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Zastavniy N. V.
ORCID ID: 0000-0003-2097-3582

National University of "Kyiv-Mohyla Academy"

Tyshchenko S. V.
ORCID ID: 0000-0003-1804-8494

National University of "Kyiv-Mohyla Academy"

Zhukovska O. A.
Cand. physics and mathematics, associate professor
ORCID ID: 0000-0003-1110-9696

National Technical University of Ukraine
"Igor Sikorsky Kyiv Polytechnic Institute"

Shchestyuk N. U.
Cand. physics and mathematics, associate professor
ORCID ID: 0000-0002-7652-8157

National University of "Kyiv-Mohyla Academy"

APPLICATION OF THE INVESTOR'S PROBLEM TO FINANCIAL MARKET OF SECURITIES

ЗАСТОСУВАННЯ ЗАДАЧІ ІНВЕСТОРА ЩОДО ТОРГІВЛІ ЦІННИМИ ПАПЕРАМИ

The article investigates the application of the investor problem to securities trading in single-period and multi-period models. The investor has to determine at stage t the sum x_t *he invests in the purchase of securities (shares or options), if from previous historical data he knows the distribution function of a random variable* S_t *– the share price, provided that with* $E(S_t) < \infty$ *in period t. The investor's income is a random variable* Y_{t+1} *at stage t. In addition, the investor knows the bank interest rates on both deposit and loan. The relevance of this study is of particular importance in a crisis; in particular, the study of the relationship between the optimal investment in securities (in terms of profit maximization) and the management and measuring of the risk of such investment comes to the fore. This problem can be solved by the existence of a close relationship between the optimal investment and the measurement of risk given by stochastic optimization. It is possible because the average value-at-risk AV@R is related to a simple stochastic optimization problem with a piecewise linear profit/cost function and maximal value is attained. The problem includes consideration of two possible scenarios: when the income* Y_{t+1} *at the end of the period is more than the investment* x_t *at the beginning of the period and when the income is less than the investment and there is a shortfall. The result of the work is the formulation of four statements about the values of optimal investments in stocks and options in the single-period and multi-period models and about the potential maximum profits for such applying such a strategy. It turned out that the value of optimal investment can be directly expressed in terms of VaR of some probability level, and this level is expressed in terms of credit and deposit interest rates an characterizes the state of the national economic* *environment. The results of the theoretical study were illustrated on real financial data using two models of financial markets. In the first model, the stock movement obeys the geometric Brownian motion (GBM model). The second model was a model in which the movement of risky assets is described by generalized diffusion with a new fractal time (FAT model). The results of optimal investments for both models were compared with the fair price obtained by the Black-Scholes formula. The results of the study conducted in this work were used to develop recommendations for making decisions about optimal investments in securities (stocks, options) to obtain speculative income in stock trading.*

Keywords: investor problem, measuring risk, value at risk, risk management.

