Dynamical variables in Gauge-Translational Gravity

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Assuming that the natural gauge group of gravity is given by the group of isometries of a given space, for a maximally symmetric space we derive a model in which gravity is essentially a gauge theory of translations. Starting from first principles we verify that a nonlinear realization of the symmetry provides the general structure of this gauge theory, leading to a simple choice of dynamical variables of the gravity field corresponding, at first order, to a diagonal matrix, whereas the non-diagonal elements contribute only to higher orders.

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1. Introduction

The establishment of a gauge symmetry lies on the empirical evidence of a locally invariant property related to a group of transformations. In this sense the existence of a continuous ten-parameter group (Poincaré) giving rise to the conservation of the fundamental dynamical variables, strongly suggests the existence of a relevant link between dynamics and the basic properties of the space-time i.e geometry.

From the geometrical point of view Poincaré can be defined as the group of isometries of Minkowsky space, so that, being essentially gravity a dynamical theory of the space-time, it seems natural to consider the group of isometries of a given space as the gauge group of such a theory.

Minkowsky is the simplest case (zero curvature) of a maximally 3+1 symmetric space and thus it excludes the presence of a cosmological constant. On the other hand, our knowledge of the geometrical properties of the space-time is only phenomenological and therefore it is approximate. Strictly speaking what we observe is that the space is approximately homogeneous and isotropic, and that it is endowed with the kinematical Lorentz group of Relativity so that we can assume that the symmetry group of space-time seems to be very close to Poincaré. Consequently we assume that the general candidate for a gauge theory of gravity is the group of isometries of a maximally symmetric space (the limit of zero curvature being the Poincaré group).

Lastly, the evidence that elementary matter is fermionic strongly supports the hypothesis that gravity couples to it through the vierbein. The assumption that the vierbein is the connection of the local translations makes it to transform as a tensor under diffeomorphisms and under the (even local) Lorentz group. These properties are obtained by defining the vierbeins as a NLR of Poincaré group (cosets with respect to lorentz).

The need to couple the fermionic matter to gravity stems from the attempts to enlarge the geometrical framework of General Relativity with the introduction of a suitable internal group [1][2][3][4][5][6][7][8][9].

The search for an unified description seems to suggest, as reasonable starting point, the adoption of a common and general gauge scheme for all interactions including the gravity itself. These ideas gave raise to the programm of finding approaches in which gravity is mediated by gauge connections as it happens for the remaining fundamental forces [10][11][12]. The appearance of tetrads, an object with holonomic and nonholonomic indices, is, to this purpose, an unavoidable requirement. We claim that tetrads are the fingerprint of the presence of translations in the Gauge Group, a natural feature in a theory like gravity which can be essentially considered as a dynamical theory of the space-time itself. As we have mentioned in a previous paper this reminds the Feynman's words "Gravity is that field which corresponds to a gauge invariance with respect the displacement transformations". We stress that, as it shall be shown in what follows, the natural way to realize a symmetry containing translations is precisely a non linear realization where the cosets have the form $e^{\mathrm{i}p_i\varphi^i}$, where the set of fields φ^i, which becomes isomorphic to the coordinates, acts as the parameters which characterize the coset. In this way the fields φ^i introduce a dynamical interpretation of an ingredient like the coordinates which is present in any field theory. On the other hand,it can be seen that, from the group theoretical point of view, they behave as the Goldstone bosons with respect the gauged translations.

To make the paper as self contained as possible we include in Section 2 a brief review sketching the general lines of the non linear local realizations of the space-time groups. In Section 3 we deduce, starting from first principles and definitions, the integrability conditions which determine the structure of the gauge theory, serving, at the same time, as a link between the gauge and the geometrical description. Section 4 is devoted to establish the structure of the gauge theory which provides us the underlying background of the canonical geometrical description, allowing, for instance, alternative and simpler choices of the dynamical variables, an essential question in gravity theories. We conclude with some final remarks on the possible extensions and open problems.

2. The structure of the tetrads

We briefly review here some fundamental tools and results from previous works.

