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AdS Robin solitons and their stability

Piotr Bizoń[1], Dominika Hunik-Kostyra[1] and Maciej Maliborski[2]

[1] Institute of Theoretical Physics, Jagiellonian University, Kraków, Poland
[2] Gravitational Physics, Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

E-mail: bizon@th.if.uj.edu.pl, dominika.hunik@uj.edu.pl and maciej.maliborski@univie.ac.at

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Abstract

We consider the four-dimensional Einstein–Klein–Gordon–AdS system with the conformal mass subject to the Robin boundary conditions at infinity. Above a critical value of the Robin parameter, at which the AdS spacetime goes linearly unstable, we prove existence of a family of globally regular static solutions (that we call AdS Robin solitons) and discuss their properties.

Keywords: asymptotically anti de-Sitter spacetime, solitons, Robin boundary condition

(Some figures may appear in colour only in the online journal)

1. Introduction

We consider the four-dimensional Einstein–Klein–Gordon–AdS system with mass μ related to the negative cosmological constant Λ through $\mu^2 = \frac{2}{3}\Lambda$. For this, and only this, value of mass the system is conformally well-behaved at null and spatial infinity and consequently the initial-boundary value problem is well-posed for a variety of different boundary conditions at infinity [1, 2]. Here, we focus on the one-parameter family of Robin boundary conditions. It has been known that along this family there is a critical parameter value at which the system undergoes a bifurcation: the (zero energy) anti-de Sitter (AdS) spacetime becomes linearly unstable above that critical value [3] and there emerges a pair of (negative energy) globally regular static solutions (henceforth called AdS Robin solitons) [4]. The main goal of this paper is to establish the existence of AdS Robin solitons rigorously and analyze the structure of the bifurcation in more detail. In preparation of future analysis of the role of solitons in dynamics, we also determine their spectrum of linearized perturbations.

2. Setup

The Einstein–Klein–Gordon–AdS system is given by

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + \mu^2 \phi^2 \right) \, g_{\alpha\beta} \right), \tag{1a}$$

$$\Box_g \phi - \mu^2 \phi = 0, \tag{1b}$$

where $\Box_g = g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ is the wave operator associated with the metric $g_{\alpha\beta}$, μ is the mass of the scalar field, and Λ is a negative constant. We assume spherical symmetry and write the metric in the form

$$g = \frac{\ell^2}{\cos^2 x} \left(-A \, \mathrm{e}^{-2\delta} \, \mathrm{d}t^2 + A^{-1} \, \mathrm{d}x^2 + \sin^2 x \, \mathrm{d}\omega^2 \right), \tag{2}$$

where $(t, x, \omega) \in (-\infty, \infty) \times [0, \pi/2) \times \mathbb{S}^2$, $\mathrm{d}\omega^2$ is the round metric on $\mathbb{S}^2$ and $\ell^2 = -3/\Lambda$. The metric functions A, δ and the scalar field ϕ depend on (t, x). We choose units such that $\ell = 1$ and $4\pi G = 1$ and introduce new variables

$$f = \frac{\phi}{\cos x} \quad \text{and} \quad B = \frac{A - 1}{\cos^2 x}. \tag{3}$$

Then the system (1) reduces to

$$(\Box_{\hat{g}} - 1) f = \frac{2 + \mu^2}{\cos^2 x} f - \left(1 - 3 \cos^2 x \right) B f - \mu^2 \sin^2 x \, f^3, \tag{4a}$$

$$\cos x \, \partial_x B = -\frac{B}{\sin x} - \sin x \, (1 + B \cos^2 x) \, \Phi - \mu^2 \, \sin x \, f^2, \tag{4b}$$

$$\partial_x \delta = - \sin x \, \cos x \, \Phi, \tag{4c}$$

$$\partial_t B = -2A \, \sin x \, (\cos x \, \partial_x f - f \, \sin x) \, \partial_t f, \tag{4d}$$

where $\Phi = (\cos x \, \partial_x f - f \, \sin x)^2 + A^{-2} \, \mathrm{e}^{2\delta} \cos^2 x \, (\partial_t f)^2$ and

$$\Box_{\hat{g}} = -\mathrm{e}^{\delta} \partial_t \left(A^{-1} \, \mathrm{e}^{\delta} \partial_t \right) + \frac{\mathrm{e}^{\delta}}{\sin^2 x} \, \partial_x \left(A \, \mathrm{e}^{-\delta} \sin^2 x \, \partial_x \right)$$

is the polar wave operator associated with the conformal metric $\hat{g}_{\alpha\beta} = \cos^2 x \, g_{\alpha\beta}$. On the right side of equation (4a) the derivatives of metric functions were eliminated using equations (4b) and (4c).

