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"A Dynamic Equilibrium Model of Search, Bargaining, and Money"

by

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Abstract

We characterize dynamic (not just steady state) equilibria in a search-theoretic model of fiat money, where buyers and sellers, upon meeting, enter bargaining games to determine prices. Equilibrium in the bargaining game is approximated in terms of a tractable dynamical system, in much the same way that the Nash solution approximates equilibrium in bargaining games in stationary environments. The model with our dynamic bargaining solution can generate outcomes (such as limit cycles) that never arise in the same model if one imposes a myopic bargaining solution, as has been done in the past.

1 Introduction

Recent papers by Shi (1995a) and Trejos and Wright (1995a) integrate bilateral bargaining into search-theoretic models of the exchange process so as to develop a new microeconomic foundation for monetary economics.[1] This approach is proving useful for studying a wide variety of applications in monetary economics from a novel perspective.[2] To this point, however, the analysis of these models has either focused exclusively on steady states or, as in Trejos and Wright (1995a), simply imposed a myopic axiomatic bargaining solution and then proceeded to discuss dynamics.

The attraction of an axiomatic bargaining model of the sort introduced by Nash (1950) is that it is simple. The drawback, from the present perspective, is that it is inherently timeless: one specifies agents' preferences over a set of possible outcomes (agreements and disagreement), and then establishes that there exists a unique outcome satisfying a number of axioms. There is no discussion whatsoever of the bargaining process, or of the resources

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[1] Earlier papers, such as Kiyotaki and Wright (1989, 1991, 1993), also present search-theoretic models of monetary economies; but these models have the deficiency that the terms of trade are fixed exogenously, and hence they have nothing to say about the determination of (real or nominal) prices.

[2] Recent examples include: Aiyagari et al. (1995), who analyze conditions that allow for the coexistence of fiat money and interest-bearing government securities; Shi (1995b), who introduces private credit; Trejos (1995), who studies the effects of asymetric information; Trejos and Wright (1995b), who analyze exchange rates; Li (1995), who analyzes the role of middlemen; Li and Wright (1995), who study the effects of government transaction policies; and Velde et al. (1995), who study issues in monetary history, including Gresham's Law and the debasement of commodity monies.

— including *time* — that may be essential in reaching an agreement. It therefore seems worth considering explicit strategic bargaining models of the sort developed by Rubinstein (1982), in which one specifies a sequence of moves and preferences over the time of agreement as well as the terms of agreement, and then looks for subgame perfect equilibria. Only in this way can one analyze how negotiations between any two individuals (say, over a particular price) are affected by the fact that economy-wide variables (such as the aggregate price level) may be changing over time.

Of course, in stationary environments there is a close relation between the models of Nash and Rubinstein; as shown by Binmore (1987) and Binmore et al. (1986), when the time between offers in the strategic model is small the unique subgame perfect equilibrium can be approximated by the Nash solution with threatpoints and bargaining power that depend on details of the strategic model. Hence, we can think of the Nash solution as a reduced form for an underlying strategic model, which is very useful in applications as long as one is interested only in steady states. We generalize this result to environments that may be nonstationary, in the sense that we provide a tractable differential equation that approximates equilibrium in the strategic model and can be used in dynamic applications in much the same way that the Nash solution can be used in static or steady state analyses.

Our differential equation representation of bargaining equilibrium has the interpretation that agents are *forward looking* when they negotiate, while imposing a simple Nash solution has the interpretation that they are myopic. This can be qualitatively important. For example, in the monetary model, it implies that there can exist limit cycles in nominal prices and real economic

activity that never arise when one imposes a myopic bargaining solution, as in Trejos and Wright (1995a). Other applications in macro and labor economics that also impose myopic bargaining solutions in dynamic models include Pissarides (1987), Drazen (1988), and Mortensen (1989). We discuss conditions under which this may actually be valid, but, in general, the myopic solution does not give the same answer as our forward looking solution.

The existence of cycles in the search model of fiat model is of interest in its own right. Although endogenous limit cycles have been derived in similar models in the past (see, e.g., Diamond and Fudenberg [1990] or Boldrin et al. [1994]), those models had to rely on increasing returns to scale (in the meeting technology). Here, the dynamics are driven purely by self-fulfilling expectations in the search and bargaining processes. Moreover, it seems clear that the existence of endogenous fluctuations depends on the fiat nature of the medium of exchange in the model. Hence, the results provide support for the long-standing notion that monetary economies are particularly susceptible to fluctuations induced by self-fulfilling prophecies, animal spirits, and so on (see the discussion and references in Wright [1994]).

The rest of the paper is organized as follows. In Section 2 we present the basic search and bargaining model of money. In Section 3 we analyze dynamic bargaining games and derive our differential equation representation of equilibrium. This analysis is presented in somewhat general terms, rather than the particular application to monetary economics, since (as suggested above) the results may prove useful in other applications. In Section 4 we imbed the solution to the dynamic bargaining game into the search model and describe equilibria. In Section 5 we present some extensions to more

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general bargaining environments. We conclude in Section 6.[3]

2 The Monetary Model

There is a $[0, 1]$ continuum of infinitely-lived individuals. These individuals meet in an anonymous random matching process where all exchange is bilateral and quid pro quo. To make exchange difficult, some notion of specialization is required. Here, we assume that there are k types of agents and k goods, $k > 2$, with the property that type j only consumes good j and only produces good $j + 1$ (modulo k). This rules out direct barter. Also, we assume the goods are nonstorable. This rules out commodity money. Hence, if trade occurs at all in this economy it requires the use of fiat currency.[4]

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[3]There exist prior analyses of nonstationary bargaining environments. For example, going back to Stahl (1972), people have considered finite horizons (which can be analyzed using backward induction). Also, in some of the search and bargaining literature, when some traders leave the market the arrival rates for other agents can change; see the survey in Osborne and Rubinstein (1990). Perry and Reny (1993) analyze a continuous-time dynamic bargaining model, and Merlo and Wilson (1993a, 1993b) analyze models where payoffs and bargaining power both follow general discrete-time stochastic processes. None of these authors pursue the main object of interest here, however, which is to derive an approximation to the equilibrium that will be useful in applications such as the search and bargaining models studied in labor, macro, and monetary economics.

