Dissipative and stochastic geometric phase of a qubit within a canonical Langevin framework

Pedro Bargueño[1,*] and Salvador Miret–Artés[2]

[1]*Departamento de Física de Materiales, Universidad Complutense de Madrid, 28040 Madrid, Spain*
[2]*Instituto de Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain*

Dissipative and stochastic effects in the geometric phase of a qubit are taken into account using a geometrical description of the corresponding open-system dynamics within a canonical Langevin framework based on a Caldeira-Leggett-like Hamiltonian. By extending the Hopf fibration $S^3 \to S^2$ to include such effects, the exact geometric phase for a dissipative qubit is obtained, whereas numerical calculations are used to include finite-temperature effects on it.

Introduction. The concept of geometric phase (GP) in quantum systems was proposed by Berry [1] when he studied the dynamics of an isolated quantum system that undergoes an adiabatic cyclic evolution. This cyclic evolution is due to the variation of parameters of the Hamiltonian and is accompanied by a change in the wave function by an additional phase factor which depends only on the geometric structure of the space of parameters. The underlying mathematical structure behind GPs was pointed out almost simultaneously by Simon [2]. Soon afterward, the generalization to non-adiabatic cyclic evolution was carried out by Aharonov and Anandan [3] and noncyclic evolution and sequential projection measurements by Samuel and Bhandary [4]. Although GPs have been observed in the laboratory [5–7], realistic quantum systems are always subject to decoherence due to their surroundings. Therefore, the extension of the GP to the case of open quantum systems becomes fundamental. The first formal extension of the GP was carried out by introducing the concept of parallel transport along density operators [8]. In a more physical context, the concept of GP was generalized for nondegenerated mixed states [9] and for degenerate mixed states under unitary evolution [10]. Using a kinematic approach [11], GPs for mixed states in nonunitary evolution were addressed [12]. Within a spin-boson model, GPs in open quantum systems have also been calculated [13] together with a study of their geometric nature [14]. A different approach was introduced in Ref. [15], where the GP was described by a distribution. In classical physics, the counterpart of the Aharonov-Anandan (or Berry) phase was discovered early by Hannay [16]. Regarding classical dissipative systems, GP shifts have been defined in dissipative limit cycle evolution [17–20], showing that they can be identified with the classical Hannay angle in an extended phase space [21]. Moreover, GPs can be constructed for a purely classical adiabatically slowly driven stochastic dynamics [22–24].

In this paper we tackle the problem of including dissipative and stochastic effects in the GP of a qubit. Our study is based on a geometrical description for a nonisolated qubit within a canonical Langevin framework (see [25] and references therein) using a Caldeira-Leggett-like Hamiltonian [26].

Mathematical preliminaries. It is well known that if $|\Psi\rangle$ represents a normalized n-level system, then $|\Psi\rangle \in S^{2n-1}$.

Thus, the geometry of odd-dimensional spheres is related to the quantum mechanics of finite-dimensional Hilbert spaces. In fact, the celebrated Hopf fibration [27] relates the quantum and classical description of qubits by means of the map $\pi : S^3 \to S^2$. This map can be understood as a composition, $\pi = \Xi \circ \Omega$, where $\Omega : S^3 \subset \mathbb{C}^2 \to \mathbb{C}P^1$ sends an element of $\mathbb{C}^2$ to its equivalence class and $\Xi : \mathbb{C}P^1(= \mathbb{C} \cup \infty) \to S^2$ is given by the stereographic projection. It can be shown that the Hopf map can be written in terms of the Pauli matrices as $\pi(|\Psi\rangle \in S^3) = (\langle\Psi|\hat{\sigma}_x|\Psi\rangle, \langle\Psi|\hat{\sigma}_y|\Psi\rangle, \langle\Psi|\hat{\sigma}_z|\Psi\rangle) \in S^2$, where $\langle\Psi|\hat{\sigma}_x|\Psi\rangle^2 + \langle\Psi|\hat{\sigma}_y|\Psi\rangle^2 + \langle\Psi|\hat{\sigma}_z|\Psi\rangle^2 = 1$. Thus, from the Hopf map it can be shown that quantum and classical mechanics may be embedded in the same formulation. Specifically, for a qubit, the Strocchi map [28] is exactly the Hopf map previously described. However, the $S^7 \to S^4$ Hopf map, which is an entanglement-sensitive fibration, does not have a classical analog [29]. As there is a map $S^{2n-1} \to \mathbb{C}P^{n-1}$ and the complex projective space has a natural symplectic structure ($\mathbb{C}P^{n-1}$ is a Kähler manifold), n-state systems have a well-defined classical correspondence, which is given by the Strocchi map [28]. Thus, one can derive a classical Hamiltonian function for an n-level system by defining, for example, appropriate n action-angle coordinates in $\mathbb{C}P^{n-1}$ [30]. For example, as shown in [31], the symplectic structure of S^2 is responsible for the Aharonov-Anandan GP.

