Stability-Preserving, Time-Efficient Mechanisms for School Choice in Two Rounds

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Abstract

We address the following dynamic version of the school choice question: a city, named City, admits students in two temporally-separated rounds, denoted $\mathcal{R}_1$ and $\mathcal{R}_2$. In round $\mathcal{R}_1$, the capacity of each school is fixed and mechanism $\mathcal{M}_1$ finds a student optimal stable matching. In round $\mathcal{R}_2$, certain parameters change, e.g., new students move into the City or the City is happy to allocate extra seats to specific schools. We study a number of Settings of this kind and give polynomial time algorithms for obtaining a stable matching for the new situations.

It is well established that switching the school of a student midway, unsynchronized with her classmates, can cause traumatic effects. This fact guides us to two types of results: the first simply disallows any re-allocations in round $\mathcal{R}_2$, and the second asks for a stable matching that minimizes the number of re-allocations. For the latter, we prove that the stable matchings which minimize the number of re-allocations form a sublattice of the lattice of stable matchings. Observations about incentive compatibility are woven into these results. We also give a third type of results, namely proofs of NP-hardness for a mechanism for round $\mathcal{R}_2$ under certain settings.

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1 Introduction

School choice is among the most consequential events in a child's upbringing, whether it is admission to elementary, middle or high school, and hence has been accorded its due importance not only in the education literature but also in game theory and economics. In order to deal with the flaws in the practices of the day, the seminal paper of Abdulkadiroglu and Sonmez [3] formulated this as a mechanism design problem. This approach has been enormously successful, especially in large cities involving the admission of tens of thousands of students into hundreds of schools, e.g., see [2, 1, 4, 18], and today occupies a key place in the area of market design in economics, e.g., see [23, 21, 22, 13].

Once the basic game-theoretic issues in school choice were adequately addressed, researchers turned attention to the next level of questions. In this vein, in a recent paper, Feigenbaum et. al. [12] remarked, "However, most models considered in this literature are essentially

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static. Incorporating dynamic considerations in designing assignment mechanisms ... is an important aspect that has only recently started to be addressed."

Our paper deals with precisely this. We define several settings for school choice in which an instance is made available in the first round $\mathcal{R}_1$ and at a later time, in the second round $\mathcal{R}_2$, some of the parameters change. Each setting asks for a pair of mechanisms, $(\mathcal{M}_1,\ \mathcal{M}_2)$ for finding matchings of students to schools in these two rounds. All our settings insist that the matchings found in both rounds are stable. It will be convenient to classify our results into three types. In Type A and B, both mechanisms $\mathcal{M}_1$ and $\mathcal{M}_2$ are required to run in polynomial time.

1. **Type A:** Mechanism $\mathcal{M}_2$ is disallowed to reassign the school of any student matched by $\mathcal{M}_1$. We present two settings, **A1** and **A2**.
2. **Type B:** Mechanism $\mathcal{M}_2$ is allowed to reassign the school of students matched by $\mathcal{M}_1$; however, it needs to (provably) minimize the number of such reassignments. We present two settings, **B1** and **B2**.
3. **Type C:** These are NP-hardness results – of mechanism $\mathcal{M}_2$ for four problems and of a fifth problem, which involves only one round.

1.1 Our model and its justification

Our solutions to Type A and B results will strictly adhere to the following tenets; we justify them below.

1. **Tenet 1:** All matchings produced by our mechanisms need to be stable.
2. **Tenet 2:** In Type A results, mechanism $\mathcal{M}_2$ is disallowed to reassign the school of any student matched by $\mathcal{M}_1$, and in Type B results, $\mathcal{M}_2$ must provably minimize the number of such reassignments.
3. **Tenet 3:** We want all our mechanisms to run in polynomial time.

The use of the classic Gale-Shapley [14] Deferred Acceptance Algorithm has emerged as a method of choice in the literature. Our mechanisms also use this algorithm. Stability comes with key advantages: First, no student and school, who are not matched to each other, will have the incentive to go outside the mechanism to strike a deal. Second, it eliminates *justified envy*, i.e., the following situation cannot arise: there is a student s_i who prefers another student s_j's school assignment, say h_k, while being fully aware that h_k preferred her to s_j.

Switching the school of a student midway, unsynchronized with her classmates – such as when the entire class moves from elementary to middle or from middle to high school – is well-known to cause traumatic effects, e.g., see [15]. It is for these reasons that in Type A results, mechanism $\mathcal{M}_2$ is disallowed to reassign the school of any student matched by $\mathcal{M}_1$ and in Type B results, $\mathcal{M}_2$ must provably minimize the number of such reassignments. For Type A results, we say that $\mathcal{M}_2$ extends M to a stable matching M'. For Type B results, we say that $\mathcal{M}_2$ computes a minimum stable re-allocation M' of M.

The strongest notion of incentive compatibility for a mechanism is *dominant strategy incentive compatible (DSIC)*, for students. This entails that regardless of the preferences reported by other students, a student can do no better than report her true preference list, i.e., truth-telling is a dominant strategy for all students. This immediately simplifies the task of students and their parents, since they don't need to waste any effort trying to game the system. Furthermore, if students are forced to adjust their choices in an attempt to gain a better matching, the mechanism, dealing with choices reported to it, may be forced to make matches that are suboptimal for students as well as schools.

