metric Lie group $H^5(c, 0, 0; 0, -\alpha)$, which corresponds to the direct product $H^2(c) \times E(1,1)$, where $E(1,1)$ is equipped with a left-invariant Riemannian metric with principal Ricci curvatures $(0, 0, -2\alpha^2)$.

Remark 6.6. As for dimensions greater than or equal to six, we recall some facts.

Let (M, g) be an n-dimensional connected Riemannian manifold (M, g) admitting a structure $S \in \mathcal{T}_1 \oplus \mathcal{T}_2$. Defining the vector field ξ on M by $\xi = (1/(n-1)) \sum_{i=1}^{n} S_{e_i} e_i$, for any local orthonormal basis $\{e_i\}$, then S can be written as $S_X Y = g(X, Y)\xi - g(Y, \xi)X + \pi(X, Y)$, where π is a tensor field of type $(1, 2)$. The one-form ω metrically dual to the vector field ξ is called the fundamental form of S.

Pastore and Verroca [11] studied proper structures S of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ (that is, S belongs neither to $\mathcal{T}_1$ nor to $\mathcal{T}_2$), having closed fundamental 1-form ω. They proved (among other results) that a connected, simply-connected Riemannian manifold which is a warped product and admits a nontrivial such structure S, is isometric to the real hyperbolic space H^n of constant curvature $-\|\xi\|^2$ and $n \geq 6$. However, the description of H^n as a Riemannian homogeneous space corresponding to the proper structure $S \in \mathcal{T}_1 \oplus \mathcal{T}_2$ does not correspond to that of a cyclic metric Lie group, since the corresponding structure on the latter is necessarily vectorial.

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