Sparse Hop Spanners for Unit Disk Graphs

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Abstract

A unit disk graph G on a given set of points P in the plane is a geometric graph where an edge exists between two points $p, q \in P$ if and only if $|pq| \leq 1$. A subgraph G' of G is a k-*hop spanner* if and only if for every edge $pq \in G$, the topological shortest path between p, q in G' has at most k edges. We obtain the following results for unit disk graphs.

- **I** Every n-vertex unit disk graph has a 5-hop spanner with at most $5.5n$ edges. We analyze the family of spanners constructed by Biniaz (2020) and improve the upper bound on the number of edges from $9n$ to $5.5n$.
- **II** Using a new construction, we show that every n-vertex unit disk graph has a 3-hop spanner with at most $11n$ edges.
- **III** Every n-vertex unit disk graph has a 2-hop spanner with $O(n \log n)$ edges. This is the first nontrivial construction of 2-hop spanners.
- **IV** For every sufficiently large n, there exists a set P of n points on a circle, such that every plane hop spanner on P has hop stretch factor at least 4. Previously, no lower bound greater than 2 was known.
- **V** For every point set on a circle, there exists a plane 4-hop spanner. As such, this provides a tight bound for points on a circle.
- **VI** The maximum degree of k-hop spanners cannot be bounded from above by a function of k.

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1 Introduction

A k-spanner (or k-*hop spanner*) of a connected graph $G = (V, E)$ is a subgraph $G' = (V, E')$, where $E' \subseteq E$, with the additional property that the distance between any two vertices in G' is at most k times the distance in G [24, 38]. (The *distance* between two vertices is the minimum number of edges on a path between them.) The graph G itself is a 1-hop spanner. The minimum k for which G' is a k-spanner of G is referred to as the *hop stretch factor* (or *hop number*) of G'. An alternative characterization of k-spanners is given in the following lemma [38].

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▶ **Lemma 1** (Peleg and Schäffer [38]). *The subgraph $G' = (V, E')$ is a k-spanner of the graph $G = (V, E)$ if and only if the distance between u and v in G' is at most k for every edge $uv \in E$.*

If the subgraph G' has only $O(|V|)$ edges, then G' is called a *sparse* spanner. In this paper we are concerned with constructing sparse k-spanners (with small k) for unit disk graphs in the plane. Given a set P of n points $p_1, \ldots, p_n$ in the plane, the unit disk graph (UDG) is a geometric graph $G = G(P)$ on the vertex set P whose edges connect points that are at most unit distance apart. A *spanner of a point set* P is a spanner of its UDG.

Recognizing UDGs was shown to be NP-Hard by Breu and Kirkpatrick [9]. Unit disk graphs are commonly used to model network topology in ad hoc wireless networks and sensor networks. They are also used in multi-robot systems for practical purposes such as planning, routing, power assignment, search-and-rescue, information collection, and patrolling; refer to [2, 18, 23, 28, 34] for some applications of UDGs. For packet routing and other applications, a bounded-degree plane geometric spanner of the wireless network is often desired but not always feasible [7]. Since a UDG on n points can have a quadratic number of edges, a common desideratum is finding sparse subgraphs that approximate the respective UDG with respect to various criteria. *Plane* spanners, in which no two edges cross, are desirable for applications where edge crossings may cause interference.

Obviously, for every $k \geq 1$, every graph $G = (V, E)$ on n vertices has a k-spanner with $|E| = O(n^2)$ edges. If G is the complete graph, a star rooted at any vertex is a 2-hop spanner with $n - 1$ edges. However, the $O(n^2)$ bound on the size of a 2-hop spanner cannot be improved; a classic example [24] is that of a complete bipartite graph with $n/2$ vertices on each side. In general, if G has girth $k + 2$ or higher, then its only k-spanner is G itself. According to Erdős' girth conjecture [21], the maximum size of a graph with n vertices and girth $k + 2$ is $\Theta(n^{1+1/\lceil k/2 \rceil})$ for $k \geq 2$. The conjecture has been confirmed for some small values of k, but remains open for $k > 9$. For any graph G with n vertices, a k-spanner with $O(n^{1+1/\lceil k/2 \rceil})$ edges can be constructed in linear time [4, 5]. We show that for unit disk graphs, we can do much better in terms of the number of edges for every $k \geq 2$.

Spanners in general and unit disk graph spanners in particular are used to reduce the size of a network and the amount of routing information. They are also used for maintaining network connectivity, improving throughput, and optimizing network lifetime [6, 22, 23, 27, 39].

Spanners for UDGs with hop stretch factors bounded by a constant were introduced by Catusse, Chepoi, and Vaxès in [11]. They constructed (i) 5-hop spanners with at most $10n$ edges for n-vertex UDGs; and (ii) plane 449-hop spanners with less than $3n$ edges. Recently, Biniaz [6] improved both these results, and showed that for every n-vertex unit disk graph, there exists a 5-hop spanner with at most $9n$ edges. The author also showed how to construct a plane 341-hop spanner for a n-vertex unit disk graph. It is straightforward to verify that the algorithms presented in [6, 11] run in time that is polynomial in n.

Our results. The following are shown for unit disk graphs.

I Every n-vertex unit disk graph has a 5-hop spanner with at most $5.5n$ edges (Theorem 4 in Section 2). We carefully analyze the construction proposed by Biniaz [6] and improve the upper bound on the number of edges from the $9n$ to $5.5n$.

II Using a new construction, we show that every n-vertex unit disk graph has a 3-hop spanner with at most $11n$ edges (Theorem 5 in Section 2). Previously, no 3-hop spanner construction algorithm was known.

III Every n-vertex unit disk graph has a 2-hop spanner with $O(n \log n)$ edges. This is the first construction with a subquadratic number of edges (Theorem 10 in Section 3) and our main result.

IV For every $n \geq 8$, there exists an n-element point set S such that every plane hop spanner on S has hop stretch factor at least 3. If n is sufficiently large, the lower bound can be raised to 4 (Theorems 11 and 12 in Section 4). A trivial lower bound of 2 can be easily obtained by placing four points at the four corners of a square of side-length 1/2.

V For every point set S on a circle C, there exists a plane 4-hop spanner (Theorem 13 in Section 4). The lower bound of 4 holds for some point-set on a circle.

VI For every pair of integers $k \geq 2$ and $\Delta \geq 2$, there exists a set S of $n = O(\Delta^k)$ points such that the unit disk graph $G = (V, E)$ on S has no k-spanner whose maximum degree is at most Δ (Theorem 14 in Section 5). An extension to dense graphs is given by Theorem 15 in Section 5. In contrast, Kanj and Perković [23] showed that UDGs admit bounded-degree *geometric* spanners.

Related work. Peleg and Schäffer [38] have shown that for a given graph G (not necessarily a UDG) and a positive integer m, it is NP-complete to decide whether there exists a 2-spanner of G with at most m edges. They also showed that for every graph on n vertices, a $(4k+1)$-spanner with $O(n^{1+1/k})$ edges can be constructed in polynomial time. In particular, every graph on n vertices has a $O(\log n)$-spanner with $O(n)$ edges. Their result was improved by Althöfer et al. [1], who showed that a $(2k-1)$-spanner with $O(n^{1+1/k})$ edges can be constructed in polynomial time; the run-time was later improved to linear [4, 8]. Kortsarz and Peleg obtained approximation algorithms for the problem of finding, in a given graph, a 2-spanner of minimum size [24] or minimum maximum degree [25].

