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Identities involving 3-variable Hermite polynomials arising from umbral method

Nusrat Raza[1], Umme Zainab[2], Serkan Araci[3*] and Ayhan Esi[4]

*Correspondence:
serkan.araci@hku.edu.tr
[3]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410, Gaziantep, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we employ an umbral method to reformulate the 3-variable Hermite polynomials and introduce the 4-parameter 3-variable Hermite polynomials. We also obtain some new properties for these polynomials. Moreover, some special cases are discussed and some concluding remarks are also given.

MSC: 05A40; 33C45

Keywords: Umbral method (umbra); 3-variable generalised Hermite polynomials; 4-parameter 3-variable Hermite polynomials; Generating function

1 Introduction

The multi-variable Hermite polynomials have been used in the analysis of charged-beam transport problems in classical mechanics as well as in the formulation of quantum-phase-space mechanics. Umbral methods have been largely exploited to study the properties of the Hermite polynomials. Recently Dattoli et al. applied the method of umbral to obtain certain results for the Hermite polynomials [8]. The study of umbral formalism provides a fairly helpful tool in many topics of practical nature concerning physics of free electron laser. In this paper, we extend the umbral treatment of the Hermite polynomials from two variables to three variables.

We begin with some umbral results on the 2-variable Hermite polynomials (2VHP) $H_n(x,y)$. We recall that 2VHP $H_n(x,y)$ are defined by means of the following generating function and series definition [2]:

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2} \tag{1}$$

and

$$H_n(x,y) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{x^{n-2k}y^k}{k!(n-2k)!}, \tag{2}$$

respectively.

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The boundary conditions for 2VHP $H_n(x,y)$ are as follows [8, 21]:

$$H_n(x,0) = x^n \tag{3}$$

and

$$H_n(0,y) = n!\frac{y^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\left|\cos\left(n\frac{\pi}{2}\right)\right|, \tag{4}$$

respectively.

In this paper, we employ the umbral method to the 3-variable Hermite polynomials. Also, we exploit the umbral method to obtain several extensions of the 3-variable Hermite polynomials. Recently, Dattoli *et al.* gave an umbral method for 2VHP $H_n(x,y)$, which plays an important role in the field of special functions and applied mathematics to obtain all the relevant properties of the other special polynomials as well as special functions [8].

In [8], Dattoli considered the idea of umbra, denoted by $\hat{b}_y$, for 2VHP $H_n(x,y)$ as follows:

$$\hat{b}_y^r\phi_0 = \frac{y^{\frac{r}{2}}r!}{\Gamma(\frac{r}{2}+1)}\left|\cos r\frac{\pi}{2}\right| \quad (\phi_0 \neq 0), \tag{5}$$

where ϕ_0 is known as polynomial vacuum and $\hat{b}_y$ acting on the state ϕ_0 yields 2VHP $H_n(x,y)$.

The exponential of umbra $\hat{b}_y$ is particulary important to derive the generating functions for 2VHP $H_n(x,y)$. The exponential of umbra $\hat{b}_y$ is as follows [8]:

$$e^{\hat{b}_y t}\phi_0 = e^{yt^2}. \tag{6}$$

In view of equation (5), 2VHP $H_n(x,y)$ can be reduced binomially as follows:

$$H_n(x,y) = (x+\hat{b}_y)^n\phi_0, \tag{7}$$

see [8].

Dattoli [8] introduced the 2-parameter 2-variable Hermite polynomials 2P2VHP $H_n(x,y|\beta,\alpha)$:

$$H_n(x,y|\beta,\alpha) = \hat{b}_y^\beta\left(x+\hat{b}_y^\alpha\right)^n\phi_0. \tag{8}$$

The generating function of 2P2VHP $H_n(x,y|\beta,\alpha)$ is as follows:

$$\sum_{n=0}^{\infty}H_n(x,y|\beta,\alpha)\frac{t^n}{n!} = e^{xt}y^{\frac{\beta}{2}}e_{(\alpha,\beta)}\left(y^{\frac{\alpha}{2}}t\right), \tag{9}$$

where

$$e_{(\alpha,\beta)}(x) = \sum_{r=0}^{\infty}\frac{\Gamma(\alpha r+\beta+1)x^r}{\Gamma(\frac{\alpha r+\beta}{2}+1)r!}\left|\cos\left(\frac{\alpha r+\beta}{2}\pi\right)\right|, \tag{10}$$

see [8], which is a generalisation of the exponential function. It is worthy to note that

$$e_{(1,0)}(x) = e^{x^2}. \tag{11}$$

Now, we recall that the 3-variable Hermite polynomials (3VHP) $H_n(x,y,z)$ are defined by means of the following generating function and series definition [6]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y,z) = \exp\bigl(xt + yt^2 + zt^3\bigr) \tag{12}$$

and

$$H_n(x,y,z) = n! \sum_{k=0}^{[\frac{n}{3}]} \frac{H_{n-3k}(x,y) z^k}{k!(n-3k)!}, \tag{13}$$

respectively.

The operational definition of 3VHP $H_n(x,y,z)$ is as follows [6]:

$$H_n(x,y,z) = e^{zD_x^3 + yD_x^2} x^n, \tag{14}$$

where

$$D_x := \frac{d}{dx}.$$

The Gould–Hopper polynomials (GHP) $H_n^{(m)}(x,y)$ are defined by means of the following generating function and series definition [13]:

$$\sum_{n=0}^{\infty} H_n^{(m)}(x,y) \frac{t^n}{n!} = e^{xt + yt^m} \tag{15}$$

and

$$H_n^{(m)}(x,y) = n! \sum_{r=0}^{[\frac{n}{m}]} \frac{x^{n-mr} y^r}{r!(n-mr)!}, \tag{16}$$

respectively.