[4]Versions of the model that allow some direct barter, commodity money, credit, and several other complications are contained in the papers discussed in the Introduction, although those papers only consider steady states, or ignore bargaining by fixing exchange ratios exogenously. A somewhat related paper by Casella and Feinstein (1990) is concerned with bargaining when the aggregate price level is changing over time. In that model,

At $t = 0$, a fraction $M \in [0,1]$ of the population are each endowed with one unit of fiat currency, and the rest with production opportunities. For simplicity we assume that when agents spend their money they spend all of it (say, because it comes in indivisible units), and that except for those initially endowed with production opportunities no agent can produce until after he consumes. This implies that at.*every* point in time there will be M agents with one unit of money (called buyers) and $1 - M$ agents with production opportunities (called sellers), since each exchange involves exactly one unit of currency going from some buyer to some seller. If a buyer gets q units of output for his money, the implied nominal price is $p = 1/q$.

Consumption of q units of one's consumption good generates utility $U(q)$, while production of one's production good generates disutility $c(q)$. We can always normalize $U(q) = q$, with no loss in generality, as long as we also renormalize the cost function $c(q)$ (since we can always let agents bargain over utility rather than physical units of output). Assume $0 < c'(0) < 1$, $c''(q) > 0$ for all $q \geq 0$, and $c(q) > q$ for large q. Initially we assume $c(0) = 0$, although we introduce a fixed cost below by allowing $c(0) > 0$.

For now, time is considered as a sequence of discrete periods of length $\Delta > 0$. Agents meet randomly according to a Poisson process with arrival rate α, which means that the probability of a meeting in a period is approximately

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however, buyers are in the market for a finite number of periods, and so the solution is easily computed using backward induction. Moreover, that is not a model of fiat money, per se, because it assumes a cash-in-advance constraint that is not needed here (in the sense that the present model allows for the existence of valued fiat money even if there is also barter, commodity money, etc.).

$\alpha\Delta$ with the approximation becoming arbitrarily good as $\Delta \to 0$. Hence, the probability that a seller meets a buyer in a period is (approximately) $\alpha\Delta M$ and the probability that a buyer meets a seller is $\alpha\Delta(1-M)$. When a buyer and seller meet, the probability is $1/k$ that the latter can produce the former's desired good. If in that event they agree to trade, they negotiate the quantity q, the buyer hands over his money, consumes and becomes a seller, while the seller takes the money and becomes a buyer.

We will present explicit details of the bargaining process below. For now, simply assume that when a buyer meets an appropriate seller they complete negotiations and trade immediately, at some (possibly random) quantity q; of course, it will have to be verified below that immediate trade is consistent with equilibrium behavior. Letting V_b and V_s denote the value functions for buyers and sellers, the standard dynamic programming equations of search theory are

$$V_b(t) = \frac{1}{1+r\Delta}\{\alpha\Delta(1-M)\tfrac{1}{k}[E_t q(t+\Delta) + V_s(t+\Delta)] \qquad (1)$$
$$+[1-\alpha\Delta(1-M)\tfrac{1}{k}]V_b(t+\Delta) + o(\Delta)\}$$

$$V_s(t) = \frac{1}{1+r\Delta}\{\alpha\Delta M\tfrac{1}{k}[-E_t c(q(t+\Delta)) + V_b(t+\Delta)] \qquad (2)$$
$$+[1-\alpha\Delta M\tfrac{1}{k}]V_s(t+\Delta) + o(\Delta)\},$$

where r is the rate of time preference.[5] Note that these functions are not

[5] For example, (1) says that between t and $t+\Delta$ a buyer meets a seller that can produce his desired good with probability $\alpha\Delta(1-M)/k$, which yields payoff $E_t q(t+\Delta) + V_s(t+\Delta)$ from consuming and becoming a seller. Otherwise, he remains a buyer. The term $o(\Delta)$ appears because of the Poisson approximation for the meeting probabilities.

indexed by agent type, because we will only consider symmetric equilibria here where all types use the same strategies and receive the same payoffs.

It remains to describe the determination of $q(t)$. One approach is to adopt the generalized Nash bargaining solution,

$$q(t) = \arg\max \ [q + V_s(t) - T_b(t)]^\theta [-c(q) + V_b(t) - T_s(t)]^{1-\theta} \qquad (3)$$

where $T_i(t)$ is the called the threatpoint of agent i and θ the bargaining power of the buyer, and the maximization is subject to

$$q(t) + V_s(t) \geq V_b(t) \qquad (4)$$

$$-c(q(t) + V_b(t) \geq V_s(t). \qquad (5)$$

Constraints (4) and (5) simply say that trade is voluntary. If they are not binding, then $q(t)$ solves the first order condition for (3)

$$\theta[-c(q) + V_b(t) - T_s(t)] - (1 - \theta)[q + V_s(t) - T_b(t)]c'(q) = 0. \qquad (6)$$

Choosing the threatpoints and θ closes the model. An equilibrium may be defined as a (nonnegative) list $[V_s(t), V_b(t), q(t)]_{t=0}^{\infty}$ satisfying (1), (2) and (3), subject to constraints (4) and (5) for all t.

Convenient threatpoints include $T_i(t) = 0$ and $T_i(t) = V_i(t)$ (see Trejos and Wright [1995a]). However, the apparent arbitrariness of $T_i(t)$, as well as θ, seems somewhat problematic. A response is to model the bargaining process explicitly as a noncooperative game, the equilibria of which may perhaps be represented as the solution to a particular Nash bargaining problem for appropriate choices of $T_i(t)$ and θ (this research agenda is the so-called *Nash program*). For example, Binmore (1987) and Binmore et al. (1986) show how

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different games lead to different versions of (6) in stationary environments (time-invariant payoffs, etc.).

In general, we cannot impose stationarity here, since part of the payoff to completing a bargain is the continuation value $V_i(t)$, which may vary *endogenously* with time. In the next section we therefore extend the type of result derived by Binmore (1987) and Binmore et al. (1986) to environments that are potentially nonstationary. Our generalization of (6) will be a forward looking dynamic (difference or differential) equation, rather than something quite as simple as (6); but, as we shall see, it turns out to be surprisingly tractable and fits very neatly into search theory.