Gale and Shapley [14] proved that if the Differed Acceptance Algorithm is run with students proposing, it will yield a *student-optimal matching*, i.e., each student will get the best possible school, according to her preference list, among all stable matchings. However, this matching may be extremely unfavorable to an individual student – it may be matched to a school which is very low on her preference list, giving her incentive to cheat, i.e., provide a false preference list, in order to get a better matching. Almost two decades after the Gale-Shapley result, Dubins and Freedman [8] proved, via a highly non-trivial analysis, that this algorithm is DSIC for students. This ground-breaking result opened up the Gale-Shapley algorithm to a host of highly consequential applications, including school choice.

In all of our results of Type A and B, mechanism $\mathcal{M}_1$ finds a student-optimal stable matching using the Gale-Shapley Differed Acceptance Algorithm and is therefore DSIC for students. For Setting B2 we provide a mechanism for round $\mathcal{R}_2$ that is DSIC. However, our mechanisms for round $\mathcal{R}_2$ for the remaining three settings do not achieve this. Our main open problem is to fix this. For completeness, and in order to motivate this open problem, we discuss incentive-compatibility for each of these setting in Section 6.

It is well known that the set of Stable Matchings of a given instance forms a finite distributive lattice [16]. By orienting the underlying partial order of this lattice appropriately, the student-optimal stable matching can be made the top element of this lattice and the school optimal matching the bottom element. For both Settings of Type B, we show that the set of minimum stable re-allocations form a sublattice of this lattice. We provide polynomial-time mechanisms for computing the top and bottom elements of this sublattice. For Setting B1, we show that the top of the sublattice is also the top of the whole lattice, i.e., it is the student-optimal stable matching; this is crucial for showing DSIC for B1.

1.1.1 Type A and B settings

The four settings involve the admission of students of a city, named City, into schools; the preference lists of both students and schools are provided to the mechanisms. $\mathcal{M}_1$ computes a student-optimal stable matching, M, over all the participants in $\mathcal{R}_1$. In $\mathcal{R}_2$ some of the parameters over which M was defined are updated. $\mathcal{M}_2$ then modifies the matching M to produce a new matching M' that is stable over the new parameters defined in $\mathcal{R}_2$.

For Settings of Type A, in round $\mathcal{R}_1$, the capacity of each school is fixed but in round $\mathcal{R}_2$, the City is happy to allocate extra seats to specific schools per the recommendation of mechanism $\mathcal{M}_2$, which in turn has to meet specified requirements imposed by the City. Let L be the set of *left-over students*, those who could not be admitted in round $\mathcal{R}_1$.

In round $\mathcal{R}_2$ of Setting A1, the problem is to maximize the number of students admitted from L, by extending M in a stability-preserving manner. In Setting A2, a set N of *new students* also arrive from other cities and their preference lists are revealed to $\mathcal{M}_2$. The requirement now is to admit as few students as possible from N and subject to that, as many as possible from L, again in a stability-preserving manner. Finally, we give a procedure that outputs all possible stability-preserving extensions of a given stable matching (which may be exponentially many) with polynomial delay.

For Settings of Type B, the capacity of each school is fixed in $\mathcal{R}_1$, but in $\mathcal{R}_2$ the City has to deal with the arrival of new students and new schools. This could lead the matching found by $\mathcal{M}_1$ to no longer be stable.

In round $\mathcal{R}_2$ of Setting B1, a set N of new students arrive and their preference lists are revealed to $\mathcal{M}_2$. The capacity of schools remain unchanged and the problem is to find a matching, M' that is stable under the arrival of new students which minimizes the number of students who are assigned to a different school in M'. In Setting B2, a set H' of new schools arrive and the City allows the capacities of the original schools to increase. The preference lists of the students are updated to reflect these new schools, we again require that $\mathcal{M}_2$ compute a new stable matching, M' over the updated preference lists that minimizes the number of students who get matched to a different school in M'.

1.2 Related work

Besides the references pointed out above on school choice, in this section, we will concentrate on recent work on dynamic matching markets, especially those pertaining to school choice. Feigenbaum et. al. [12] study the following issue that arises in NYC public high schools, which admits over 80,000 students annually: after the initial centralized allocation, about 10% of the students choose not attend the school allocated to them, instead going to private or charter schools. To deal with this, [12] give a two-round solution which maintains truthfulness and efficiency and minimizes the movement of students between schools.

An interesting phenomena that has been observed in matching markets is *unraveling*, under which matches are made early to beat the competition, even though it leads to inefficiencies due to unavailability of full information. A classic case, indeed one that motivated the formation of centralized clearing houses, is that of the market for medical interns in which contracts for interns were signed two years before the future interns would even graduate [19]. A theoretical explanation of this phenomena was recently provided by [11].

[17] point out that stable pairings may not necessarily last forever, e.g., a student may switch from private to public school or a married couple may divorce. They study dynamic, multi-period, bilateral matching markets and they define and identify sufficient conditions for the existence of a dynamically stable matching.

[7] develops a notion of stability that applies in markets where matching opportunities arrive over time, much like the seats in our work. One of the things shown in this paper is that agents' incentive to wait for better matching opportunities can make achieving stability very difficult. Indeed, the notion of dynamic stability given in this paper is a necessary condition which a matching must satisfy in order that agents do not to find it profitable to game a mechanism by showing up in later rounds.