Since

$${}_m\hat{b}_y^r \phi_0 = \frac{y^{\frac{r}{m}} r!}{\Gamma(\frac{r}{m}+1)} A_{m,r}, \tag{17}$$

$$A_{m,r} = \begin{cases} 1 & r = mp, p \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{18}$$

Dattoli [8] defined GHP $H_n^{(m)}(x,y)$ in terms of the nth power of the binomial given by

$$H_n^{(m)}(x,y) = (x + {}_m\hat{b}_y)^n \phi_0. \tag{19}$$

The 3-variable generalised Hermite polynomials (3VgHP) $H_n^{(s,m)}(x,y,z)$ are defined by means of the following generating function [11]:

$$\sum_{n=0}^{\infty} H_n^{(s,m)}(x,y,z)\frac{t^n}{n!} = e^{(xt+yt^m+zt^s)} \tag{20}$$

and equivalently by

$$H_n^{(s,m)}(x,y,z) = n!\sum_{r=0}^{[\frac{n}{s}]} \frac{H_{n-sr}^{(m)}(x,y)z^r}{(n-sr)!r!}. \tag{21}$$

The operational definition of 3VgHP $H_n^{(s,m)}(x,y,z)$ are as follows [11]:

$$H_n^{(s,m)}(x,y,z) = \exp\bigl(zD_x^s + yD_x^m\bigr)x^n \tag{22}$$

and

$$H_n^{(s,m)}(x,y,z) = \exp\bigl(zD_x^s\bigr)H_n^{(m)}(x,y). \tag{23}$$

In this paper, motivated by the work of Dattoli on the umbral behaviour of the Hermite polynomials [8, 10], we extend the umbral formalism to the 3-variable Hermite polynomials. In Sect. 2, we define an umbra for the 3-variable Hermite polynomials and obtain umbral definition for 3-variable Hermite polynomials and 3-variable generalised Hermite polynomials. In Sect. 3, we introduce an extension of 3-variable Hermite polynomials to 4-parameter 3-variable Hermite polynomials by using the umbral formalism and establish certain results involving these polynomials. In Sect. 4, we discuss some special cases of 4-parameter 3-variable Hermite polynomials. Some concluding remarks are given in Sect. 5.

2 Umbra and 3-variable Hermite polynomial

In [3, 7, 8, 16], it is established that the umbral method serves as an important tool to deal with certain properties of special functions. In this paper, by making use of their method, we introduce the umbral definition of the 3-variable Hermite polynomials $H_n(x,y,z)$. In this section, we also obtain the umbra for 3VHP $H_n(x,y,z)$ and study some of its new properties.

Taking $x = 0$ and $y = 0$ in equation (13), we obtain the boundary condition for the 3-variable Hermite polynomials $H_n(x,y,z)$:

$$H_n(0,0,z) = \frac{z^{\frac{n}{3}}n!}{\Gamma(\frac{n}{3}+1)}\left(\left|2\cos\frac{n\pi}{3}\right| - |\cos n\pi|\right). \tag{24}$$

In view of equation (24), we introduce the following umbra:

$$\hat{c}_z^r\psi_0 = \frac{z^{\frac{r}{3}}r!}{\Gamma(\frac{r}{3}+1)}\left(\left|2\cos r\frac{\pi}{3}\right| - |\cos r\pi|\right) \quad (\psi_0 \neq 0), \tag{25}$$

where $\hat{c}_z$ acts on the vacuum ψ_0.

It follows from Eq. (25) that

$$e^{\hat{c}_z t}\psi_0 = e^{zt^3}. \tag{26}$$

Using equations (6) and (26) in equation (12), we get the following umbral form of the generating function of 3VHP $H_n(x,y,z)$:

$$\sum_{n=0}^{\infty} H_n(x,y,z)\frac{t^n}{n!} = e^{(x+\hat{b}_y+\hat{c}_z)t}\phi_0\psi_0, \tag{27}$$

which on expanding the exponential function in the right-hand side and then comparing the equal powers of t from both sides of the resultant equation gives the following umbral definition of the 3-variable Hermite polynomials $H_n(x,y,z)$:

$$\begin{aligned} H_n(x,y,z) &= (x+\hat{b}_y+\hat{c}_z)^n\phi_0\psi_0 \\ &= e^{(\hat{c}_z+\hat{b}_y)D_x}x^n\phi_0\psi_0 \\ &= e^{\hat{c}_z D_x}H_n(x,y)\psi_0, \end{aligned}$$

where $\hat{b}_y$ is acting on ϕ_0 and $\hat{c}_z$ is acting on ψ_0.

The use of the above equation allows a significant simplification of the theory of 3VHP $H_n(x,y,z)$, and it would be largely exploited in the field of special functions. We note that such a point of view has opened new avenues in the derivation of lacunary generating functions and for the relevant combinatorial interpretation [12].

Now, we obtain the umbral definition and umbral operational definition of the 3-variable generalised Hermite polynomials (3VgHP) $H_n^{(s,m)}(x,y,z)$.

In view of equation (25), we introduce the following generalised form of umbra $\hat{c}_z$:

$${}_s\hat{c}_z^r\psi_0 = \frac{z^{\frac{r}{s}}r!}{\Gamma(\frac{r}{s}+1)}A_{s,r}, \tag{28}$$

where

$$A_{s,r} = \begin{cases} 1 & r = sp, p \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{29}$$

If we take $A_{3,r} = (|2\cos r\frac{\pi}{3}| - |\cos r\pi|)$, then for $s = 3$ equation (28) gives (25) and ${}_s\hat{c}_z\psi_0$ reduces to $\hat{c}_z\psi_0$.

By equation (28), we have

$$e^{{}_s\hat{c}_z t}\psi_0 = e^{zt^s}. \tag{30}$$

Using equations (23) and (30), we get

$$H_n^{(s,m)}(x,y,z) = \exp({}_s\hat{c}_z D_x)H_n^{(m)}(x,y)\psi_0, \tag{31}$$

which on further simplification gives umbral operational definition of 3VgHP $H_n^{(s,m)}(x,y,z)$ as follows:

$$\begin{aligned} H_n^{(s,m)}(x,y,z) &= e^{({}_s\hat{c}_z + {}_m\hat{b}_y)D_x} x^n \phi_0 \psi_0 \\ &= \exp({}_s\hat{c}_z D_x)(x + {}_m\hat{b}_y)^n \phi_0 \psi_0, \end{aligned}$$

where ${}_m\hat{b}_y{}^r \phi_0$ and ${}_s\hat{c}_z{}^r \psi_0$ are defined in equations (17) and (28), respectively.

By using Crofton identity given in [9]

$$e^{\lambda D_x^m} f(x) = f\bigl(x + m\lambda D_x^{m-1}\bigr), \tag{32}$$

we obtain 3VgHP $H_n^{(s,m)}(x,y,z)$ binomially as follows:

$$H_n^{(s,m)}(x,y,z) = (x + {}_m\hat{b}_y + {}_s\hat{c}_z)^n \phi_0 \psi_0. \tag{33}$$

In the next section, we generalise the 3-variable Hermite polynomial to 4-parameters 3-variables Hermite polynomials arising from umbral method.