Using the pair of action-angle coordinates (I, Φ) on S^2, the Hopf map can be expressed as $\pi(|\Psi\rangle) = (\sqrt{1 - I^2} \cos\Phi, \sqrt{1 - I^2} \sin\Phi, I)$, where $|\Psi\rangle = a_1|1\rangle + a_2|2\rangle$ ($a_j = |a_j|e^{i\phi_j} \in \mathbb{C}$), $I \equiv |a_1|^2 - |a_2|^2$, and $\Phi \equiv \phi_1 - \phi_2$. Thus, the Hamiltonian operator $\hat{H} = \sum_i \eta_i \hat{\sigma}_i$, where $\hat{\sigma}_i$ are the Pauli matrices and $\eta_i \in \mathbb{R}$ can be Hopf-mapped to a Hamiltonian function, H_0, given by $H_0 = -2\sqrt{1 - I^2}(\eta_1 \cos\Phi + \eta_2 \sin\Phi) + 2\eta_3 I$ (this is the Meyer-Miller-Stock-Thoss Hamiltonian [32,33], widely used in molecular physics).

Dissipative-stochastic Hopf fibration of S^3. Our study is based on a Caldeira-Leggett-like [26] Hamiltonian for a qubit in the Langevin framework (see [25] and references therein), which can be expressed as

$$H = H_0 + \frac{1}{2}\sum_i \left(p_i^2 + x_i^2\omega_i^2\right) - \Phi\sum_i c_i x_i + \sum_i \Phi^2 c_i^2, \tag{1}$$

where the oscillator mass has been taken to be 1 and c_i are the system-bath coupling constants. This model takes into

account a renormalization term due to the interaction with the environment.

Let us start with pure and normalized quantum states. Using the (I,Φ) action-angle coordinates, it is easy to see that the unit-radius sphere S^2 is the set of points satisfying $(-\sqrt{1-I^2}\cos\Phi)^2 + (-\sqrt{1-I^2}\sin\Phi)^2 + I^2 = 1$. The radius of this sphere remains constant along the time due to energy conservation. That is, the corresponding Hamiltonian function, given by $H_0 = -\sqrt{1-I^2}(\cos\Phi + \sin\Phi) + I$, remains constant in time (we have taken $\eta_i = 1$ for simplicity). When ohmic dissipation is assumed, and after removing the bath variables, the corresponding equations of motion issued from Eq. (1) are

$$\dot{I} = -\sqrt{1-I^2}(\sin\Phi + \cos\Phi) - 2\gamma\dot{\Phi}(t) + \xi(t),$$

$$\dot{\Phi} = \frac{I}{\sqrt{1-I^2}}(\cos\Phi + \sin\Phi) + 1, \qquad (2)$$

which correspond to the effective Hamiltonian function $H(t) = -\sqrt{1-I^2}(\cos\Phi + \sin\Phi) + I + 2\gamma\Phi\dot{\Phi} - \xi(t)\Phi$. This stochastic dynamics can be interpreted in terms of a stochastic Bloch sphere $S^2(\gamma,\xi)$ defined by

