Sección Especial en Óptica Cuántica / Special Section on Quantum Optics

Theoretical pathways towards experimental quantum simulators

Vías teóricas hacia simuladores cuánticos experimentales

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ABSTRACT:

Recent progress in quantum optics and quantum information has brought the long-standing dreams of quantum simulators and even quantum computers (almost) within reach. Here, we review some current theoretical pathways supporting experimental progress towards quantum simulators. In this, we focus mainly on topological aspects and numerical studies of quantum computation.

Keywords: Quantum Simulators, Quantum Computers, Coherent Control, Bose-Hubbard Model, Topology, Quantum Hall Effect, Artificial Gauge Fields, Ultra-Cold Atoms.

RESUMEN:

Los recientes avances en óptica cuántica e información cuántica han permitido alcanzar el sueño de muchos años de los simuladores cuánticos e incluso los ordenadores cuánticos están (casi) al alcance. Aquí, revisamos algunos vías teóricas actuales que sustentan el progreso experimental hacia simuladores cuánticos. Para ello, nos centramos principalmente en los aspectos topológicos y los estudios numéricos de la computación cuántica.

Palabras clave: Simuladores Cuánticos, Computadoras Cuánticas, Control Coherente, Modelo de Bose-Hubbard, Topología, Efecto Hall Cuántico, Campos Artificiales de Calibre, Átomos Ultra-Fríos.

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1. Introduction

The simple laws of nature often give rise to beautifully complex situations that are enormously difficult to grasp, let alone predict. Quantum systems, in particular, are notoriously difficult to study because their complexity grows exponentially with the number of particles they contain. Thus the largest simple (spin one-half) quantum systems that can currently be simulated exactly contain approximately 40 spins (depending on symmetries), and at the current rate of computational progress, it would only get to about 100 by the year 2100. A radically different approach is therefore needed to tackle such complex systems –current methods are simply not powerful enough.

Nature, of course, "solves" the equations governing its evolution seemingly effortlessly. Many scientists have therefore embraced the idea of building a device, in which model processes could be used to simulate highly complex problems. After all, if we were able to fully control and manipulate a sufficient number of quantum particles, we could make them perform the evolution that corresponds to another (possibly interesting and difficult to solve) quantum system. The idea of building quantum simulators, or even general purpose quantum computers was famously proposed by Richard P. Feynman in 1982 [1]. However, neither theoretical understanding nor experimental control were sufficient to consider building such a device at that time. Tremendous scientific progress of the past ten to twenty years however, has approached different disciplines working on quantum systems –condensed matter physics, quantum information theory, atomic, molecular and optical physics– is now bringing us closer to this long-held vision.

Nevertheless, the experimental progress has been so sudden that there are still many basic questions about quantum simulators in general –and specific models in particular – that need to be explored. After participating in the worldwide efforts to advancing quantum optics in general and quantum information and ultracold atomic gases in particular for over a decade, we therefore present some current topics towards quantum simulators, on which we are working.

First, we review our recent studies concerning quantum communication networks. Second, we will discuss control possibilities of quantum states and studies of the Bose-Hubbard model. Then, we discuss numerical and mathematical studies of such systems and finally show slightly more exotic yet highly active current trends of the field before concluding this article.

2. Quantum communication networks

Because not only communication networks, but the theory of information itself, is grounded in physical theories, and quantum theory is the most fundamental physical description of the world, it has become apparent that in the future communication networks will be based on quantum information theory. We have obtained promising results on the distribution of entanglement (the principle resource in quantum information) across large networks, a fundamental subject about which very little is known. The problems in the design of quantum networks stem from the various imperfections in prepared quantum states. One fruitful avenue is to exploit classical bond percolation theory which is the most basic description of classical regular networks with imperfect connections. In previous work, we showed how a typical quantum network prepared with pure but imperfectly entangled states can be mapped directly to a well understood classical percolation model. In particular, we demonstrated several methods by which a quantum network corresponding to a certain bond percolation model can be pre-processed with quantum operations, transforming it into a different network that corresponds to a different percolation model with dramatically improved connectivity properties [2-4]. Recently, we extended this idea with preprocessing steps that produce multi-partite states [5]. These multi-partite methods have a much wider range of applicability, and also introduce mappings to different classes of percolation models, such as the site percolation models. Another direction we have explored is entanglement concentration on complex networks, beginning with the Erdös-Reyni model [6]. Here we found that a different

sequence of quantum operations produces rare subgraphs at initial entanglement densities that are much lower than those densities required after a direct mapping to classical graph theory. We are currently extending these studies in two directions. On hand we are considering more realistic complex networks, for instance models exhibiting scale-free and smallworld properties. On the other hand it is important to learn how to treat fairly general classes of mixed states because approximating states on real networks by pure states will not yield even qualitatively correct results in the foreseeable future. We have found that combining these two elements has produced new optimization questions that we have successfully analyzed. These results will be published soon.