3 An extension of the 3-variable Hermite polynomials

It is realised that the advantage of umbral method is that this method serves as an important extension of certain special functions that cannot be extended by using classical operational method; see for example [14, 15]. In this section, by using the fact that the power of these umbras can be any real numbers, we extend the 3-variable Hermite polynomials to 4-parameter 3-variable Hermite polynomials by using the Hermite umbras given as $\hat{b}_y$ and $\hat{c}_z$ in equations (5) and (25), respectively.

Further, we study the properties of the 4-parameter 3-variable Hermite polynomials $H_n(x,y,z|\beta,\alpha;p,q)$ and apply the umbral method to aforementioned polynomial.

We introduce the 4-parameter 3-variable Hermite polynomials (4P3VHP)
$H_n(x,y,z|\beta,\alpha;p,q)$ given by

$$H_n(x,y,z|\beta,\alpha;p,q) = \hat{b}_y^\beta \hat{c}_z^p \bigl(x + \hat{b}_y^\alpha + \hat{c}_z^q\bigr)^n \phi_0 \psi_0, \tag{34}$$

where α, β, p and $q \in \mathbb{N} \cup \{0\}$.

By equation (34), we have the following generating function for 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$.

Theorem 3.1 *The generating function of* 4*P*3*VHP* $H_n(x,y,z|\beta,\alpha;p,q)$ *is given by*

$$\sum_{n=0}^{\infty} H_n(x,y,z|\beta,\alpha;p,q)\frac{t^n}{n!} = e^{xt} y^{\frac{\beta}{2}} e_{(\alpha,\beta)}\bigl(y^{\frac{\alpha}{2}} t\bigr) z^{\frac{p}{3}} \mathcal{E}_{p,q}\bigl(z^{\frac{q}{3}} t\bigr), \tag{35}$$

where

$$\mathcal{E}_{p,q}(x) = \sum_{r=0}^{\infty} \frac{\Gamma(p+qr+1)x^r}{\Gamma(\frac{p+qr}{3}+1)r!}\left(\left|2\cos\frac{(p+qr)\pi}{3}\right| - \left|\cos(p+qr)\pi\right|\right). \tag{36}$$

Proof From equation (34), we have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y,z|\beta,\alpha;p,q)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\hat{b}_y^{\beta}\hat{c}_z^{p}\left(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}\right)^n\phi_0\psi_0$$
$$=\hat{b}_y^{\beta}\hat{c}_z^{p}e^{(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q})t}\phi_0\psi_0.$$

Since it is obvious that $[x+\hat{b}_y^{\alpha},\hat{c}_z^{q}]=0$ and $[x,\hat{b}_y^{\alpha}]=0$ and using the Weyl decoupling identity [9]

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}}e^{\hat{B}}e^{\frac{-k}{2}},\quad k=[\hat{A},\hat{B}],(k\in\mathbb{C}) \tag{37}$$

in the above equation, we find

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y,z|\beta,\alpha;p,q)=\hat{b}_y^{\beta}\hat{c}_z^{p}e^{xt}e^{\hat{b}_y^{\alpha}t}e^{\hat{c}_z^{q}t}\phi_0\psi_0, \tag{38}$$

which, on expanding the exponentials in the right-hand side, gives

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y,z|\beta,\alpha;p,q)=e^{xt}\sum_{r=0}^{\infty}\frac{\hat{b}_y^{\alpha r+\beta}t^r}{r!}\sum_{s=0}^{\infty}\frac{\hat{c}_z^{p+qs}t^s}{s!}\phi_0\psi_0.$$

Now, using equations (5) and (25), we get

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y,z|\beta,\alpha;p,q)$$
$$=e^{xt}y^{\frac{\beta}{2}}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha r+\beta+1)y^{\frac{\alpha r}{2}}t^r}{\Gamma(\frac{\alpha r+\beta}{2}+1)r!}\left|\cos\left(\frac{\alpha r+\beta}{2}\pi\right)\right|$$
$$\times z^{\frac{p}{3}}\sum_{s=0}^{\infty}\frac{\Gamma(p+qs+1)z^{\frac{qs}{3}}t^s}{\Gamma(\frac{p+qs}{3}+1)s!}\left(\left|2\cos\pi\frac{(p+qs)}{3}\right|-\left|\cos(p+qs)\pi\right|\right).$$

Using equations (10) and (36) in the right-hand side of the above equation, we get assertion (35). □

Remark 3.1 The function $\mathcal{E}_{p,q}(x)$ is a generalisation of e^x, as for $p=0$ and $q=1$ in equation (36), we get $\mathcal{E}_{0,1}(x)=e^{x^3}$.

Next, we obtain the following series definition for 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$.

Theorem 3.2 *The series definition for* 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$ *is given by*

$$H_n(x,y,z|\beta,\alpha;p,q)=n!\sum_{r=0}^{n}\frac{\Gamma(p+qr+1)z^{\frac{p+qr}{3}}H_{n-r}(x,y|\beta,\alpha)}{\Gamma(\frac{p+qr}{3}+1)r!(n-r)!}$$
$$\times\left(\left|2\cos\frac{(p+qr)\pi}{3}\right|-\left|\cos(p+qr)\pi\right|\right), \tag{39}$$

where $H_{n-r}(x,y|\beta,\alpha)$ denotes 2P2VHP given by means of the following generating function:

$$\sum_{n=0}^{\infty} H_n(x,y|\beta,\alpha)\frac{t^n}{n!} = e^{xt} y^{\frac{\beta}{2}} e_{(\alpha,\beta)}\big(y^{\frac{\alpha}{2}} t\big).$$

Proof From equation (38), we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y,z|\beta,\alpha;p,q) = \hat{b}_y^{\beta} \hat{c}_z^{p} e^{(x+\hat{b}_y^{\alpha})t} e^{\hat{c}_z^{qt}} \phi_0\psi_0,$$

which, on expanding exponentials in the right-hand side of the above equation, gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y,z|\beta,\alpha;p,q) &= \sum_{n=0}^{\infty} \hat{b}_y^{\beta} \big(x+\hat{b}_y^{\alpha}\big)^n \frac{t^n}{n!} \sum_{r=0}^{\infty} \hat{c}_z^{p+qr} \frac{t^r}{r!} \phi_0\psi_0 \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{\hat{b}_y^{\beta}(x+\hat{b}_y^{\alpha})^{n-r} \hat{c}_z^{p+qr} t^n}{(n-r)!r!} \phi_0\psi_0.\end{aligned}$$

Using equations (8) and (25), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y,z|\beta,\alpha;p,q) &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{\Gamma(p+qr+1) z^{\frac{p+qr}{3}} H_{n-r}(x,y|\beta,\alpha)}{\Gamma(\frac{p+qr}{3}+1) r!(n-r)!} \\ &\quad\times \left(\left|2\cos\frac{(p+qr)\pi}{3}\right| - \big|\cos(p+qr)\pi\big|\right) t^n.\end{aligned}$$

Comparing the equal powers of t from both sides of the above equation, we get assertion (39). □

Further, we discuss an alternative formulation of the theory of the generalised Hermite polynomials using umbral formalism, which will be embedded with the technique developed in this paper.