In the following we set $\mu^2 = -2$. For this value of mass the wave equation (4a) is regular at $x = \pi/2$ because the first term on the right side (which is the only singular term) vanishes[3]. Thanks to this fact, the initial-boundary value problem for the system (4) is well posed for a variety of boundary conditions at the conformal boundary (both reflective and dissipative) [1, 2].[4] In this paper we focus our attention on the one-parameter family of Robin boundary

[3] This cancellation is due to the fact that for $\mu^2 = -2$ the left sides of equations (1b) and (4a) are asymptotically conformal, that is $(\Box_g + 2)\phi \approx (\cos x)^{-3} (\Box_{\hat{g}} - 1) f$ near $x = \pi/2$. However, the constraint equation (4b) has a singularity at $x = \pi/2$ so it does not appear possible to extend the solutions 'beyond infinity' (cf [5] where an extension of solutions across the conformal boundary at *timelike* infinity was analyzed for the system (1) with $\mu^2 = \frac{2}{3}\Lambda > 0$).

[4] This should be contrasted with the widely studied massless case for which only the Dirichlet boundary condition is compatible with the basic requirement of finite total mass [6].

conditions

$$\partial_x f - b\, f|_{x=\frac{\pi}{2}} = 0, \tag{5}$$

where b is a constant (hereafter referred to as the Robin parameter). For $b = 0$ the Robin condition reduces to the Neumann condition $\partial_x f|_{x=\pi/2} = 0$.

Assuming (5) and expanding the fields in power series in $z = \pi/2 - x$ we obtain the following asymptotic behavior near $z = 0$

$$f(t, x) = \alpha - b\alpha z + \mathcal{O}(z^2), \tag{6}$$

$$B(t, x) = \alpha^2 - (3b\alpha^2 + M)z + \mathcal{O}(z^2), \tag{7}$$

$$\delta(t, x) = \delta_\infty + \frac{1}{2}\alpha^2 z^2 + \mathcal{O}(z^3), \tag{8}$$

where $\alpha(t)$ and $\delta_\infty(t)$ are free functions[5] and M is a constant. To see the physical meaning of M, let us define the renormalized mass function

$$m = -B\,\tan x + \frac{\sin^3 x}{\cos x}\, f^2. \tag{9}$$

The first term on the right side is the Misner–Sharp mass function defined by $m_{\mathrm{MS}} = r(1 + r^2 - g^{\mu\nu}\partial_\mu r \partial_\nu r)$, where $r = \tan x$ is the areal radial coordinate. This function diverges as $x \to \pi/2$ and the purpose of the second term (called the counterterm) is to cancel this divergence. The leading order behavior of the counterterm is determined by the asymptotics (6) and (7) but otherwise can be chosen freely. Using equation (4b) we get

$$\partial_x m = \rho \sin^2 x, \tag{10}$$

where

$$\rho = A^{-1}\, \mathrm{e}^{2\delta}(\partial_t f)^2 + (\partial_x f)^2 + f^2 + B(\cos x\, \partial_x f - f \sin x)^2, \tag{11}$$

hence

$$m(t, \pi/2) = \int_0^{\pi/2} \rho \sin^2 x\, \mathrm{d}x. \tag{12}$$

This quantity can be interpreted as the bulk energy. From the asymptotic expansions (6) and (7) it follows that

$$M = m(t, \pi/2) - b\alpha^2(t), \tag{13}$$

where the second term on the right side can be viewed as the energy stored on the boundary. Although both the bulk and boundary energies are time dependent, their sum M is conserved. In what follows, we will refer to M as the total energy (mass). The exchange of energy between the bulk and the boundary is a characteristic feature of systems subject to the Robin boundary conditions. Note that some of the bulk energy is 'lost' to the boundary if $b < 0$ and 'gained' from the boundary if $b > 0$.