3. Coherent control of many-body physics by strong time-periodic forcing

When asking how to control the quantum state or the dynamics of a small quantum system like an atom or a single spin, a common answer will be to use coherent radiation, that is time-periodic forcing. A powerful and intuitive theoretical framework in this context is given by the dressed atom picture [7]. In contrast, when talking about the quantum physics of strongly correlated many-body systems, time-periodic forcing is usually considered merely as a probe within the regime of linear response.[1] However, within the last years it has been shown that strong time-periodic forcing is also a powerful and versatile tool for the coherent control of the quantum state of a many-body system. A striking example is a recent experiment in the group of Ennio Arimondo in Pisa, where a quantum phase transition has been induced solely by dressing a many-body lattice systems of ultra-cold atoms by a fast lattice oscillation [8,9]. It is not by chance that these new developments emerge in

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[1] Exceptions like superconducting devices or in the future also nanomechanical resonators are in fact constructed such that only a few degrees of freedom, those of the order parameter or a single oscillatory mode, are relevant within an effective description. Thus the many-body character of the system is not relevant on the level where coherent control schemes are considered.

the field of ultracold atoms where quantum optics and condensed matter physics fuse in a fruitful manner. Here we will give a brief overview of the subject, clearly with a focus on contributions of the members of the quantum optics theory group at ICFO.

A single particle that can hop between neighboring sites m of a one-dimensional tight-binding lattice resembles a spin degree of freedom with quantum numbers m, subjected to a static magnetic field transverse to the chosen quantization corresponding to the tunnel coupling. From the spin model it is well known that a second, rapidly sinusoidally oscillating magnetic field oriented along the quantization axis will "dress" the system such that the strengths of the static field appears to be modified by factor depending on the amplitude of the time-periodic field (effective modification of atomic g factors [10]). Accordingly, in the tight-binding model a rapidly oscillating force (akin to an "ac-voltage" for a charged particle) will cause a modification of the tunnel coupling between neighbouring sites. Such a modification of tunnelling in a driven lattice has been predicted by Dunlap and Kenkre [11] who pointed out that a complete suppression of tunnelling is possible such that the spreading of wave packets is hindered (dynamic localization). Independently, the phenomenon has been described by Haenggi and co-workers (coherent destruction of tunneling) [12] as well as by Holthaus [13].

When considering a system of ultra-cold atoms in an optical lattice a tight-binding description is usually well justified. However, in competition to tunnelling, the particles will also interact with each other, giving rise to intriguing many-body physics [14]. Thus, one can ask: (i) Is it still possible to dress the lattice by time-periodic forcing such that tunnelling processes are modified in a coherent way? And, if yes: (ii) Can one use this kind of dressing as a tool to reliably manipulate the many-body state of a lattice system in a non-trivial way? (iii) Does the dressing procedure even allow the realization of many-body quantum states or dynamics that is qualitatively different from the physics encountered already the undriven system? And, finally: (iv) Can the scenario generalized to other forms of time-periodic forcing giving rise to a greater variety of dressed lattice models? All these questions can be answered affirmatively.

By employing quantum Floquet theory on the manybody level, it has been worked out in [9] that the tunnel modification stays intact, even in the presence of strong on-site interaction between the particles, while the interaction among the particles is not altered by the dressing. The only condition to be fulfilled is that the frequency of the forcing is large compared to the energy scales given by both interaction and tunneling such that resonant excitation is suppressed on a time-scale being large compared to the experiment. By separation of time-scales, a fast motion induced by the forcing can be integrated out. (A systematic description of processes beyond the simple modification of tunnelling as they become relevant when the frequency is lowered and their application for spectroscopy are described in [15].). In a cold atom system the required kind of forcing can be achieved inertially, simply by shaking the lattice, i.e. by moving it back and forth. Recently, in the Arimondo group in Pisa the expansion of a BEC in a shaken lattices has been observed to be modified as predicted, without significant heating [16,17]. This answers question (i).