A number of recent papers [24, 6, 5, 10] consider the consequences of having a mechanism that repeats the Gale-Shapley Deferred Acceptance algorithm multiple times, similar to our work. Note that Deferred Acceptance is not *consistent* in that if one runs it, then removes some agents and their assignments, and runs it again on the remaining agents, one does not obtain the same assignment restricted to the left-over agents. In these papers, the authors show that there is room for manipulation by submitting empty lists in the first round. However, unlike our model in which changes are introduced in round $\mathcal{R}_2$, in all these papers, there is nothing that motivates running Deferred Acceptance twice, namely no arrivals of new students, no change in capacities, no changes in preferences, etc.

1.3 Overview of structural and algorithmic ideas

The main idea for obtaining a stability-preserving mechanism in round $\mathcal{R}_2$ for Settings $A1$ and $A2$ lies in the notion of a *barrier* which ensures that students admitted in $\mathcal{R}_2$ do not form blocking pairs. A crucial issue is to place barriers optimally to ensure that the number of students admitted is optimized (minimized or maximized) appropriately.

The algorithm for enumerating stable extensions of a stable matching, given in Section 4.3, relies heavily on the fundamental structural property of stable matchings. Enumerated matchings are extended by only one student in an iteration. At each step, the algorithm finds all such feasible extensions by one student in a way such that there must be at least one feasible assignment, for any student, at each step. This assurance is crucial in guaranteeing that the delay between any two enumerated matchings is polynomial.

For Settings B1 and B2, the mechanism proceeds by iteratively resolving blocking pairs. Structurally, we show that the set of all minimum stable re-allocations forms a sublattice of the stable matching lattice. Our analysis relies on the fact that the set of students who are assigned to a different school in round $\mathcal{R}_2$ cannot be matched to their original school in any minimum stable re-allocation. This lets us divide the set of students into two groups, students matched to the same school (fixed students), and students matched to different schools (moving students). We then construct a smaller stable matching instance, I, over the set of moving students. By appropriately placing barriers for each student and school in I we can ensure that the union of any stable matching in I and the matching restricted to the fixed students will also be stable. This stable matching is a minimum stable re-allocation and defines a bijection between the set of minimum stable re-allocations and set of stable matchings in I, we exploit the lattice structure of the latter.

2 Preliminaries

2.1 The stable matching problem for school choice

The stable matching problem takes as input a set $H = \{h_1, h_2, \ldots, h_m\}$ of m public schools and a set $S = \{s_1, s_2, \ldots, s_n\}$ of n students who are seeking admission to the schools. Each school $h_j \in H$ has an integer-valued *capacity*, $c(j)$, stating the maximum number of students that can be assigned to it. If h_j is assigned $c(j)$ students, we will say that h_j is *filled*, and otherwise it is *under-filled*.

Each student $s_i \in S$ has a strict and complete preference list, $l(s_i)$, over $H \cup \{\emptyset\}$. If s_i prefers $\emptyset$ to h_j, then it prefers remaining unassigned rather being assigned to school h_j. We will assume that the list $l(s_i)$ is ordered by decreasing preferences. Therefore, if s_i prefers h_j to h_k, we can equivalently say that h_j appears *before* h_k or h_k appears *after* h_j on s_i's preference list. Clearly, the order among the schools occurring after $\emptyset$ on s_i's list is immaterial, since s_i prefers remaining unassigned rather than being assigned to any one of them. Similarly, each school $h_j \in H$ has a strict and complete preference list, $l(h_j)$, over $S \cup \{\emptyset\}$. Once again, for each student s_i occurring after $\emptyset$, h_j prefers remaining under-filled rather than admitting s_i, and the order among these students is of no consequence.

Given a set of schools, $H' \subseteq H$, by the *best school for s_i in H'* we mean the school that s_i prefers the most among the schools in H'. Similarly, given a set of students, $S' \subseteq S$, by the *best student for h_j in S'* we mean the student whom h_j prefers the most among the students in S'.

A *matching* M is a function, $M : S \to H \cup \{\emptyset\}$ such that if $M(s_i) = h_j$ then it must be the case that s_i prefers h_j to $\emptyset$ and h_j prefers s_i to $\emptyset$; if so, we say that student s_i is *assigned to* school h_j. If $M(s_i) = \emptyset$, then s_i is not assigned to any school. The matching M also has to ensure that the number of students assigned to each school h_j is at most $c(j)$.

For a matching M, a student-school pair (s_i, h_j) is said to be a *blocking pair* if s_i is not assigned to h_j, s_i prefers h_j to $M(s_i)$ and one of the following conditions holds:

1. h_j prefers s_i to one of the students assigned to h_j, or
2. h_j is under-filled and h_j prefers s_i to $\emptyset$.

The blocking pair is said to be *type 1* (*type 2*) if the first (second) condition holds. A matching M is said to be *stable* if there is no blocking pair for it.

▶ **Theorem 1** (Rural Hospitals Theorem [20]).
1. *Over all the stable matchings of the given instance: the set of matched students is the same and the number of students matched to each school is also the same.*
2. *Assume that school h is not matched to capacity in a stable matching. Then, the set of students matched to h is the same over all stable matchings.*

2.2 The Stable Matching Lattice

▶ **Definition 2.** *A Lattice $\mathcal{L} = (S, \succeq)$, is defined over a finite set S, and a partial order $\succeq$, if for every pair of elements $a, b \in S$, there exists a unique least upperbound and a unique greatest lowerbound. We call the least upperbound the join of a and b and denote it by $a \vee b$, and anagolously call the least lowerbound the meet of a and b and denote it by $a \wedge b$*

▶ **Definition 3.** *Let SM denote the set of stable matchings over given instance (S, H, c), then for two stable matchings $M, M' \subseteq SM$, $M \succeq M'$, if and only if $\forall\ s_i \in S,\ s_i$ weakly prefers $M(s_i)$ to $M'(s_i)$*

Given two stable matchings M and M' consider two new maps M_U and M_L, defined as follows:
- $M_U(s_i) = max(M(s_i), M'(s_i))$
- $M_L(s_i) = min(M(s_i), M'(s_i))$

where max is the partner s_i weakly prefers between M and M', and min is the complement of max.