Now, we obtain the following result.

Theorem 3.3 *The following formula for 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$ holds*:

$$\begin{aligned}&H_{n+k}(x,y,z|\beta,\alpha;p,q)) \\ &\quad= \sum_{r=0}^{k}\sum_{s=0}^{r} \binom{k}{r}\binom{r}{s} x^s H_n\big(x,y,z|\alpha(r-s)+\beta,\alpha;q(k-r)+p,q\big).\end{aligned} \tag{40}$$

Proof From equation (34), we have

$$\begin{aligned}H_{n+k}(x,y,z|\beta,\alpha;p,q) &= \hat{b}_y^{\beta}\hat{c}_z^{p}\big(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}\big)^{n+k}\phi_0\psi_0 \\ &= \hat{b}_y^{\beta}\hat{c}_z^{p}\big(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}\big)^{k}\big(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}\big)^{n}\phi_0\psi_0.\end{aligned}$$

Expanding the first bracket of the right-hand side of the above equation binomially, we have

$$H_{n+k}(x,y,z|\beta,\alpha;p,q)) = \hat{b}_y^{\beta}\hat{c}_z^{p}\sum_{r=0}^{k}\binom{k}{r}\left(x+\hat{b}_y^{\alpha}\right)^r\hat{c}_z^{q(k-r)}\left(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}\right)^n\phi_0\psi_0.$$

Again, expanding the first bracket of the right-hand side of the above equation binomially, we find

$$H_{n+k}(x,y,z|\beta,\alpha;p,q)) = \sum_{r=0}^{k}\sum_{s=0}^{r}\binom{k}{r}\binom{r}{s}x^s\hat{b}_y^{\alpha(r-s)+\beta}\hat{c}_z^{q(k-r)+p}\left(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}\right)^n\phi_0\psi_0.$$

Using equation (34) in the right-hand side of the above equation, we get assertion (40). □

For $k=n$, Theorem 3.3 gives the following result.

Corollary 3.1 *The following index duplication formula for* 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$ *holds*:

$$\begin{aligned} &H_{2n}(x,y,z|\beta,\alpha;p,q) \\ &\quad= \sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n\bigl(x,y,z|\alpha(r-s)+\beta,\alpha;q(n-r)+p,q\bigr). \end{aligned} \tag{41}$$

Further, we obtain the following result.

Theorem 3.4 *The following argument duplication formula for* 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$ *holds*:

$$\begin{aligned} H_n(2x,y,z|\beta,\alpha;p,q)) &= \sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}} \\ &\quad\times H_{n-s}\left(x,\frac{y}{2},\frac{z}{2}|\alpha(r-u)+\beta,\alpha;q(s-r)+p,q\right). \end{aligned} \tag{42}$$

Proof From equation (34), we have

$$H_n(2x,y,z|\beta,\alpha;p,q) = \hat{b}_y^{\beta}\hat{c}_z^{p}\left[\left(x+\frac{\hat{b}_y^{\alpha}}{2}+\frac{\hat{c}_z^{q}}{2}\right)+\left(x+\frac{\hat{b}_y^{\alpha}}{2}+\frac{\hat{c}_z^{q}}{2}\right)\right]^n\phi_0\psi_0,$$

which on simplification gives

$$H_n(2x,y,z|\beta,\alpha;p,q) = \hat{b}_y^{\beta}\hat{c}_z^{p}\sum_{s=0}^{n}\binom{n}{s}\left(x+\frac{\hat{b}_y^{\alpha}}{2}+\frac{\hat{c}_z^{q}}{2}\right)^{n-s}\left(x+\frac{\hat{b}_y^{\alpha}}{2}+\frac{\hat{c}_z^{q}}{2}\right)^s\phi_0\psi_0.$$

Expanding the second bracket in right-hand side of the above equation binomially, we find

$$H_n(2x,y,z|\beta,\alpha;p,q)$$
$$=\hat{b}_y^{\beta}\hat{c}_z^{p}\sum_{s=0}^{n}\sum_{r=0}^{s}\binom{n}{s}\binom{s}{r}\left(x+\frac{\hat{b}_y^{\alpha}}{2}+\frac{\hat{c}_z^{q}}{2}\right)^{n-s}\left(x+\frac{\hat{b}_y^{\alpha}}{2}\right)^{r}\left(\frac{\hat{c}_z^{q}}{2}\right)^{s-r}\phi_0\psi_0,$$

which on further simplification gives

$$H_n(2x,y,z|\beta,\alpha;p,q)=\hat{b}_y^{\beta}\hat{c}_z^{p}\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^{u}\left(\frac{\hat{b}_y^{\alpha}}{2}\right)^{r-u}\left(\frac{\hat{c}_z^{q}}{2}\right)^{s-r}$$
$$\times\left(x+\frac{\hat{b}_y^{\alpha}}{2}+\frac{\hat{c}_z^{q}}{2}\right)^{n-s}\phi_0\psi_0.$$

Using equation (34) in the right-hand side of the above equation, we get assertion (42). □

Now, we find the following series representation of the 4-parameter 2-variable Hermite polynomials in terms of the 4-parameter 3-variable Hermite polynomials.