A further prediction results directly from the fact that the tunneling can be suppressed strongly by shaking: It should be possible to bring a bosonic system adiabatically from its superfluid ground state having delocalized particles (found in the tunneling-dominated regime) through a quantum phase transition to a Mott insulator state with the particles localized (being the ground state when tunneling is weak) and back [9]. This effect has also been observed experimentally in Pisa [8], which answers question (ii).

Apart from quenching the tunneling matrix element, one can even change its sign by dressing the lattice by shaking. This will have intriguing consequences in a non-bipartite lattice such as the triangular lattice [18]. Here, a sign-change in the hopping induces frustration to the kinetics of the particles. As a consequence, in the weakly interacting regime the dressed system features a superfluid "ground state" with staggered currents around the triangular lattice

plaquettes, breaking time-reversal symmetry. In the hard-core boson limit of strong interaction, the dressed system resembles a frustrated quantum antiferromagnet that is conjectured to host an exotic spin-liquid ground state. Referring to (iii), by lattice shaking novel physics is expected to emerge that is hardly found in standard bosonic lattice systems.

In order to address question (iv), we wish to point out that there are also other interesting dressing scenarios based on oscillatory potential scenarios. E.g. Hemmerich and Moirais Smith have proposed a scheme allowing to create an artificial staggered magnetic field pattern in a cubic lattice [19,20]. Another possibility is to use the forcing in order to coherently couple energetically distant single-particle levels. Such a technique can be used to re-establish coherent tunneling in a lattice subjected to a strong static potential tilt suppressing tunneling and leading to a localization of single particle states (Wannier-Stark effect) [21]. This effect has been observed experimentally in Pisa by observing the expansion of a BEC in a tilted lattice [22].

4. Bose-Hubbard model with occupation dependent parameters

Systems of ultra-cold atoms provide a unique setup for the controlled realization of many-body physics with possible applications ranging from quantum metrology to realizations of quantum computers and quantum simulators [23]. A very important and ubiquitous tool for the manipulation of cold atom systems are practically defect-free spatially periodic potentials as they can be created by using standing laser waves. These periodic potentials, also known as optical lattices, can be used to trap atoms at each minima of the periodic potential or sites. Due to the unprecedented experimental control over the motion of atoms in the optical lattices, these systems are playground for realization of quantum simulators. So a crucial aspect for achieving this goal is to theoretically model these systems and investigate their properties. Such a model which quantitatively describe ultra-cold bosonic atoms in optical lattices is called Bose-Hubbard model. This model is characterized by two parameters:

1) the tunneling rate of an atom between two adjacent sites, denoted by J 2) interaction strength at each site, denoted by U. The atoms are usually interacting via repulsive force so U is positive When the lattice is deep, i.e $U/J \to \infty$ each site has fixed number of atoms, a state known as Mott-insulator. In the opposite limit, each site can have different number of atoms and the corresponding state is called a superfluid state. The transition from Mott-insulator to superfluid state is characterized by delocalization of single particle and hole over the entire optical lattice. One aspect of this model is independence of J and U on the number of atoms present in each site. This assumption breaks down when the interaction between the atoms increases. At present we are studying novel physics arises from the effect of strong interactions. We find that, with increasing interaction strength a new state arises where superfluidity is due to motion of two particles always moving together as shown in Fig. 1, like a two-person forward roll in a circus. As a result the two particles behave as a single object [24]. Subsequently they live in a exotic lattice, which is different than the original one. We also find that with increasing interaction, the building constituents of the superfluid phase changes to a object containing two particles which now lives in the same site.

Fig. 1: The leftmost picture displays the pair of particles. They first tunnel to a state where one particle is at the top of another as shown in the middle picture. Then one of them tunnel again like a two-person forward roll in a circus as shown in the rightmost picture.

5. Mathematical foundations and fundamental problems of quantum mechanics

On top of projects directly related to experimental studies, there are topics that are more mathematical in nature, and often related to our most fundamental understanding of quantum mechanics itself. Typical examples falling into this category are entanglement and separability of quantum states. Entanglement [25] is receiving a lot of attention due to its