▶ **Theorem 4** ([16]). *The set of stable matchings $(SM, \succeq)$ characterizes a finite distributive lattice. Morever M_L, M_U represent the meet and join of any two stable matchings in the lattice.*

3 Our Results for the Four Settings

In round $\mathcal{R}_1$, the setup defined in Section 2.1 prevails and $\mathcal{M}_1$ simply computes the student-optimal stable matching respecting the capacity of each school, namely $c(j)$ for h_j. Let this matching be denoted by M, $S_M \subseteq S$ be the set of students assigned to schools by M and $L = (S - S_M)$ be the set of *left-over students*. As shown in [9], $\mathcal{M}_1$ is DSIC for students.

For Settings of Type A, in round $\mathcal{R}_2$, the City has decided to extend matching M in a stable manner without any restrictions on extra capacity added to each school.

For Settings of Type B, in round $\mathcal{R}_2$ a change is made to the sets of participants, which may cause M to no longer be a valid or stable matching. $\mathcal{M}_2$ then updates M to M' in order to ensure a stable matching. By allowing *updates*, we let some students in M get unmatched in M', or get matched to different schools. The City would like to minimize the number of students who would have to change schools, or no longer be matched to a school, in going from M to M'. We call M' a **minimum stable re-allocation** of M. Formally, M' is a minimum stable re-allocation of M if M' is a stable matching over all participants and the number of students $s_i \in S_M$ where $M(s_i) \neq M'(s_i)$ is minimized.

3.1 Type A and B Settings

Setting A1. In this setting, in round $\mathcal{R}_2$, the City wants to admit as many students from L as possible in a stablity-preserving manner. We will call this problem Max_L. We will prove the following:

▶ **Theorem 5.** *There is a polynomial time mechanism $\mathcal{M}_2$ that extends matching M to M' so that M' is stable w.r.t. students S and schools H. Furthermore, $\mathcal{M}_2$ yields the largest matching that can be obtained by a mechanism satisfying the stated conditions.*

Let k be the maximum number of students that can be added from L, as per Theorem 5. Next, suppose that the City can only afford to add $k' < k$ extra seats. We show in Section 4.1 how this can be achieved while maintaining all the properties stated in Theorem 5.

Setting A2. In this setting, in round $\mathcal{R}_2$, in addition to the leftover set L, a set N of *new students* arrive from other cities and their preference lists are revealed to mechanism $\mathcal{M}_2$. Additionally, the schools also update their preference lists to include the new students. In this setting, the City wants to give preference to students who were not matched in round $\mathcal{R}_1$, i.e., L, over the new students, N. Thus it seeks the subset of N that *must* be admitted to avoid blocking pairs and subject to that, maximize the subset of L that can be added, again in a stability-preserving manner. We will call this problem $Min_N Max_L$. We will prove the following:

▶ **Theorem 6.** *There is a polynomial time mechanism $\mathcal{M}_2$ that accomplishes the following:*
1. *It finds smallest subset $N' \subseteq N$ with which the current matching M needs to be extended in a stability-preserving manner.*
2. *Subject to the previous extension, it finds the largest subset $L' \subseteq L$ with which the matching can be extended further in a stability-preserving manner.*

Setting B1. In this setting, a set N of *new students* arrive from other cities in round $\mathcal{R}_2$. The preference lists of schools are also updated to include students in N, though their relative preferences between students in $S \cup \{\emptyset\}$ are unchanged. The City wants to find a stable matching over students $S \cup N$ and schools H that minimizes the number of students who are re-allocated from their original school in M.

▶ **Theorem 7.** *There is a polynomial time mechanism $\mathcal{M}_2$ that finds a minimum stable reallocation with respect to Round $\mathcal{R}_1$ matching M, students $S \cup N$, and schools H.*

Setting B2. In this setting, the City has some new schools H' that have opened up in $\mathcal{R}_2$. The preference lists of students are updated to include schools in H', though their relative preferences between schools in $H \cup \{\emptyset\}$ are unchanged. The City also allows schools in H to increase their capacity in round $\mathcal{R}_2$. The City wants to find a stable matching over students S and schools $H \cup H'$ that minimizes the number of students who are re-allocated from their original school in M.

▶ **Theorem 8.** *There is a polynomial time mechanism $\mathcal{M}_2$ that finds a minimum stable reallocation with respect to Round $\mathcal{R}_1$ matching M, students S, and schools $H \cup H'$.*

4 Mechanisms for Type A Settings

4.1 Setting A1

We will first characterize situations under which a matching is not stable, i.e., admits a blocking pair. This characterization will be used for proving stability of matchings constructed in round $\mathcal{R}_2$. For this purpose, assume that M is an arbitrary matching, not necessarily stable nor related to the matching computed in round $\mathcal{R}_1$. For each school $h_j \in H$, define the *least preferred student assigned to* h_j, denoted LPS-Assigned(h_j), to be the student whom h_j prefers the least among the students that are assigned to h_j.