$$(\sqrt{1-I^2}\cos\Phi)^2 + (\sqrt{1-I^2}\sin\Phi)^2 + I^2$$

$$= 1 - \gamma\frac{d}{dt}(\Phi^2) + \xi(t)\Phi, \qquad (3)$$

where γ is the friction constant and $\xi(t)$ is a stochastic Gaussian process. The time-dependent radius is given by $R_t^2(\gamma,\xi) = 1 - \gamma\frac{d}{dt}(\Phi^2) + \xi\Phi$. Thus, dissipative and stochastic effects make the Bloch sphere breathe, Eq. (3), by changing its radius in time. This radius is bounded for $\xi = 0$, $R_t(\gamma,0) \leqslant 1$. The equality is reached at $t = 0$ and at asymptotic times, where thermal equilibrium is reached (in this case, a point in the unit radius sphere represents a pure state). Otherwise, mixed states are represented at each instant of time as points in different spheres of variable radius, as shown in Fig. 1. When $\xi \neq 0$, the square radius becomes a stochastic variable which can reach $R_t(\gamma,\xi) \geqslant 1$ for a particular phase-space trajectory.

The breathing of the Bloch sphere can be geometrized by extending its round metric by adding both dissipation and noise. If these terms are included, the metric of $S^2(\gamma,\xi)$ can be written as $ds^2[S^2(\gamma,\xi)] = R_t^2(\gamma,\xi)ds^2[S^2]$, where $ds^2[S^2] = \frac{dI^2}{1-I^2} + (1-I^2)d\Phi^2$ is the round metric for S^2 in action-angle coordinates.

In order to calculate the stochastic GP, let us start with pure states. If we choose an orthonormal moving frame field $\sigma^1 = \frac{dI}{\sqrt{1-I^2}}$, $\sigma^2 = \sqrt{1-I^2}d\Phi$, then $d\sigma^2 = -\frac{I}{\sqrt{1-I^2}}dI \wedge d\Phi = -I\sigma^1 \wedge d\Phi = Id\Phi \wedge \sigma^1$. Using Cartan's first structure equation, we obtain the only nonvanishing connection one-form, $\omega = Id\Phi$. Thus, the dynamic phase is obtained as $\phi_d = \oint \omega = \oint Id\Phi = \pi I = \pi\cos\theta$. If S^2 is considered as an embedded submanifold of S^3, taking into account that the spin connection of S^3 is $\tilde{\omega} = d\Xi + Id\Phi$, where Ξ is the extra Euler angle which parametrizes the third dimension, it can be shown [29] that S^2 inherits from S^3 the connection one-form $\tilde{\omega} = (I-1)d\Phi$. Thus, the GP is expressed as $\phi_g = \oint \tilde{\omega}$. Now, if dissipation and noise are taken into account, the orthonormal moving frame field is given by $R_t(\gamma,\xi)\sigma^i$ with $i = 1,2$. In this case, the only nonvanishing connection
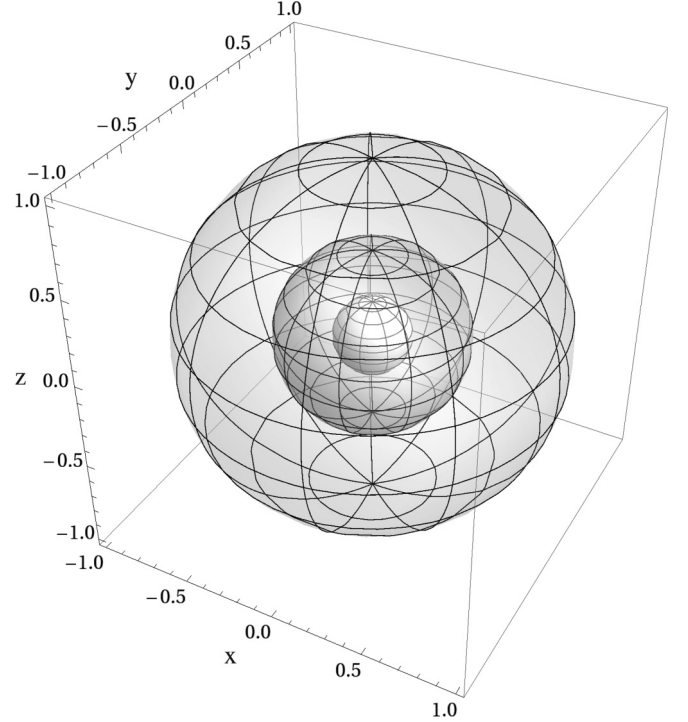