We remark that the expression (13) can be obtained in a systematic way within the diffeomorphism covariant Hamiltonian framework of Wald and Zoupas [7] (see section 2.2 in

[5] We use the normalization $\delta(t, 0) = 0$, hence t is the proper time at the center.

[8]). Nonetheless, we believe that our hands-on approach, based solely on the analysis of the system (4), is helpful in getting insight into not so widely known physics of the Robin boundary conditions.

3. AdS Robin solitons

For time-independent solutions the system (4) with $\mu^2 = -2$ takes the form

$$(1 + B\cos^2 x)f'' + \cot x\left(2 + (1 - 4\sin^2 x)B + 2\sin^2 x\, f^2\right) f' - f + \left(1 - 3\cos^2 x\right) Bf - 2\sin^2 x\, f^3 = 0, \tag{14a}$$

$$\cot x\, B' + \frac{B}{\sin^2 x} + (1 + B\cos^2 x)(\cos x\, f' - \sin x\, f)^2 - 2f^2 = 0, \tag{14b}$$

$$\delta' + \sin x \cos x(\cos x\, f' - \sin x\, f)^2 = 0, \tag{14c}$$

where the derivatives of metric functions were eliminated from equation (14a) using equations (14b) and (14c). It is routine to prove that this system has local solutions near $x = 0$ which behave as follows

$$f(x) \sim c + \frac{1}{6}cx^2, \quad B(x) \sim \frac{2}{3}c^2x^2, \quad \delta(x) \sim -\frac{1}{9}c^2x^4, \tag{15}$$

where c is a free parameter.

Lemma. *For any c the local solution (15) extends smoothly up to $x = \pi/2$ and fulfills the boundary conditions (6)–(8).*

Proof. To prove this lemma it is convenient to use the radial coordinate $r = \tan x$ and return to the original field variables

$$\phi(r) = f(x)\cos x, \quad A(r) = 1 + B(x)\cos^2 x. \tag{16}$$

Then, equations (14a) and (14b) become

$$(1 + r^2)A\phi'' + \left(r(1 + r^2)A\phi'^2 + (1 + r^2)A' + \frac{2 + 4r^2}{r}A\right)\phi' + 2\phi = 0, \tag{17a}$$

$$(1 + r^2)A' - \frac{1 + 3r^2}{r}(1 - A) - 2r\phi^2 + r(1 + r^2)A\phi'^2 = 0. \tag{17b}$$

The local solutions (15) translate to

$$\phi(r) \sim c - \frac{1}{3}cr^2, \quad A(r) \sim 1 + \frac{2}{3}c^2r^2. \tag{18}$$

We first observe that the function $B = (A - 1)(1 + r^2)$ is monotone increasing. To see this, suppose that $B(r)$ has a maximum at some point $r_0 > 0$. Differentiating equation (17b), substituting $B'(r_0) = 0$, and eliminating $\phi''(r_0)$ and $B(r_0)$ using equations (17a) and (17b), respectively, we get after simplifications

$$B''(r_0) = 2\phi'^2 + 4r^2\phi^2\phi'^2 + 4(r\phi' + \phi)^2|_{r=r_0}, \tag{19}$$

which is manifestly positive, contradicting that the point r_0 exists. Since $B(r)$ is positive for small $r > 0$, this implies that $A(r) \geqslant 1$ for all r.

Next, we define a function

$$H = \frac{1}{2}(1 + r^2)A\phi'^2 + \phi^2. \tag{20}$$

Using the system (17) we obtain

$$H' = -\frac{(1 + r^2)A(3 + r^2\phi'^2) + 2r^2\phi^2 + 3r^2 + 1}{2r}, \tag{21}$$

which is manifestly negative, hence $H(r)$ is a monotonically decreasing Lyapunov function. Since $A \geqslant 1$, it follows that $r^2\phi'^2$ and ϕ^2 remain bounded for all r.