У статті досліджено застосування проблеми інвестора до торгівлі цінними паперами в одноперіодній і багатоперіодній моделях. Інвестор повинен визначити на етапі t суму x_t*, яку він вкладає в купівлю цінних паперів (акцій або опціонів), якщо з попередніх історичних статистичних даних йому відомий розподіл випадкової величини* S_t *– ціни акцій, за умови, що* $E(S_t) < \infty$ *в період t, а дохід інвестора визначається випадковою величиною* Y_{t+1} *на етапі t. Крім цього інвестору відомо банківські процентні ставки як на депозит, так і на кредит. Актуальність даного дослідження набуває особливого значення в умовах кризових явищ; зокрема, на перший план виходить дослідження зв'язку між оптимальним інвестування у цінні папери (з точки зору максимізації прибутку) та управлінням і обчисленням ризику такого інвестування. Дане завдання може бути вирішено за рахунок існування близького зв'язку між оптимальним інвестуванням та вимірюванням ризику, що дається стохастичною оптимізацією. Це можливо бо середнє значення ризику AVaR (як для одноперіодної так і для багатоперіодної моделей) виявилось пов'язаним із простою стохастичною проблемою із кусково-неперервною функцією доходу/втрат та, як раніше було доведено, максимальне значення цієї функції досягається. Постановка задачі включає розгляд двох можливих сценаріїв: коли дохід* Y_{t+1} *в кінці періоду більше вкладених інвестицій* x_t *на початку періоду та коли дохід менше вкладених інвестицій і відбувається дефіцит. Результатом роботи стало формулювання чотирьох тверджень щодо величин оптимальних інвестицій в акції і опціони в одноперіодній та багатоперіодній моделях та щодо потенційних максимальних прибутків при застосуванні такої стратегії. Виявилось, що величину оптимальних інвестицій можна напряму виразити через VaR деякого рівня альфа, а цей рівень виражається через величини кредитних і депозитних ставок і є характеристикою стану національного економічного середовища. Результати теоретичного дослідження було проілюстровано на реальних фінансових даних для двох моделей фінансових ринків. У першій моделі рух акцій підкорюється геометричному броунівському руху (GBM модель). Другою моделлю стала модель, у якій рух ризикованих активів описується узагальненою дифузією з новим фрактальним часом (FAT модель). Результати оптимальних інвестицій для обох моделей було порівняно із справедливою ціною, отриманою за формулою Блека - Шоулза. Результати дослідження, проведеного в даній роботі, було використано для розробки рекомендацій щодо прийняття рішень відносно оптимальних інвестувань у цінні папери (акції, опціони) для отримання спекулятивних доходів у біржовій торгівлі.*

Ключові слова: проблема інвестора, управління ризиком, значення ризику, ризик менеджмент.

Introduction. For the effective organization of the securities trading, the investor needs to correctly plan the possible investments in order to get maximal speculative profits and to ensure minimal risk. In general, the problem is to find an investment strategy for which $EU(X(T))$ is to be maximized, where E denotes the mathematical expectation, $U(\cdot)$ is a utility function, and where $X(T)$ represents the wealth at the final time T.

There are many works devoted to different modifications of this problem (see, e.g., Merton (1969) and survey in Hakansson (1997) and Karatzas and Shreve (1998)). In the setting generally assumed in finance, see Merton (1990), the coefficients are assumed to satisfy an Ito equation. Then the solution of the optimal investment problem can be obtained via dynamic programming approach. However, it is not easy to find the explicit solution by this method, because the corresponding Bellman equation is usually degenerate. Explicit formulas for optimal strategies have been obtained only for a few cases where appreciation rates are assumed to be non-random and known. In paper [2] the optimal investment problem was considered for a diffusion market consisting of a finite number of risky assets and stated as a problem with a maximin performance criterion. This criterion is to ensure that a strategy is found such that the minimum of utility over all distributions of parameters is maximal.

Setting objectives. The purpose of this paper is to get solution of investment problem using such probability functionals for risk measuring as VaR and $AVaR$ because following the financial tsunami experiences of 2008 and crisis Covid19, the risk controls of risky assets and derivative instruments on stocks have become tremendously important for investors. Due to our approach, the investor gets a tool, which allows him to integrate the investment problem with risk management.

Problem statement. We consider multi-stage decision problem, which is a multi-period generalization of the simple investment problem [1].

An investor has to determine at stage $\boldsymbol{t}$ the amount $\boldsymbol{x_t}$ he will invest in a good opportunity at stage $\boldsymbol{t+1}$. From the regular business, he gets an income $\boldsymbol{Y_t}$ with $\boldsymbol{E|Y_t| <\ \infty}$ at time $\boldsymbol{t}$.

If an investor has a speculative profit from stock trading, his investment $\boldsymbol{x_t}$ at time $\boldsymbol{t}$ is equal to the price of shares $\boldsymbol{S_t}$, and the income from the sale of shares $\boldsymbol{Y_{t+1}}$ at time $\boldsymbol{t+1}$ coincides with the value $\boldsymbol{S_{t+1}}$.

In the case of options trading, its investment $\boldsymbol{x_t}$ at time $\boldsymbol{t}$ is equal to $\boldsymbol{C_t}$ - the premium of the option (market price of the option) for a given strike price $\boldsymbol{K_t}$, and the income from the realization of options $\boldsymbol{Y_{t+1}}$ at the moment $\boldsymbol{t+1}$ is defined as $\boldsymbol{Y_{t+1} = [S_{t+1} - K_t]^+}$ in the case of call options and $\boldsymbol{Y_{t+1} = [S_{t+1} - K_t]^- -}$ in the case of put options.