In the geometric setting, where the vertices are embedded in a metric space, spanners have been studied in [3, 10, 12, 13, 26, 28] and many other papers. In particular, plane geometric spanners were studied in [7, 8, 17, 16]. The reader is also referred to the surveys [8, 20, 30] and the monograph [33] dedicated to this subject.

Notation and terminology. For two points $p, q \in \mathbb{R}^2$, we denote the Euclidean distance by $d(p, q)$ or sometimes by $|pq|$. The distance between two sets, $A, B \subset \mathbb{R}^2$, is defined by $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. For a set A, its boundary and interior are denoted by ∂A and $\text{int}(A)$, respectively. The diameter of a set A, denoted $\mathtt{diam}(A)$, is defined by $\mathtt{diam}(A) = \sup\{d(a, b) : a, b \in A\}$.

Given a graph, $N(u)$ denotes the set of vertices adjacent to u. For $p, q \in V$, let $\rho(p, q)$ denote a shortest path in G', i.e., a path containing the fewest edges; and $h(p, q)$ denote the corresponding hop distance (number of edges). For brevity, a hop spanner for a point set $P \subset \mathbb{R}^2$ is a hop spanner for the UDG on P.

A geometric graph is *plane* if any two distinct edges are either disjoint or only share a common endpoint. Whenever we discuss plane graphs (plane spanners in particular), we assume that the points (vertices) are in *general position*, i.e., no three points are collinear.

A *unit disk* (resp., *circle*) is a disk (resp., circle) of unit radius. The complete bipartite graph with parts of size m and n is denoted by $K_{m,n}$; in particular, $K_{1,n}$ is a star on $n+1$ vertices. We use the shorthand notation $[n]$ for the set $\{1, 2, \ldots, n\}$.

2 Sparse (possibly nonplane) hop spanners

In this section we construct hop spanners with a linear number of edges that provide various trade-offs between the two parameters of interest: number of hops and number of edges.

2.1 Construction of 5-hop spanners

We start with a short outline of the 5-hop spanner constructed by Biniaz [6, Theorem 3]; it is based on a hexagonal tiling with cells of unit diameter. Note that the UDG contains every edge between points in the same cell. In every nonempty cell, a star is formed rooted at an arbitrarily chosen point in the cell. Then, for every pair of cells, exactly one edge of the UDG is chosen, if such an edge exists. The author showed that the resulting graph is a 5-hop spanner with at most $9n$ edges.

We next provide a more detailed description and an improved analysis of the construction. Consider a regular hexagonal tiling $\mathcal{T}$ in the plane with cells of unit diameter; refer to Fig. 1 (left). We may assume that no point lies on a cell boundary. Every point in P lies in the interior of some cell of $\mathcal{T}$ (and so the distance between any two points inside a cell is less than 1). Let $p \in P$ be a point in a cell σ. Denote by $H_1, \ldots, H_6$ the six cells adjacent to σ in counterclockwise order; these cells form the *first layer* around σ. Let $H_7, \ldots, H_{18}$ be the twelve cells at distance two from σ in counterclockwise order, forming the *second layer* around σ, such that H_7 is adjacent to only H_1 in the first layer.

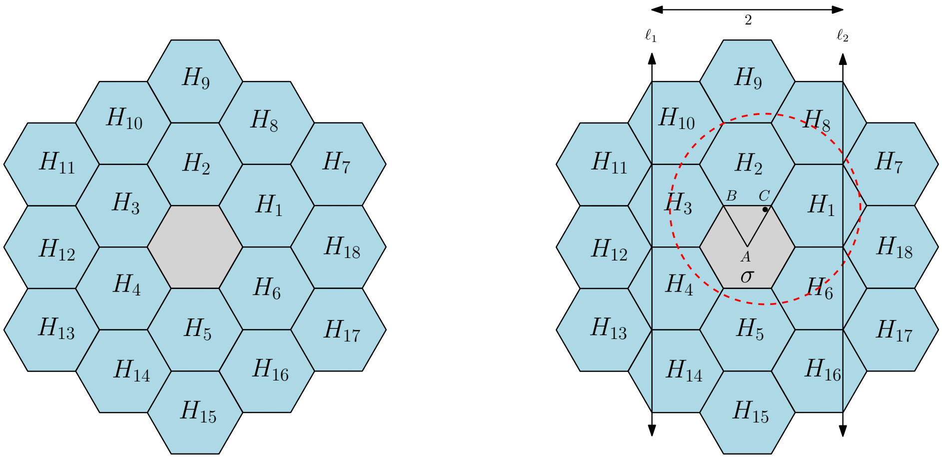

Figure 1 Left: Hexagonal grid with cells of unit diameter; the figure shows the two layers of cells around the lightly shaded cell in the middle. Right: The unit disk centered at p intersects 11 cells $H_1, \ldots, H_{10}, H_{18}$.

For every two distinct cells $\sigma, \tau \in \mathcal{T}$, take an arbitrary edge $pq \in E$, $p \in \sigma$, $q \in \tau$, if such an edge exists; we call such an edge a *bridge*. Each cell σ can have bridges to at most 18 other cells, namely those in the two layers around σ. A bridge is *short* if it connects points in adjacent cells and *long* otherwise.

▶ **Lemma 2.** *Let $p \in P$ be a point that lies in cell σ. The unit disk D centered at p intersects at most five cells from the second layer around σ.*

Proof. Let A be the center of σ (shaded gray in Fig. 1 (right)). Subdivide σ into six regular triangles incident to A. By symmetry, we can assume that $p \in \Delta ABC$, where $BC = \sigma \cap H_2$.

Note that $d(\Delta ABC, H_i) > 1$ for $i \in \{13, 14, 15, 16, 17\}$, and D is disjoint from the five cells H_{13}, H_{14}, H_{15}, H_{16}, and H_{17}. Now, observe that $d(H_7 \cup H_{18}, H_{11} \cup H_{12}) = 2$. Hence, D intersects at most one of $H_7 \cup H_{18}$ and $H_{11} \cup H_{12}$. Consequently, D intersects at most $12 - 5 - 2 = 5$ cells from the second layer around σ. ◀

Obviously, any two points in a cell σ are at most unit distance apart. Further, observe that the unit disk D centered at p intersects all six cells $H_1, \ldots, H_6$. As such, Lemma 2 immediately yields the following.

▶ **Corollary 3.** *All neighbors of each point $p \in \sigma$ lies in σ and at most 11 cells around σ.*

▶ **Theorem 4.** *The (possibly nonplane) 5-hop spanner constructed by Biniaz [6, Theorem 3] has at most $5.5n$ edges.*

Proof. Let P be a set of n points and $G = (V, E)$ be the corresponding UDG. Let $x \geq 1$ be the number of points in a hexagonal cell $\sigma \in \mathcal{T}$. The construction has $x - 1$ inner edges that make a star and at most 18 outer edges (bridges) connecting points in σ with points in other cells. We analyze the situation depending on x.

If $x = 1$, there are no inner edges and at most 11 outer edges by Corollary 3. As such, the degree of the (unique) point in σ is at most 11.

If $x = 2$, there is one inner edge and at most 16 outer edges. Indeed, by Lemma 2, each point $p \in P \cap \sigma$ has neighbors in at most five cells from the second layer around σ (besides points in P in the six cells in the first layer). Two points in $P \cap \sigma$ can jointly have neighbors in at most $6 + 5 + 5 = 16$ other cells. As such, the average degree for points in σ is at most $(2 + 16)/2 = 9$.