various applications outperforming classical physics with the most prominent examples being quantum teleportation [26] or entanglement-based cryptography [27]. The problem of judging if a given composite quantum state is entangled, the so-called separability problem [28] arises naturally and turns out to be very difficult to solve. A well-known way to tackle the separability problem is, for example, based on the notion of one and many-copy entanglement witnesses. The latter, introduced in a series of papers [29-31] and thoroughly investigated in the literature (cf. [25,28]), are quantum observables whose mean values on one or many copies (in a collective sense) of a given quantum state can judge if the state is entangled, but also they allow to estimate the amount of entanglement the state has. Such criteria are then not only useful as a theoretical entanglement detectors, but also are promising from the experimental point of view. Using the theory of majorization it was recently possible to develop methods allowing for construction of such many-copy entanglement witnesses, which, in some instances, give necessary and sufficient separability criterion provided that sufficiently many copies of a state are at our disposal [32]. Interestingly, first experiments involving two-copy entanglement witnesses were already performed [33-35]. A different ways of dealing with the separability problem could be to study the geometry of the sets of density matrices or to characterize the entanglement witnesses with respect to their optimality in entanglement detection (cf. [25,28]).

Another line of research lying at a heart of quantum information concerns distillation of secure cryptographic key from quantum states. On the one hand, interesting questions are those concerning relations between the latter and entanglement, in particular bound entanglement. This phenomenon, discovered in 1998 by the Horodecki family [36], is the existence of entangled quantum states (called the "black holes" of quantum information [37]) which need entanglement to be created, but from which no entanglement can be extracted within the distant laboratories paradigm. For instance, only recently, the longstanding open problem about possibility of secure key distillation from bound entanglement was solved

[38,39]. They provided a paradigm of secure key distillation within which even bound entangled states can be used to establish secret cryptographic key. Basing on these seminal results, analogous problem was later analyzed in the scenario of many communicating users and a similar conclusion was drawn, namely, there exist multipartite bound entangled states allowing to establish secure key in such scenario [40,41]. Still, however, a question from which entangled states we can or cannot draw cryptographic key has not been answered. It is already known that any state from which one may distill bits of secure key has to be entangled [42,43], but whether any entangled state may be used to obtain secure bits remains an open problem. On the other hand, by exploring the aforementioned paradigm of secure key distillation, it was recently possible to prove that quantum states which represent perfectly secure bits of cryptographic key (the so–called private states) violate Bell inequalities [44], i.e., they allow to obtain correlations that cannot be reproduced by classical means. This allowed for establishing new link (beyond the already known as e.g. [45]) between privacy and another resource known in quantum physics –nonlocality.

Yet another highly interesting line of research focuses on the relationship between two peculiar properties of quantum mechanics: contextuality and nonlocality. These properties are expressed by the celebrated Kochen-Specker and Bell Theorems respectively. Both Theorems are of a fundamental importance for our understanding of quantum mechanics, as they touch upon the questions: Is quantum mechanics fundamental and definite or is there a deeper level? Is quantum mechanics non-local and if so in which sense? Apart from that, there are also fascinating practical applications of the mentioned theorems to quantum cryptography, e.g. Bell inequalities are used for secure data transmission. Let us recall that contextuality of quantum mechanics, the subject of the Kochen-Specker theorem, means that one cannot reproduce predictions of quantum mechanics by assuming that observables have some definite, preascribed values that would not depend on the other observables, i.e. would be non-contextual, context independent. On the other hand, Bell

Theorem states that one cannot reproduce certain quantum correlations using classical statistics, obeying the principle of locality. Connecting the two Theorems is thus a step towards building a more coherent understanding of quantum theory, with some possible importance for practical use. The interesting question here is in what sense non-locality can be understood as a form of contextuality with spatially separated contexts. More precisely, for which classes of Bell inequalities one can construct a Kochen-Specker type of paradox, and vice versa: how to construct a Bell inequality for a given Kochen-Specker paradox and what properties it possesses. So far the most promising have been Bell inequalities with a so called algebraic violation. These are the inequalities which single out the "most non-local" probability distributions in a sense that a probability distribution maximally violating such an inequality can have no local part.

On an even more mathematical level there are studies that connect quantum mechanics and harmonic analysis. The latter is an important branch of mathematics, which deals with groups and their representations. The developed formalism establishes a new link between physics (quantum mechanics, theory of entanglement) and mathematics (harmonic analysis). The core of the formalism is a non-commutative analogue of Fourier Transform, which changes the mathematical representation of quantum states and operations. Quantum mechanical states, for example, become functions on some non-commutative group, having a specific positivity properties—so called positive definiteness. The resulting mathematical framework is slightly general than the standard quantum mechanical one—it is a kind of a "generating function formalism" for the latter. This is the most interesting of its features, discovered so far. Interestingly, such properties as e.g. entanglement are elegantly represented within the formalism and we were able to prove a harmonic-analytical version of Peres-Horodecki Theorem. This is a fundamental theorem in the theory of entanglement, providing necessary and sufficient conditions for entanglement in terms of certain quantum operations (positive maps). The research currently done within this harmonic analytical

framework consists of: i) searching for new entanglement criteria, ii) studying quantum and general non-signaling correlations, and, at a more fundamental level, iii) an attempt to reformulate quantum mechanics in terms of the developed framework. That is, starting from a group, interpreted as a group of changes of a reference frame, and representations of states through positive definite functions on the group. This approach could be called a "transformational" interpretation of quantum mechanics, as opposed to e.g. C*-algebraic one, which is "observational".