If the available funds are less than the committed amount $\boldsymbol{x_t}$, a shortfall occurs, which causes unit costs of $\boldsymbol{u_t > 1}$. If however the funds are more than $\boldsymbol{x_t}$, the surplus can be carried over to the next period, but it loses $\boldsymbol{1-t}$ of its value, where $\boldsymbol{0 \le l_t \le 1}$.

If the funds $\boldsymbol{Y_{t+1}}$ received from the sale of securities are less than the invested amount $\boldsymbol{x_t}$, there is a shortfall, which increases according to the coefficient $\boldsymbol{u_t > 1}$. If $\boldsymbol{Y_t}$ exceeds $\boldsymbol{x_t}$, the surplus can be carried forward, but it loses $\boldsymbol{1-l_t}$ per unit of its value, where the discount rate is $\boldsymbol{0 \le l_t \le 1}$.

Denote by $\boldsymbol{k_t}$ – (random) surplus carried over from period $\boldsymbol{t}$ to period $\boldsymbol{t+1}$. And $\boldsymbol{k_0 = 0}$, then:

$$k_t = [l_{t-1}k_{t-1} + Y_t - x_{t-1}]^+, t = 1, \dots, T.$$

And let the shortfall $\boldsymbol{m_t}$:

$$m_t = [l_{t-1}k_{t-1} + Y_t - x_{t-1}]^-.$$

Both equations can be combined into one

$$l_{t-1}k_{t-1} + Y_t - x_{t-1} = k_t - M_t;\ k_t \geq 0, M_t \geq 0. \quad (1)$$

Then the Present Value of income:

$$H(x_0, Y_1, \dots, x_{T-1}, Y_T) = \sum_{t=1}^{T}(x_{t-1} - u_t M_t) + l_T k_T. \quad (2)$$

The problem is to maximize the expected profit (**2**) $\boldsymbol{E\big(H(x_0, Y_1, \dots, x_{T-1}, Y_T)\big)}$ subject to (**1**).

Methodology. For solving this stochastic problem, we use probability functionals VaR, $AVaR$ and their properties.

For one period model and for a given portfolio, time horizon T, and probability p, the p-VaR can be defined informally as the maximum possible loss during that time after we exclude all worse outcomes whose combined probability is at most p. More formally, p-VaR is defined such that the probability of a loss greater than VaR is (at most) p while the probability of a loss less than VaR is (at least) $1 - p$. Common parameters for standard VaR are 1% and 5% probabilities and one day and two weeks horizons, although other combinations are in use.

In context of our problem the α-quantile of the profit distribution

$$VaR_\alpha(Y) = G^{-1}(\alpha),\ 0 < \alpha < 1 \quad (3)$$

is called value-at-risk of level α. Although VaR is a very popular measure of risk, it has undesirable mathematical characteristics such as a lack of subadditivity and convexity. As an alternative measure of risk, the average value-at-risk $AVaR$ is known to have better properties than VaR.

The average value-at-risk at level α, $0 < \alpha \leq 1$ of Y is defined as

$$AVaR_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha G^{-1}(u)du \quad (4)$$

where G is the distribution function of Y.

The average value-at-risk has very useful property, it may be represented as the optimal value of the following optimization problem [1], [8]:

$$AVaR_\alpha(Y) = \max\left\{x - \frac{1}{\alpha}\mathrm{E}([Y - x]^-) : x \in R\right\}. \quad (5)$$

Multi-period average value-at-risk. Let $Y = (Y_1, \dots, Y_T)$ be an integrable stochastic process.