If $x \geq 3$, there are $x - 1$ inner edges and at most 18 outer edges. As such, the average degree for points in σ is at most

$$\frac{2(x-1)+18}{x} = \frac{2x+16}{x} \leq \frac{22}{3}.$$

Summation over all cells implies that the average degree in the resulting 5-hop spanner G' is at most 11, thus G' has at most $5.5n$ edges. ◀

2.2 Construction of 3-hop spanners

Here we show that every point set in the the plane has a 3-hop spanners of linear size. This brings down the hop-stretch factor of Biniaz's construction from 5 to 3 at the expense of increasing in the number of edges (from $5.5n$ to $11n$).

▶ **Theorem 5.** *Every n-vertex unit disk graph has a (possibly nonplane) 3-hop spanner with at most $11n$ edges.*

Proof. Our construction is based on a hexagonal tiling $\mathcal{T}$ with cells of unit diameter (as in Subsection 2.1). Let G' be the 5-hop spanner described in Section 2.1. We construct a new graph G'' that consists of all bridges from G' and, for each nonempty cell $\sigma \in \mathcal{T}$, a spanning star of the points in σ defined as follows.

Let $\sigma \in \mathcal{T}$ be a nonempty cell. Let $p_i \in P \cap \sigma$. For every cell $\tau \in \mathcal{T}$ in the two layers around σ, if $d(p_i, \tau) \leq 1$ and G' contains a bridge pq, where $p \in \sigma \setminus \{p_i\}$ and $q \in \tau$, then we add the edge $p_i p$ to G''. Since $\texttt{diam}(\sigma) = 1$, if pq is a short bridge, then p is the center of a spanning star on $P \cap \sigma$. In addition, if no short bridge is incident to any point in σ, then we add a spanning star of $P \cap \sigma$ (centered at the endpoint of a long bridge, if any) to G''.

It is easy to see that the hop distance between any two points within a cell is at most 2. Indeed, by construction, the points in each nonempty cell are connected by a spanning star. Consider now a pair of points $p_i \in \sigma_i$, $p_j \in \sigma_j$, $i \neq j$, where $p_i p_j \in E$. By construction, there is a bridge $pq \in G''$ between the cells σ_i and σ_j. As such, p_i is connected to p_j by a 3-hop path p_i, p, q, p_j. Refer to Fig. 2 for an illustration.

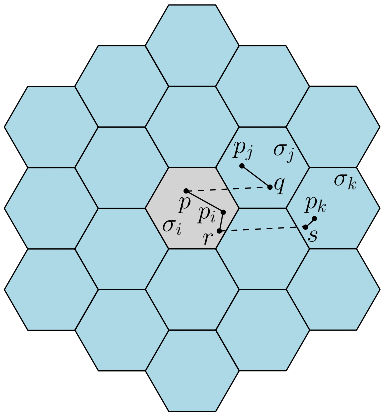

Figure 2 Three points in P, $p_i \in \sigma_i$, $p_j \in \sigma_j$, and $p_k \in \sigma_k$ where $p_ip_j, p_ip_k \in E$. Edge pq is a short bridge connecting σ_i and σ_j and edge rs is a long bridge connecting σ_i and σ_k.

We can bound the average degree of the points in σ as follows. Let x be the number of points in σ. By Corollary 3, the neighbors of each point $p_i \in \sigma$ lie in σ and at most 11 cells around σ. If p_i is not incident to any bridge, we add at most 11 edges between p_i and other points in σ; these edges increase the sum of degrees in σ by $2 \cdot 11 = 22$. Otherwise assume that p_i is incident to b_i bridges, for some $1 \leq b_i \leq 11$. Then we add edges from p_i to at most $11 - b_i$ other points in σ. The b_i bridges each have only one endpoint in σ. Overall, these edges contribute $2(11 - b_i) + b_i = 22 - b_i < 22$ to the sum of degrees in σ.

If no short bridge has an endpoint in σ, then by Lemma 2 we add at most 5 edges between each point $p_i \in \sigma$ and endpoints of long bridges; these edges increase the sum of degrees in σ by $2 \cdot 5 = 10$. However, we also add a spanning star that contributes $2(x-1)$ to the same sum. Overall, the sum of degrees in σ is bounded from above by

$$\begin{cases} 2 \cdot 11x = 22x, & \text{if some short bridge has an endpoint in } \sigma \\ 2(x-1) + 10x < 12x, & \text{otherwise.} \end{cases}$$

Thus, the average vertex degree is at most 22 in all $\sigma \in \mathcal{T}$. Consequently, the 3-hop spanner G'' has at most $11n$ edges. ◀

3 Construction of 2-hop spanners

In this section, we construct a 2-hop spanner with $O(n \log n)$ edges for a set of n points in the plane. We begin with a construction in a bipartite setting (cf. Lemma 9), and then extend it to the general setup.

We briefly review the concept of ε-nets [32], which is crucial for our construction. Let $(P, \mathcal{R})$ be a set system (a.k.a. *range space*), where P is a finite set in an ambient space and $\mathcal{R}$ is a collection of subsets of that space (called *ranges*). For $\varepsilon > 0$, an *ε-net* for $(P, \mathcal{R})$ is a set $N \subset P$ such that for every $R \in \mathcal{R}$, $|P \cap R| \geq \varepsilon \cdot |P|$ implies $N \cap R \neq \emptyset$. When the ambient space is $\mathbb{R}^d$ for some $d \in \mathbb{N}$, and $\mathcal{R}$ is a collection of semi-algebraic sets, there exists an ε-net of size $O(\frac{d}{\varepsilon} \log \frac{d}{\varepsilon})$, and this bound is best possible in many cases [36]. However, for some geometric set systems, ε-nets of size $O(\frac{1}{\varepsilon})$ are possible. For example, if P is a set of points in the plane and $\mathcal{R}$ consists of halfplanes, then there exists an ε-net of size $O(\frac{1}{\varepsilon})$ [37]. We adapt this results to unit disks in a somewhat stronger form (cf. Lemma 8).

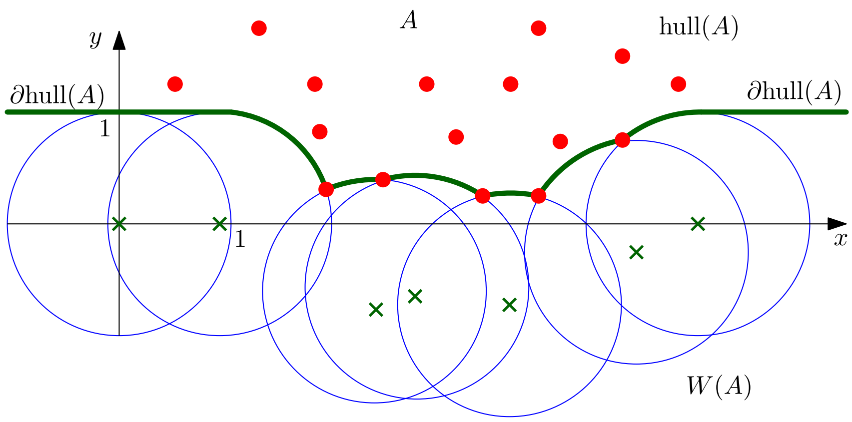

Figure 3 A set A of 16 points above the x-axis, $W(A)$, and hull(A). The boundary ∂hull(A) is an x-monotone curve, which consists of horizontal segments and arcs of unit circles centered on or below the x-axis (the centers are marked with crosses).