A maximally four dimensional symmetric space admits a maximal number of Killing vectors supporting a semisimple Lie algebra described by the ten generators:

$$P_i = \mathrm{i}\,\{\partial_i + \frac{k}{4}(2x_i x^j - \delta_i^j r^2)\partial_j\} \tag{1}$$

$$L_{ij} = \mathrm{i}\,(\delta_i^k x_j - \delta_j^k x_i)\partial_k \tag{2}$$

where $r^2 = \eta_{ij}x^i x^j$ and being k the sectional curvature. The commutation relations can be written in the form:

$$[P_i, P_j] = \mathrm{i}\,k L_{ij} \tag{3}$$

$$[L_{ij}, P_k] = \mathrm{i}\,\eta_{k[i}P_{j]} \tag{4}$$

$$[L_{ij}, L_{kl}] = -\mathrm{i}\,\{\eta_{i[k}L_{l]j} - \eta_{j[k}L_{l]i}\} \tag{5}$$

which reduces to Poincaré when $k \to 0$. The occurrence of "translational-like"transformations rises the problem of realizing a local symmetry of this kind in which the Lorentz subgroup H still be linearly represented, as dictated by the particle phenomenology. The natural choice is given by a local non linear realization with cosets defined as $e^{\mathrm{i}\,\varphi^i P_i}$ [13][14][15][16], which is the most general one preserving the linear action of the subgroup H.

The non linear gauge realizations of space-time symmetry groups containing translations have been the object of several papers [17][19][20][21][22][23][24] in the past. Nevertheless, in order to make this work more readable we include here a brief review of the methods and main results.

Let G be a Lie group having a subgroup H, we assume that the elements $C(\varphi)$ (cosets) of the quotient space G/H can be characterized by a set of parameters say φ. Let us denote by ψ an arbitrary linear representation of the subgroup H.

The non linear realization can be derived from the action of a general element "g" of the whole group on the coset representatives defined in the form:

$$g\,C(\varphi) = C(\varphi')h(\varphi, g) \tag{6}$$

where $h(\varphi, g) \in H$. It acts linearly on the representation space ψ according to:

$$\Psi' = \varrho[h(\varphi, g)]\,\Psi, \tag{7}$$

being $\varrho[h]$ a representation of the subgroup H.

The next step to construct a non linear local theory is to define suitable gauge connections. They can be obtained by substituting the ordinary Cartan 1-form $\omega = C^{-1}dC$ by a generalized expression of the form:

$$\Gamma = C^{-1}DC \tag{8}$$

where $D = d + \Omega$ is the covariant differential built with the 1-form connection Ω defined on the algebra of the whole group and having the canonical transformation law:

$$\Omega' = g\Omega g^{-1} + g d g^{-1} \tag{9}$$

The generalized local Cartan 1-form is:

$$\Gamma = C^{-1}\mathbf{D}C = e^{-\mathrm{i}\,\varphi^i P_i}(d + \mathrm{i}\,T^i P_i + \frac{\mathrm{i}}{2}A^{ij}L_{ij})e^{\mathrm{i}\,\varphi^i P_i} \tag{10}$$

where T is the linear translational connection and A^{ij} the corresponding one for the Lorentz group.

Using Hausdorff-Campbell formulas to deal with exponentials, after a little algebra (the details can be found in [25][26]) we obtain:

$$\Gamma = \mathrm{i}\,\hat{e}^i P_i + \frac{i}{2}\hat{A}^{ij} L_{ij} \tag{11}$$

where $\hat{e}^i$ and $\hat{A}^{ij}$ are the 1-form non linear local connections given by the following expressions:

$$\hat{e}^i = N D\varphi^i + \frac{1-N}{\mu^2}(\varphi^j D\varphi_j)\varphi^i + M\,T^i + \frac{1-M}{\mu^2}(T^j\varphi_j)\varphi^i \tag{12}$$

and

$$\hat{A}^{ij} = A^{ij} + \frac{1-M}{\mu^2}\varphi^{[i} D\varphi^{j]} + k\,N\varphi^{[i}T^{j]} \tag{13}$$

where $D\varphi^i \equiv (d\varphi^i + A^i_j\varphi^j)$ is the Lorentz covariant differential, $\mu^2 \equiv \eta_{ij}\,\varphi^i\varphi^j \equiv \varphi_i\varphi^i$, and M and N are given by the following series:

$$M = 1 - \frac{k\mu^2}{2!} + \frac{(k\mu^2)^2}{4!} + ... \sim \cos\sqrt{k\mu^2} \tag{14}$$

and

$$N = 1 - \frac{k\mu^2}{3!} + \frac{(k\mu^2)^2}{5!} + ... \sim \frac{1}{\sqrt{k\mu^2}}\sin\sqrt{k\mu^2} \tag{15}$$