For a given sequence of constants $c = (c_1, \dots, c_T)$, probabilities $\alpha = (\alpha_1, \dots, \alpha_T)$, and a filtration $F = (F_0, \dots, F_{T-1})$, the multi-period average value-at-risk is defined as in [1]:

$$AVaR_{\alpha,c}(Y; F) = \sum_{t=1}^{T} c_t E\big[AVaR_{\alpha_t}(Y_t | F_{t-1})\big].$$

Results of the research. *One-period model*. Let A be the maximum value of the expected present value of income. If we use property (5) to the optimization problem (1)–(2) for $t = 1$, then we have:

$$A = maxE(H) = (1 - l_1)AVaR_\alpha(Y_1) + l_1E(Y_1),$$

where $\alpha = \frac{1-l}{u-l}$, l_1 is a discount factor.

In this case, the optimal amount of investment is

$$x^* = G^{-1}(\alpha) = VaR_\alpha(G), \quad (6)$$

where G is the distribution function of Y [1].

Suppose we have a market, where S evolves GBM and there are two banking processes with interest rate R for deposit and r for landing. Then the following propositions are true.

Proposition 1: Under the 1-period model, the optimal decision for investment in share is the value $x^* = G^{-1}(\alpha) = VaR_\alpha(G)$, where G is the lognormal distribution function for stock price movements. The maximum value of speculative income is

$$A = (1 - l)AVaR_\alpha(G) + lE(G), \quad (7)$$

where $\alpha = \frac{1-e^{-rT}}{e^{RT}-e^{-rT}}$, $l_1 = e^{-rT}$.

Proposition 2: Under the 1-period model, the optimal decision for investment in option is the value is $x^* = G^{-1}(\alpha)$, where the G is distribution function for pay off $Y = [S_1 - K_1]^+$ for the Call option and for $Y = [S_1 - K_1]^-$ for the Put-option. The maximum value of speculative income:

for Call-option:

$$A = (1 - l_1)AVaR([S_1 - K_1]^+) + l_1E([S_1 - K_1]^+),$$

for Put-option:

$$A = (1 - l_1)AVaR([S_1 - K_1]^-) + l_1E([S_1 - K_1]^-).$$

Multi-period model

For $t = 2$:

$$k_2 = [l_1k_1 + Y_2 - x_1]^+,$$
$$M_2 = [l_1k_1 + Y_2 - x_1]^-.$$

Both equations can be combined into one:

$$l_1k_1 + Y_2 - x_1 = k_2 - M_2, \quad k_2 \geq 0, \quad M_2 \geq 0.$$

For $t = 1$ we have:

$$Y_1 - x_0 = k_1 - M_1$$

Total income function for two periods:

$$H(x_0, Y_1, x_1, Y_2) = x_0 - u_1M_1 + x_1 - u_2M_2 + l_2k_2$$

According to the first period:

$$H(x_0, Y_1) = x_0 - u_1M_1 + l_1k_1 = x_0 - u_1[Y_1 - x_0]^- + l_1[Y_1 - x_0]^+$$

then the total income can be written as

$$H(x_0, Y_1, x_1, Y_2) = (x_0 - u_1M_1 + l_1k_1) + x_1 - u_2M_2 + l_2k_2 - l_1k_1.$$

Out problem is

$$\max \mathrm{E}\big(H(x_0, Y_1, x_1, Y_2)\big) =$$
$$= \max\mathrm{E}(x_0 - u_1M_1 + l_1k_1) + \max\mathrm{E}(x_1 - u_2M_2 + l_2k_2) - l_1k_1.$$

This can be rewritten as:

$$\max\mathrm{E}\big(H(x_0, Y_1, x_1, Y_2)\big) = \max\mathrm{E}(x_0, Y_1) + \max\mathrm{E}(x_1, Y_2) - l_1\mathrm{E}(k_1).$$

So, the maximum value of the mathematical expectation for the total period is not equal to the sum of the mathematical expectations for the individual periods. Let's try to write this in terms of $AVaR$

$$\max \mathrm{E}(x_0, Y_1) = (1 - l_1) AVaR_{\alpha_1}(Y_1) + l_1 E(Y_1).$$

Easy to show that

$$\max \mathrm{E}(x_1, Y_2) = (1 - l_2) AVaR_{\alpha_2}(Y_2) + l_2 E(Y_2).$$

Then

$$\max \mathrm{E}\big(H(x_0, Y_1, x_1, Y_2)\big) =$$
$$= (1 - l_1) AVaR_{\alpha}(Y_1) + (1 - l_2) AVaR_{\alpha}(Y_2) + l_1 E(Y_1) + l_2 E(Y_2) - l_1 E(k_1).$$