Theorem 3.5 *The series definition of* 4P2VHP $H_n(x,y|\beta,\alpha;p,q)$ *is given by*

$$H_n(x,y|\beta,\alpha;p,q)=\sum_{r=0}^{n}\binom{n}{r}(-1)^{r}H_{n-r}(x,y,z|\beta,\alpha;p+qr,q). \tag{43}$$

Proof From equation (34), we have

$$H_n(x,y|\beta,\alpha;p,q)=\hat{b}_y^{\beta}\hat{c}_z^{p}\left(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}-\hat{c}_z^{q}\right)^{n}\phi_0\psi_0, \tag{44}$$

from which, on expanding binomially, we get

$$H_n(x,y|\beta,\alpha;p,q)=\hat{b}_y^{\beta}\hat{c}_z^{p}\sum_{r=0}^{n}\binom{n}{r}(-1)^{r}\left(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q}\right)^{n-r}\hat{c}_z^{qr}\phi_0\psi_0. \tag{45}$$

Using equation (34) in the above equation we get assertion (43). □

Remark 3.2 For taking $p=0$ and $q=1$ in equation (44) of Theorem 3.5, we get the following series representation of 2P2VHP $H_n(x,y,|\beta,\alpha)$:

$$H_n(x,y|\beta,\alpha)=\sum_{r=0}^{n}\binom{n}{r}(-1)^{r}H_{n-r}(x,y,z|\beta,\alpha;r,1). \tag{46}$$

Remark 3.3 For taking $\beta=0$, $\alpha=1$, $p=0$ and $q=1$ in equation (44) of Theorem 3.5, we get the following series representation of 2VHP $H_n(x,y)$:

$$H_n(x,y)=\sum_{r=0}^{n}\binom{n}{r}(-1)^{r}H_{n-r}(x,y,z|-,1;r,1). \tag{47}$$

Now, we obtain the operational definition of 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$.

Since $D_x H_n(x,y) = nH_{n-1}(x,y)$ and $D_x H_n(x,y,z) = nH_{n-1}(x,y,z)$, it can be verified that

$$H_n(x,y,z|\beta,\alpha;p,q) = y^{\frac{\beta}{2}} e_{(\alpha,\beta)}\big(y^{\frac{\alpha}{2}} D_x\big) z^{\frac{p}{3}} \mathcal{E}_{p,q}\big(z^{\frac{q}{3}} D_x\big) x^n,$$

and for taking $\alpha = 1$ and $\beta = 0$, we have

$$H_n(x,y,z|-,1;p,q) = z^{\frac{p}{3}} \mathcal{E}_{p,q}\big(z^{\frac{q}{3}} D_x\big) H_n(x,y).$$

In the next section, we consider some special cases of the results established in this section.

4 Special cases

In this section, we obtain certain new as well as known special polynomials by using suitable choices for parameters and variable z in equations (34), (35) and (39) as special cases of 4-parameter 3-variable Hermite polynomials.

In the following table, the umbral definitions, generating functions and series definitions of certain polynomials are listed.

For the same choices of parameters α, β, p and q considered in Table 1, equations (41) and (42) give the index duplication and argument duplication formulas for the special polynomials mentioned in the same table. The respective formulas are listed in Table 2.

In the concluding remarks, we present further argument supporting the effectiveness of the umbral method.

Table 1 Some new and known special polynomials

S. No.	Parameters	Polynomials	Umbral definition	Generating function	Series definition
I.	$q=1$	$H_n(x,y,z\vert\beta,\alpha;p,1)$	$\hat{b}_y^{\beta}\hat{c}_z^{p}(x+\hat{b}_y^{\alpha}+\hat{c}_z)^n\phi_0\psi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(\alpha,\beta)}(y^{\frac{\alpha}{2}}t)z^{\frac{p}{3}}\mathcal{E}_{p,1}(z^{\frac{1}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(p+r+1)z^{\frac{p+r}{3}}H_{n-r}(x,y\vert\beta,\alpha)}{\Gamma(\frac{p+r}{3}+1)r!(n-r)!}\times$ $(\vert 2\cos\frac{(p+r)\pi}{3}\vert-\vert\cos(p+r)\pi\vert)$
II.	$\alpha=1$	$H_n(x,y,z\vert\beta,1;p,q)$	$\hat{b}_y^{\beta}\hat{c}_z^{p}(x+\hat{b}_y+\hat{c}_z^{q})^n\phi_0\psi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(1,\beta)}(y^{\frac{1}{2}}t)z^{\frac{p}{3}}\mathcal{E}_{p,q}(z^{\frac{q}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(p+qr+1)z^{\frac{p+qr}{3}}H_{n-r}(x,y\vert\beta,1)}{\Gamma(\frac{p+qr}{3}+1)r!(n-r)!}\times$ $(\vert 2\cos\frac{(p+qr)\pi}{3}\vert-\vert\cos(p+qr)\pi\vert)$
III.	$\alpha=1$; $q=1$	$H_n(x,y,z\vert\beta,1;p,1)$	$\hat{b}_y^{\beta}\hat{c}_z^{p}(x+\hat{b}_y+\hat{c}_z)^n\phi_0\psi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(1,\beta)}(y^{\frac{1}{2}}t)z^{\frac{p}{3}}\mathcal{E}_{p,1}(z^{\frac{1}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(p+r+1)z^{\frac{p+r}{3}}H_{n-r}(x,y\vert\beta,1)}{\Gamma(\frac{p+r}{3}+1)r!(n-r)!}\times$ $(\vert 2\cos\frac{(p+r)\pi}{3}\vert-\vert\cos(p+r)\pi\vert)$
IV.	$p=0$	$H_n(x,y,z\vert\beta,\alpha;-,q)$	$\hat{b}_y^{\beta}(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q})^n\phi_0\psi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(\alpha,\beta)}(y^{\frac{\alpha}{2}}t)\mathcal{E}_{0,q}(z^{\frac{q}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(qr+1)z^{\frac{qr}{3}}H_{n-r}(x,y\vert\beta,\alpha)}{\Gamma(\frac{qr}{3}+1)r!(n-r)!}\times$ $(\vert 2\cos\frac{(qr)\pi}{3}\vert-\vert\cos(qr)\pi\vert)$
V.	$p=0$; $q=1$	$H_n(x,y,z\vert\beta,\alpha;-,1)$	$\hat{b}_y^{\beta}(x+\hat{b}_y^{\alpha}+\hat{c}_z)^n\phi_0\psi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(\alpha,\beta)}(y^{\frac{\alpha}{2}}t)e^{zt^3}$	$n!\sum_{r=0}^{\frac{n}{3}}\frac{z^r H_{n-3r}(x,y\vert\beta,\alpha)}{r!(n-3r)!}$
VI.	$\alpha=1$; $p=0$	$H_n(x,y,z\vert\beta,1;-,q)$	$\hat{b}_y^{\beta}(x+\hat{b}_y+\hat{c}_z^{q})^n\phi_0\psi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(1,\beta)}(y^{\frac{1}{2}}t)\mathcal{E}_{0,q}(z^{\frac{q}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(qr+1)z^{\frac{qr}{3}}H_{n-r}(x,y\vert\beta,1)}{\Gamma(\frac{qr}{3}+1)r!(n-r)!}\times$ $(\vert 2\cos\frac{(qr)\pi}{3}\vert-\vert\cos(qr)\pi\vert)$
VII.	$\alpha=1$; $p=0$; $q=1$	$H_n(x,y,z\vert\beta,1;-,1)$	$\hat{b}_y^{\beta}(x+\hat{b}_y+\hat{c}_z)^n\phi_0\psi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(1,\beta)}(y^{\frac{1}{2}}t)e^{zt^3}$	$n!\sum_{r=0}^{[\frac{n}{3}]}\frac{z^r H_{n-3r}(x,y\vert\beta;1)}{r!(n-3r)!}$
VIII.	$\beta=0$	$H_n(x,y,z\vert-,\alpha;p,q)$	$\hat{c}_z^{p}(x+\hat{b}_y^{\alpha}+\hat{c}_z^{q})^n\phi_0\psi_0$	$e^{xt}e_{(\alpha,0)}(y^{\frac{\alpha}{2}}t)z^{\frac{p}{3}}\mathcal{E}_{p,q}(z^{\frac{q}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(p+qr+1)z^{\frac{p+qr}{3}}H_{n-r}(x,y\vert-,\alpha)}{\Gamma(\frac{p+qr}{3}+1)r!(n-r)!}\times$ $(\vert 2\cos\frac{(p+qr)\pi}{3}\vert-\vert\cos(p+qr)\pi\vert)$