In (10), the translational connection 1-form T^i has dimensions of length. In order to have a dimensionless connection γ^i homogeneous with the ordinary Lorentz connection A^{ij}, we introduce a constant characteristic length, say λ, and define $T^i = \lambda\gamma^i$. We notice that the occurrence of a fundamental length is a common feature in gravity theories [27][28][29][30][31][32], as for instance the Planck scale, at the basis of string theory, or the spacing parameter in lattice theories. In our scheme, we claim that a characteristic length finds its natural place in the translational connection above. Then the smallness of λ emphasizes the interpretation of gravity as a perturbation of a background (usually supposed flat) metric.

In order to express all the objects in terms of only the non linear connections we introduce in (12) the value of A_{ij} worked out from (13) in terms of $\hat{A}_{ij}$, obtaining

$$\hat{e}^i = \frac{N}{M}\hat{D}\varphi^i + \frac{1}{\mu^2}(1 - \frac{N}{M})(\varphi^j d\varphi_j)\varphi^i + \lambda(\frac{1}{M}\,\bar{\gamma}^i + \frac{\varphi^i\varphi_j}{\mu^2}\gamma^j) \tag{16}$$

where $\hat{D}$ stands for the Lorentz covariant differential in terms of $\hat{A}_{ij}$, and $\bar{\gamma}^i = (\delta_{ij} - \frac{\varphi^i\varphi_j}{\mu^2})$.

Our focus now is on the structure of the non-linear vierbein (16). In its limit for $\lambda \to 0$ one has

$$\hat{e}(0)^i = \frac{N}{M}\hat{D}\varphi^i + \frac{1}{\mu^2}(1 - \frac{N}{M})(\varphi^j d\varphi_j)\varphi^i \ . \tag{17}$$

whereas in the limit $k \to 0$ we have $M = N = 1$, so that it becomes

$$e^i = D\varphi^i + \lambda\gamma^i \ \equiv e(0)^i + \lambda\gamma^i \ , \tag{18}$$

which is the expression for the Poincaré case, where e^i and its non-linear version $\hat{e}^i$ coincide.

Now we remind that the essential feature of a tetrad is given by its double character, transforming as general vector in the Greek indices and as a Lorentz vector in the Latin ones, so that it provides a link between both spaces.

Now we observe that the covariant derivative of a Lorentz vector like $\hat{e}(0)^i_\mu$ in (17) and (18) is the minimal structure able to take the role of a tetrad, so we shall call it "minimal tetrad". We stress that the difference between $\hat{e}(0)^i_\mu$ and $\hat{e}^i_\mu$ regards the behavior under local translations, due to the presence of the connection γ^i_μ .

3. Integrability conditions

The passage to the geometrical description can be made with the help of the general tetrad (16) defining, as usual, a metric tensor of the form:

$$g_{\mu\nu} = \hat{e}_{i\mu}\hat{e}^i_\nu \ = g(0)_{\mu\nu} + \lambda\gamma_{(\mu\nu)} + \lambda^2\gamma_{\mu\rho}\gamma_{\nu\sigma}\, g(0)^{\rho\sigma} \ , \tag{19}$$

where $g(0)_{\mu\nu} = \hat{e}(0)_{i\mu}\hat{e}(0)^i_\nu$ is the corresponding "minimal metric tensor", and we have used $\hat{e}(0)_{i\mu}$ and its formal inverse $\hat{e}(0)^\nu_j$ to transform indices.