FIG. 1. Representation of the dissipative-induced time dependence of the Bloch sphere for $\xi = 0$. Smaller spheres correspond to later times.

one-form is $\omega(\gamma,\xi) = R_t(\gamma,\xi)Id\Phi = R_t(\gamma,\xi)\omega$. This allows us to define the stochastic dynamic phase acquired after a cycle of period T as $\phi_d(\gamma,\xi) = \oint \omega(\gamma,\xi) = \oint R_t(\gamma,\xi)I(T)d\Phi = \cos\theta(T)\oint \sqrt{1 - \gamma\frac{d}{dt}(\Phi^2) + \xi\Phi}\,d\Phi$. Therefore, the stochastic connection one-form, which can be defined as $\tilde{\omega}(\gamma,\xi) = [IR_t(\gamma,\xi) - 1]d\Phi$, leads to the corresponding GP:

$$\phi_g(\gamma,\xi) = \oint \tilde{\omega}(\gamma,\xi) = \oint [I(T)R_t(\gamma,\xi) - 1]\,d\Phi$$

$$= \oint \left(I(T)\sqrt{1 - \gamma\frac{d}{dt}(\Phi^2) + \xi\Phi} - 1\right)d\Phi. \qquad (4)$$

Note that the dynamics driven by this kind of Caldeira-Leggett-like coupling could be interpreted in terms of a mapping between conformal spheres with conformal unitary group fibers at each instant. Moreover, the GP defined by Eq. (4) becomes $\tilde{\phi}_g(\gamma,\xi) = \oint (I\sqrt{1 - \gamma\frac{d}{dt}(\tilde{\Phi}^2) + \xi\tilde{\Phi}} - 1)d\tilde{\Phi}$ under the transformation $|\tilde{\Psi}\rangle = e^{i\alpha}|\Psi\rangle$ (or $\tilde{\Phi} = \Phi + \alpha$). Thus, $\phi_g(\gamma,\xi)$ is a gauge-invariant quantity.

Dissipative qubit (zero temperature). Let us consider a simple qubit which can be represented by the Hamiltonian operator $\hat{H} = \epsilon\hat{\sigma}_z$. The corresponding dissipative dimensionless Hamiltonian function $(t \to 2\epsilon t)$, which can be written as $H(t) = I + \frac{\gamma}{2\epsilon}\Phi\dot{\Phi}$, leads to the pair of coupled equations $\dot{I} = -\frac{\gamma}{2\epsilon}\dot{\Phi}$ and $\dot{\Phi} = 1$ with solutions $I(t) = I_0 - \frac{\gamma}{2\epsilon}t$ and $\Phi(t) = \Phi_0 + t$, where I_0 and Φ_0 are the initial conditions of the action-angle variables. After a cycle of period $T = 2\pi$ (the scaled Rabi frequency is $\omega_0 = 1$), the dissipative GP can

be computed as

$$\phi_g(\gamma) = \oint [R_t(\gamma)I - 1] \, d\Phi$$

$$= \oint \left[I(T)\sqrt{1 - \frac{\gamma}{2\epsilon}\Phi\dot{\Phi}} - 1 \right] d\Phi$$

$$= -\pi + \frac{4}{3}\left[\left(1 - \frac{\pi\gamma}{2\epsilon}\right)^{3/2} - 1 \right]\left(\pi - \frac{\epsilon}{\gamma}\cos\theta_0\right), \tag{5}$$

where $I_0 = \cos\theta_0$. Note that the nondissipative GP is recovered when $\gamma \to 0$.