Table 1 (*Continued*)

S. No.	Para-meters	Polynomials	Umbral definition	Generating function	Series definition
IX.	$\beta=0$; $q=1$	$H_n(x,y,z\|-,\alpha;p,1)$	$\hat{c}_z^p(x+\hat{b}_y^{\alpha}+\hat{c}_z)^n\phi_0\psi_0$	$e^{xt}e_{(\alpha,0)}(y^{\frac{\alpha}{2}}t)z^{\frac{p}{3}}\mathcal{E}_{p,1}(z^{\frac{1}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(p+r+1)z^{\frac{p+r}{3}}H_{n-r}(x,y\|-,\alpha)}{\Gamma(\frac{p+r}{3}+1)r!(n-r)!}\times (\|2\cos\frac{(p+r)\pi}{3}\|-\|\cos(p+r)\pi\|)$
X.	$\alpha=1$; $\beta=0$	$H_n(x,y,z\|-,1;p,q)$	$\hat{c}_z^p(x+\hat{b}_y+\hat{c}_z^q)^n\phi_0\psi_0$	$e^{xt+yt^2}z^{\frac{p}{3}}\mathcal{E}_{p,q}(z^{\frac{q}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(p+qr+1)z^{\frac{p+qr}{3}}H_{n-r}(x,y)}{\Gamma(\frac{p+qr}{3}+1)r!(n-r)!}\times (\|2\cos\frac{(p+qr)\pi}{3}\|-\|\cos(p+qr)\pi\|)$
XI.	$\alpha=1$; $\beta=0$; $q=1$	$H_n(x,y,z\|-,1;p,1)$	$\hat{c}_z^p(x+\hat{b}_y+\hat{c}_z)^n\phi_0\psi_0$	$e^{xt+yt^2}z^{\frac{p}{3}}\mathcal{E}_{p,1}(z^{\frac{1}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(p+r+1)z^{\frac{p+r}{3}}H_{n-r}(x,y)}{\Gamma(\frac{p+r}{3}+1)r!(n-r)!}\times (\|2\cos\frac{(p+r)\pi}{3}\|-\|\cos(p+r)\pi\|)$
XII.	$\beta=0$; $p=0$	$H_n(x,y,z\|-,\alpha;-,q)$	$(x+\hat{b}_y^{\alpha}+\hat{c}_z^q)^n\phi_0\psi_0$	$e^{xt}e_{(\alpha,0)}(y^{\frac{\alpha}{2}}t)\mathcal{E}_{0,q}(z^{\frac{q}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(qr+1)z^{\frac{qr}{3}}H_{n-r}(x,y\|-,\alpha)}{\Gamma(\frac{qr}{3}+1)r!(n-r)!}\times (\|2\cos\frac{(qr)\pi}{3}\|-\|\cos(qr)\pi\|)$
XIII.	$\beta=0$; $p=0$; $q=1$	$H_n(x,y,z\|-,\alpha;-,1)$	$(x+\hat{b}_y^{\alpha}+\hat{c}_z)^n\phi_0\psi_0$	$e^{xt}e_{(\alpha,0)}(y^{\frac{\alpha}{2}}t)e^{zt^3}$	$n!\sum_{r=0}^{[\frac{n}{3}]}\frac{z^rH_{n-3r}(x,y\|-,\alpha)}{r!(n-3r)!}$
XIV.	$\alpha=1$; $\beta=0$; $p=0$	$H_n(x,y,z\|-,1;-,q)$	$(x+\hat{b}_y+\hat{c}_z^q)^n\phi_0\psi_0$	$e^{xt+yt^2}\mathcal{E}_{0,q}(z^{\frac{q}{3}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(qr+1)z^{\frac{qr}{3}}H_{n-r}(x,y)}{\Gamma(\frac{qr}{3}+1)r!(n-r)!}\times (\|2\cos\frac{(qr)\pi}{3}\|-\|\cos(qr)\pi\|)$
XV.	$\alpha=1$; $\beta=0$; $p=0$; $q=1$	$H_n(x,y,z)$	$(x+\hat{b}_y+\hat{c}_z)^n\phi_0\psi_0$	$e^{(xt+yt^2+zt^3)}$ [6]	$n!\sum_{r=0}^{[\frac{n}{3}]}\frac{z^rH_{n-3r}(x,y)}{r!(n-3r)!}$ [6]
XVI.	$p=0$; $q=0$; $z=0$	$H_n(x,y\|\beta,\alpha)$	$\hat{b}_y^{\beta}(x+\hat{b}_y^{\alpha})^n\phi_0$ [8]	$e^{xt}y^{\frac{\beta}{2}}e_{(\alpha,\beta)}(y^{\frac{\alpha}{2}}t)$ [8]	$n!\sum_{r=0}^{n}\frac{\Gamma(\alpha r+\beta+1)x^{n-r}y^{\frac{\alpha r+\beta}{2}}}{\Gamma(\frac{\alpha r+\beta}{2}+1)r!(n-r)!}\times \|(\cos\frac{\alpha r+\beta}{2}\pi)\|$
XVII.	$p=0$; $q=0$; $\alpha=1$; $z=0$	$H_n(x,y\|\beta,1)$	$\hat{b}_y^{\beta}(x+\hat{b}_y)^n\phi_0$	$e^{xt}y^{\frac{\beta}{2}}e_{(1,\beta)}(y^{\frac{1}{2}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(r+\beta+1)x^{n-r}y^{\frac{r+\beta}{2}}}{\Gamma(\frac{r+\beta}{2}+1)r!(n-r)!}\times \|(\cos\frac{r+\beta}{2}\pi)\|$
XVIII.	$\beta=0$; $p=0$; $q=0$; $z=0$	$H_n(x,y\|-,\alpha)$	$(x+\hat{b}_y^{\alpha})^n\phi_0$	$e^{xt}e_{(\alpha,0)}(y^{\frac{\alpha}{2}}t)$	$n!\sum_{r=0}^{n}\frac{\Gamma(\alpha r+1)x^{n-r}y^{\frac{\alpha r}{2}}}{\Gamma(\frac{\alpha r}{2}+1)r!(n-r)!}\times \|(\cos\frac{\alpha r}{2}\pi)\|$
XIX.	$\beta=0$; $p=0$; $\alpha=1$; $q=0$; $z=0$	$H_n(x,y)$	$(x+\hat{b}_y)^n\phi_0$ [8]	e^{xt+yt^2} [2]	$n!\sum_{r=0}^{\frac{n}{2}}\frac{x^{n-2r}y^r}{r!(n-2r)!}$ [2]