3 Bargaining Theory

Since the central aim here is to obtain a solution to a strategic bargaining game that can be used in applications, we do not purport to study the most general case. In particular, we are concerned with cases where in equilibrium there is no delay (players always reach immediate agreement), even though it is the threat of delay that drives the solution. Also, out of equilibrium, if no agreement is reached in a period, we assume the players prefer to continue bargaining rather than break up to search for a new partner. The latter is automatically satisfied in models with constraints like (4) and (5). The former is less automatic but is made in the interests of tractability; indeed, unless it holds, (1) and (2) do not describe the value functions. We will refer to an equilibrium of this class as an *Immediate Trade Equilibrium* (ITE). Of course, even if we are only interested in ITE, we still have to check that

immediate trade is consistent with equilibrium behavior.[6]

We depart from the particular model in the previous section to study a slightly more general situation. Time is an infinite sequence of periods of length $\Delta > 0$. There are two agents, labeled $i = 1, 2$, and 1 is interested in obtaining some portion of a "cake" owned by 2 in exchange for a fixed amount of something else (like money). The cake may change over time. If at time t agent 1 trades with agent 2 for q units of the cake, their instantaneous utilities are $u_1(q, t)$ and $u_2(q, t)$. Additionally, they discount the future at rate $r > 0$, so that the payoff from this trade for i, discounted back to the present, is $e^{-rt} u_i(q, t)$. Assume $u_i \in C^2$, $\partial u_1 / \partial q > 0$, and $\partial u_2 / \partial q < 0$, for all t. Also, assume that u_1 and u_2 are concave in q for all t. For example, in the model of the previous section, the payoff to the buyer is $u_1(q, t) = q + V_s(t)$ and the payoff to the seller is $u_2(q, t) = -c(q) + V_b(t)$.

Agents derive some utility from not trading at all, normalized to 0. Define $\mathcal{A}(t)$ as the set

$$\mathcal{A}(t) = \{q : u_i(q, t) \geq 0, i = 1, 2\}, \tag{7}$$

and assume $\mathcal{A}(t)$ is non-empty for all t and uniformly bounded in t; i.e., gains to trade always exists, and $u_1(q, t) < 0$ for q sufficiently small and $u_2(q, t) < 0$ for q sufficiently large.[7] Finally, assume that $u_i(q, t)$ is bounded in t, and that

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[6]Much of the bargaining literature has focussed on situations where delay may occur in equilibrium; see, for example, Binmore (1987) and Merlo and Wilson (1993a, 1993b), among others. Although this is an important and interesting issue, it is not our concern here.

[7]This insures that q is bounded. In principle, we could also impose a constraint such as $q \in [0, \hat{q}]$, but we simply assume that such contraints are not binding in most of what

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the time derivative $\partial u_i(q,t)/\partial t$ is also bounded for all (q,t).

The bargaining procedure assumes random alternating offers. Suppose at time t the agents have not yet reached agreement. With equal probability, nature chooses either player 1 or 2 to propose a value of q. Given that offer, the other agent decides either to accept or reject. If he accepts, exchange takes place, payoffs are realized, and the players part company. If he rejects, they realize no instantaneous utility that period, and the game moves to the next period where nature again randomly chooses a proposer with equal probability. This continues until an offer is accepted.[8]

Our goal is to characterize subgame-perfect equilibria in strategies that are history independent although potentially nonstationary. History independent strategies do not depend on offers made at previous points in time. However, the equilibrium strategies generally must depend on time because the payoffs generally vary over time.

In any equilibrium with history independent strategies, if 1 is willing to accept q at t and $q' > q$, then 1 must also be willing to accept q' at t. Similarly, if 2 is willing to pay q at t and $q' < q$ then 2 must also be willing to pay q' at t. Hence, there exist *reservation values*, $q_1(t)$ and $q_2(t)$, such that at time t agent 1 accepts any $q \geq q_1(t)$ and agent 2 accepts any $q \leq q_2(t)$. Moreover, in any ITE, the best proposal is always the reservation value of the other agent. This implies that we can identify an ITE strategy profile with $[q_1(t), q_2(t)]$, where at time t agent 1 proposes $q_2(t)$ if it is his turn to

follows.

[8]In Section 5 we generalize things to allow different probabilities of making the next offer, different rates of time preference, breakdowns in the negotiations, etc.

make an offer and accepts any $q \geq q_1(t)$ if it is his turn to respond, while agent 2 proposes $q_1(t)$ if it is his turn to make an offer and accepts any $q \leq q_2(t)$ if it is his turn to respond.

In any ITE, therefore, the reservation values satisfy the following recursive relations:

$$u_1[q_1(t), t] = \frac{1}{1 + r\Delta} \left\{ \tfrac{1}{2} u_1[q_1(t + \Delta), t + \Delta] + \tfrac{1}{2} u_1[q_2(t + \Delta), t + \Delta] \right\} \quad (8)$$

$$u_2[q_2(t), t] = \frac{1}{1 + r\Delta} \left\{ \tfrac{1}{2} u_2[q_1(t + \Delta), t + \Delta] + \tfrac{1}{2} u_2[q_2(t + \Delta), t + \Delta] \right\} \quad (9)$$

For example, (8) says that agent 1 is indifferent between accepting his reservation value at t, or delaying until $t + \Delta$ when a new proposer will be determined. These equations are forward looking, in the sense that reservation values at t are defined in terms of reservation values at $t + \Delta$. If nothing changes over time (8) and (9) determine a pair of numbers (q_1, q_2). Here they constitute a dynamical system that determines paths $[q_1(t), q_2(t)]$. A solution to (8) and (9) is actually an ITE if and only if $q_1(t) \leq q_2(t)$ for all t, as this ensures that at each stage player 1 gets a higher payoff by trading at $q_2(t)$ than by deferring agreement, and similarly for player 2.

We begin the analysis with some preliminary technical results. Proofs are in the Appendix.[9]

Lemma 1 *In ITE, $q_i(t)$ is bounded and lies in $A(t)$ for all t, $i = 1, 2$.*

Lemma 2 *Let $q(t) = \tfrac{1}{2}[q_1(t) + q_2(t)]$. As $\Delta \to 0$, $q_i(t)$ converges to $q(t)$ and $q_i(t) - q(t) = O(\Delta)$, $i = 1, 2$.*

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[9]In the statement of Lemma 2, the notation $q_i(t) - q(t) = O(\Delta)$ means that, as $\Delta \to 0$, $q_i(t) - q(t) \to k\Delta$ for some finite k.

The last result indicates that when the time between offers is small $q_1(t)$ and $q_2(t)$ can be well approximated by the average $q(t)$. The goal now is to characterize the behavior of $q(t)$.