Alpha-shapes. As a generalization of convex hulls of a set of points, Edelsbrunner, Kirkpatrick and Seidel [19] introduced α-shapes, using balls of radius $1/\alpha$ instead of halfplanes. We introduce a similar concept, in the bipartite setting, as follows; see Fig. 3 for an illustration. We consider the set system $(A, \mathcal{D})$, where A is a finite set of points in the plane above the x-axis and $\mathcal{D}$ is the set of all unit disks centered on or below the x-axis. Let $W(A)$ be the union of all unit disks $D \in \mathcal{D}$ such that $A \cap \text{int}(D) = \emptyset$; and let $\text{hull}(A) = \mathbb{R}^2 \setminus \text{int}(W(A))$.

The following easy observation shows that disks in $\mathcal{D}$, restricted to the upper halfplane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, behave similarly to halfplanes in $\mathbb{R}^2$.

▶ **Lemma 6.** *For any two points $p_1, p_2 \in \mathbb{R}^2$ above the x-axis, there is at most one unit circle centered at a point on or below the x-axis that is incident to both p_1 and p_2. Consequently, for any two unit disks $D_1, D_2 \in \mathcal{D}$, at most one point in $\partial D_1 \cap \partial D_2$ lies above the x-axis.*

Proof. Suppose that two unit circles, c_1 and c_2, are incident to both p_1 and p_2. Then the centers of c_1 and c_2 are on the orthogonal bisector of segment p_1p_2, on opposite sides of the line through p_1p_2. Hence one of the circle centers is above the x-axis. Therefore at most one of the circles is centered at a point on or below the x-axis. ◀

We continue with a few basic properties of the boundary of hull(A), which exhibit the same behavior as convex hulls with respect to lines in the plane.

▶ **Lemma 7.** *The set system $(A, \mathcal{D})$ defined above has the following properties:*
1. *∂hull(A) lies above the x-axis;*
2. *every vertical line intersects ∂hull(A) in one point, thus ∂hull(A) is an x-monotone curve;*
3. *for every unit disk $D \in \mathcal{D}$, the intersection $D \cap (\partial\text{hull}(A))$ is connected (possibly empty);*
4. *for every unit disk $D \in \mathcal{D}$, if $A \cap D \neq \emptyset$, then $A \cap D$ contains a point in ∂hull(A).*

Proof. Let h be the minimum of the y-coordinates of the points in A. If $h \geq 1$, then $W(A) = \{(x, y) : y \leq 1\}$ is a halfplane bounded by the line $y = 1$, so the lemma trivially holds. In the remainder of the proof, assume that $0 < h < 1$.

(1) Since $0 < h < 1$, the halfplane below the horizontal line $y = h$ lies in the interior of $W(A)$ (as every point below this line is in the interior of a unit disk whose center is below the x-axis and whose interior is disjoint from A). Property 1 follows.

(2) Let $p \in \partial\text{hull}(A)$. Then p lies on the boundary of a unit disk D_p whose center is below the x-axis (and whose interior is disjoint from A). In particular $D_p \subset W(A)$. The vertical line segment from p to the x-axis lies in D_p, hence in $W(A)$. Consequently, $W(A)$ contains the vertical downward ray emanating from p. Property 2 follows.

(3) Let $D \in \mathcal{D}$. Suppose, to the contrary, that the intersection $D \cap (\partial\text{hull}(A))$ has two or more components. By property 1, the x-coordinates of the components are disjoint intervals, and the components have a natural left-to-right ordering. Let p_1 be the rightmost point in the first component, and let p_2 be the leftmost point in the second component. Clearly $p_1, p_2 \in \partial D$. Let q be an arbitrary point in $\partial\text{hull}(A)$ between p_1 and p_2. Then q lies on the boundary of a unit disk D_q whose center is below the x-axis (and whose interior is disjoint from A). Since $D_q \subset W(A)$, neither p_1 nor p_2 is in the interior of D_q. Since the center of D_q is below the x-axis, ∂D_q contains two interior-disjoint circular arcs between q and the x-axis; and both arcs must cross ∂D. We have found two intersection points in $\partial D \cap \partial D_q$ above the x-axis, contradicting Lemma 6. This completes the proof of Property 3.

(4) Let $D \in \mathcal{D}$ such that $A \cap D \neq \emptyset$. By continuously translating D vertically down until its interior is disjoint from A, we obtain a unit disk D' such that $A \cap \text{int}(D') = \emptyset$ but $A \cap \partial D' \neq \emptyset$. Since the center of D' is vertically below the center of D, we have $A \cap \partial D' \subset A \cap D$ and $D' \subset W(A)$. This implies that $A \cap \partial D' \subset \partial\text{hull}(A)$, as required. ◀

▶ **Lemma 8.** *Consider the set system $(A, \mathcal{D})$ defined above. For every $\varepsilon \in (0, \frac{2}{3})$, we can construct an ε-net $N = \{v_1, \dots, v_k\} \subset A$, labeled by increasing x-coordinates, such that*

(i) $|N| \leq \lfloor 2/\varepsilon \rfloor$;

(ii) $N \subset \partial hull(A)$;

(iii) *for every $D \in \mathcal{D}$, the points in $D \cap N$ are consecutive in N; and*

(iv) *for every $D \in \mathcal{D}$, $|N \cap D| \geq 5$ implies $|A \cap D| \geq 2\varepsilon|A|$.*

Proof. Let $M = A \cap \partial\text{hull}(A)$ be the set of points in A lying on the boundary of hull(A). By Lemma 7(4), if a unit disk $D \in \mathcal{D}$ contains any point in A, it contains a point from M. Consequently M is an ε-net for $(A, \mathcal{D})$ for every $\varepsilon > 0$. For a given $\varepsilon > 0$, let $N = N_\varepsilon$ be a minimal subset of M that is an ε-net for $(A, \mathcal{D})$ (obtained, for example, by successively deleting points from M while we maintain an ε-net).

Let $N = \{v_1, \dots, v_k\}$, where we label the elements in N by increasing x-coordinates. For notational convenience, we introduce a point $v_0 \in \partial\text{hull}(A)$ on a vertical line one unit left of v_1, and $v_{k+1} \in \partial\text{hull}(A)$ on a vertical line one unit right of v_k. For $i = 1, \dots k$, the minimality of N implies that $N \setminus \{v_i\}$ is not an ε-net, and so there exists a unit disk $D \in \mathcal{D}$ such that $|A \cap D| \geq \varepsilon|A|$ and $D \cap N = \{v_i\}$. Let $D_i \in \mathcal{D}$ be such a disk, with $|A \cap D_i| \geq \varepsilon|A|$ and $D_i \cap N = \{v_i\}$. By Lemma 7(3), D_i contains a connected arc of the x-monotone curve $\partial\text{hull}(A)$, but D_i contains neither v_{i-1} nor v_{i+1}. In particular, the x-coordinate of every point in $A \cap D_i$ lies between that of v_{i-1} and v_{i+1}. Consequently, every point in A lies in at most two disks D_i, $1 \leq i \leq k$. It follows that

$$k \cdot \varepsilon|A| = \sum_{i=1}^{k} \varepsilon|A| \leq \sum_{i=1}^{k} |A \cap D_i| \leq 2|A|,$$

hence $k \leq \lfloor 2/\varepsilon \rfloor$. This proves (i).