5 Concluding remarks

Gaussian integral representation of Hermite polynomials as well as specific umbral methods play an important role in classical problems arising in quantum optics, quantum mechanics, biomathematics and engineering (see for example [1, 17–20]). They are exploited to calculate the optical mode overlapping and transition rates between quantum eigenstates of the harmonic oscillator. A general method allowing the direct evaluation of these integrals has not been developed. Babusci et al. described a unifying method, flexible for generalisation, which provides a direct method for the evaluation of this class of integrals [4, 5].

Table 2 Index and argument duplication formulas

S. No.	Polynomials	Index duplication formula	Argument duplication formula
I.	$H_n(x,y,z\vert\boldsymbol{\beta},\boldsymbol{\alpha};p,1)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert\boldsymbol{\alpha}(r-s)+\boldsymbol{\beta},\boldsymbol{\alpha};p+n-r,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert\boldsymbol{\alpha}(r-u)+\boldsymbol{\beta},\boldsymbol{\alpha};s-r+p,1)$
II.	$H_n(x,y,z\vert\boldsymbol{\beta},1;p,q)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s+\boldsymbol{\beta},1;q(n-r)+p,q)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u+\boldsymbol{\beta},1;q(s-r)+p,q)$
III.	$H_n(x,y,z\vert\boldsymbol{\beta},1;p,1)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s+\boldsymbol{\beta},1;n-r+p,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u+\boldsymbol{\beta},1;s-r+p,1)$
IV.	$H_n(x,y,z\vert\boldsymbol{\beta},\boldsymbol{\alpha};-,q)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert\boldsymbol{\alpha}(r-s)+\boldsymbol{\beta},\boldsymbol{\alpha};q(n-r),q)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert\boldsymbol{\alpha}(r-u)+\boldsymbol{\beta},\boldsymbol{\alpha};q(s-r),q)$
V.	$H_n(x,y,z\vert\boldsymbol{\beta},\boldsymbol{\alpha};-,1)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert\boldsymbol{\alpha}(r-s)+\boldsymbol{\beta},\boldsymbol{\alpha};n-r,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^{s-r-u}\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert\boldsymbol{\alpha}(r-u)+\boldsymbol{\beta},\boldsymbol{\alpha};s-r,1)$
VI.	$H_n(x,y,z\vert\boldsymbol{\beta},1;-,q)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s+\boldsymbol{\beta},1;q(n-r),q)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u+\boldsymbol{\beta},1;q(s-r),q)$
VII	$H_n(x,y,z\vert\boldsymbol{\beta},1;-,1)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s+\boldsymbol{\beta},1;n-r,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u+\boldsymbol{\beta},1;s-r,1)$
VIII.	$H_n(x,y,z\vert-,\boldsymbol{\alpha};p,q)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert\boldsymbol{\alpha}(r-s),\boldsymbol{\alpha};q(n-r)+p,q)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert\boldsymbol{\alpha}(r-u),\boldsymbol{\alpha};q(s-r)+p,q)$
IX.	$H_n(x,y,z\vert-,\boldsymbol{\alpha};p,1)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert\boldsymbol{\alpha}(r-s),\boldsymbol{\alpha};n-r+p,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert\boldsymbol{\alpha}(r-u),\boldsymbol{\alpha};s-r+p,1)$
X.	$H_n(x,y,z\vert-,1;p,q)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s,1;q(n-r)+p,q)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u,1;q(s-r)+p,q)$
XI.	$H_n(x,y,z\vert-,1;p,1)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s,1;n-r+p,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u,1;s-r+p,1)$
XII.	$H_n(x,y,z\vert-,\boldsymbol{\alpha};-,q)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert\boldsymbol{\alpha}(r-s),\boldsymbol{\alpha};q(n-r),q)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert\boldsymbol{\alpha}(r-u),\boldsymbol{\alpha};q(s-r),q)$
XIII.	$H_n(x,y,z\vert-,\boldsymbol{\alpha};-,1)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert\boldsymbol{\alpha}(r-s),\boldsymbol{\alpha};n-r,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert\boldsymbol{\alpha}(r-u),\boldsymbol{\alpha};s-r,1)$
XIV.	$H_n(x,y,z\vert-,1;-,q)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s,1;q(n-r),q)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u,1;q(s-r),q)$
XV.	$H_n(x,y,z)$	$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^sH_n(x,y,z\vert r-s,1;n-r,1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{u=0}^{r}\binom{n}{s}\binom{s}{r}\binom{r}{u}x^u\frac{1}{2^{s-u}}\times H_{n-s}(x,\frac{y}{2},\frac{z}{2}\vert r-u,1;s-r,1)$
XVI.	$H_n(x,y\vert\boldsymbol{\beta},\boldsymbol{\alpha})$	$\sum_{r=0}^{n}\binom{n}{r}x^rH_n(x,y\vert\boldsymbol{\alpha}(n-r)+\boldsymbol{\beta},\boldsymbol{\alpha})$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\binom{n}{s}\binom{s}{r}x^r\frac{1}{2^{s-r}}\times H_{n-s}(x,\frac{y}{2}\vert\boldsymbol{\alpha}(s-r)+\boldsymbol{\beta},\boldsymbol{\alpha})$
XVII.	$H_n(x,y\vert\boldsymbol{\beta},1)$	$\sum_{r=0}^{n}\binom{n}{r}x^rH_n(x,y\vert n-r+\boldsymbol{\beta},1)$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\binom{n}{s}\binom{s}{r}x^r\frac{1}{2^{s-r}}\times H_{n-s}(x,\frac{y}{2}\vert s-r+\boldsymbol{\beta},1)$
XVIII.	$H_n(x,y\vert-,\boldsymbol{\alpha})$	$\sum_{r=0}^{n}\binom{n}{r}x^rH_n(x,y\vert\boldsymbol{\alpha}(n-r),\boldsymbol{\alpha})$	$\sum_{s=0}^{n}\sum_{r=0}^{s}\binom{n}{s}\binom{s}{r}x^r\frac{1}{2^{s-r}}\times H_{n-s}(x,\frac{y}{2}\vert\boldsymbol{\alpha}(s-r),\boldsymbol{\alpha})$
XIX.	$H_n(x,y)$	$\sum_{r=0}^{n}\binom{n}{r}x^{n-r}H_n(x,y\vert r)$ [8]	$\sum_{s=0}^{n}\sum_{r=0}^{s}\binom{n}{s}\binom{s}{r}x^r\frac{1}{2^{s-r}}\times H_{n-s}(x,\frac{y}{2}\vert s-r)$ [8]