To determine the asymptotic behavior of solutions for $r \to \infty$ it is convenient to use the logarithmic radial variable $\tau = \log r$. In terms of τ the system (17) is asymptotically autonomous for $\tau \to \infty$ and the limiting autonomous system is

$$A\ddot{\phi} + (3 + 2\phi^2)\dot{\phi} + 2\phi = 0, \tag{22a}$$

$$\dot{A} - 3(1 - A) - 2\phi^2 + A\dot{\phi}^2 = 0, \tag{22b}$$

where dot denotes the derivative with respect to τ (by an abuse of notation, we use the same symbols for the original and limiting systems). From the general theory of asymptotically autonomous dynamical system [9, 10] and the existence of the Lyapunov function H, it follows that the asymptotic behavior of solutions of the system (17) for $\tau \to \infty$ is governed by the above limiting system. Elementary analysis gives the attracting fixed point $\phi = 0$, $\dot{\phi} = 0$, $A = 1$ with the leading order behavior

$$\phi(\tau) = c_1 \mathrm{e}^{-\tau} + c_2 \mathrm{e}^{-2\tau} + \mathcal{O}(\mathrm{e}^{-3\tau}), \quad A(\tau) - 1 = c_1^2 \mathrm{e}^{-2\tau} + c_3 \mathrm{e}^{-3\tau} + \mathcal{O}(\mathrm{e}^{-4\tau}), \tag{23}$$

where c_k are free parameters, which are related to the free parameters α, b, and M in the expansions (6) and (7) by

$$c_1 = \alpha, \quad c_2 = -b\alpha, \quad c_3 = -3b\alpha^2 - M. \tag{24}$$

This completes the proof. □

The above lemma ensures that for each c the solution starting with the initial conditions (15) automatically satisfies the Robin condition $f'(\pi/2) = bf(\pi/2)$ for some parameter b (which depends on c). We will refer to these globally regular static solutions as the AdS Robin solitons (or just solitons for short) and denote them by (f_s, B_s, δ_s). The profiles of solitons can be easily determined numerically by integrating the system (14) with the boundary conditions (15). We note in passing that an analogous reasoning leads to a two-parameter family of hairy black holes (where the second parameter is the horizon radius).

As far as we know, the AdS Robin solitons and hairy black holes were first studied in the literature in the context of so called 'designer gravity' [4, 11], however, to the best of our knowledge, their existence remained unproven.

4. Bifurcation analysis

It is illuminating to look at the solitons from the viewpoint of the local bifurcation theory. To this end, consider the perturbation expansion of solitons for small c

$$b = b_* + c^2 b_2 + \mathcal{O}(c^4), \quad f = cf_1 + c^3 f_3 + \mathcal{O}(c^5), \quad B = c^2 B_2 + c^4 B_4 + \mathcal{O}(c^6). \tag{25}$$

Inserting this expansion into the system (14) and requiring regularity at $x = 0$, at the lowest order we get

$$b_* = \frac{2}{\pi}, \quad f_1 = \frac{x}{\sin x}, \quad B_2 = \frac{x}{\sin x}\left(-\cos x + \frac{x}{\sin x}\right). \tag{26}$$

Calculations of higher orders are tedious and we relegate them to the appendix. In particular, in (A9) we obtain the explicit expression for the coefficient b_2

$$b_2 = \frac{\pi}{6}(16 \ln 2 - 1) + \frac{1}{\pi}(1 - 12\zeta(3)) \approx 1.010\,09. \tag{27}$$

The fact that b_2 is positive means that at b_* we have a supercritical pitchfork bifurcation where the AdS solution bifurcates into a pair of solitons $(\pm f_s, B_s, \delta_s)$. As usual, this kind of bifurcation is associated with exchange of linear stability and, indeed, in the next section we will show that for $b > b_*$ the AdS space becomes linearly unstable whereas the solitons are linearly stable.

Using (7) and the expansion (25), we get the approximation for the mass

$$M_{\mathrm{s}} = B'(\pi/2) - 3b\alpha^2 \simeq \frac{3\pi}{2}c^2 + B_4'(\pi/2)c^4 - 3\left(\frac{2}{\pi} + b_2 c^2\right)\left(\frac{\pi}{2}c + f_3(\pi/2)c^3\right)^2,$$

which upon substitution of (A4) and (A8) yields

$$M_{\mathrm{s}} \simeq -\frac{\pi^2 b_2}{8}c^4 = -\frac{\pi^2}{8b_2}(b - b_*)^2. \tag{28}$$