Theorem 1 *In ITE, in the limit as $\Delta \to 0$, $q(t)$ satisfies*

$$\dot{q} = \frac{1}{2}\left[\frac{ru_1(q,t) - \partial u_1(q,t)/\partial t}{\partial u_1(q,t)/\partial q}\right] + \frac{1}{2}\left[\frac{ru_2(q,t) - \partial u_2(q,t)/\partial t}{\partial u_2(q,t)/\partial q}\right]. \tag{10}$$

Proof: Let $\varepsilon(t) = q_1(t) - q(t) = q(t) - q_2(t)$, where $\varepsilon(t) = O(\Delta)$ by Lemma 2. A first order Taylor approximation on (8) and (9) implies

$$u_1[q(t),t] + \varepsilon(t)\frac{\partial u_1[q(t),t]}{\partial q} = \frac{u_1[q(t+\Delta),t+\Delta]}{1+r\Delta} + o(\Delta) \tag{11}$$

$$u_2[q(t),t] - \varepsilon(t)\frac{\partial u_2[q(t),t]}{\partial q} = \frac{u_2[q(t+\Delta),t+\Delta]}{1+r\Delta} + o(\Delta) \tag{12}$$

where $o(\Delta)$ satisfies $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. If we multiply (11) by $\partial u_2[q(t),t]/\partial q$ and (12) by $\partial u_1[q(t),t]/\partial q$, add the equations and simplify, we get

$$\left\{u_1[q(t),t] - \frac{u_1[q(t+\Delta),t+\Delta]}{1+r\Delta}\right\}\frac{\partial u_2[q(t),t]}{\partial q}$$

$$+ \left\{u_2[q(t),t] - \frac{u_2[q(t+\Delta),t+\Delta]}{1+r\Delta}\right\}\frac{\partial u_1[q(t),t]}{\partial q} = o(\Delta). \tag{13}$$

Now multiply by $1 + r\Delta$, divide by Δ, and take the limit as $\Delta \to 0$ to get

$$\left(ru_1 - \frac{\partial u_1}{\partial q}\dot{q} - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial q} + \left(ru_2 - \frac{\partial u_2}{\partial q}\dot{q} - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial q} = 0,$$

where for ease of notation we drop the arguments of $u_i[q(t),t]$. Solving for $\dot{q}$ yields (10). $\square$

The above result says that in a dynamic environment we can approximate the trades that occur in ITE by solutions to differential equation (10). It is instructive to consider what happens if the payoff functions u_i settle down over time.

Theorem 2 *Suppose $u_i(q,t) \rightarrow \bar{u}_i(q)$ as $t \rightarrow \infty$, $i = 1, 2$, where $\bar{u}_i$ satisfies all of the assumptions on u_i. Then, in the limit as $\Delta \rightarrow 0$, if an ITE exists it is unique, and $q(t) \rightarrow \bar{q}$ as $t \rightarrow \infty$ where*

$$\bar{q} = \arg\max \ \bar{u}_1(q)\bar{u}_2(q).$$

Proof: If $u_i(q,t) = \bar{u}_i(q)$ then (10) becomes

$$\dot{q} = \frac{1}{2}\left(\frac{r\bar{u}_1}{\bar{u}_1'} + \frac{r\bar{u}_2}{\bar{u}_2'}\right) \equiv \Upsilon(q).$$

The solution to $\Upsilon(q) = 0$ is $\bar{q}$. Moreover, $\Upsilon'(q) > 0$. This implies that if $q(t) > \bar{q}$ in the limit then $q(t)$ increases without bound, and if $q(t) < \bar{q}$ in the limit then $q(t)$ decreases without bound. But Lemma 1 says that q is bounded, and so it must converge to $\bar{q}$. As the dynamics exhibit an unstable root as $t \rightarrow \infty$ while $q(t) \rightarrow \bar{q}$ as $t \rightarrow \infty$, there is a unique solution to (10) given this boundary condition, and therefore a unique ITE. □

Hence, in steady state, an ITE outcome is the same as the Nash bargaining solution with zero threatpoints and equal bargaining power (as previously established by Binmore [1987] and Binmore et al. [1986]). But this does *not* generally hold outside of steady state. To see this, consider what we call the myopic Nash solution:

$$q(t) = \arg\max[u_1(q,t) - T_1(t)]^\theta [u_2(q,t) - T_2(t)]^{1-\theta}. \qquad (14)$$

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With $T_i(t) = 0$ and $\theta = \frac{1}{2}$, this implies

$$u_1(q, t)\frac{\partial u_2(q, t)}{\partial q} + u_2(q, t)\frac{\partial u_1(q, t)}{\partial q} = 0. \tag{15}$$

In general, and as we show below with an example, this generates a different time path for q than our forward looking solution.[10]

There is, however, a special case in which (10) and (15) do generate the same path, and not just the same steady state: the case where agents are risk neutral. This is useful because some previous authors, including Pissarides (1987), Drazen (1988), and Mortensen (1989), have imposed the myopic Nash solution in dynamic models with risk neutral agents, and this can now be justified as an approximation to a strategic bargaining game. However, to the extent that one wants to generalize those models to include risk averse agents, the myopic Nash solution would not give the same answer as our forward looking solution.

Theorem 3 *Suppose $u_i(q, t) = \eta_i q + \varphi_i(t)$, where $\eta_1 > 0 > \eta_2$ and $\varphi_i(t)$ is bounded for all t. Then if an ITE exists it is unique, and q satisfies the myopic Nash condition with $T_i(t) = 0$ and $\theta = \frac{1}{2}$ for all t; that is, q satisfies (15).*

Proof: Without loss in generality, normalize $\eta_1 = 1$ and $\eta_2 = -1$. Then (10) reduces to

$$\dot{q} = \tfrac{1}{2}\left(rq + r\varphi_1 - \dot{\varphi}_1\right) - \tfrac{1}{2}\left(-rq + r\varphi_2 - \dot{\varphi}_2\right),$$

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[10]One might be tempted to say that the myopic Nash solution gives the "wrong" answer outside of steady state because it uses the wrong threat points or bargaining power. We address this in detail below; the point here is simply that a Nash solution that gives the right steady state does not generally give the right path.