Two comments are now in order. The first one concerns equation(19) that imitates a weak field expansion over a background metric $g(0)_{\mu\nu}$. It must be emphasized however that it is not a perturbation approach but an exact result derived from the underlying gauge structure, which is apparent only at the vierbein level. Secondly, the decomposition (16) implies a non-trivial structure for the formal inverse $\hat{e}^\mu_i$ present in the definition of the contravariant metric tensor. We explicitly assume that the theory is analytical in the characteristic length λ, so that the formal inverses are given by an expansion in powers of λ. Strictly speaking a similar question arises with the definition of $g(0)^{\rho\sigma}$ present in (19), and we are to show that the structure and properties of this minimal metric tensor can be derived from general integrability conditions.

In a previous work [25] we have seen that the field equations of gravity in the vacuum can be interpreted as a gauge theory of translations defined in the metric of a maximally symmetric background space. Now we are going to show that this result holds without making recourse to the field equations even in the presence of matter, or, in other words, as a consequence of the underlying gauge structure of the theory which is previous to any dynamics.

The analyticity in λ lets to work out the existence conditions and ensuing properties of the solutions in the limit $\lambda \to 0$ (minimal tetrads). To this end we first redefine the Lorentz connection $\hat{A}^{ij}_{\mu}$ as follows:

$$\hat{A}^{ij}_{\mu} = \hat{e}^{\alpha i} \mathcal{D}_{\mu} \hat{e}^{j}_{\alpha} + B^{ij}_{\mu}, \tag{20}$$

where $\mathcal{D}_{\mu}$ is the ordinary Christoffel covariant derivative acting on the coordinate index α of the tetrad $\hat{e}^{j}_{\alpha}$. The first term of this redefinition, usual in gauge theories of gravity, describes the value of the Lorentz connection in the absence of matter, whereas the second one B^{ij}_{μ} takes into account the coupling with the spin densities present in the matter terms, so that $B^{ij}_{\mu} = 0$ in the vacuum.

Now we shall see that the background metric can be derived from integrability conditions which are previous to the equations of motion. This requires some rather involved algebra that we briefly outline in the following.

For $\lambda = 0$, contracting (20) with φ^{j} we get

$$\hat{A}^{i}_{\mu j} \varphi^{j} = [e(0)^{\alpha i} \mathcal{D}(0)_{\mu} [e(0)_{\alpha j} \varphi^{j}] - \partial_{\mu} \varphi^{i} + B^{i}_{\mu}], \tag{21}$$

where $\mathcal{D}(0)_{\mu}$ is the Christoffel covariant derivative constructed with the metric tensor $g(0)_{\mu\nu}$, and $B^{i}_{\mu} = B^{i}_{\mu j} \varphi^{j}$. Using (21) and the notation $\chi \equiv k\, \mu^{2}$, (17) becomes

$$\hat{e}(0)^{i}_{\mu} = \frac{N}{M} [\hat{e}(0)^{\alpha i} \frac{1}{2k} \mathcal{D}(0)_{\mu} \mathcal{D}(0)_{\alpha} \chi + B^{i}_{\mu}] + (1 - \frac{N}{M}) \frac{1}{2k} \mathcal{D}(0)_{\mu} \chi\, \varphi^{i} \tag{22}$$

Contracting it with $\hat{e}(0)_{i\nu}$ one obtains

$$g(0)_{\mu\nu} = \frac{1}{2k} \frac{N}{M} [\mathcal{D}(0)_{\mu} \mathcal{D}(0)_{\nu} \chi + (\frac{N}{M} - 1) \frac{1}{2k} \mathcal{D}(0)_{\mu} \chi \mathcal{D}(0)_{\nu} \chi] + \frac{N}{M} B_{\mu\nu} \tag{23}$$

The square bracket in (23)can be brought to the form $H(\chi) \mathcal{D}(0)_{\mu} \mathcal{D}(0)_{\nu} F(\chi)$. Exploiting the explicit values (14) and (15), the function F and the integration factor H turn out to be M and $(cN)^{-1}$ respectively, where c is a constant. With $c = (2k)^{-1}$ we obtain

$$g(0)_{\mu\nu} = \frac{1}{M}\, \mathcal{D}(0)_{\mu} \mathcal{D}(0)_{\nu}\, M + \frac{N}{M}\, B_{\mu\nu} \tag{24}$$

Taking the trace of (24) we have:

$$M = \frac{1}{4}\left(\Box(0)M + B\right) , \tag{25}$$

where B is the trace of $B_{\mu\nu}$ and $\Box(0)$ is the covariant d'Alembertian corresponding to $g(0)_{\mu\nu}$. Substituting in (24) we finally obtain

$$D(0)_\mu D(0)_\nu M = \frac{1}{4}\,\Box(0)M - \bar{B}_{\mu\nu}, \tag{26}$$

being $\bar{B}_{\mu\nu} \equiv B_{\mu\nu} - \frac{1}{4}\,g(0)_{\mu\nu}B$.