It is instructive to rederive this result along the lines of designer gravity [4]. Letting $\alpha = f(\pi/2)$ and $\beta = f'(\pi/2)$, we get from (25) (in this paragraph ‘ = ’ means equality up to order $\mathcal{O}(c^4)$)

$$\alpha = f_1(\pi/2)\,c + f_3(\pi/2)\,c^3, \quad \beta = f_1'(\pi/2)\,c + f_3'(\pi/2)\,c^3, \tag{29}$$

which can be viewed as the parametric equation of the curve in the (α, β) plane. Eliminating c we get the function

$$\beta_{\mathrm{s}}(\alpha) = \frac{2}{\pi}\alpha + \frac{4b_2}{\pi^2}\alpha^3, \tag{30}$$

where the subscript ‘s’ indicates that the function is associated with solitons. Following the approach used in designer gravity we introduce the effective potential

$$\mathcal{V}(\alpha) = 2\int_0^\alpha \beta_s(\alpha')\mathrm{d}\alpha' - b\alpha^2 = -(b - b_*)\alpha^2 + \frac{2b_2}{\pi^2}\alpha^4. \tag{31}$$

By construction, critical points of the effective potential correspond to solitons. The key observation, made by Hertog and Horowitz in [4], is that the value of the effective potential at

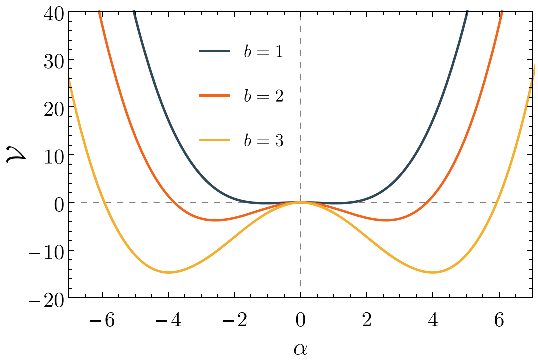

Figure 1. Effective potentials for sample values of b.

the critical point is equal to the soliton mass. In our case, $\mathcal{V}'(\alpha)=0$ for $\alpha_s^2=\frac{\pi^2}{4b_2}(b-b_*)$ (and, of course, for $\alpha=0$ corresponding to the AdS space). Substituting this into (31) we get $M_s=\mathcal{V}(\alpha_s)$ which reproduces the formula (28). Note that $\mathcal{V}''(\alpha_s)>0$.

Further from the bifurcation point, the soliton function $\beta_s(\alpha)$ and the corresponding effective potential $\mathcal{V}(\alpha)$ can be determined numerically[6]. We find that for each $b>b_*$ the effective potential has the shape of a Mexican hat (see figure 1) with exactly three critical points: the local maximum at zero and two global minima at $\pm\alpha_s$. This implies that the soliton solution is unique (modulo reflection symmetry) and suggests that it is stable.

Remark. It is natural to expect that for any given $b>b_*$ the soliton is the ground state, i.e. for any regular initial data satisfying the Robin condition (5) there holds the inequality $M\geqslant M_s$ which saturates if and only if the data correspond to the soliton [4]. Our numerical constructions of initial data corroborate this conjecture but we have not been able to prove it (see [8] for partial results in this direction).

5. Linear stability analysis

Linearizing the system (4) around the AdS solution ($f=B=\delta=0$) and separating time $f(t,x)=\mathrm{e}^{i\omega t}v(x)$ we get the eigenvalue problem[7]

$$Lv=\omega^2 v, \quad \text{where } L=-\frac{1}{\sin^2 x}\,\partial_x\left(\sin^2 x\,\partial_x\right)+1. \tag{32}$$

The operator L (which is just the polar conformal Laplacian on the 3-sphere) is symmetric on the Hilbert space $L^2\left([0,\pi/2],\,\sin^2 x\,\mathrm{d}x\right)$ and the Robin boundary condition

$$v'-bv|_{x=\frac{\pi}{2}}=0 \tag{33}$$

provides a one-parameter family of its self-adjoint extensions.

[6] For large values of c, there develops a boundary layer near $x=\pi/2$ with exponentially shrinking width. Using the method of matched asymptotics one can show that both α and b grow as e^{c^2} for $c\to\infty$ which makes the numerics (in compactified variable x) cumbersome.