6. Numerical simulation of quantum systems

Numerical studies of quantum optical systems have turned out to be a powerful tool for progress. Identifying suitable models for target quantum simulators, and developing experimental proposals to carry out the simulations are therefore important aspects of numerical studies. Ideally, a system needs to be rich and hard enough so that it is worth devoting all the effort that it requires a quantum simulation. An example of such systems are those with long-ranged interactions [46], which lead to novel states of matter and are quite hard to study with traditional methods [48-51]. A core part of our work is to test and characterize models suitable for simulations [47] using traditional numerical approaches. For this, we employ a battery of state-of-the-art techniques, which includes the novel class of algorithms based on tensor networks. Generally speaking, these algorithms approximate a given system with a state of a predetermined maximal amount of entanglement. In this way, it is, for example, possible to find novel ordering effects in disordered quantum systems [52].

Another crucial property for in systems is that they must be simple enough to be within experimental reach with the technology available today, or in the near future. At the stage of designing a quantum simulation, it is critical to have frequent contact with experimentalists to receive feedback and improve the proposals. Recently, for example, we proposed and then implemented (with

colleagues in the Institute for Quantum Computing in Canada) a quantum simulation that demonstrated the effect that an environment has on the geometric phase of a system – the first time that this "open-system" phase has been ever measured [53]. Furthermore, our previous experience with these specific quantum simulations (done in nuclear magnetic resonance experiments) allowed us to tune our environment near an effective quantum critical point. Now, we are building on this experience and designing further experiments that will allow us to explore how to use geometric phases for quantum computing.

7. Exotic magnetic phases for spin −3/2 and 5/2 ultra-cold fermions

Ultra-cold atoms loaded into optical lattices can simulate different quantum many body models with well controllable parameters. Above all they can mimic the celebrated and fundamental models of several areas of physics: the Bose- as well as the fermion Hubbard model, and the Heisenberg model. [14,54,55] The experimental realization of high spin-F Bose-Einstein condensates [56] and Fermi gases [57-59] highly stimulates the studying of the high spin versions of these models. A series of novel phases of these model were predicted in earlier works like valence bond solids, two orbital SU(N) magnetism, chiral spin liquid.

The spin −3/2 fermion system as the simplest case beyond F=1/2 was intensively studied: such systems exhibit a generic SO(5) or isomorphically, Sp(4) symmetry and based on this symmetry property a general analysis was made by Wu [60]. Here, among others the half-filled system was analyzed in detail within a mean-field approximation.

In the strong coupling limit at quarter filling the low energy properties of the system can be described by high spin magnetic exchange. This effective model was investigated [61-63] for special values of the couplings along the high symmetric SU(4) lines: the ground state of the system in these two part of the parameter space is Neel order and plaquette order, respectively.

However, far away from these high symmetric lines the effective spin model contains multispin interactions, too, which makes difficult to handle it even on mean-field level. Starting from a usual Hubbard model in order to remain within the two particle representation, we made a convenient rearrangement of the interaction terms based on the decomposition of the total (two-particle) spin space into its symmetric and antisymmetric part with respect to the exchange of two spin indices of the scattering particles. [64] This treatment allows us to collect and treat adequately the usual two-particle interaction terms to describe the different processes in the spin channel within the concept of site- and bond spin. The effectiveness of the treatment does not depend on the occurrent fermionic or bosonic statistic of the interacting particles.

Applying this method to F=3/2 fermions at quarter filling on a two-dimensional square lattice we determined the complete ground state phase diagram of the system on mean-field level (see Fig. 2). In absence of external magnetic field the ground state is either antiferromagnetically ordered (Neel phase) or bond centered density waves appear forming disconnected boxes. The U(1) flux passing through the boxes either 0 or π, and these two gauge-non-equivalent states have the same energy. However, by applying external magnetic field, bond centered spin density waves also appear in addition to the particle density waves, and similarly to them form disconnected boxes. The two types of density waves coexist with weak ferromagnetic order. This SU(2) flux phase is stable in an extended region of the parameter space in the experimentally reachable regime.