It is also interesting to remark that Eq. (5) is only valid for $\gamma\pi/2\epsilon \leqslant 1$, which reflects nothing but the Caldeira-Leggett energy renormalization in an Ohmic environment. Moreover, this energy renormalization can be reinterpreted in terms of a Bloch sphere whose radius develops harmonic oscillations. In order to show this statement in simple terms, let us introduce the damping factor by means of the phenomenological Caldirola-Kanai Hamiltonian [34–36] for a harmonic oscillator (which is equivalent to the corresponding Caldeira-Leggett Hamiltonian for the oscillator at zero temperature). The Hamiltonian reads $H = (1/2)p^2 e^{-\frac{\gamma}{2\epsilon}\pi t} + q^2 e^{\frac{\gamma}{2\epsilon}\pi t}$ (the π factor has been introduced to note that time evolution is cyclic). It is straightforward to derive the corresponding equations of motion, leading to $\ddot{y} + [1 - (\frac{\gamma\pi}{2\epsilon})^2]y = 0$, where $y = p,q$. Thus, the renormalized frequency due to the damping term is $\bar{\omega}^2 = \omega_0^2 - (\frac{\gamma\pi}{2\epsilon})^2$, which has physical sense only when $\gamma\pi/2\epsilon \leqslant 1$, as Eq. (5) shows. Therefore, the radius of the Bloch sphere develops harmonic oscillations with a renormalized frequency (another interpretation is that $|I(t)| \leqslant 1$ requires $\gamma\pi/2\epsilon \leqslant 1$).

The weak-coupling limit of the GP given by Eq. (5) can be expressed as

$$\phi_g(\gamma) = -\pi(1 - \cos\theta_0) - \frac{\gamma}{\epsilon}\left(\frac{\pi}{2}\right)^2(\cos\theta_0 + 4) + O(\gamma^2). \tag{6}$$

A direct comparison with other authors [10,12,15] is not pertinent since different models of dissipation and other approaches to calculating the GP were used.

On the other hand, for this dissipative dynamics, information on interference experiments can be straightforwardly extracted from the probability density itself. The typical interference intensity, J_t, evolves in time according to $J_t \propto |\Psi(t)|^2 \propto 1 + \sqrt{1 - I^2(t)}\cos\Phi(t) = 1 + \sqrt{1 - (I_0 - \frac{\gamma}{2\epsilon}t)^2}\cos(t + \Phi_0)$, depending critically on the ratio $\gamma/2\epsilon$, which is directly related to the GP given by Eq. (5).

Stochastic qubit (nonzero temperature). Noise effects can be included in a simple way by assuming a Gaussian stochastic process with distribution function $\rho(\xi;\beta) = \sqrt{\frac{\beta}{2\pi}}\exp(-\frac{\beta\xi^2}{2})$, where β is the inverse of the temperature, which is given in units of 2ϵ. Thus, for a qubit, the squared radius is also a stochastic process given by $R_t^2(\gamma,\xi) = 1 + (\xi - \frac{\gamma}{2\epsilon})\Phi$. Therefore, the stochastic GP acquired after a cycle of period

T is given by

$$\phi_g(\gamma;\beta) = \left(\cos\theta(T) - \frac{\gamma\pi}{2\epsilon}\right)\left[\oint\int_{-\infty}^{\infty}\sqrt{1 - \Phi\left(\frac{\gamma}{2\epsilon} - \xi\right)}\right.$$

$$\left. \times \sqrt{\frac{\beta}{2\pi}}\exp\left(-\frac{\beta\xi^2}{2}\right)d\xi\,d\phi\right] - \pi. \tag{7}$$

