Brief Reports

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On the Tolman-Hawking wormhole

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The Tolman-Hawking wormhole $a=(R^2+\tau^2)^{1/2}$ can be obtained as a solution of the Einstein equations with a conformally coupled scalar field. However, this procedure gives rise to negative values of the effective gravitational constant. Working in a pure-gravity minisuperspace isotropic model, I propose a much simpler method to obtain this instanton based on the introduction of a cutoff in the scale factor which is equivalent to introducing a three-sphere with minimum constant radius in the Euclidean metric. The resulting wormhole model no longer has any negative effective gravitational constant.

Euclidean gravity has now opened up a very promising way for getting rid of the cosmological constant.[1] This has been made possible by introducing gravitational instantons that lead to changes in topology.[2] Essentially there are two kinds of Euclidean four-dimensional wormholes which turn out to be solutions to Einstein equations for gravity coupled to suitable scalar fields. The Giddings-Strominger-Myers wormhole[3,4] is the solution obtained from gravity coupled to an axion field. In the Robertson-Walker minisuperspace conformal-time language, this instanton is given by the scale factor

$$a=K_0\cosh^{1/2}(2\eta)\ , \tag{1}$$

where K_0 is a constant representing the radius of the throat and we have assumed a vanishing cosmological constant. Such an instanton has been generalized to the case in which the gravitational action contains terms which are quadratic in the scalar curvature.[5]

The other type of gravitational Euclidean wormhole has a simpler structure. It can be obtained by solving the Euclidean Einstein equations with or without a cosmological term λ and a conformally coupled scalar field. For a Robertson-Walker metric, the case $\lambda=0$ leads to a scale factor

$$a=(R_0^2+\tau^2)^{1/2} \tag{2}$$

which we hereafter denote as a Tolman-Hawking wormhole.[2,6]

For $\lambda>0$ the solution has the nonsingular, periodic form

$$a=(2\lambda)^{-1/2}[1-(1-4\lambda R_0)^{1/2}\cos(2\lambda^{1/2}\tau)]^{1/2} \tag{3}$$

which was discovered by Halliwell and Laflamme.[7] These authors have shown[7] that the crucial difference between the Giddings-Strominger-Myers wormhole and the Tolman-Hawking wormhole is that the latter can induce a change of sign in the effective gravitational constant. This would ultimately lead to negative energies and instabilities for perturbations about solutions (2) and (3).

The aim of this paper is to present a new simple procedure for obtaining the four-dimensional Tolman-Hawking wormhole. This procedure does not use any conformally coupled scalar field and avoids the above-mentioned problem about a negative effective gravitational constant. For a Robertson-Walker isotropic metric, the Euclidean action integral of pure gravity in four dimensions with a positive cosmological constant λ is

$$\tilde{I}=-\frac{1}{16\pi G}\int d\tau\, Na\left[1+\frac{\dot{a}^2}{N^2}-a^2\lambda\right] \tag{4}$$

with $\dot{a}=da/d\tau$.

We simply introduce a cutoff in the scale factor a squared, $a^2\rightarrow a^2-m^2$, where m is an arbitrary constant. The original Euclidean manifold with metric

$$ds^2=d\tau^2+a^2d\Omega_3^2$$

is then provided with an additional three-sphere of constant radius m which we interpret as the minimum metric on the three-sphere compatible with the Euclidean manifold; i.e., we introduce a transformed Euclidean time

$$d\tau'=\left[1-\frac{m^2}{a^2}\right]^{1/2}d\tau\ .$$

The Euclidean action is now

$$\tilde{I}(a,m)=-\frac{1}{16\pi G}\int d\tau' Na\left[1+\frac{\dot{a}^2}{N^2}-a^2\lambda+m^2\lambda\right], \quad (5)$$

where $\dot{a}$ denotes now $da/d\tau'$.