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which implies

$$\dot{q} - rq = \tfrac{1}{2}\left[(\dot{\varphi}_2 - \dot{\varphi}_1) - r(\varphi_2 - \varphi_1)\right].$$

Solutions to this equation are of the form

$$q = \tfrac{1}{2}(\varphi_2 - \varphi_1) + \eta_0 e^{rt},$$

where η_0 is a constant. Since q is bounded by Lemma 1, we have $\eta_0 = 0$ and $q = \tfrac{1}{2}(\varphi_2 - \varphi_1)$. In the case under consideration, this is also the myopic Nash solution. $\square$

Note that until now we have been assuming immediate trade. As remarked above, it must be checked that this is consistent with equilibrium behavior. Let $\Pi_i(t) = e^{-rt}u_i[q(t), t]$ be the equilibrium discounted payoff to i if agreement is made at time t, given q solves (10). Then immediate trade for all t constitutes an equilibrium if

$$\Pi_i(t) \geq 0 \quad \text{and} \quad \Pi_i'(t) \leq 0 \tag{16}$$

for all t and $i = 1, 2$, and the inequalities are strict for at least one agent. The first inequality says the agents always want to trade, while the second says they always want to trade sooner rather than later.[11]

Rather than verifying (16) directly in each application, note that $\Pi_i'(t) <$

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[11]By inspection of (11), a sufficient condition that $\varepsilon(t) \leq 0$ for all t in the limit as $\Delta \to 0$, and hence $q_1(t) \leq q_2(t)$ for all t, is that $u_1[q(t), t] - u_1[q(t + \Delta), t + \Delta]/(1 + r\Delta)$ is strictly positive and is $O(\Delta)$ for all t. By definition of $\Pi_1(t)$, a sufficient condition for existence of an ITE in the limit as $\Delta \to 0$ is that $\Pi_1'(t) < 0$ for all t. Similarly for $i = 2$.

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0 for at least one agent if and only if

$$\left(ru_1 - \frac{\partial u_1}{\partial t}\right)\frac{\partial u_2}{\partial q} - \left(ru_2 - \frac{\partial u_2}{\partial t}\right)\frac{\partial u_1}{\partial q} < 0. \tag{17}$$

A simple sufficient condition which guarantees (17) is the "shrinking cake" assumption that $e^{-rt}u_i(q,t)$ is decreasing in t for all $q \in A(t)$ for all t, and is strictly decreasing for one agent (see Binmore [1987]). However, as these payoff functions may be endogenously determined as part of a bigger model (as is the previous section), it may not be possible to establish whether the "shrinking cake" condition holds before solving the entire model. In that case, it is necessary to generate a candidate ITE and then check that (17) holds along the equilibrium path.[12]

To close this section we present a simple example, where $u_1(q,t) = q^\beta$, with $0 < \beta \leq 1$, and $u_2(q,t) = e^{-\delta t} - q$. One interpretation is that the cake is depreciating at rate δ (or, if $\delta < 0$, appreciating, but as long as $r > 0$, the agents may still want to trade sooner rather than later). For these functional forms, (10) can be written

$$\dot{q} = \frac{r(1+\beta)q - \beta(r+\delta)e^{-\delta t}}{2\beta}. \tag{18}$$

Since the u_i functions settle down over time, Theorem 2 implies $q(t) \to \bar{q}$, where in this case $\bar{q} = 0$.

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[12]If it fails for some t, and if a non-degenerate equilibrium exists, then there will be some periods where trade is delayed. The researcher's problem would then be to match trading phases to deferred-trading phases. Equation (10) would describe q in the trading phase.

The solution to (18) subject to this boundary condition is

$$q^* = \frac{\beta \left(r + \delta\right) e^{-\delta t}}{r(1 + \beta) + 2\delta\beta},$$

and the implied payoffs are $\Pi_1 = A^\beta e^{-(r+\beta\delta)t}$ and $\Pi_2 = Ae^{-(r+\delta)t}$ where $A = \frac{r+\delta}{r(1+\beta)+2\delta\beta}$. If $r + \delta > 0$ then $\Pi_i'(t) < 0$, which is sufficient for ITE. By way of comparison, the myopic Nash solution with zero threatpoints and equal bargaining powers is

$$q^n = \frac{\beta e^{-\delta t}}{1 + \beta}.$$

It is easy to check that $q^* > q^n$ if and only if $\delta > 0$. The intuition is that Π_1 falls more slowly than Π_2 along the equilibrium path when $\delta > 0$, effectively making agent 1 less averse to delay. The myopic Nash solution ignores this, while the forward looking solution takes it into account and therefore gives a bigger payoff to agent 1.[13]

4 Monetary Equilibria

We are now in a position to characterize equilibrium in the model of monetary trade with forward looking strategic bargaining. Using the results in the last section, for small $\Delta > 0$, assuming an ITE exists in the bargaining game that buyers and sellers play once they meet, the value functions are given by (1) and (2), where $q(t + \Delta)$ is a random variable which takes values $q_1(t + \Delta)$ and $q_2(t + \Delta)$ with equal probabilities, and the values of $q_1(t + \Delta)$ and $q_2(t + \Delta)$ are defined by the recursive equations (11) and (12) with $u_1(q, t) = q + V_s(t)$ and $u_2(q, t) = -c(q) + V_b(t)$.

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[13]These results hold for $\beta < 1$; if $\beta = 1$, then $q^n = q^*$, as predicted by Theorem 3.

Now consider the limiting case as $\Delta \rightarrow 0$. Rearranging and taking the limit of (1) and (2), we arrive at the continuous time dynamic programming equations,

$$rV_b = \alpha(1 - M)\tfrac{1}{k}(q + V_s - V_b) + \dot{V_b} \qquad (19)$$

$$rV_s = \alpha M \tfrac{1}{k}[-c(q) + V_b - V_s] + \dot{V_s}, \qquad (20)$$

where q is the limiting value of both q_1 and q_2. Theorem 1 says that (if an ITE exists) q satisfies (10), which in this model specializes to

$$\dot{q} = \frac{rq + rV_s - \dot{V_s}}{2} - \frac{-rc(q) + rV_b - \dot{V_b}}{2c'(q)}. \qquad (21)$$

An (immediate trade) equilibrium with forward looking strategic bargaining is now a list of functions $[V_b(t), V_s(t), q(t)]$ satisfying the three differential equations (19), (20), and (21) subject to constraints (4) and (5), as well as the immediate trade conditions (17) for all t.

Fortunately, the structure of (21) is quite simple, with the natural interpretation that q changes over time to split the flow surplus between the buyer and seller in each (arbitrarily small) period. It is this property of the bargaining outcome that makes it particularly tractable in a search context, since these flow surpluses are given by the dynamic programming equations (19) and (20). In particular, if we define $x = V_b - V_s$, then subtracting (19) and (20) yields

$$rx = \alpha(1 - M)\frac{1}{k}(q - x) - \alpha M \frac{1}{k}[-c(q) + x] + \dot{x},$$

and inserting (19) and (20) into (21) yields

$$\dot{q} = \frac{rq + \alpha M \frac{1}{k}[x - c(q)]}{2} + \frac{rc(q) + \alpha(1 - M)\frac{1}{k}(x - q)}{2c'(q)}.$$

Moreover, constraints (4) and (5) can be rewritten $(q, x) \in \mathcal{B}$, where $\mathcal{B} = \{(q, x) : c(q) \leq x \leq q\}$. This reduces the model to two dimensions.