Taking now symmetric and antisymmetric parts of (26) we get:

$$D(0)_\mu D(0)_\nu M = \frac{1}{4}\,g(0)_{\mu\nu}\Box(0)M - \frac{1}{2}\,\bar{B}_{(\mu\nu)}, \tag{27}$$

and

$$\bar{B}_{[\mu\nu]} = B_{[\mu\nu]} = 0 , \tag{28}$$

which in Lorentz indices gives

$$B_{[\mu\nu]} = e(0)^k_{[\mu}e(0)^i_{\nu]}B_{kij}\varphi^j = 0 \qquad \Rightarrow \qquad B_{[ki]j}\varphi^j = 0 \tag{29}$$

We recall that the fields φ^j are by definition independent functions as long as, being the Goldstone bosons of the gauged translations, there are not dynamical relations among them. Otherwise stated, their motion equations are satisfied identically since they yield the null covariant divergence of the Einstein tensor. Therefore (29) is verified only when $B_{[ki]j} = 0$. Taking into account the antisymmetry of B_{kij} in the last two indices we can write

$$B_{kij} = B_{ikj} = -B_{ijk}, \tag{30}$$

so that symmetrizing in ij one gets finally

$$B_{(ij)k} = 0 . \tag{31}$$

As a consequence, at least at zeroth order in λ , equation (27) reduces to

$$\mathcal{D}(0)_\mu \mathcal{D}(0)_\nu\, M = \frac{1}{4}\,g(0)_{\mu\nu}\Box(0)M . \tag{32}$$

It must be emphasized here that the question is not finding a solution M to (32), which we know *a priori*, but acknowledging that its mere existence implies, as a well known integrability condition, the maximally symmetric character of the space. Thus $g(0)_{\mu\nu}$ is determined from first principles and previously to any dynamics. As a particular case, for $k \to 0$ equation (32) is replaced by [25][26]

$$g(0)_{\mu\nu} = \mathcal{D}(0)_\mu \mathcal{D}(0)_\nu \, \sigma \qquad (\sigma \equiv \frac{1}{2}\mu^2) \; , \tag{33}$$

which leads us to a Minkowskian metric. As we shall see, it suffices to adopt φ^i as coordinates to verify that $g(0)_{\mu\nu}$ reduces to the flat metric η_{ij} .

Consequently the geometrical description is given in terms of the finite expansion (19), which depends on the translational connection (the true gravitational dynamical variable) $\gamma_{\mu\nu}$ and a background maximally symmetric metric tensor $g(0)_{\mu\nu}$.

4. Gauge structure and dynamical variables

In this scheme, the dynamical gravitational variables in the geometrical approach are embodied in the translational connection $\gamma_{\mu\nu}$ defined on a background metric $g(0)_{\mu\nu}$. Once $g(0)_{\mu\nu}$ has been determined *prior* to any dynamics by integrability conditions, it is of uppermost interest to pinpoint the structure of the minimal tetrad $e(0)^i_\mu$ associated to it, which shall provide us with very useful tools for the identification of the dynamical variables, a fundamental problem in gravity theories.

The passage from the gauge description to the geometrical one is canonically accomplished by using (20) for $B^{ij}_\mu = 0$ in the Field Strength Tensor, which becomes:

$$F^{ij}_{\mu\nu} = e^i_\alpha e^j_\beta R^{\alpha\beta}_{\mu\nu} \tag{34}$$

Starting from this relationship we first consider the Poincaré case where the cancelation of the Riemann tensor at zero order in λ stems from the integrability conditions, so that, being $R(0)^{\alpha\beta}_{\mu\nu} = 0$, we conclude that also $F(0)^{ij}_{\mu\nu} = 0$ and then A^{ij}_μ must be a pure gauge. The structure of such a connection is given by the inhomogeneous part of the formal variation of a gauge connection, thus we write:

$$A(0)^{ij}_\mu = U^{ik} \partial_\mu U^j_k, \tag{35}$$

where U^{ik} is an arbitrary pseudo-orthogonal matrix describing a general Lorentz transformation. Putting this in the zeroth order of (18) we obtain:

$$e(0)^i_\mu = \partial_\mu \varphi^i + U^{ik} \partial_\mu U_{kj} \varphi^j \; , \tag{36}$$

so we can write:

$$e(0)^i_\mu = \partial_\mu \varphi^i + U^{ik} \partial_\mu [U_{kj} \varphi^j] - \partial_\mu \varphi^i = U^{ik} \partial_\mu [U_{kj} \varphi^j] = U^{ik} \partial_\mu \hat{\varphi}_k \tag{37}$$

where $\hat{\varphi}_k = U_{kj}\varphi^j$. Then the background metric may be written as follows:

$$g(0)_{\mu\nu} = U^{ik}\partial_\mu\hat{\varphi}_k U^{il}\partial_\nu\hat{\varphi}_l = \partial_\mu\hat{\varphi}_k\partial_\nu\hat{\varphi}^k \tag{38}$$

It is immediate to check that (38) satisfies the condition (33). In a non linear realization of the Poincaré group the fields φ^i transform as the cartesian coordinates, thus in a Minkowskian space they can be properly used as coordinates.

To reproduce the usual geometrical approach we note that the translational connection $\gamma_{\mu\nu} = e^i_\mu\gamma_{ij}e^j_\nu$ exhibits an underlying invariance under Lorentz transformations U in the Latin indexes, so that we can fix the gauge to render γ_{ij} (and consequently $\gamma_{\mu\nu}$) symmetrical, thus recovering the usual ten degrees of freedom of canonical gravity.

Now we recover the expression (19) of the general metric tensor, taking the symmetric and antisymmetric parts of $\gamma_{\mu\nu}$:

$$\gamma_{\mu\nu} = \frac{1}{2}\, s_{\mu\nu} + \frac{1}{2}\, a_{\mu\nu}, \tag{39}$$

being $s_{\mu\nu} = \gamma_{(\mu\nu)}$ and $a_{\mu\nu} = \gamma_{[\mu\nu]}$, so (19) becomes:

$$g_{\mu\nu} = g(0)_{\mu\nu} + \lambda\, s_{\mu\nu} + \frac{\lambda^2}{4}[s_{\mu\rho}s_{\nu\rho} + s_{(\mu\rho}a_{\nu)\sigma} + a_{\mu\rho}a_{\nu\sigma}]g(0)^{\rho\sigma}. \tag{40}$$

Steering to Lorentz indices, namely $s_{\mu\nu} = e(0)^i_\mu\, s_{ij}\, e(0)^j_\nu$ and $a_{\mu\nu} = e(0)^i_\mu\, a_{ij}\, e(0)^j_\nu$, and adopting the coordinates x^μ for the cartesian Goldstone ones φ^i, the metric tensor $g(0)_{\mu\nu}$ reduces to η_{ij} . Now we can choose U such that s_{ij} becomes $U^k_i s_{kl} U^l_j = d_{ij}$ diagonal, and $a_{ij} \to U^k_i a_{kl} U^l_j = \hat{a}_{ij}$, obtaining:

$$g_{ij} = \eta_{ij} + \lambda\, d_{ij} + \frac{\lambda^2}{4}[d_{ik}d_{jl} + d_{(ik}\hat{a}_{j)l} + \hat{a}_{ik}\hat{a}_{jl}]\eta^{kl}. \tag{41}$$

We then have the usual ten degrees of freedom of canonical gravity, albeit in quite a different arrangement: the four eigenvalues of the symmetric part of $\gamma_{\mu\nu}$ and the six elements of an antisymmetric matrix. We stress that these d.o.f. appear in (41) at different orders in λ so that, for instance, the calculations at first order get highly simplified.