The previous integral in the noise variable has no analytic solution. Thus, we first integrate in Φ, obtaining

$$\phi_g(\gamma;\beta) = \left(\cos\theta(T) - \frac{\gamma\pi}{2\epsilon}\right)\left[\int_{-\infty}^{\infty} -\frac{2}{3}\left(\frac{\gamma}{2\epsilon} - \xi\right)^{-1}\right.$$

$$\times \left[\left(1 - \left\{\frac{\gamma}{2\epsilon} - \xi\right\}\pi\right)^{3/2} - 1\right]\sqrt{\frac{\beta}{2\pi}}$$

$$\left. \times \exp\left(-\frac{\beta\xi^2}{2}\right)d\xi\right] - \pi. \tag{8}$$

To illustrate the computation of this integral (which does not have any analytical solution either), only the GP for a nondissipative qubit at finite temperature is considered. Moreover, as the noise term becomes more important for high temperatures, we assume that the mean of the Gaussian process is β^{-1}. Therefore, the GP can be factorized as

$$\phi_g(\beta) = \cos\theta_0\left[\int_{-\infty}^{\infty}\frac{2}{3\xi}[(1 + \xi\pi)^{3/2} - 1]\sqrt{\frac{\beta}{2\pi}}\right.$$

$$\times \exp\left\{-\frac{\beta(\xi - \beta^{-1})^2}{2}\right\}d\xi\right] - \pi$$

$$\equiv \cos\theta_0 f(\beta) - \pi, \tag{9}$$

where $f(\beta)$ encodes the thermal information of the GP acquired by an isolated qubit. This function is depicted in Fig. 2 within a large range of temperatures, displaying a linear behavior at high temperatures and a crossover at a critical temperature given by $T_c \approx 2\epsilon$ ($T_c \approx 1$ if adimensional temperatures are used). This temperature corresponds to the energy difference of the two levels of the qubit. Moreover, as
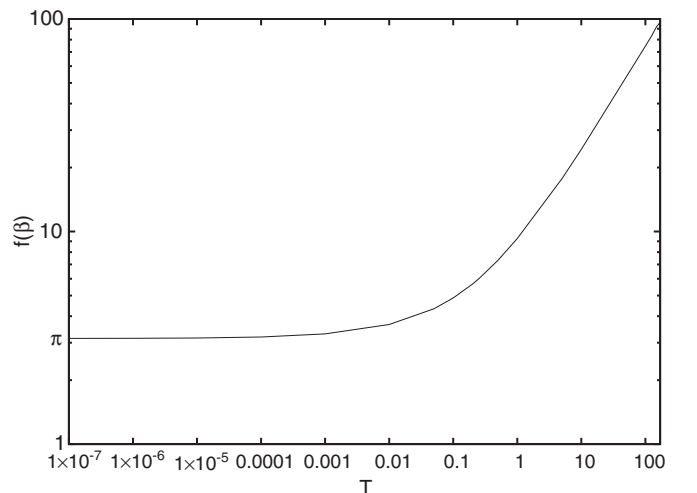

FIG. 2. Thermal behavior of the GP obtained by numerical integration of Eq. (9) (a double-logarithmic scale is used). Note that $f(\beta) \to \pi$ at very low temperatures.

$f(\beta) \to \pi$ at very low temperatures, the GP acquired for the pure state at zero temperature is recovered. Finally, note that this type of calculation can be straightforwardly extended to any dissipation value.

In summary, a geometrical description of the dissipative and stochastic dynamics of a qubit within a Langevin formalism (with a Caldeira-Leggett-like coupling) has been developed. The Hopf fibration $S^3 \to S^2$ has been extended to include both stochastic and dissipative effects in terms of a Bloch sphere which develops harmonic oscillations (in the dissipative case). This procedure has allowed us to define a gauge-invariant GP, which has been computed for both dissipative and stochastic cases.

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