In order to simplify the notation, rescale time by letting $\alpha/k = 1$ (which requires, of course, renormalizing r). Then the model can be represented as the dynamical system

$$\begin{bmatrix} \dot{q} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dfrac{rq + M[x - c(q)]}{2} + \dfrac{rc(q) + (1 - M)(x - q)}{2c'(q)} \\ (1 + r)x - Mc(q) - (1 - M)q \end{bmatrix}, \qquad (22)$$

and an (immediate trade) equilibrium is any solution to (22) that stays in $\mathcal{B}$ and also satisfies the sufficient conditions for immediate trade, $\Pi_b = e^{-rt}(q + V_s)$ and $\Pi_s = e^{-rt}[-c(q) + V_b]$ strictly decreasing in t. A special case is a steady state, which is an equilibrium where q and x are constant. We also distinguish between monetary and nonmonetary equilibria, where the latter entails $q = 0$ for all t.

Previously, Shi (1995a) and Trejos and Wright (1995a) studied steady states by imposing the myopic Nash bargaining solution with zero threat-points and equal bargaining power.[14] Additionally, Trejos and Wright stud-

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[14]They also analyzed the Nash solution with nonzero threatpoints. Consistent with the discussion in the next section, zero threatpoints are correct if there are no exogenous breakdowns in the bargaining, which can be motivated by the assumption that agents do not meet other potential trading partners between rounds in the bargaining game. If, alternatively, agents do meet other potential trading partners between rounds, and they have to leave with them and abandon their current partners, then there can be exogenous breakdowns and the appropriate threatpoints are $T_i = V_i$. We have also analyzed the dynamics in the model with exogenous breakdowns using the generalized version of our

ied dynamic equilibria by looking for nonconstant solutions to (19) and (20) that satisfy the myopic Nash solution at each point in time. While this yields the same q as our forward looking solution in steady state, it does not yield the same q outside of steady state. To the extent that one thinks of the Nash solution as a reduced form for strategic bargaining, clearly agents ought to be forward looking in the bargaining game as they are in other aspects of the model. Hence, the forward looking solution seems the preferred alternative.

We first look for steady states. It is obvious that $(q,x) = (0,0)$ is a nonmonetary steady state. Following the method in Trejos and Wright, it is easy to show that whenever a monetary steady state exists it is unique, and it exists if and only if $c'(0)$ is below some threshold $\tilde{c}$.[15] For the remainder of the analysis we impose $c'(0) < \tilde{c}$. We now proceed to consider dynamic equilibria with forward looking bargaining.

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forward looking solution; the results are available upon request.

[15]From (22), $\dot{q} = \dot{x} = 0$ is equivalent to $\Psi(q) = 0$, where $\Psi(q)$ is defined in (23) in the text below. Moreover, a necessary and sufficient for $(q,x) \in \mathcal{B}$ is that $q \leq \hat{q}$, where $\hat{q}$ is defined by $(1-M)\hat{q} = (r+1-M)c(\hat{q})$. Note that $\hat{q} > 0$ as long as $c'(0) < \tilde{c}$, where $\tilde{c}$ is the smaller root of the quadratic

$$\tilde{c}^2 - \frac{2(r+M)}{M}\tilde{c} + \frac{(r+M)(1-M)}{M(r+1-M)} = 0.$$

One can show $\Psi(0) = 0$, $\Psi(\hat{q}) < 0$, and $\Psi'(q) < 0$ at any $q \in (0,\hat{q}]$ such that $\Psi(q) = 0$. One can also show that $\Psi'(0) > 0$ if and only if $c'(0) < \tilde{c}$. Hence, if $c'(0) < \tilde{c}$ then there is a unique $q^* \in (0,\hat{q})$ such that $\Psi(q^*) = 0$, and therefore a unique monetary steady state; otherwise, there is no such q and no monetary steady state.

The Jacobian of (22) is

$$
J = \begin{bmatrix} r - \frac{Mc'}{2} - \frac{1-M}{2c'} - \frac{[rc+(1-M)(x-q)]c''}{2(c')^2} & \frac{M}{2} + \frac{1-M}{2c'} \\[2ex] -Mc' - (1-M) & 1 + r \end{bmatrix}.
$$

It is routine (if tedious) to show that $\det(J) = -\Psi'(q)/2c'(q)$ in steady state, where

$$
\Psi(q) = (r+M)[(1-M)q - (r+1-M)c(q)] \tag{23}
$$
$$
-(r+1-M)[(r+M)q - Mc(q)]c'(q).
$$

Since $\Psi'(0) > 0$ (see footnote 15), $\det(J) < 0$ at the nonmonetary steady state, and it is a saddle point. Since $\Psi'(q^*) < 0$, $\det(J) > 0$ at the monetary steady state, and it is either a sink or a source. One can show that in steady state

$$
\text{trace}(J) = 1 + r + \frac{(1 - M - Mc')^2 - \Psi'}{2(1+r)c'}.
$$

As $\Psi'(q^*) < 0$, $\text{trace}(J) > 0$ at the monetary steady state, it must be a source.

One can also show that along the boundary of the set $\mathcal{B}$ the flow is outward; thus, as shown in Figure 1, orbits never enter $\mathcal{B}$ from outside of $\mathcal{B}$. Since $(0,0) \in \mathcal{B}$, the saddle path leading to the nonmonetary steady state lies entirely in $\mathcal{B}$, and so any orbit beginning on the saddle path is an equilibrium. Furthermore, since it cannot come from outside of $\mathcal{B}$, the saddle path must either emanate from (q^*, x^*) or from a cycle surrounding (q^*, x^*).

Using the myopic Nash solution, Trejos and Wright (1995) also find that the monetary steady state is a source and the nonmonetary steady state is a saddle point. However, that model cannot exhibit cyclical dynamics: in

equilibrium, $q(t)$ is always decreasing in t. This is not true in the model with forward looking bargaining. It is easy to construct examples where the saddle path emanates from (q^*, x^*) and converges monotonically to $(0, 0)$, but also to choose parameter values where the monetary steady state is an unstable focus and $[q(t), x(t)]$ first spirals around (q^*, x^*) before converging to $(0, 0)$. Hence, prices and quantities can oscillate over time, at least initially.