Now we generalize formula (7) for the case $t = T$. Recall that A is the maximum expected present value of the entire operation, provided that the filtering F_{t-1} is the result of all previous decisions x_{t-1} at the time of period $t - 1$.

$$A(Y_1, \dots, Y_T, F_0, \dots, F_T) = \max\{\mathrm{E}[\sum_{t=1}^{T} (x_{t-1} - u_t M_t) + l_T k_T] : x_t\} \qquad (8)$$

Problem (8) has its dual representation

$$A(Y_1, \dots, Y_T, F_0) = \inf\{\sum_{t=1}^{T} \mathrm{E}(Y_t Z_t : Z \in Z_*)\}$$

and can be rewritten as [1]:

$$A(Y_1, \dots, Y_T, F_0, \dots, F_T) = \sum_{t=1}^{T} l_t \mathrm{E}(Y_t) + \sum_{t=1}^{T} (1 - l_t) \mathrm{E}\big[AVaR_{\alpha_t}(Y_t | F_{t-1})\big]$$

Proposition 3. In a multi-period model, the optimal price (in terms of income) is the value

$$x_t^* = VaR_{\beta_{t+1}}(Y_{t+1} | F_t) - l_t k_t.$$

And the maximum value of speculative income for options, where

$$Y = [S_1 - K_1]^+ \qquad \text{for Call-option}$$

and

$$Y = [S_1 - K_1]^- \qquad \text{for Put-option},$$

and for Call-option

$$\sum_{t=1}^{T} l_t \mathrm{E}([S_1 - K_1]^+) + \sum_{t=1}^{T} (1 - l_t) \mathrm{E}\big[AVaR_{\beta,c}([S_1 - K_1]^+, \dots, [S_t - K_t]^+, F)\big],$$

for Put-option

$$\sum_{t=1}^{T} l_t \mathrm{E}([S_1 - K_1]^-) + \sum_{t=1}^{T} (1 - l_t) \mathrm{E}\big[AVaR_{\beta,c}([S_1 - K_1]^-, \dots, [S_t - K_t]^-, F)\big]$$

Remark 1. Assume that the decision maker is a clairvoyant for each period F_t. Then $x_{t-1} = Y_t$ is the optimal solution, $M_t = 0$ and $k_t = 0$ for all t.

Numerical results for investment in European call options on the Apple Inc. stocks are demonstrated. We considered spot price $S_0 = 277.0$ for March 14, 2020. The strike price for call options with maturity $T = 1/12$ year is set at $K = 255$; 260; 265; 270, the yearly volatility for returns of the underlying asset is computed at $\sigma =$ 33.7 percents, the yearly riskless interest rate is set at $i =$ 5.8%. To illustrate the approach we propose, we consider now the case where the yearly interest rates for borrowing is $R =$ 5.4 and for lending is $r =$ 1.2%. Then $\alpha =$ 0.82 due (6) for one month. For finding optimal investment we need to build empirically or theoretically distribution function for payoff and then calculate a quantile for this distribution of level α.

On the picture 1 you can see histogram of $\vartheta_i = \frac{n_i}{n \times h}$, (where n is the total amount of data in the time series; h is the width of one column of the histogram), histogram of ϑ_i for $S - K$ and histogram ϑ_i for $[S - K]^+$. On the picture 2 cumulative distribution function for $[S - K]^+$ was built.