[7] The eigenvalue problem (32) is a particularly simple case of the master eigenvalue problem for linear perturbations of AdS space that was solved by Ishibashi and Wald in full generality using the properties of hypergeometric functions [3]. For the reader's convenience we reproduce their results in our special case using more elementary tools.

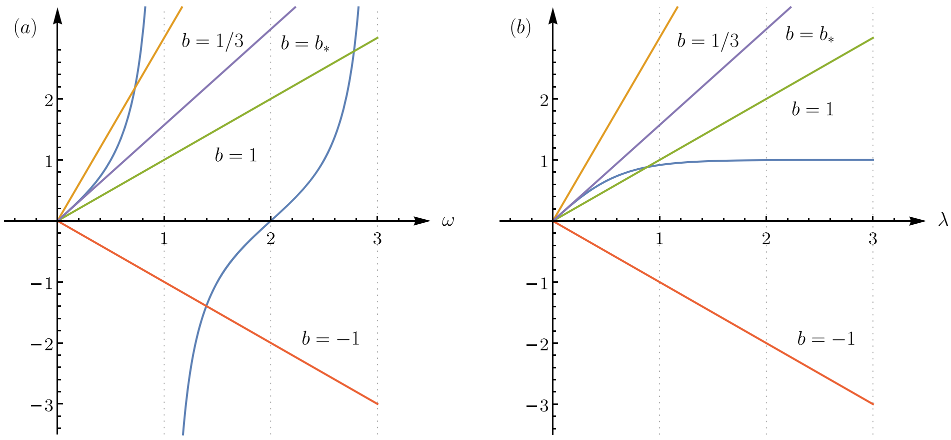

Figure 2. Graphical solutions of the quantization conditions (35) and (38).

For $\omega^2 > 0$ the regular solution of (32) is

$$v(x) = \frac{\sin(\omega x)}{\sin\, x}. \tag{34}$$

Imposing the Robin condition (33) we obtain the quantization condition for the eigenfrequencies

$$\omega = b\, \tan\left(\omega\pi/2\right). \tag{35}$$

From the graphical analysis shown in figure 2(a) we see that for each non-negative integer n there is exactly one eigenfrequency ω_n such that

$$2n + 1 < \omega_n < 2n + 2 \quad \text{if } b < 0,$$

$$2n < \omega_n < 2n + 1 \quad \text{if } 0 < b < \frac{2}{\pi}.$$

For large n the quantization condition (35) gives the asymptotically resonant spectrum

$$\omega_n = 2n + 1 - \frac{b}{\pi n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{36}$$

The lowest eigenvalue ω_0^2 vanishes at $b = b_* = 2/\pi$; the corresponding eigenfunction is the linearized static solution f_1 given in (26). An elementary perturbative calculation gives near b_*

$$\omega_0^2 \approx \frac{6}{\pi}(b_* - b). \tag{37}$$

For general $b > b_*$ there is an exponentially growing mode $e^{\lambda_0 t} v_0(x)$, where the exponent $\lambda_0 = \sqrt{-\omega_0^2}$ is given by the unique positive root of the equation (see figure 2(b))

$$\lambda = b\, \tanh\left(\lambda\pi/2\right) \tag{38}$$

and the corresponding eigenfunction is $v_0(X) = \sinh(\lambda_0 x)/\sin x$.

Next, we look at the linear stability of solitons. Linearizing the system (4) around the soliton and separating time, we get the eigenvalue problem

$$L_s v = \tilde{\omega}^2 \, v, \tag{39}$$

where

$$L_s = -\frac{A_s e^{-\delta_s}}{\sin^2 x} \, \partial_x \left(A_s e^{-\delta_s} \sin^2 x \, \partial_x \right) + A_s e^{-2\delta_s} \, U \tag{40}$$

and

$$\begin{aligned} U =& 1 + (3\cos^2 x - 1)B_s + 4\sin^2 x\,(2 - \sin^2 x) f_s^2 \\ &+ \sin x \, \cos x\,(8\sin^2 x - 4) f_s f_s' - \cos^2 x\,(2 + 4\sin^2 x) f_s'^2 \\ &+ 8\sin^3 x \cos^3 x f_s^3 f_s' - 4\sin^4 x \cos^2 x f_s^4 - 4\sin^2 x \cos^4 x f_s^2 f_s'^2. \end{aligned} \tag{41}$$