$Max_L(M, L)$:
Input: Stable matching M and set L.
Output: Stable, Max_L extension of M.

1. $\forall s_i \in S_M : M'(s_i) \leftarrow M(s_i)$
2. $\forall h_j \in H :$ Barrier$(h_j) \leftarrow$ BS-Preferring(h_j).
3. $L' \leftarrow \{s_i \in L \mid \exists h_j$ s.t. s_i appears before Barrier(h_j) in $l(h_j)$, and h_j appears before $\emptyset$ in $l(s_i)\}$.
4. $\forall s_i \in L' :$ Feasible-Schools$(s_i) \leftarrow \{h_j \mid s_i$ appears before Barrier(h_j) in $l(h_j)\}$.
5. $\forall s_i \in L' :$ $M'(s_i) \leftarrow$ Best school for s_i in Feasible-Schools(s_i).
6. $\forall s_i \in (L - L') :$ $M'(s_i) \leftarrow \emptyset$.
7. Return M'.

Figure 1 Mechanism for round $\mathcal{R}_2$ for problem Max_L in Setting A1.

Next, for each student $s_i \in S_M$, define the *set of schools preferred by* s_i, denoted Preferred-Schools(s_i) by $\{h_j \mid s_i$ prefers h_j to $M(s_i)\}$; note that $M(s_i) = \emptyset$ is allowed in this definition. Further, for each school $h_j \in H$, define the *set of students that prefer* h_j *over the school they are assigned to*, denoted Preferring-Students(h_j) to be $\{s_i \mid h_j \in$ Preferred-Schools$(s_i)\}$. Finally, define *best student preferring* h_j, denoted BS-Preferring(h_j), to be the student whom h_j prefers the best in the set Preferring-Students(h_j). If Preferring-Students$(h_j) = \emptyset$ then we will define BS-Preferring$(h_j) = \emptyset$; in particular, this happens if h_j is under-filled.

The mechanism $\mathcal{M}_2$ for round $\mathcal{R}_2$ for Max_L in Setting A1 is given in Figure 1. Step 1 simply ensures that the matching found by $\mathcal{M}_2$ extends the round $\mathcal{R}_1$ matching. Step 2 defines the Barrier for each school to be BS-Preferring(h_j); observe that this could be $\emptyset$. Step 3 determines the set $L' \subseteq L$ that can be assigned schools in a stability-preserving manner and Step 5 computes the school for each student in this subset.

For the problem of admitting fewer students, we give the following:

▶ **Proposition 9.** *Let k be the total number of students added from L in round $\mathcal{R}_2$ in the previous theorem and let $k' < k$. There is a polynomial time mechanism $\mathcal{M}_2$ that is stability-preserving, and extends matching M to M' so that $|M'| - |M| = k'$.*

4.2 Setting A2

The mechanism for round $\mathcal{R}_2$ for $Min_N Max_L$ in Setting A2 is given in Figure 2 provided in the appendix. Suppose there is a school h_j, student $s_k \in S_M$ is assigned to it and there is a student $s_i \in N$ such that h_j prefers s_i to s_k. Now, if s_i is kept unmatched, (s_i, h_j) will form a blocking pair of type 1. Next suppose h_j is under-filled and there is a student $s_i \in N$ such that h_j and s_i prefer each other to $\emptyset$. This time, if s_i is kept unmatched, (s_i, h_j) will form a blocking pair of type 2. Motivated by this, for a student s_i, define the *set of schools forming blocking pairs with* s_i, denoted Schools-FBPairs(s_i), to be:

$$\text{Schools-FBPairs}(s_i) = \{h_j \in H \mid h_j \text{ prefers } s_i \text{ to LPS-Assigned}(h_j), s_i \text{ prefers } h_j \text{ to } \emptyset\}$$
$$\bigcup \{h_j \in H \mid h_j \text{ is under-filled and } h_j \text{ and } s_i \text{ prefer each other to } \emptyset\}.$$

$Min_N Max_L(M, N, L)$:
Input: Stable matching M, and sets N and L.
Output: Stable, $Min_N Max_L$ extension of M.

1. $\forall s_i \in S_M : M'(s_i) \leftarrow M(s_i)$
2. $\forall h_j \in H :$ Barrier1$(h_j) \leftarrow$ BS-Preferring(h_j).
3. $N' \leftarrow \{s_i \in N \mid$ Schools-FBPairs(s_i) is non-empty$\}$.
4. $\forall h_j \in H :$ Barrier2$(h_j) \leftarrow$ Best student for h_j in $(N - N')$.
5. $\forall h_j \in H :$ Barrier$(h_j) \leftarrow$ Best student for h_j in $\{$Barrier1(h_j), Barrier2(h_j) $\}$.
6. $L' \leftarrow \{s_i \in L \mid \exists h_j$ s.t. s_i appears before Barrier(h_j) in $l(h_j)$, and h_j appears before $\emptyset$ in $l(s_i)\}$.
7. $\forall s_i \in (N' \cup L') :$ Feasible-Schools$(s_i) \leftarrow \{h_j \mid s_i$ appears before Barrier(h_j) in $l(h_j)$ $\}$
8. $\forall s_i \in (N' \cup L') :$ $M'(s_i) \leftarrow$ Best school for s_i in Feasible-Schools(s_i).
9. $\forall s_i \in ((L - L') \cup (N - N')) :$ $M'(s_i) \leftarrow \emptyset$.
10. Return M'.

Figure 2 Mechanism for round $\mathcal{R}_2$ for $Min_N Max_L$ in Setting A2.