Similar analysis can be easily made for F=5/2 fermions for the special values of the coupling constants $G_4 \sim (-7G_0+10G_2)/3$. G_4 is the coupling of the interaction with 9-fold spin multiplicity that appears in the Hamiltonian for spin -5/2 system in addition to the singlet (G_0) and quintet (G_2) scatterings. The phase diagram (see Fig. 3) is clearly richer than that of the F=3/2 system, even though that our preliminary analysis for the spin −5/2 fermions is restricted to a single value of the quintet coupling. We found three stable phases: the Neel order, and two different types of quasi-plaquette orders. In phase

plaquette I the particle density wave order parameter is nonzero on every links, but the stronger and weaker links alternate and form plaquette configuration. In phase plaquette II ferromagnetic order also appears and leads to a similar SU(2) phase as in case of the $F=3/2$ with applied magnetic field. The only difference that this is also a quasi-plaquette order since the particle just as the spin density order parameters are nonzero on every links. Note that in both plaquette phase the flux passing through the plaquettes is zero.
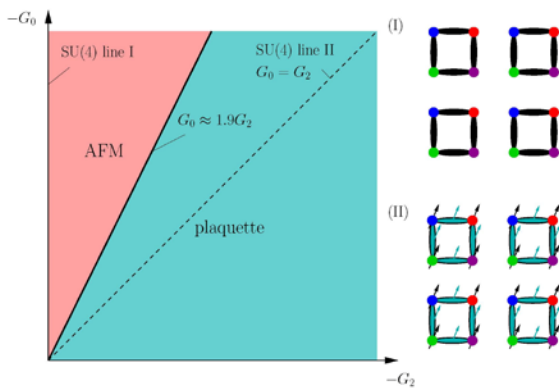
Fig. 2. The ground state phase diagram of the $F=3/2$ fermion system in the strong coupling limit on 2D square lattice at quarter filling. G_0 and G_2 are the couplings in the singlet and quintet channel, respectively. The phase boundary is around $G_0 \sim 1.9 G_2$. The lattice was splitted into 4 sublattices denoted by different colours on the configuration subfigures: (I) the U(1) flux state, (II) the SU(2) flux state.
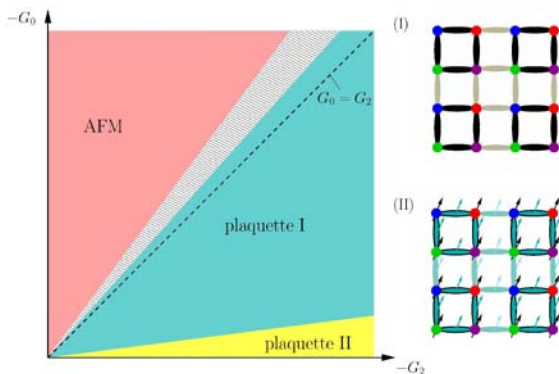

Fig. 3. The phase diagram of the spin -5/2 fermion system with strong on-site repulsion on 2D square lattice at 1/6 filling based on preliminary calculations. Also in this case, the lattice was splitted into 4 sublattices. On subfigure (I) the structure of the plaquette I phase is shown, on subfigure (II) the structure of the plaquette II phase.

We see that by the convenient rearranging of the two particle interaction terms based on their symmetry properties, the low energy effective Hamiltonian of high spin systems turns easier to handle. Applying this method in case of $F=3/2$, and 5/2 fermion system appearance of novel, exotic phases was predicted, like different SU(2) plaquette orders.

8. Layered quantum Hall insulators with ultra-cold atoms

Probably the simplest paradigmatic example of topological transport is the integer quantum Hall (IQH) effect [65], which takes place at low enough temperatures. By varying the magnetic field perpendicular to a 2-dimensional semiconductor layer, which also has an electric potential difference between two opposite edges, one can notice a transverse current. The transverse conductivity can take only quantized values and is proportional to the winding number of the Berry's curvature of the many-body wave function [66].