By construction, we have $N \subset M \subset \partial\text{hull}(A)$, which confirms (ii), and (iii) follows from Lemma 7(3). It remains to prove (iv); refer to Fig. 4. Assume that $D \in \mathcal{D}$ and $|N \cap D| \geq 5$. By (iii), we may assume that D contains five consecutive points in N, say, $v_i, \dots, v_{i+4}$. For $j \in \{i+1, i+2, i+3\}$, consider the disk $D_j \in \mathcal{D}$ defined above,

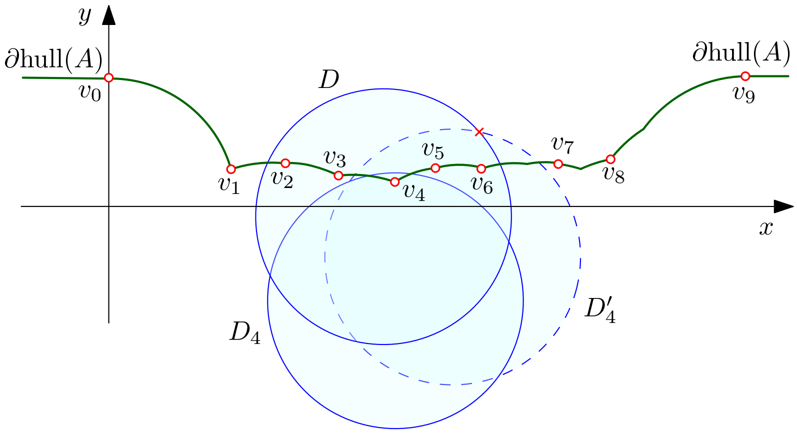

Figure 4 Illustration for the proof of Lemma 8(iv) with $i = 2$ and $j = 4$. A unit disk D with $D \cap N = \{v_2, v_3, v_4, v_5, v_6\}$, and a unit disk D_4 with $v_4 \in D_4$ and $v_3, v_5 \notin D_4$. A hypothetical unit disk D'_4 (dashed) such that $v_4 \in D'_4$, and $\partial D'_4 \cap \mathrm{hull}(A)$ crosses $\partial D \cap \mathrm{hull}(A)$.

where $v_j \in D_j$ but $v_{j-1}, v_{j+1} \notin D_j$. In particular, $D_j \cap (\partial\mathrm{hull}(A))$ lies between v_{j-1} and v_{j+1}. By Lemma 6, the circular arcs $\partial D \cap \mathrm{hull}(A)$ and $\partial D_j \cap \mathrm{hull}(A)$ cross at most once. However, if they cross once, then D_j contains one of the endpoints of $D \cap (\partial\mathrm{hull}(A))$, and by Lemma 7(3) it contains $\{v_i, \ldots, v_j\}$ or $\{v_j, \ldots, v_{i+4}\}$, which is a contradiction. We conclude that $\partial D \cap \mathrm{hull}(A)$ and $\partial D_j \cap \mathrm{hull}(A)$ do not cross. Consequently, $D_j \cap \mathrm{hull}(A) \subset D \cap \mathrm{hull}(A)$, hence $A \cap D_j \subset A \cap D$. As noted above, $|A \cap D_j| \geq \varepsilon|A|$. Furthermore, $A \cap D_{i+1}$ and $A \cap D_{i+3}$ are disjoint as they are on opposite sides of the vertical line passing through v_{i+2}. Thus we obtain $|A \cap D| \geq |A \cap (D_{i+1} \cup D_{i+3})| \geq |A \cap D_{i+1}| + |A \cap D_{i+3}| \geq 2\varepsilon|A|$, as claimed. ◀

Let A and B be two disjoint point sets above and below the x-axis, respectively. Denote by $U(A, B)$ the unit disk graph on $A \cup B$ and by $G(A, B)$ the bipartite subgraph of $U(A, B)$ consisting of all edges between A and B.

▶ **Lemma 9.** *Let $P = A \cup B$ be a set of n points in the plane such that $\mathtt{diam}(A) \leq 1$, $\mathtt{diam}(B) \leq 1$, and A (resp., B) is above (resp., below) the x-axis. Then there is a subgraph H of $U(A, B)$ with at most $O(n \log n)$ edges such that for every edge ab of $G(A, B)$, H contains a path of length at most 2 between a and b.*

Proof. Our proof is constructive. For every point $b \in B$, let D_b be the unit disk centered at b. Consider the set system $(A, \mathcal{B})$, where $\mathcal{B} = \{D_b : b \in B\}$. We partition the set of disks $\mathcal{B}$ into $O(\log n)$ subsets based on the number of points of A contained in the disks. For every $i = 1, \ldots, \lceil \log n \rceil$, let

$$\mathcal{B}_i = \left\{ D \in \mathcal{B} : \frac{|A|}{2^i} \leq |A \cap D| < \frac{|A|}{2^{i-1}} \right\}.$$

For every $i = 1, \ldots, \lceil \log n \rceil$, let $\varepsilon_i = \frac{1}{2^i}$. Lemma 8 yields an ε_i-net $N_i \subset A$ of size at most $\lfloor 2/\varepsilon_i \rfloor = 2^{i+1}$ for $(A, \mathcal{B}_i)$.

We construct the graph H as a union of stars; see Fig. 5 for an illustration. For every $i = 1, \ldots, \lceil \log n \rceil$ and every $v \in N_i$, we create a star centered at v as follows. Let $B_i(v)$ be the set of points $b \in B$ such that $D_b \in \mathcal{B}_i$ (that is, $|A|/2^i \leq |A \cap D_b| < |A|/2^{i-1}$), $v \in D_b$, and v is the leftmost point in $N_i \cap D_b$. Let $A_i(v)$ be the set of points $a \in A$ contained in unit

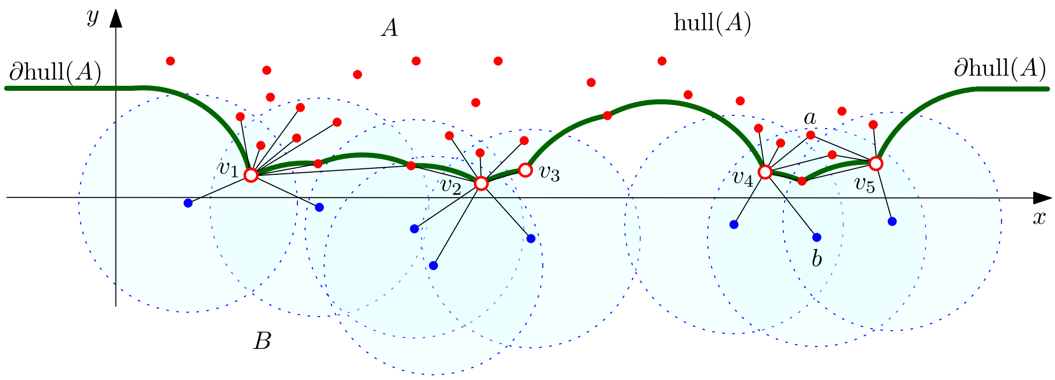

Figure 5 Set A (resp., B) is above (resp., below) the x-axis. The points in an ε_i-net $N_i = \{v_1 \dots, v_5\}$ are marked with hollow dots. The graph H_i is a union of stars centered at $v_1, \dots, v_5$. (To avoid clutter, the depicted point set does not meet conditions $\texttt{diam}(A) \leq 1$ and $\texttt{diam}(B) \leq 1$ of Lemma 9.)

disks centered at some point in $B_i(v)$. Let $S_i(v)$ be the star on $A_i(v) \cup B_i(v)$ centered at v. By construction, every point in $B_i(v)$ is at distance at most 1 from v, and $\texttt{diam}(A) \leq 1$; this implies that $S_i(v)$ is a subgraph of $U(A, B)$. Let H_i be the union of all stars centered at vertices in N_i; and let H be the union of the graphs H_i for $i = 1, \dots, \lceil \log n \rceil$. Note that H is a union of stars in $U(A, B)$, hence a subgraph of $U(A, B)$.