For b slightly above b_* (i.e. for small c), we have

$$L_s = L + c^2 \, P + \mathcal{O}(c^4), \tag{42}$$

where the operator P can be calculated using the expansions (25). To calculate the perturbations of eigenvalues we assume the following ansatz

$$v_n = c v_n^* + c^3 u_n + \mathcal{O}(c^5), \quad \tilde{\omega}_n^2 = {\omega_n^*}^2 + \gamma_n c^2 + \mathcal{O}(c^4), \quad b = b_* + b_2 c^2 + \mathcal{O}(c^4), \tag{43}$$

where ${\omega_n^*}^2$ and v_n^* are the eigenvalues and normalized eigenfunctions of the operator L at the bifurcation point and b_2 is given in (27). Substituting this ansatz into the Robin boundary condition we get at the first and third order in c

$${v_n^*}'(\pi/2) = b_* v_n^*(\pi/2), \quad u_n'(\pi/2) = b_* u_n(\pi/2) + b_2 v_n^*(\pi/2). \tag{44}$$

Substituting the ansatz (43) into (39), we get at the third order in c

$$L u_n + P v_n^* = {\omega_n^*}^2 u_n + \gamma_n v_n^*. \tag{45}$$

Projecting this equation on v_n^* and noting, via (44), that

$$(v_n^*, L u_n) = ({v_n^*}' u_n - v_n^* u_n')|_{x=\frac{\pi}{2}} + (u_n, L v_n^*) = -b_2 {v_n^*}^2(\pi/2) + {\omega_n^*}^2 (u_n, v_n^*), \tag{46}$$

we obtain the leading order approximation for the eigenvalues

$$\tilde{\omega}_n^2(c) \approx {\omega_n^*}^2 + \gamma_n c^2, \quad \gamma_n = (v_n^*, P v_n^*) - b_2 {v_n^*}^2(\pi/2). \tag{47}$$

In particular, for the lowest eigenvalue we obtain

$$\tilde{\omega}_0^2 \approx \gamma_0 c^2, \tag{48}$$

where

$$\gamma_0 = 32 \, \log \, 2 - 2 + \frac{1}{\pi^2} \, (12 - 144\zeta(3)) \approx 3.8582. \tag{49}$$

The positivity of γ_0 confirms the expectation that the solitons are linearly stable near the bifurcation point. Solving the eigenvalue problem numerically, we find that the eigenvalues $\tilde{\omega}_n^2$

Table 1. The first six eigenfrequencies of linear perturbations around the soliton for the parameter $c = 0.1$. In the second row the approximate eigenfrequencies given by (47) are shown for comparison.

n	0	1	2	3	4	5
$\tilde{\omega}_n$	0.197 35	2.870 65	4.930 28	6.957 14	8.973 63	10.985 49
$\tilde{\omega}_n^{\text{pert}}$	0.196 42	2.870 62	4.930 25	6.957 11	8.973 60	10.985 46

grow monotonically with c. The numerical values of the first few eigenfrequencies (as measured by the central observer) for a small parameter $c = 0.1$ (corresponding to $b \approx 0.6467$) are displayed in table 1.

From the leading order WKB approximation [12] it follows that for large n

$$\tilde{\omega}_n = \frac{2n+1}{a} + \mathcal{O}\left(\frac{1}{n}\right), \quad a = \frac{2}{\pi}\int_0^{\pi/2} A_s^{-1} e^{\delta_s}\, dx, \tag{50}$$

which compares well with numerical results even if n is not very large.

6. Discussion

The Einstein–Klein–Gordon–AdS system with mass $\mu^2 = \frac{2}{3}\Lambda < 0$ is well-behaved at the conformal boundary which makes it a good toy model for studying the role of boundary conditions in dynamics of asymptotically AdS spacetimes [13]. In this paper we focused on the Robin boundary conditions and proved existence of a one-parameter family of solitons for $b > b_*$. We also demonstrated that the linearized perturbations around these solitons have no growing modes. A natural question is: are the AdS Robin solitons nonlinearly stable? Numerical simulations, to be reported in [14], indicate a positive answer and provide evidence for existence of plethora of time-periodic and quasiperiodic solutions, not only in the perturbative regime (which is expected in view of the non-resonant spectrum) but also, somewhat surprisingly, for large perturbations.