We have considered a generalization of the IQH effect by changing the gauge group from the Abelian U(1) group of standard electrodynamics to the non-Abelian and non-compact Heisenber-Weyl group (HWG), which is generated by 3 elements, say $\hat{z}$, $\hat{p}$ and the identity operator, with the canonical commutation relation $[\hat{z}, \hat{p}] = i\hat{1}$. As a consequence of non-compactness this commutator cannot be represented in finite dimensions. We have shown that in spite of the above difficulty this gauge group can be realized relatively simply with today's technology of ultra-cold fermions and optical lattices by identifying the role of the HWG in layered 3-dimensional lattice systems.

The 2-dimensional, tight binding lattice Hamiltonian of non-interacting fermions in an external gauge potential takes the form

$$H_{2d} = T_x e^{i\theta_x} + T_x^\dagger e^{-i\theta_x} + T_y e^{i\theta_y} + T_y^\dagger e^{-i\theta_y}. \quad (1)$$

where T_x (T_y) is the translation operator in the x (y) direction acting on the wave function as $T_x \psi_{m,n} = \psi_{m-1,n}$ ($T_x \psi_{m,n} = \psi_{m,n-1}$). The energy is measured in units of the tunneling strength. The external gauge potential modifies the nearest

neighbor tunneling by unitary operators, which are elements of the HWG and depend also on the position m and n. They can be expressed with the help the "phase factors" θ_x and θ_y, which are elements of the Lie algebra of the HWG and are Hermitian matrices. We have chosen the external gauge potential in a similar manner than the Landau gauge used for describing the IQH effect.

$$\theta_x = \alpha \hat{p} , \qquad (2a)$$

$$\theta_y = 2\pi\beta m + 2\pi\gamma \hat{z} . \qquad (2b)$$

The gauge coordinate $\hat{z}$ is mapped to the real external z direction and we have assumed, that the z direction also supports a sufficiently deep optical lattice with lattice spacing b, therefore $z = b\, l$, with $l \in Z$. Note, that the parameter can take only specific values, since $e^{i\alpha z} f(z) = f(z+\alpha)$ for any function $f(z)$. On a lattice $z = b\, l$, therefore has to be integer times the lattice spacing. If we consider only the simplest choice, $\alpha = 1$, then $e^{i\theta_x} = T_z^{\dagger}$, which is the translation operator in the z direction. Hence the 2-dimensional problem with infinite internal states is mapped to a true 3-dimensional problem, where the indices of the wave function $\psi_{m,n,l}$ refer to lattice positions in the real 3-dimensional space.

The tunneling processes in the xz plane are special; if the particle tunnels in the x direction by one to the left (right), it has to tunnel one position in the z direction up (down) too. Therefore, it is convenient to introduce new coordinates: $\xi = (m-l)/2$, and $\eta = m+l$, and a new tunneling operator $T_{\xi} = T_x T_z^{\dagger}$ replacing the first two terms in the Hamiltonian to give

$$H_{\mathrm{H}} = T_{\xi} + T_{\xi}^{\dagger} + T_y e^{i\theta_y} + T_y^{\dagger} e^{-i\theta_y} . \qquad (3)$$

The phase factor acquired by tunneling along the y-direction reads in the new coordinates as

$$\theta_y = 2\pi(\beta - \gamma)\xi + \pi(\beta + \gamma)\eta . \qquad (4)$$

As there is no tunneling along the η-direction, Eq. (3) describe independent 2-dimensional IQH systems layered on top of each other in an external Abelian gauge potential given by Eq. (4). Each layer behaves similarly, except for η even ξ takes integer values and for η odd ξ takes half integer values, as illustrated in (Fig. 4).
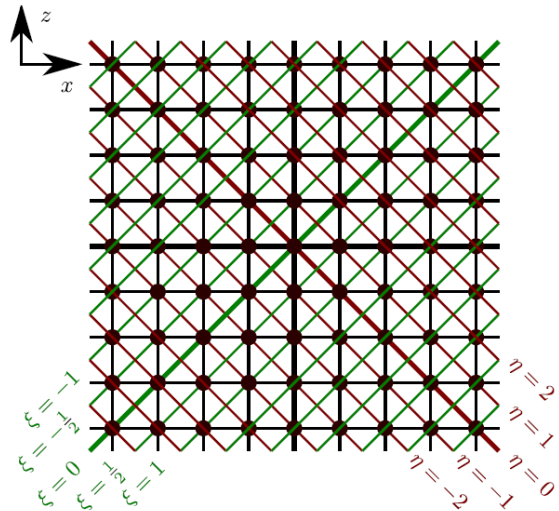

Fig. 4. The new coordinates ξ and η. Due to the special structure of the tunneling η is a conserved quantity.