To construct an example of a stable limit cycle, consider augmenting the model slightly by introducing a fixed cost, $c(0) > 0$. As shown in Figure 2, if the fixed cost is not too big, it shifts the $\dot{q} = 0$ and $\dot{x} = 0$ curves so that the intersection at the origin moves to $(q^0, x^0) \in int(\mathcal{B})$. Hence, there are now two monetary steady states, (q^0, x^0) and (q^*, x^*) (as well as a nonmonetary steady state at the origin, not shown in the figure). With a fixed cost, if we impose the myopic Nash bargaining solution, it is still the case that (q^*, x^*) is a source, (q^0, x^0) is a saddle, and the saddle path converges monotonically from the former to the latter. That is, there are no spirals, let alone cycles. Therefore, if we find cycles in our model, it must be due to the forward looking nature of the bargaining solution.

Our strategy is to proceed as follows. First fix M, and set $r = \bar{r}$. Then let $c(q) = a_0 + a_1 q + a_2 q^2$ in the neighborhood of (q^*, x^*), and choose the coefficients a_j so that $(q^*, x^*) \in int(\mathcal{B})$ and $\text{trace}(J) = 0$ at (q^*, x^*). Note that this is impossible when $a_0 = 0$, as we argued earlier that $\text{trace}(J) > 0$ at the monetary steady state under the assumption $c(0) = 0$, but it is possible if $a_0 > 0$ (this is where the fixed cost comes in). Now for a range of values of r in the neighborhood of $\bar{r}$, we study the system numerically.

For $r < \bar{r}$, (q^*, x^*) is a source and (q^0, x^0) a saddle; see Figure 2. The

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important thing about the global dynamics in this case is that the unstable manifold of (q^0, x^0) loops around the branch of the stable manifold connecting (q^*, x^*) and (q^0, x^0). As we increase r, these branches of the stable and unstable manifolds get closer together until, at some $r = \hat{r} < \bar{r}$, they coalesce to form a homoclinic orbit that starts at (q^0, x^0), loops around (q^*, x^*), and returns to (q^0, x^0).

For $r > \hat{r}$, the branch of the unstable manifold lies inside of a region formed by the two branches of the stable manifold and the vertical axis in Figure 3. Notice that orbits that start in this region cannot escape. Hence, the branch of the unstable manifold in this region must either converge to (q^*, x^*) or to a cycle around (q^*, x^*). But for $r < \bar{r}$, we have trace$(J) > 0$, and (q^*, x^*) is a source. Applying the Poincare − Bendixson Theorem (see, e.g., Guckhenheimer and Holmes 1983), there exists a stable limit cycle around (q^*, x^*) for all $r \in (\hat{r}, \bar{r})$. The size of the cycle is decreasing in r, and for $r > \bar{r}$, it collapses into (q^*, x^*), as shown in Figure 4.

The essential point is that for all $r \in (\hat{r}, \bar{r})$ any orbit that starts in the region formed by the stable manifold and the vertical axis depicted in Figure 3 converges to a limit cycle. To argue that such a path is an equilibrium, we need to verify two more things: that it stays within $\mathcal{B}$, and that it satisfies the immediate trade condition, $\Pi_i'(t) < 0$ for all t. Since $(q^*, x^*) \in int(\mathcal{B})$, at least for r near $\bar{r}$, the cycles are sufficiently small they must stay in $int(\mathcal{B})$. We then verified numerically in examples that $\Pi_i'(t) < 0$ along the cycle. Hence, there exist stable limit cycles that satisfy all of the conditions for monetary equilibria.

To reiterate, price cycles are possible with forward looking bargaining

but not possible using the myopic Nash solution. The intuition is relatively straightforward. The above analysis implies there are two relevant "shadow prices": x (the added value of holding money while searching) and q (the value of money when trading). In both versions of this model, the dynamic programming equations imply x is a forward looking variable which depends on future values of q. In the version with the myopic Nash solution, implicitly q is a function of the current value of x only, and this precludes cyclic phenomena. In the version with strategic bargaining, q depends on future values of x and it is this additional feedback which generates the possibility of cycles.

5 Generalizations

As we suggested earlier, our forward looking bargaining solution may prove useful in other applications. Hence, here we analyze some extensions to the basic model, including different rates of time preference, different probabilities of getting to make the next offer, and exogenous breakdowns. All proofs are in the Appendix.

Let r_i be the discount rate of agent i. Let γ_i be a flow utility that i gets while bargaining is in progress. Let λ_i be the Poisson arrival rate with which i believes exogenous breakdowns in the bargaining will occur, and b_i his utility in this event (we do not necessarily impose $\lambda_1 = \lambda_2$, since a breakdown effectively occurs for one agent but not the other if the former is replaced by an identical new agent). If there is no breakdown, the next offer is made by agent i with probability π_i, where $\pi_1 + \pi_2 = 1$. Note that we allow r_i, γ_i, π_i,

λ_i, and b_i to depend on time, although to save space we do not make this dependence explicit in the notation. First we generalize Theorem 1 (again, proofs are in the Appendix).

Theorem 4 *In ITE, in the limit as $\Delta \to 0$, $q(t)$ satisfies*

$$\dot{q} = \pi_2 \left[\frac{(r_1 + \lambda_1)u_1 - \gamma_1 - \lambda_1 b_1 - \partial u_1/\partial t}{\partial u_1/\partial q} \right]. \qquad (24)$$

$$+ \pi_1 \left[\frac{(r_2 + \lambda_2)u_2 - \gamma_2 - \lambda_2 b_2 - \partial u_2/\partial t}{\partial u_2/\partial q} \right].$$

One reason for considering the extended model is that we can provide some insight into the relationship between strategic bargaining and the Nash solution. In particular, we can ask, are there paths for $T_i(t)$ and $\theta(t)$ that make the Nash solution reproduce the path for $q(t)$ described by (24) at each point in time, and not just in steady state?

The solution to (14) satisfies

$$(1 - \theta)[u_1(q,t) - T_1]\frac{\partial u_2}{\partial q} + \theta[u_2(q,t) - T_2]\frac{\partial u_1}{\partial q} = 0. \qquad (25)$$

Suppose we have an ITE, and consider the case where Δ is arbitrarily small. Then, comparing (25) and (36) from the proof of Theorem (4) in the Appendix, it is immediate that we must choose $\theta = \pi_1$ and

$$T_i = \frac{\gamma_i \Delta + \lambda_i \Delta b_i + (1 - \lambda_i \Delta)u_i[q(t + \Delta), t + \Delta]}{1 + r_i \Delta}.$$

Outside of a steady state, the appropriate T_i is i's expected payoff should they not reach agreement this period. In an ITE, this is the agent's utility $\gamma_i \Delta$ while there is no agreement, plus the probability of a breakdown times

b_i, plus the probability of no breakdown times the equilibrium payoff from settling next period, appropriately discounted.