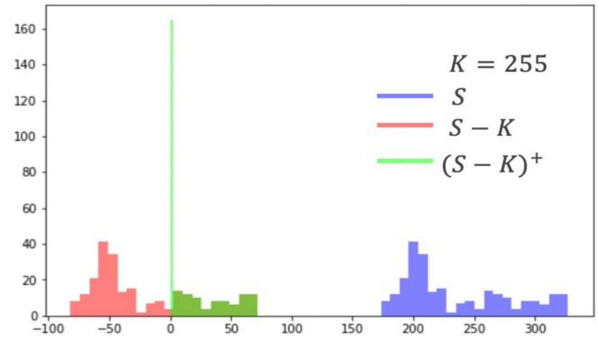

Figure 1 – Histogram for S, $S - K$, $[S - K]^+$

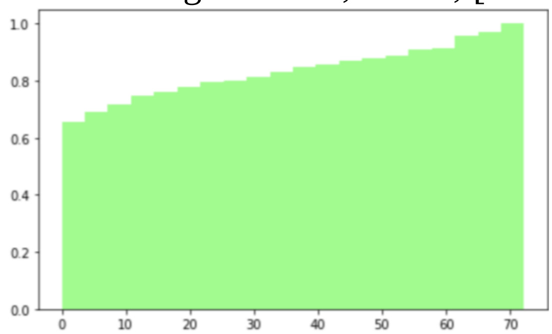

Figure 2 – Cumulative distribution function for $[S - K]^+$

Now we can calculate a quantile for this distribution of level $\alpha =$ 0.82 and the optimal solution according () $x^* =$ 32.49. For comparing optimal solutions for different strike prices, we constructed Table 1.

Table 1 – The optimal solutions for different strike prices.

K	\$255.00	\$260.00	\$265.00	\$270.00	\$275.00
x^*	\$32.49	\$28.56	\$23.34	\$18.59	\$13.05

GBM model. Sometimes it is possible to assume a known distribution function for S. For example, for the Black-Scholes model it is known that returns involve as GBM (Geometrical Brownian Motion) and stock prices S have a lognormal distribution (see, for example [3], [4]). The lognormal density function is as follows:

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(ln ln\, x - \mu)^2 / 2\sigma^2},$$

where μ – expectation; σ – standard deviation.

For $\mu = 5.35$, $\sigma = 33.7\%$ we construct an integral function and calculate the quantile for $\alpha = 0.82$. Its value is $x^* = VaR_\alpha([S-K]^+) = 32.0$.

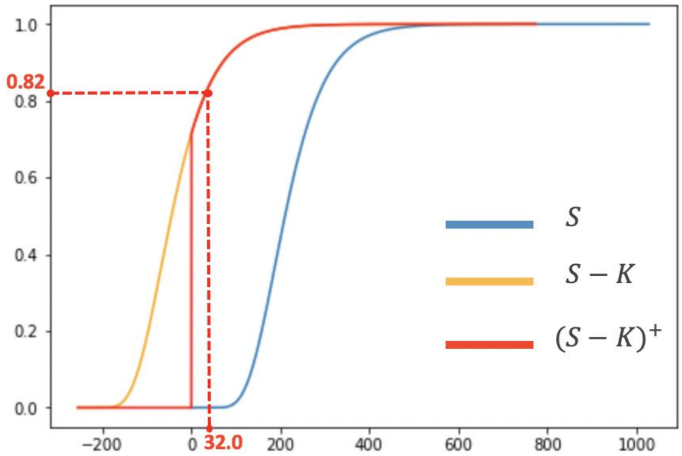

Figure 3 – Integration distribution function and quantile for $[S-K]^+$ for GBM. Quantiles for other values of the strike price K are given in Table 2.

Table 2 – The optimal solutions for different strike prices for GBM

K	\$255.00	\$260.00	\$265.00	\$270.00	\$275.00
x^*	\$32.02	\$27.02	\$22.02	\$17.02	\$12.02

FAT model. Consider a model for stock prices with fractal active time $T_t, t \geq 0$:

$$P_t = P_0 e^{\mu t + \theta T_t + \sigma W_{T_t}},\ t \geq 0,$$

where the parameters $\mu \in R$ and $\sigma > 0$ reflect the shift and volatility and $\theta \in R$.

Then if the time increments τ_t are $R\Gamma\left(\frac{v}{2}, \frac{\delta^2}{2}\right)$, where $\delta > 0$, $v > 0$, then the log returns have a Student's distribution [5], [6]. Thus, prices have a Student's log distribution with density

$$f_{logSt}(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{x\sigma\delta\sqrt{\pi}\Gamma\left(\frac{v}{2}\right)} \cdot \frac{1}{\left[1+\left(\frac{lnx-\mu}{\delta}\right)^2\right]^{\frac{v+1}{2}}}, x \in R,$$

where μ is the local parameter; σ is the standard deviation; σ is the scaling parameter ($\delta > 0$); v is the number of degrees of freedom ($v > 0$); Γ is a gamma function [7].