The IQH physics takes places for every plane simultaneously. Depending on the Fermi energy each independent plane can be in a metallic phase with nonzero longitudinal conductivity, or each can be in an insulating phase characterized by zero longitudinal conductivity but a nonzero transverse current. Just as for the IQH effect the transverse conductivity depends on the number of filled bands, which are the lattice counterparts of the Landau levels. The transverse conductivities of the separate planes add together, and since the behavior of each plane is exactly the same, the total transverse conductivity is proportional to the number of planes.

The transverse conductivity of each plane has been calculated with the Thouless-Kohomoto-Nightingaleden Nijs formula [63]:

$$\sigma_{\eta} = \frac{1}{2\pi}\int d^2k \, F_{\eta} , \qquad (5)$$

with $F_{\eta} = \nabla_k \times A_{\eta}$ the Berry's curvature calculated for plane η with the help of Berry's connection

$$A_{\eta} = i\sum_{r=1}^{N} \left\langle u_{k_{\xi},k_y}^{(r)} \middle| \nabla_k \middle| u_{k_{\xi},k_y}^{(r)} \right\rangle , \qquad (6)$$

in momentum space. The sum runs over the occupied one particle states $\left| u_{k_{\xi},k_y}^{(r)} \right\rangle$ where k_{ξ} and k_y are the quasimomenta in Bloch representation, and r indexes the different bands. The transverse conductivity (5) takes

only integer values, since it is the first Chern class of the U(N) principal bundle over the Brillouin-zone torus (k_ξ, k_y).

By introducing another external potential with strength λ that alternates between the neighboring η layers one can locally modify the positions of the Landau bands and manipulate the net transverse conductivity accordingly. (Fig. 5) shows the phase diagram of the system in the β'≡(β−γ)=1/4 flux phase. We have identified phases when all of the planes are metallic, and phases where some or all of the planes are insulating but with a possibly nonzero transverse conductivity.

We have also studied the effects arising from the finite depth of the optical lattice resulting in nonzero interlayer tunneling. In this case η is not a good quantum number any longer and the system becomes 3-dimensional. As an effect of the inter-layer hopping the eigenstates of the different η planes get hybridized and the conducting regions get widened. However, while the interlayer tunneling is sufficiently small, the gaps persist and the net transverse conductivity keeps its significance. As the inter-plane tunneling amplitude is raised the IQH regions shrink and give way to the metallic phase. There is a competition between the staggered potential strength and the tunneling strength, therefore the high transverse conductivity IQH regions vanish first since they are located around λ=0.

In this work [67] we have shown that the problem of 2-dimensional lattice fermions in an external non-compact Heisenberg gauge group is equivalent to an experimentally realizable, layered 3-dimensional Abelian quantum Hall system. We have considered its advantages in engineering different IQH phases by varying the gauge potential and another external staggered potential. As an example we have calculated the phase diagram for the flux of 1/4. Layered fermion systems are encountered in many corners of condensed matter physics and our motivation was to find new insulating phases with nontrivial topologies to get a better understanding on the possible complicated phase diagrams found in such systems.
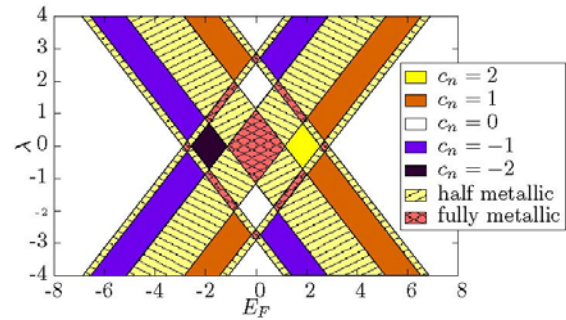

Fig. 5. The phase diagram of the system in the 1/4 flux phase in the presence of a staggered potential. The red hatched regions indicate metallic phases, while the solid filled regions represent insulating phases with c_n=0,±1,±2 transverse conductivities.

9. Conclusions

Quantum physics in general and quantum information in particular are tremendously fascinating fields of research. In this paper, we have reviewed some aspects of these vast ongoing research towards quantum simulation, quantum computation, as well as the foundations of quantum theory. Recent theoretical and experimental progress in these fields now allows for such interdisciplinary studies at the frontiers of quantum optics, atomic physics, condensed matter, information theory, and even high-energy physics. National and international scientific collaboration thus allows not only to advance research, but also to share the passion for the understanding of deep conceptual issues as well as highly complex experimental challenges.