Of course, the analogous question of nonlinear stability arises for the AdS spacetime for $b < b_*$. However here, in contrast to the Dirichlet case [6], the numerical simulations are as yet not conclusive and we leave this question to future investigations.

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Appendix

Here we give the details of the higher orders of perturbation expansion (25). At the third order equation (14a) becomes

$$\begin{aligned} f_3'' + 2\cot x\, f_3' - f_3 = -B_2\cos^2 x\, f_1'' - \cot x\, \big((1-4\sin^2 x)B_2 \\ + 2\sin^2 x f_1^2\big)\, f_1' - (1-3\cos^2 x)B_2 f_1 + 2\sin^2 x f_1^3. \end{aligned} \tag{A1}$$

Substituting f_1 and B_2, given in (26), into the right hand side and imposing $f_3(0) = 0$, we find

$$f_3(x) = \frac{1}{12\sin x}\Big((5 - 24\zeta(3))\,x + 3x\cos(2x) - \frac{2x^3}{\sin^2 x} - 3\sin(2x) + 16x^3 C_1(x) - 48x^2 S_2(x) - 72x C_3(x) + 48 S_4(x)\Big), \tag{A2}$$

where ζ is the Riemann zeta function and we defined the functions

$$S_n(x) = \sum_{k=1}^{\infty} \frac{\sin(2kx)}{k^n}, \qquad C_n(x) = \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k^n}. \tag{A3}$$

From (A2) we read off

$$f_3\left(\frac{\pi}{2}\right) = \frac{\pi}{48}\left(4 - \pi^2(8\ln 2 + 1) + 60\zeta(3)\right) \approx 0.754\,316, \tag{A4}$$

$$f_3'\left(\frac{\pi}{2}\right) = \frac{2}{3} + \frac{\pi^2}{8}(8\ln 2 - 1) - \frac{7}{2}\zeta(3) \approx 2.066\,86. \tag{A5}$$

At the fourth order equation (14b) becomes

$$\cot x\, B_4' + \frac{1}{\sin^2 x} B_4 = -B_2\cos^2 x\,(\cos x\, f_1' - \sin x\, f_1)^2 - 2(\cos x\, f_1' - \sin x\, f_1)(\cos x\, f_3' - \sin x\, f_3) + 4 f_1 f_3. \tag{A6}$$

Substituting (26) and (A2) into the right hand side and requiring regularity at $x = 0$, we find

$$\begin{aligned} B_4(x) = {} & \frac{1}{6}x\left(\cot x - \frac{x}{\sin^2 x}\right)(24\zeta(3) - 5) + 4x\left(2\cot x - \frac{3x}{\sin^2 x}\right)C_3(x) \\ & + 4x^2\left(\cot x - \frac{2x}{\sin^2 x}\right)S_2(x) - \left(6\cot x - \frac{8x}{\sin^2 x}\right)S_4(x) + \frac{3}{4}\cos^2 x \\ & + \left(\frac{3}{4} + \frac{1}{3}x^2\right)\cot^2 x + \frac{8x^4}{3\sin^2 x}C_1(x) + x^2\left(\frac{1}{4\sin^2 x} - \frac{3}{4} - \frac{1}{3}x^2\right) \\ & - x\cot x\left(1 - x^2 + \frac{1}{4}\cos(2x) + \frac{x^2}{2\sin^2 x}\right), \end{aligned} \tag{A7}$$

from which we read off

$$B_4'\left(\frac{\pi}{2}\right) = \frac{\pi}{48}\left(54 + \pi^2(32\ln 2 - 11)\right) \approx 10.7566. \tag{A8}$$

Using (A4) and (A5) and imposing the Robin condition in the expansion (25), we get

$$b_2 = \frac{2}{\pi}\left.\left(f_3' - \frac{2}{\pi}f_3\right)\right|_{\pi/2} = \frac{\pi}{6}(16\ln 2 - 1) + \frac{1}{\pi}(1 - 12\zeta(3)). \tag{A9}$$

ORCID iDs

Maciej Maliborski https://orcid.org/0000-0002-8621-9761

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