The case of a maximally symmetric space is slightly more complicated because A^{ij}_μ is not a pure gauge. We again start from the limit $\lambda \to 0$ of (12) that can be written as:

$$e(0)^i_\mu = ND_\mu\varphi^i + \frac{1-N}{2\chi}\,\partial_\mu\chi\,\varphi^i \tag{42}$$

where $\chi \equiv k\mu^2$. Taking into account (37) and using the relation $\hat{\varphi}^k = U^k_j \varphi^j$ in (42) we obtain:

$$e(0)^i_\mu = U^i_k (N\partial_\mu \hat{\varphi}^k + \frac{1-N}{2\chi}\, \partial_\mu \chi\, \hat{\varphi}^k) \ . \tag{43}$$

A more compact and convenient notation is obtained by redefining the fields $\hat{\varphi}^k$ according to

$$\tilde{\varphi}^k = F(\chi)\hat{\varphi}^k \ , \tag{44}$$

which substituted in (43) lead us to

$$e(0)^i_\mu = U^i_k \frac{N}{F}\, \partial_\mu \tilde{\varphi}^k + (\frac{1-N}{2\chi} - \frac{NF'}{F})\partial_\mu \chi\, \tilde{\varphi}^k. \tag{45}$$

The value of F is chosen so as to cancel the second term on the right hand side of (45), namely:

$$\frac{1-N}{2x} - \frac{NF'}{F} = 0, \tag{46}$$

where the prime denotes differentiation with respect to χ. Using (15) we obtain $F = \frac{c}{\sqrt{\chi}}\ \mathrm{Tan}\frac{\sqrt{\chi}}{2}$, so that

$$e(0)^i_\mu = U^i_k \frac{1+M}{c}\, \partial_\mu \tilde{\varphi}^k. \tag{47}$$

where c is an integration constant.

Writing now $F(\chi)$ in terms of the redefined fields $\tilde{\varphi}^k$ and choosing $c = 2$, the tetrad finally reads

$$e(0)^i_\mu = U^i_k (1 + \frac{1}{4}\, \tilde{\chi})^{-1} \partial_\mu \tilde{\varphi}^k \ . \tag{48}$$

Taking again, as in the Poincaré case, the Goldstone fields $\tilde{\varphi}^k$ as coordinates, we derive the metric tensor

$$g(0)_{ij} = (1 + \frac{1}{4}\tilde{\chi})^{-2} \eta_{ij} \ , \tag{49}$$

in which we recognize the so called Riemannian form of the metric for a space of constant curvature.

5. Concluding remarks and first order equations

The choice of the dynamical variables given in equation (41) simplifies the structure of the theory. This allows us to find out the general form and properties of the vacuum equations of gravity at first order in λ . To do this we start from the Einstein's equations $G_{ij} = \frac{\Lambda}{4} g_{ij}$ in the presence of a cosmological constant, that can be alternatively written:

$$R_{ij} + \frac{\Lambda}{4} g_{ij} = 0 \tag{50}$$

There exists,when two different metric tensors $g_{\mu\nu}$ and $g(0)_{\mu\nu}$ are involved, a useful relation between the Christophel's connections which highly simplify the calculations, namely:

$$\Gamma^{\rho}_{\mu\alpha} = \Gamma(0)^{\rho}_{\mu\alpha} + \Delta^{\rho}_{\mu\alpha}, \tag{51}$$

where $\Gamma^{\varrho}_{\mu\alpha}$ is the Christophel symbol constructed with $g_{\mu\nu}$ and $\Gamma(0)^{\varrho}_{\mu\alpha}$ the corresponding one to $g(0)_{\mu\nu}$, being

$$\Delta^{\rho}_{\mu\alpha} = \frac{1}{2} g^{\lambda\rho} [D(0)_{\mu} g_{\lambda\alpha} + D(0)_{\alpha} g_{\lambda\mu} - D(0)_{\lambda} g_{\mu\alpha}] \tag{52}$$

with $D(0)_{\mu}$ the covariant derivative in terms of $\Gamma(0)^{\rho}_{\mu\alpha}$.

According with (41) we are going to use in the following Latin indexes, being $g(0)_{ij}$ the background metric and $g_{ij} = g(0)_{ij} + \lambda d_{ij}$ the first order expansion of the general metric.