To prove correctness, we show that for every edge ab of $G(A, B)$ (with $a \in A$, $b \in B$), H contains a path of length 2 between a and b. Since ab is an edge of $G(A, B)$, we have $|ab| \leq 1$ hence $a \in D_b$. There exists an index $i \in \{1, \dots, \lceil \log n \rceil\}$ for which $D_b \in \mathcal{B}_i$. As $|A \cap D_b| \geq |A|/2^i = \varepsilon_i |A|$, and N_i is an ε_i-net for $(A, \mathcal{B}_i)$, we have $D_b \cap N_i \neq \emptyset$. Let v be the leftmost point in $D_b \cap N_i$. Then by construction $a \in A_i(v)$ and $b \in B_i(v)$. If $a = v$, then the star $S_i(v)$ contains the edge ab, otherwise $S_i(v)$ contains the path a, v, b of length 2.

It remains to derive an upper bound on the number of edges in H. We claim that H_i has $O(n)$ edges for all $i = 1, \dots, \lceil \log n \rceil$, which implies that H has $O(n \log n)$ edges overall.

Let $b \in B$. There is a unique index i such that $|A|/2^i \leq |A \cap D_b| < |A|/2^{i-1}$; and there is a unique leftmost point v in $N_i \cap D_b$. Therefore, b is a leaf of only one star $S_i(v)$, and so its degree is at most 1 in H_i, hence in H.

Let $i \in \{1, \dots, \lceil \log n \rceil\}$. Assume that $N_i = \{v_1, \dots, v_k\}$ is sorted by increasing x-coordinates. We also introduce points v_0 and v_{k+1} on ∂hull(A) as specified previously.

Let $a \in A$; refer to Fig. 5. Assume that a is in a star $S_i(v_j)$ for some $v_j \in N_i$. Assume further that the x-coordinate of a is between that of $v_{\ell-1}$ and v_ℓ for some $\ell \in \{1, \dots, k+1\}$. Since a is in $S_i(v_j)$, there exists a point $b \in B$ such that $a \in D_b$, $D_b \in \mathcal{B}_i$, and v_j is the leftmost point in $D_b \cap N_i$. Since $D_b \in \mathcal{B}_i$, we have $|A \cap D_b| < 2\varepsilon_i |A|$.

By Lemma 8(iv), D_b contains at most 4 points from the net N_i. In particular, the unit circle ∂D_b intersects ∂hull(A) in two points: once between v_{j-1} and v_j, and once between v_j and v_{j+4}. Consequently, $0 \leq \ell - j \leq 4$, thus a is in at most 5 possible stars $S_i(v_j)$, $v_j \in N_i$. It follows that H_i has at most $5|A| + |B| \leq 5n$ edges, as required. ◀

We now consider the general case.

▶ **Theorem 10.** *Every n-vertex unit disk graph has a (possibly nonplane) 2-hop spanner with $O(n \log n)$ edges.*

Proof. Let P be a set of n points in the plane. Consider a tiling of the plane with regular hexagons of unit diameter; and assume that no point in P lies on the boundary of any hexagon. Let $\mathcal{T}$ be the set of nonempty hexagons. Then P is partitioned into $O(n)$ sets $\{P \cap \sigma : \sigma \in \mathcal{T}\}$. As noted in Section 2.1, for every $\sigma \in \mathcal{T}$, there are 18 other cells within unit distance; see Fig. 1 (left).

For each cell $\sigma \in \mathcal{T}$, choose an arbitrary vertex $v_\sigma \in P \cap \sigma$, and create a star S_σ centered at v_σ on the vertex set $P \cap \sigma$. The overall number of edges in all stars S_σ, $\sigma \in \mathcal{T}$, is

$$\sum_{\sigma \in \mathcal{T}} (|P \cap \sigma| - 1) = n - |\mathcal{T}| \leq n.$$

For every pair of cells $\sigma_i, \sigma_j \in \mathcal{T}$, where $d(\sigma_i, \sigma_j) \leq 1$, consider the bipartite graph $G_{i,j} = G(P \cap \sigma_i, P \cap \sigma_j)$. By Lemma 9, there is a graph $H_{i,j}$ of size

$$O\big((|P \cap \sigma_i| + |P \cap \sigma_j|) \log(|P \cap \sigma_i| + |P \cap \sigma_j|)\big) = O\big((|P \cap \sigma_i| + |P \cap \sigma_j|) \log n\big).$$

Since every vertex appears in at most 18 such bipartite graphs, the total number of edges in these graphs is at most $O\left(\sum_{\sigma \in \mathcal{T}} |P \cap \sigma| \log n\right) = O(n \log n)$.

We show that the union of the stars S_σ, $\sigma \in \mathcal{T}$, and the graphs $H_{i,j}$ is a 2-hop spanner. Let ab be an edge of the unit disk graph. If both a and b are in the same cell, say $\sigma \in \mathcal{T}$, then ab is an edge in the star or the star S_σ contains the path a, v_σ, b. Otherwise, a and b lie in two distinct cells, say $\sigma_i, \sigma_j \in \mathcal{T}$, such that $d(\sigma_i, \sigma_j) \leq |ab| \leq 1$. By Lemma 9 (where the role of the x-axis is taken by any separating line), $H_{i,j}$ contains a path of length at most 2 between a and b, as required. ◀

4 Lower bounds for plane hop spanners

A trivial lower bound of 2 for the hop stretch factor of plane subgraphs of UDGs can be easily obtained by taking the four corners of a square of side-length $\frac{1}{2}$. In this case, the UDG is the complete graph but a plane spanner cannot contain both diagonals of the square. Our main result in this section is a lower bound of 4 for sufficiently large n (cf. Theorem 12). We begin with a lower bound of 3 that holds already for $n = 8$. Due to space constraints, the proof of Theorem 11 is omitted and can be found in the full version.

▶ **Theorem 11.** *For every $n \geq 8$, there exists an n-element point set S on a circle such that every plane hop spanner on S has hop stretch factor at least* 3.

We next derive a better bound assuming that n is sufficiently large.

▶ **Theorem 12.** *For every sufficiently large n, there exists an n-element point set S on a circle such that every plane hop spanner on S has hop stretch factor at least* 4.

Proof. Consider a set S of n points that form the vertices of regular n-gon R inscribed in a circle C, where the circle is just a bit larger than the circumscribed circle of an equilateral triangle of unit edge length. Formally, for a given $\varepsilon \in (0, 1/50)$, set $n = \lceil 2\varepsilon^{-1} \rceil$ and choose the radius of C such that every sequence of $\left(\frac{1}{3} - \varepsilon\right) n$ consecutive points from S makes a subset of diameter at most 1; and any larger sequence makes a subset of diameter larger than 1. Note that $\varepsilon n \geq 2$. (We may set $\varepsilon = 0.02$, which yields $n = 100$.)