Therefore, all students in N', computed in Step 3, need to be matched. Our mechanism keeps all students in $N - N'$ unmatched, thereby minimizing the number of students matched from N.

We next describe the various barriers that need to be defined. The first one, defined in Step 2, plays the same role as that in Figure 1. As before, if h_j is under-filled, Barrier1$(h_j) = \emptyset$. If a student $s_i \in (N' \cup L')$ appears after Barrier1(h_j) in $l(h_j)$ and is assigned to h_j, then (Barrier1$(h_j), h_j$) will form a blocking pair. The second one, Barrier2(h_j) in $(N - N')$ defined in Step 4. Again, if $s_i \in (N' \cup L')$ appears after Barrier2(h_j) in $l(h_j)$ and is assigned to h_j, then (Barrier2$(h_j), h_j$) will form a blocking pair. In step 5, Barrier(H_j) is defined to be the more stringent of these two barriers.

The final question is which school should $s_i \in N'$ be matched to? One possibility is to compute for each student s_i the set

$$T(s_i) = \{h_j \in H \mid \exists s_k \text{ s.t. } M(s_k) = h_j,\ h_j \text{ prefers } s_i \text{ to } s_k, \text{ and } s_i \text{ prefers } h_j \text{ to } \emptyset\},$$

and match s_i to her best school in $T(s_i)$.

Assume that s_i is matched to h_j under this scheme. A blocking pair may arise as follows: Assume s_i prefers school h_k to h_j (of course, $h_k \notin T(s_i)$), some student $s_l \in L'$ has been assigned to h_k and h_k prefers s_i to s_l. If so, (s_i, h_k) will form a blocking pair. One remedy is to redefine the barrier for h_k so s_l is not assigned to h_k. However, this will make the barrier more stringent and the resulting mechanism will, in general, match fewer students from L than our mechanism. The latter is as follows: simply match s_i to the best school which prefers her to the Barrier of that school.

▶ **Theorem 10.** *There is a polynomial time mechanism $\mathcal{M}_2$ that finds the largest subset of $(N \cup L)$ that can be matched to schools and added to the current matching while maintaining stability. This mechanism also solves $Max_N Max_L$ and $Max_L Max_N$.*

StableExtension(M, c, N):
Input: Stable matching M, capacity c, new students $N = \{s_1, s_2 \dots s_k\}$.
Output: Stable extensions of M, with polynomial delay.
$M_0 \leftarrow M$
$A_1 = \textsc{FeasibleAssignment}(M_0, c, s_1)$
For i_1 in A_1:
  $M_1 \leftarrow$ Starting from M_0, match s_1 to i_1.
  $A_2 = \textsc{FeasibleAssignment}(M_1, c, s_2)$.
  For i_2 in A_2:
    $\vdots$
      $A_k = \textsc{FeasibleAssignment}(M_{k-1}, c, s_k)$.
      For i_k in A_k:
        $M_k \leftarrow$ Starting from M_{k-1}, match s_k to i_k.
        Enumerate M_k.

Figure 3 Algorithm for enumerating stable extensions of M.

4.3 Enumeration of Stable Extensions

In this section we show how to enumerate all the possible stable extensions of a given stable matching with polynomial delay between any two enumerated matchings. Specifically, the algorithm takes as input a stable matching M from S to H satisfying capacity c and a set of new students $N = \{s_1, s_2 \dots s_k\}$ that can be added to the schools. Here the preference lists of all schools and students are also given. The algorithm enumerates all solutions M' from $S \cup N$ to $H \cup \{\emptyset\}$ such that:

- all assignments in M are preserved in M', and
- M' is stable with respect to capacity c' where

$$c'(j) = \begin{cases} \left|M'^{-1}(h_j)\right| & \text{if } \left|M'^{-1}(h_j)\right| > c(j), \\ c(j) & \text{otherwise.} \end{cases} \tag{1}$$

Note that $M'^{-1}(h_j)$ is the set of students assigned to h_j under M'. We say that M' is a *stable extension of M with respect to N*.

The complete algorithm StableExtension(M, c, N) is given in Figure 3 (Appendix). At a high level, the algorithm maintains a stable extension M_e of M with respect to a subset N' of N. At each step, a student s_i is added to N' and all possible assignments A of s_i that are compatible to M_e are identified. In other words, adding each assignment in A to M_e gives a stable extension of M with respect to $N' \cup \{s_i\}$. The algorithm branches to an assignment in A and continues to the next student. When $N' = N$, the current matching is returned. The algorithm then backtracks to a previous branching point and continues.

Figure 4 gives the subroutine for finding compatible assignments. Initially, A_i is set to be an empty set. The subroutine then goes through the preference list of s_i one by one in decreasing order. The considered school h is added to A_i and the subroutine terminates if at least one of the following happens:

- h is $\emptyset$,
- h is under-filled,
- h prefers s_i to LPS-Assigned(h) with respect to M_e.

FeasibleAssignment(M_e, c, s_i):
Input: Stable matching M_e, capacity c, student s_i.
Output: Set A_i of all possible assignments for s_i. Adding any assignment in A_i to M_e preserves stability.