The relation between the corresponding Ricci tensors is then given by the following expression:

$$R_{ij} = R(0)_{ij} + D(0)_j \Delta^k_{ki} - D(0)_k \Delta^k_{ij} + \Delta^l_{ik} \Delta^k_{jl} - \Delta^l_{ij} \Delta^k_{kl}, \tag{53}$$

where $D(0)_i$ and $R(0)_{ij}$ are respectively the covariant derivative and the Ricci tensor constructed with the background metric.

A brief calculation leads to the value of Δ^i_{jk} which reads:

$$\Delta^i_{jk} = \frac{\lambda}{2} (D(0)_j d^i_k + D(0)_k d^i_j - D(0)^i d_{jk}), \tag{54}$$

where the indexes are raised and lowered using the background metric.

From (56) we see that the zero order terms are satisfied when $R = -\Lambda$. So that the first order equations becomes:

$$\Box(0) d_{ij} + D(0)_j D(0)_i d^k_k - D(0)_k D_{(i} d^k_{j)} + \frac{\Lambda}{2} d_{ij} = 0, \tag{55}$$

and its trace:

$$\Box(0)d^k_k - D(0)_k D(0)_i d^{ki} + \frac{\Lambda}{4} d^k_k = 0 \tag{56}$$

It is not the aim of this paper to include a general survey of the first order solutions, which merit by themselves a more detailed and specific study. Notwithstanding we are going to briefly comment some features of the problem relevant in this choice of the dynamical variables. In fact being d_{ij} a diagonal matrix equation (58) contains, when $i \neq j$ additional information with respect to the usual treatments,namely the second order analytical restrictions:

$$D_k D_{(i} \hat{d}^k_{j)} = 0, \tag{57}$$

where $i \neq j$ and:

$$\hat{d}^k_j = d^k_j - \frac{1}{2}\delta^k_j d^l_l. \tag{58}$$

Obviously these restrictions are absent in any other choice of the dynamical variables and gives an important input in the search of the general scheme of the first order solutions.

Summarizing, our proposal describes the space-time physics by a twofold assumption, one is the gauge nature of the translations, which introduces a characteristic length λ interpretable as the "true"gravitational interaction, and the other is the structure of empty space, attained in the limit $\lambda \to 0$, which is taken to be a maximally symmetric background space. This geometrical assumption is equivalent to adopt the existence of a cosmological constant Λ as a phenomenological observation on the same footing of the approximate planarity (homogeneity and isotropy) of space-time.

The role played by λ in this approach, together with a principle of economy, strongly suggests to relate this characteristic length to the gravitational constant. In fact, in natural units, having a dimensionless vacuum gravity action requires the introduction of a constant factor with dimensions L^{-2} , namely the inverse Newton constant, and it becomes natural its identification with λ^{-2}.

Therefore the free gravitational lagrangian $e^\mu_i e^\nu_j F^{ij}_{\mu\nu}$ should include a depressing factor λ^2 with respect to the matter terms. Thus we observe that when (19) is used, the motion equations stemming from the variation δA^{ij}_μ become purely algebraical so that $B^{ij}_{\mu\nu} = 0$ in the absence of matter, while B^{ij}_μ becomes equal to the matter spin densities coupled linearly to A^{ij}_μ . Being the matter terms the only ones contributing to the value of B^{ij}_μ, they are evidently depressed by at least the same factor λ^2 existing between the matter Lagrangian and the vacuum term. This obviously implies that the term $B_{(\mu\nu)}$ do not contributes to the zeroth order equation (28), the structure of which agrees with the results of the ordinary dynamical treatment of gravity as a local field theory. In a gauge field theory only the fermions give rise to these kind of contributions, and the fermion spin densities are completely antisymmetric when all indexes are of the same (either tensor or Lorentz) nature, so that a symmetric part $B_{(\mu\nu)}$ should be absent in any case.

Therefore we conclude that the identification of the characteristic length λ with the gravitational constant is not only an economical and natural assumption but, at the same time, fully consistent with well established theoretical results.

As it is seen, the appearance of the cosmological constant is, in this scheme, an initial condition related to the gauge space-time group considered, so that it constitutes an initial ingredient of the background space of the theory. From this point of view revisiting the Quantum Field Theory on maximally symmetric spaces appears as a very promising topic.

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