From the previous statistics we obtain the following values of parameters: $\mu = 5.35$, $\sigma = 0.34$, $v = 5$, $\delta = 0.8$. The graph of the probability density function is shown in Figure 4, the integral function in Figure 5. The value of the quantile for is $x^* = VaR_\alpha$ $([S-K]^+) = 37.8$.

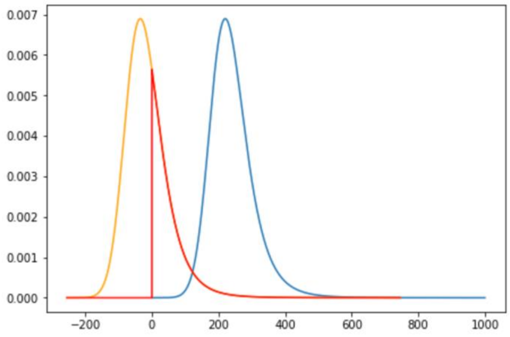

Figure 4 – Density distribution function $[S - K]^+$ for FAT model

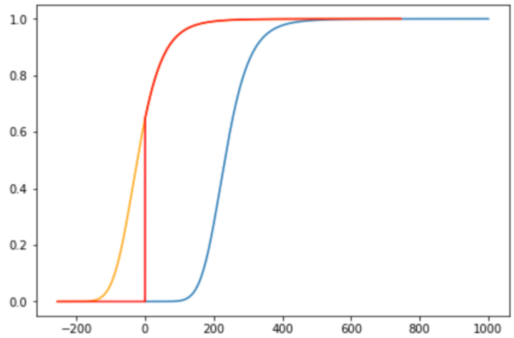

Figure 5 – Integral function distribution $[S - K]^+$ for FAT model

Quantiles for other values of the strike price K are given in Table 3.

Table 3 – The optimal solutions for different strike prices for FAT model.

K	\$255.00	\$260.00	\$265.00	\$270.00	\$275.00
x^*	\$37.77	\$32.77	\$27.77	\$22.77	\$17.77

The following fig. 6 shows graphs of the optimal investment in options for different strike prices under the 1-period model. Similar results were obtained for the n-period model.

Conclusions. In the course of this work, the application of the investor problem to securities trading in single-period and multi-period models were analyzed. Numerical results for optimal investment in European call options on the Apple Inc. stocks are demonstrated.

The results of the paper were used to develop recommendations for making decisions about optimal investments in the financial market of securities.

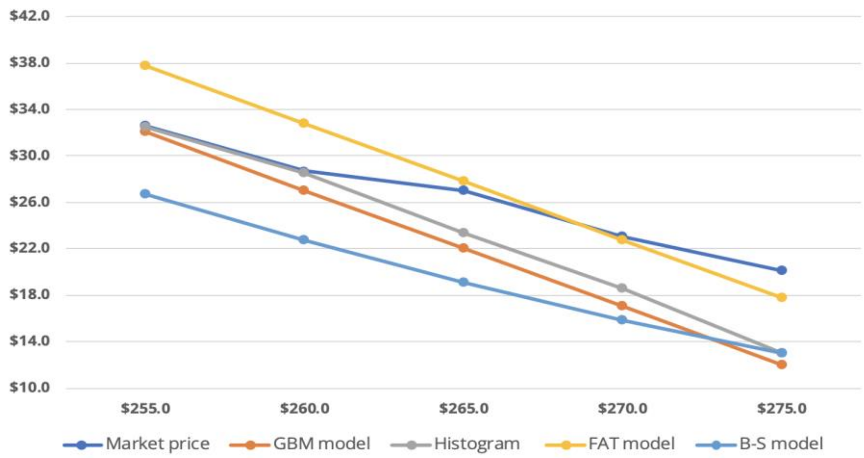

Figure 6 – Comparing optimal investments for different models of financial markets

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