We consider the following integral:

$$I_n=\int_{-\infty}^{\infty}H_n(ax+b,y,z|\beta,\alpha;p,q)e^{-cx^2+\xi x}\,dx. \tag{48}$$

By equation (48), we have

$$\sum_{n=0}^{\infty}I_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}H_n(ax+b,y,z|\beta,\alpha;p,q)\frac{t^n}{n!}e^{-cx^2+\xi x}\,dx,$$

which on using equation (35) gives

$$\sum_{n=0}^{\infty} I_n \frac{t^n}{n!} = e^{bt} y^{\frac{\beta}{2}} e_{(\alpha,\beta)}\big(y^{\frac{\alpha}{2}} t\big) z^{\frac{p}{3}} \mathcal{E}_{p,q}\big(z^{\frac{q}{3}} t\big) \int_{-\infty}^{\infty} e^{(at+\xi)x-cx^2}\, dx. \tag{49}$$

Since

$$\int_{-\infty}^{\infty} e^{bx-ax^2+c}\, dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{b^2}{4a}+c},$$

see [5], presenting the Gaussian integral, we find

$$\sum_{n=0}^{\infty} I_n \frac{t^n}{n!} = e^{bt} y^{\frac{\beta}{2}} e_{(\alpha,\beta)}\big(y^{\frac{\alpha}{2}} t\big) z^{\frac{p}{3}} \mathcal{E}_{p,q}\big(z^{\frac{q}{3}} t\big) \frac{\sqrt{\pi}}{\sqrt{c}} \exp\left(\frac{a^2}{4c} t^2 + \frac{\xi^2}{4c} + \frac{a\xi}{2c} t\right). \tag{50}$$

Using equations (1) and (35) in the right-hand side of the above equation, we obtain

$$\sum_{n=0}^{\infty} I_n \frac{t^n}{n!} = \frac{\sqrt{\pi}}{\sqrt{c}} \exp\left(\frac{\xi^2}{4c}\right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} H_n(b,y,z|\beta,\alpha;p,q) H_r\left(\frac{a\xi}{2c}, \frac{a^2}{4c}\right) \frac{t^{n+r}}{n!r!}.$$

Next, comparing the equal powers of t from both sides of the above equation, we get

$$I_n = \frac{\sqrt{\pi}}{\sqrt{c}} \exp\left(\frac{\xi^2}{4c}\right) \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(b,y,z|\beta,\alpha;p,q) H_r\left(\frac{a\xi}{2c}, \frac{a^2}{4c}\right). \tag{51}$$

In view of equations (48) and (51), we get the following result:

$$\begin{aligned} &\int_{-\infty}^{\infty} H_n(ax+b,y,z|\beta,\alpha;p,q) e^{-cx^2+\xi x}\, dx \\ &\quad = \frac{\sqrt{\pi}}{\sqrt{c}} \exp\left(\frac{\xi^2}{4c}\right) \times \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(b,y,z|\beta,\alpha;p,q) H_r\left(\frac{a\xi}{2c}, \frac{a^2}{4c}\right). \end{aligned}$$

Again, using equation (35) in the right-hand side of equation (50), we find

$$\sum_{n=0}^{\infty} I_n \frac{t^n}{n!} = \frac{\sqrt{\pi}}{\sqrt{c}} \exp\left(\frac{\xi^2}{4c}\right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} H_n\left(b + \frac{a\xi}{2c}, y, z|\beta,\alpha;p,q\right) \frac{a^{2r}}{(4c)^r} \frac{t^{n+2r}}{n!r!}.$$

Comparing the equal powers of t from both sides of the above equation, we get

$$I_n = \frac{\sqrt{\pi}}{\sqrt{c}} \exp\left(\frac{\xi^2}{4c}\right) n! \sum_{r=0}^{[\frac{n}{2}]} \frac{1}{(n-2r)!r!} H_{n-2r}\left(b + \frac{a\xi}{2c}, y, z|\beta,\alpha;p,q\right) \frac{a^{2r}}{(4c)^r}. \tag{52}$$

In view of equations (48) and (52), we get the following result:

$$\begin{aligned} &\int_{-\infty}^{\infty} H_n(ax+b,y,z|\beta,\alpha;p,q) e^{-cx^2+\xi x}\, dx \\ &\quad = \frac{\sqrt{\pi}}{\sqrt{c}} \exp\left(\frac{\xi^2}{4c}\right) n! \times \sum_{r=0}^{[\frac{n}{2}]} \frac{1}{(n-2r)!r!} H_{n-2r}\left(b + \frac{a\xi}{2c}, y, z|\beta,\alpha;p,q\right) \frac{a^{2r}}{(4c)^r}. \end{aligned}$$

Similarly, for the same choices of parameters α, β, p and q considered in Table 1, we can evaluate the integrals involving the special polynomials mentioned in the same table.

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Author details
[1]Mathematics Section, Women's College, Aligarh Muslim University, 202002, Aligarh, India. [2]Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India. [3]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410, Gaziantep, Turkey. [4]Department of Basic Engineering Sciences, Engineering Faculty, Malatya Turgut Ozal University, 44040, Malatya, Turkey.

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