1. Initilize A_i to the empty set.
2. For each h in $l(s_i)$, in decreasing order of preferences, do:
   a. If $h = \emptyset$ then Return $A_i \cup \{\emptyset\}$.
   b. Else $h = h_j$:
      i. If $\left|M_e^{-1}(h_j)\right| < c(j)$ then Return $A_i \cup \{h_j\}$.
      ii. If s_i appears before LPS-Assigned(h_j) then Return $A_i \cup \{h_j\}$.
      iii. If s_i appears after LPS-Assigned(h_j) and before BS-Preferring(h_j) then $A_i \leftarrow A_i \cup \{h_j\}$.

Figure 4 Algorithm for finding feasible matches of s_i w.r.t. current matching M_c.

Notice that in the last two scenarios above, if s_i was assigned to any school after h in her preference list, (s_i, h) would form a blocking pair. Assume none of the above scenarios happens. The subroutine adds h to A and continues if h prefers s_i to BS-Preferring(h). Otherwise, h prefers BS-Preferring(h) to s_i. Hence, assigning s_i to h would create a blocking pair. The subroutine continues to the next school in this case. The following lemma says that FeasibleAssignment correctly finds all possible assignments of a student, given the current matching, at each step.

▶ **Lemma 11.** *Let N' be the set of students assigned (possibly to $\emptyset$) in M_e, i.e., M_e is a stable extension of M with respect to N'. FeasibleAssignment(M_e, c, s_i) finds all possible assignments of s_i to $H \cup \{\emptyset\}$ such that adding each assignment to M_e gives a stable extension of M with respect to $N' \cup \{s_i\}$.*

▶ **Lemma 12.** *FeasibleAssignment(M_e, c, s_i) returns at least one possible assignment.*

From Lemmas 11 and 12, we can prove the main theorem of this section:

▶ **Theorem 13.** *StableExtension(M, c, N) enumerates all possible stable extension of M with respect to N. Moreover, the time between any two enumerations is $O((k+n)m)$.*

5 Mechanisms for Type B Settings

5.1 Setting B1

We first show some structural properties of minimum stable re-allocations in this setting. $N' \subseteq N$, defines the set of students who form blocking pairs with the current matching M. SM denote the set of stable matchings over the instance $I = (S \cup N, H, c)$, and MSR represents the set of all minimum stable re-allocations of M. For all $s_i \in N$ we set $M(s_i) = \emptyset$.

▶ **Definition 14.** *$s_i \in S \cup N$ is* ***moved*** *in $M' \in MSR$, if $M(s_i) \neq M'(s_i)$*

▶ **Lemma 15.** *All minimum stable re-allocations of M move the same set of students, S_R.*

Let H_R, be the set of schools that students in S_R are matched to in some minimum stable re-allocation then as an application of the Rural Hospitals Theorem we have:

▶ **Corollary 16.** *All students $s_i \in S \cup N - S_R$ are matched to the same school in all minimum stable re-allocations. All minimum stable re-allocations will match students in S_R to schools in H_R. Moreover, if k students from S_R are matched to a school $h_j \in H_R$, then all minimum stable re-allocations will have k students from S_R matched to H_R.*

We denote the students in $S \cup N - S_R$ as S_F, and let M_F represent the matching restricted to these students. Then for all $M' \in MSR$ and $s_i \in S_F$, $M_F(s_i) = M(s_i) = M'(s_i)$.

Consider the stable matching instance I', defined below:

(a) $\forall s_i \in S_R,$ Barrier(s_i) = Best $h_j \in$ Schools-FBPairs(s_i) over all $h_j \in H - H_R$

(b) $\forall h_j \in H_R,$ Barrier(h_j) = BS-Preferring(h_j) among students in S_F

(c) $\forall s_i \in S_R$, $l'(s_i) = l(s_i)$. Place the $\emptyset$ to the immediate left of Barrier(s_i)

(d) $\forall h_j \in H_R$ $l'(h_j) = l(h_j)$. Place the $\emptyset$ to the immediate left of Barrier(h_j)

(e) Let M' be some MSR, then $c'(h_j) = |\{s_i \in S_R \mid M'(s_i) = h_j\}|$

- $I' = (S_R, H_R, c')$ with preference lists $l'(s_i), l'(h_j)$ defines a stable matching instance, with $SM_{I'}$ denoting the set of all stable matchings over I'.

▶ **Lemma 17.** $\forall$ *$M_{I'} \in SM_{I'}$, $M' = M_{I'} \cup M_F$ is a minimum stable re-allocation. Moreover any $M' \in MSR$ can be decomposed into $M_{I'} \cup M_F$, where $M_{I'} \in SM_{I'}$.*

▶ **Lemma 18.** *$(MSR, \succeq)$ defines a sublattice of $(SM, \succeq)$.*

Adding New Students(M, N):
Input: Stable matching M and set N.
Output: Minimum stable re-allocation of M.

1. $\forall s_i \in S_M : M'(s_i) \leftarrow M(s_i)$
2. While $\exists s_i$ unmatched and (s_i, h_j) form a blocking pair do
   a. $h \leftarrow$ Best possible h_i in Schools-FBPairs(s_i)
   b. if h is filled to capacity then unmatch LPS-Assigned(h)
   c. $M'(s_i) \leftarrow h$
3. Return M'.

Figure 5 Mechanism $\mathcal{M}_2$ for adding new students in round $\mathcal{R}_2$.

The proof that $\mathcal{M}_2$ finds a MSR is provided in the Appendix. As a corollary of our proof we get:

▶ **Corollary 19.** *$\mathcal{M}_2$ produces a student-optimal minimum stable re-allocation.*

▶ **Lemma 20.** *There exists a mechanism $\mathcal{M}_3$, that finds a school-optimal minimum stable re-allocation in polynomial time.*

5.2 Setting B2

A first approach to finding a minimum stable re-allocation in Setting $B2$ would be to run Gale-Shapley over the whole instance. However unlike Setting B1, Example 21 shows that this could require as many as $|S|$ possible re-allocations.