The short circular arc between two consecutive vertices of R is referred to as an *elementary arc*. (Its center angle is $2\pi/n$.) If A is a set of elementary arcs, $X(A)$ denotes its set of endpoints; obviously $|X(A)| \geq |A|$, with equality when A covers the entire circle C.

Suppose, for the sake of contradiction, that the unit disk graph G has a plane subgraph G' with hop number at most 3. First, augment G' to a maximal noncrossing subgraph of G, by successively adding edges from $G \setminus G'$ that do not introduce crossings. Adding edges does not increase the hop number of G', which remains at most 3.

We define *maximal* edges in G' as follows. Associate every edge of G' with the shorter circular arc between its endpoints. Observe that containment between arcs is a partial order (poset). An edge of G' is *maximal* if the associated arc is maximal in this poset. Due to planarity, if two arcs overlap, then one of the arcs contains the other. Hence the maximal edges correspond to nonoverlapping arcs. As such, the maximal edges form a convex cycle, i.e., a convex polygon $P = p_1, p_2, \ldots, p_k$. Refer to Fig. 6. By the choice of C, we have $k \geq 4$. Each edge of the polygon P determines a set of points, called *block*, that lie on the associated circular arc (both endpoints of the edge are included). Since the length of each edge of P is at most 1, the restriction of G' to the vertices in a block is a triangulation.

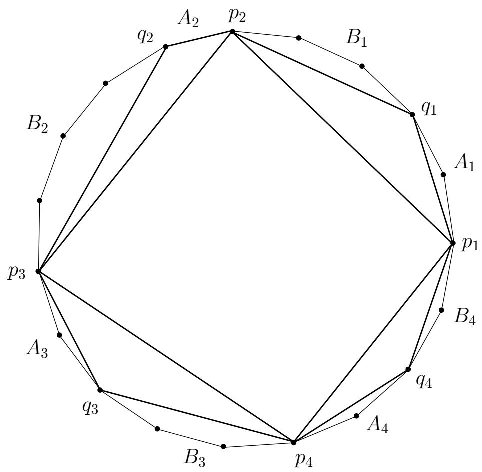

Figure 6 The partition induced by the blocks for $n = 19$ and $k = 4$. The edges p_ip_{i+1} are maximal edges of G' and $\Delta p_ip_{i+1}q_i$ is the unique triangle adjacent to p_ip_{i+1} in the triangulation of the ith block. Since $n = 19$ is small, the figure only illustrates the notation used in the proof of Theorem 12; $|A_1| = 2$, $|B_1| = 3$, $|A_2| = 1$, $|B_2| = 4$, etc.

Let $A_i \cup B_i$ be the sets of elementary arcs in counterclockwise order covering the ith block such that A_i and B_i are separated by a common vertex q_i, where the triangle $\Delta p_ip_{i+1}q_i$ is the (unique) triangle adjacent to the chord p_ip_{i+1} in the triangulation of the ith block (where addition is modulo k, so that $k + 1 = 1$). In particular, q_i is the last endpoint of an elementary arc in A_i and the first endpoint of an elementary arc in B_i, in counterclockwise order. As such, we have

$$\sum_{i=1}^{k} (|A_i| + |B_i|) = n. \tag{1}$$

By definition, we have

$$|A_i| + |B_i| \leq \left(\frac{1}{3} - \varepsilon\right) n, \quad \text{for } i = 1, \ldots, k. \tag{2}$$

By the maximality of the blocks in G', we have

$$(|A_i| + |B_i|) + (|A_{i+1}| + |B_{i+1}|) \geq \left(\frac{1}{3} - \varepsilon\right) n, \quad \text{for } i = 1, \ldots, k. \tag{3}$$

By the maximality of G', we also have $k \leq 6$, since otherwise an averaging argument would yield two adjacent blocks, say, i and $i+1$, that can be merged by adding one chord of length at most 1 and so that the merged sequence of points has size at most

$$|A_i| + |B_i| + |A_{i+1}| + |B_{i+1}| \leq \frac{2n}{7} < \left(\frac{1}{3} - \varepsilon\right) n,$$

which would be a contradiction. We claim that

$$|B_i| + |A_{i+1}| \geq \left(\frac{1}{3} - 3\varepsilon\right) n, \quad \text{for } i = 1, \dots, k. \tag{4}$$

Suppose for contradiction that $|B_i| + |A_{i+1}| \leq \left(\frac{1}{3} - 3\varepsilon\right) n$ holds for some i. Consider the εn elementary arcs preceding the arcs in B_i and the εn elementary arcs following the arcs in A_{i+1}, in counterclockwise order. Denote these sets of arcs by U_i and V_i, respectively ($|U_i| = |V_i| = \varepsilon n$). Recall that $\varepsilon n \geq 2$ and thus $|X(U_i)|, |X(V_i)| \geq |U_i| = \varepsilon n \geq 2$.

We claim that there exist $u \in X(U_i)$ and $v \in X(V_i)$ such that $|uv| \leq 1$ and $h(u, v) \geq 4$. Indeed, $\mathtt{diam}(X(U_i \cup B_i \cup A_{i+1} \cup V_i)) \leq 1$ since $X(U_i \cup B_i \cup A_{i+1} \cup V_i)$ contains at most

$$\left(\frac{1}{3} - 3\varepsilon\right) n + 2\varepsilon n \leq \left(\frac{1}{3} - \varepsilon\right) n$$

consecutive points. This proves the first part of the claim for any $u \in X(U_i)$ and $v \in X(V_i)$. For the second part, we can take u as one of the two vertices preceding q_i that is not p_i, and similarly we can take v as one of the two vertices following q_{i+1} that is not p_{i+2}. With this choice, we have $h(u, p_{i+1}) \geq 2$ and $h(p_{i+1}, v) \geq 2$, and $\rho(u, v)$ passes through p_{i+1}. Consequently,

$$h(u, v) \geq h(u, p_{i+1}) + h(p_{i+1}, v) \geq 2 + 2 = 4.$$

We have reached a contradiction, which proves (4). The summation of (4) over all $i = 1, \dots, k$, in combination with (1) and the inequality $k \geq 4$ yields

$$n = \sum_{i=1}^{k} (|A_i| + |B_i|) = \sum_{i=1}^{k} (|B_i| + |A_{i+1}|) \geq k \left(\frac{1}{3} - 3\varepsilon\right) n \geq 0.27\, kn \geq 1.08\, n.$$

This last contradiction completes the proof. ◀

An upper bound for points on a circle. For many problems dealing with finite point configurations in the plane, points in convex position or on a circle may allow for tighter bounds; see, e.g., [14, 15, 31, 40]. We show that the lower bound of 4 for points on a circle is tight in this case. Due to space constraints, the proof of Theorem 13 is omitted and can be found in the full version.

▶ **Theorem 13.** *For every point set S on a circle C, there exists a plane 4-hop spanner.*

5 The maximum degree of hop spanners cannot be bounded

A standard counting argument shows that dense (abstract) graphs do not admit constant bounded degree hop spanners (irrespective of planarity). We start with an observation regarding the complete UDG K_n and then extend it and show that the maximum degree of hop spanners of sparse UDGs is also unbounded.