The Euclidean region is defined then for $a>m$. Variation with respect to a and N yields the field equations and constraint

$$[1-3(a^2-m^2)\lambda-2\ddot{a}a](a^2-m^2)=\dot{a}^2(a^2-2m^2), \quad (6a)$$

$$a^2\dot{a}^2-(1+2m^2\lambda)a^2+a^4\lambda+m^2(1+m^2\lambda)=0, \quad (6b)$$

where we have taken the gauge $N=1$. Expressing Eqs. (6) in terms of conformal time $\eta=\int d\tau/a$, we have

$$\tfrac{1}{2}a'^2+U(a,m)=0, \quad (7a)$$

$$a''=-dU(a,m)/da, \quad (7b)$$

where $a'=da/d\eta$, and

$$U(a,m)=\tfrac{1}{2}[m^2(1+m^2\lambda)-(1+2m^2\lambda)a^2+\lambda a^4]. \quad (8)$$

Equations (7) represent the motion of a particle in a potential $U(a,m)$ with zero total energy. Consider first the case $\lambda=0$. In this case, the solution to (6) is

$$a=(m^2+\tau^2)^{1/2} \quad (9)$$

or

$$a=m\cosh\eta \quad (10)$$

in conformal time η. This is the Euclidean version of the Tolman universe, i.e., the Tolman-Hawking wormhole. It represents two asymptotically flat regions connected by a tube of minimum radius m. For $\lambda>0$, we obtain the solutions

$$a=\lambda^{-1/2}[m^2\lambda+\sin^2(\lambda\tau)]^{1/2}, \quad (11a)$$

$$a=\lambda^{-1/2}[m^2\lambda+\cos^2(\lambda\tau)]^{1/2}, \quad (11b)$$

which are given in terms of Robertson-Walker time τ and lie in the region $a_-<a<a_+$, with

$$a_\pm=\left[\frac{1+2m^2\lambda\pm 1}{2\lambda}\right]^{1/2}. \quad (12)$$

Solutions (9)–(11) are formally the same as those obtained by Halliwell and Laflamme[7] and admit therefore a similar interpretation. There is however a crucial difference with the instantons obtained by using a conformally coupled scalar field: Our solutions no longer lead to any problem about the sign of the effective gravitational constant as can be readily checked by inspecting action (5). Thus, it can be interpreted that the instanton used by Hawking in his spacetime theory of wormholes[2] is not just a solution of the Einstein equations, but a consistent one with a simpler structure than the instanton discovered by Giddings, Strominger, and Myers.

In the Lorentzian region we would have

$$dt'=\left[\frac{m^2}{a^2}-1\right]^{1/2}dt,$$

so that this region is defined here for $a<m$. The Lorentzian field equations take the form

$$-\tfrac{1}{2}a'^2+U(a,m)=0, \quad (13a)$$

$$a''=dU(a,m)/da. \quad (13b)$$

We obtain now solutions to (13), expressing them in terms of time t. For $\lambda=0$, we get a Tolman universe

$$a=(m^2-t^2)^{1/2}. \quad (14)$$

The unique Lorentzian solution for $\lambda>0$ is

$$a=\lambda^{-1/2}[m^2\lambda\sinh^2(\lambda^{1/2}t)]^{1/2}, \quad (15)$$

which is defined for $0<a<a_-$.

In principle, there is another solution to (13):

$$a=\lambda^{-1/2}[m^2\lambda+\cosh^2(\lambda^{1/2}t)]^{1/2}. \quad (16)$$

However, such a de Sitter-type solution is defined for $a>a_+$ and, therefore, it is not allowed in the Lorentzian regime $a<m$.

In summary, we have achieved a method to obtain the Tolman-Hawking wormhole which does not imply any unphysical change of sign in the effective gravitational constant. Such a method is based in introducing a three-sphere with minimum constant radius in the usual Robertson-Walker isotropic manifold. Apart from the fact that it achieves the desired result, the inclusion of such a cutoff is motivated by recent work done in quantum gravity[8] which predicts that it is altogether impossible to measure the position or size of any object with a precision larger than about the Planck length. Thus, since the most probable value of the radius of the wormhole throat is precisely about the Planck length,[3,4] one could well interpret a wormhole as a topological consequence from the existence of a maximum physical resolution limit.

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