▶ **Example 21.** Let there be $n+1$ students and schools. The preference lists (mod $n+1$) for any student s_i is $(h_{i-1}, h_i, ..., h_{i-2})$ and the preference list for any school h_j is $(s_j, s_{j+1}, ..., s_{j-1})$. In round $\mathcal{R}_1$, all participants but h_{n+1} are present and each school has 1 seat. In round $\mathcal{R}_2$, h_{n+1} arrives with capacity 1. The only stable matching from round $\mathcal{R}_1$ would match each s_i to h_i and s_{n+1} would remain unmatched. Assigning s_{n+1} to h_{n+1} would result in a stable matching requiring no re-allocations. However, running Gale-Shapley over all participants would yield a matching of each s_i to h_{i-1}, but this matching requires n re-allocations.

▶ **Lemma 22.** *Each student weakly improves in any minimum stable reallocation.*

▶ Remark 23. The lattice structure shown in the previous section carries over to this Setting as well. This follows since both Settings B1 and B2 can be reduced to an instance where schools have unit capacity. Consider the unit capacity setting: a stable matching is found in round $\mathcal{R}_1$, and in round $\mathcal{R}_2$ a set of new participants arrive on one side. Since each school has unit capacity, schools and students become interchangeable.

▶ **Lemma 24.** *$(MSR, \succeq)$ is a sublattice of $(SM, \succeq)$. Moreover $\mathcal{M}_2$ finds the school-optimal minimal stable re-allocation.*

Adding New Schools(M, H'):
Input: Stable matching M and set H'.
Output: Minimum stable re-allocation of M.

1. $\forall s_i \in S_M : M'(s_i) \leftarrow M(s_i)$
2. While $\exists h_j \in H \cup H'$ with unmet-capacity and BS-Preferring$(h_j) \neq \emptyset$:
   a. Break current match if exists of BS-Preferring(h_j)
   b. $M' \leftarrow M' \cup (\text{BS-Preferring}(h_j), h_j)$
3. Return M'.

Figure 6 Mechanism $\mathcal{M}_2$ for adding new schools in round $\mathcal{R}_2$.

▶ **Lemma 25.** *There exists a mechanism $\mathcal{M}_3$, that finds a student-optimal minimum stable re-allocation in polynomial time.*

6 Incentive Compatibility

For the four settings discussed, it would be highly desirable if we could prove that mechanism $\mathcal{M}_2$ in round $\mathcal{R}_2$ is DSIC. We show that for Setting B1 that this truly is the case. Unfortunately for Settings $A1, A2$ and $B2$ we show that the current mechanisms outlined above are not incentive compatible. We relax DSIC and consider the weaker notion of a mechanism for which *incentive compatibility is a Nash equilibrium (ICNE)*. Under such a mechanism, a student cannot gain by misreporting her choices, if all other students are truthful. We show that no mechanisms in round $\mathcal{R}_2$ for Setting A1, A2 and B2 can be even ICNE.

▶ **Lemma 26.** *$\mathcal{M}_2$ in Setting B1 is DSIC for students.*

▶ **Lemma 27.** *There is no pair of stability-preserving, ICNE mechanism $(\mathcal{M}_1, \mathcal{M}_2)$ for Setting A1 and A2.*

▶ **Lemma 28.** *There is no pair of stability-preserving, ICNE mechanism $(\mathcal{M}_1, \mathcal{M}_2)$ for Setting B2.*

The key distinction for incentive compatibility between Setting A1,A2 and B1, is that in B1 if a student is unmatched after round $\mathcal{R}_1$ it will remain unmatched after round $\mathcal{R}_2$. However in Settings A1, A2 we try to accommodate students who were unmatched after round $\mathcal{R}_1$, so they still have a chance to get matched in $\mathcal{R}_2$. This providies the possibility of affecting the matching produced in $\mathcal{R}_1$ by misreporting their preference list so as to make the Barriers computed by $\mathcal{M}_2$ more favorable for them.

7 NP-Hardness Results

▶ **Problem 29.** *A different version of Setting A2, the City wants to extend original matching M so that it maximizes the number of students who get matched from L, and subject to this, minimize the number of students who get matched from N. ($max_L min_N$)*

▶ **Problem 30.** *Same setting as Problem 29, but the City wants to maximize the number of students who get matched from N, and subject to this, minimize the number of students who get matched from L. ($max_N min_L$)*

▶ **Problem 31.** *A set of new students N arrive in round $\mathcal{R}_2$. The City wants to extend the matching to include k students from N, such that it maximizes the number of students matched from L.*

▶ **Problem 32.** *In round $\mathcal{R}_2$, we are allowed to re-allocate some students matched in round $\mathcal{R}_1$ in order to match more students from L. Find a stable matching that maximizes the number of students matched from L, and subject to this, minimizes the number of re-allocations made.*

▶ **Problem 33.** *In the single round setting, given a set of students, and schools with strictly ordered preference lists $l(s), l(h)$ respectively, and a weight function $w(j)$ over the edges of students to schools, find a vector of capacities for the schools and a stable matching with respect to this vector that maximizes the total weight.*

▶ **Theorem 34.** *Problems 29, 30, 31, 32, and 33 are NP-hard.*

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