▶ **Theorem 14.** *For every pair of integers $k \geq 2$ and $\Delta \geq 2$, there exists a set S of $n = O(\Delta^k)$ points such that the unit disk graph $G = (V, E)$ on S has no k-spanner whose maximum degree is at most Δ.*

Proof. Consider a set S of n points so that the unit disk graph $G = (V, E)$ on S is the complete graph K_n (e.g., n points in a disk of unit diameter). Choose a point $p \in S$. Let $S_0 = \{p\}$. Let $N(u)$ denote the set of vertices adjacent to u in G'. Since the degree of p in G' is at most Δ, $|N(p)| \leq \Delta$. Let $S_1 := N(p)$. The points in S_1 have edges to a set S_2 of at most $\Delta(\Delta - 1)$ points in $S \setminus (S_0 \cup S_1)$. In general, the set S_i contains at most $\Delta(\Delta - 1)^{i-1}$ points in $S \setminus \cup_{j=0}^{i-1} S_j$. Consider the sets $S_0, \ldots, S_k$. Now it is easy to check that

$$\sum_{i=0}^{k} |S_i| \leq 1 + \Delta \frac{(\Delta - 1)^k - 1}{\Delta - 2} = O(\Delta^k).$$

Let M denote the above expression in Δ, q be a point in $S \setminus (\cup_{i=0}^{k} S_i)$ and $n = M + 1$. Observe that $h(p, q) \geq k + 1$, whereas $pq \in E$, and so G' is not a k-spanner for S. ◀

An alternative argument is included below – in a form that we use later in this section. We arrange n points so that the unit disk graph $G = (V, E)$ on S is the complete graph K_n (e.g., n points in a disk of unit diameter). Assume that the points are labeled from 1 to n; and assume there exists a subgraph G' whose maximum degree is at most Δ that is a k-spanner of G. For each vertex $v \in V$, label the elements in $N(v)$ by $1, 2, \ldots$ (the maximum label is $\leq \Delta$), in some arbitrary fashion. For every edge $uv \in E$, $u < v$, there is a connecting path of at most k edges in G'. Such a path can be uniquely encoded by a string of length $k + 1$ over the alphabet $[n] \cup \{0, 1, 2, \ldots, \Delta\}$: by specifying the start vertex u followed by an encoding of the edges in the path. There are at most Δ choices for the first edge in the path; and at most $\Delta - 1$ choices for any subsequent edge and zero for indicating the end of a path whose length is shorter than k; the encoding of a path whose length ℓ is shorter than k has $k - \ell$ trailing zeros at the end. As such, there are at most $(n - 1)\Delta^k$ encodings. If

$$(n - 1)\Delta^k < \binom{n}{2},$$

i.e., if $n > 2\Delta^k$, some edge has no encoding, which is a contradiction, and this completes the proof.

▶ **Theorem 15.** *Let $t \colon \mathbb{N} \to \mathbb{N}$, $t(n) \leq (n - 1)/2$, be an integer function that tends to ∞ with n. For every pair of integers $k \geq 2$ and $\Delta \geq 2$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, there is a set S of n points in the plane such that*

(i) *the unit disk graph $G = (V, E)$ on S has $\Theta(n \cdot t(n))$ edges, and*

(ii) *G has no k-spanner whose maximum degree is at most Δ.*

Proof. Observe that $2t(n) + 1 \leq n$. Let n be large enough so that $t(n) > 2\Delta^k$ (we can choose infinitely many n with this property). Write $t = t(n)$. For a given t, arrange n points into $\left\lfloor \frac{n}{2t+1} \right\rfloor$ groups of size $2t + 1$ and a remaining group (if any) of size $n - \left\lfloor \frac{n}{2t+1} \right\rfloor (2t + 1)$. Place the groups in disjoint disks of unit diameter in the plane, so that the UDG of each group is a complete graph; and arrange the disks along a line such that the UDG has exactly one edge between any two consecutive groups. Assume that there exists a subgraph G' whose maximum degree is at most Δ that is a k-spanner of G. Encode paths in G' as in the proof of Theorem 14. The number of edges in G is bounded from above and from below as follows:

$$|E| \le \left\lfloor \frac{n}{2t+1} \right\rfloor \binom{2t+1}{2} + \binom{2t}{2} + \left\lfloor \frac{n}{2t+1} \right\rfloor$$
$$\le \frac{n}{2t+1}(2t+1)t + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor = nt + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor, \quad (5)$$
$$|E| \ge \left\lfloor \frac{n}{2t+1} \right\rfloor \binom{2t+1}{2} \ge \binom{2t+1}{2} = t(2t+1), \quad (6)$$
$$|E| \ge \left\lfloor \frac{n}{2t+1} \right\rfloor \binom{2t+1}{2} \ge \left(\frac{n}{2t+1} - 1\right)(2t+1)t = nt - (2t+1)t. \quad (7)$$

We distinguish two cases: t is large or t is small as specified below.
If $\frac{n-1}{4} \le t \le \frac{n-1}{2}$, by (5) and (6) we have

$$|E| \le nt + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor \le nt + (n-2)t + 1 \le 2nt.$$
$$|E| \ge t(2t+1) \ge \frac{n-1}{4}(2t+1) > (n-1)\Delta^k.$$

If $t \le \frac{n-1}{4}$, by (5) and (7) we have

$$|E| \le nt + t(2t-1) + \left\lfloor \frac{n}{2t+1} \right\rfloor \le nt + \frac{(n-3)t}{2} + \frac{n}{3}.$$
$$|E| \ge nt - (2t+1)t = t(n - (2t+1)) \ge t\left(n - \frac{n+1}{2}\right) = \frac{n-1}{2}t > (n-1)\Delta^k.$$

Note that $|E| = O(nt)$ and $|E| = \Omega(nt)$ in both cases; consequently, $|E| = \Theta(nt)$. Since $|E|$ exceeds the number of encodings (analogous to the proof of Theorem 14) in both cases, we reach a contradiction and this completes the proof. ◀

6 Conclusion

Observe that if G is a UDG that is triangle-free, then the only 2-hop spanner of G is the graph itself; recall the bipartite case mentioned in Section 1. Thus if G has a superlinear number of edges and is triangle-free, then by the above observation, every 2-hop spanner of G (and there is only one, G) has a superlinear number of edges. This direction does not materialize in a superlinear lower bound for 2-hop spanners because of the following.

▶ **Proposition 16.** *Let G be a UDG that is triangle-free. Then G has at most $2.5n$ edges.*

Proof. It suffices to show that the degree of every vertex is at most five. Assume for contradiction that a point p has degree at least six and let q and r be two consecutive neighbors of p in order of visibility, where $|pq|, |pr| \le 1$. Put $\alpha = \angle qpr$; we have $\alpha \le \pi/3$. Since at least another interior angle of the triangle Δpqr is at least $\pi/3$, the Law of Sines implies $|qr| \le \max\{|pq|, |pr|\} \le 1$ and thus Δpqr is a triangle in G, a contradiction. ◀

We conclude with two remaining open problems:

1. Are there point sets for which every plane hop-spanner has hop stretch factor at least 5?
2. Can our $O(n \log n)$ upper bound on the size of 2-hop spanners on n points in the plane be improved? Are there n-element point sets for which every 2-hop spanner has $\omega(n)$ edges? Recent results show that unit disks may exhibit surprising behavior [29, 35].

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