Although this result clarifies what the appropriate threatpoints are, in general, it is not of great practical use as T_i depends on the endogenous $u_i[q(t+\Delta), t+\Delta]$. However, there are alternative Nash representations where T_i depends only on exogenous variables *if* we restrict attention to steady states.

Theorem 5 *Suppose $u_i(q,t) = \bar{u}_i(q)$ and all parameters are constant with respect to t; then as $\Delta \to 0$, (24) and (25) generate the same q if we set*

$$T_i = \frac{\gamma_i + \lambda_i b_i}{r_i + \lambda_i}$$

$$\theta = \frac{\pi_1(r_2 + \lambda_2)}{\pi_1(r_2 + \lambda_2) + \pi_2(r_1 + \lambda_1)}.$$

As a special case, if $r_i = r$ and $\lambda_i = \lambda$, then $\theta = \pi_1$.

The threatpoint T_i has a simple interpretation as i's expected discounted payoff should they continue bargaining but never reach agreement. The next theorem shows that this threatpoint is also appropriate outside of steady state if both players are risk neutral (and given some additional conditions).

Theorem 6 *Suppose an ITE exists in the limit as $\Delta \to 0$, and $u_i(q,t) = \eta_i q + \varphi_i(t)$, where $\eta_1 > 0 > \eta_2$ and $\varphi_i(t)$, $b_i(t)$, and $\gamma_i(t)$ are bounded for all t. Without loss of generality, let $\eta_1 = 1$ and $\eta_2 = -1$. Then, if π_i does not depend on t and $r_i + \lambda_i = r + \lambda$ does not depend on i (but may depend on t) there is a Nash representation for q given by $\theta = \pi_1$ and*

$$T_i = \int_t^\infty e^{-R(t,\tau)} \left[\gamma_i(\tau) + \lambda_i(\tau) b_i(\tau) \right] d\tau$$

where $R(t, \tau) = \int_t^\tau [r(\sigma) + \lambda(\sigma)]d\sigma$. In particular, if γ_i, λ_i, b_i, and r_i are constants then

$$T_i = \frac{\gamma_i + \lambda_i b_i}{r_i + \lambda_i}.$$

With these results in hand, it is worth mentioning that several earlier results in the literature can be obtained as special cases. First, let $\lambda_i = \gamma_i = 0$; then $T_i = 0$ and $\theta = \pi_1 r_2 / (\pi_1 r_2 + \pi_2 r_1)$, which is the Nash representation derived in Binmore et al. (1986) for what they call their time preference model. Now set $r_i = \gamma_i = 0$ but relax the assumption $\lambda_i = 0$; then $T_i = b_i$ and $\theta = \pi_1 \lambda_2 / (\pi_1 \lambda_2 + \pi_2 \lambda_1)$, which is the representation in Binmore et al. (1986) for what they call their model with exogenous breakdowns.

An important case is the one where after a breakdown agent i may meet a new partner, as in many models of search and bargaining. Let the arrival rate of new partners for i be α_i, and for simplicity set $\gamma_i = 0$; then the usual dynamic programming equation from search theory implies that (in steady state)

$$r_i b_i = \alpha_i (\bar{u}_i - b_i). \tag{26}$$

In this case, equilibrium in the bargaining game satisfies (see equation 35 in the Appendix)

$$\pi_2 \left[(r_1 + \lambda_1)(\bar{u}_1 - b_1) + r_1 b_1 \right] \bar{u}_2' + \pi_1 \left[(r_2 + \lambda_2)(\bar{u}_2 - b_2) + r_2 b_2 \right] \bar{u}_1' = 0. \tag{27}$$

Inserting (26) into (27), we have

$$\pi_2 (r_1 + \lambda_1 + \alpha_1)(\bar{u}_1 - b_1)\bar{u}_2' + \pi_1 (r_2 + \lambda_2 + \alpha_1)(\bar{u}_2 - b_2)\bar{u}_1' = 0. \tag{28}$$

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This implies that q has a Nash representation with $T_i = b_i$ and[16]

$$\theta = \frac{\pi_1(r_2 + \alpha_2 + \lambda_2)}{\pi_1(r_2 + \alpha_2 + \lambda_2) + \pi_2(r_1 + \alpha_1 + \lambda_1)}. \qquad (29)$$

A typical case in the literature is the one where the only source of breakdowns is that new agents may arrive during the bargaining, and when a new type i agent arrives he replaces the incumbent (see, e.g., Rubinstein and Wolinsky [1985], Wolinsky [1987], or Binmore and Herarro [1988]). This implies that the breakdown rate for type 1 is the arrival rate for type 2 and vice-versa: $\lambda_1 = \alpha_2$ and $\lambda_2 = \alpha_1$. If we let $r_1 = r_2$ and $\pi_1 = \pi_2$, for example, then $\theta = \frac{1}{2}$, and different arrival rates show up in terms of different threat points but the same bargaining power.[17]

Many search and bargaining models of the labor market can be interpreted as special cases of this framework (see, e.g., Mortensen and Pissarides [1994] and the references contained therein). The monetary model in the previous section can also be modified to include breakdowns in bargaining.

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[16]Alternatively, we can eliminate b_i entirely from (27) and write

$$\frac{\pi_2(r_1 + \lambda_1 + \alpha_1)r_1}{r_1 + \alpha_1}\bar{u}_1\bar{u}_2' + \frac{\pi_1(r_2 + \lambda_2 + \alpha_1)r_2}{r_2 + \alpha_2}\bar{u}_2\bar{u}_1' = 0.$$

This implies that q has an equivalent Nash representation with $T_i = 0$ and

$$\theta = \frac{\pi_1 r_2(r_1 + \alpha_1)(r_2 + \alpha_2 + \lambda_2)}{\pi_1 r_2(r_1 + \alpha_1)(r_2 + \alpha_2 + \lambda_2) + \pi_2 r_1(r_2 + \alpha_2)(r_1 + \alpha_1 + \lambda_1)}.$$

[17]Alternatively, following the previous footnote, there is an equivalent Nash representation with $T_i = 0$ and $\theta \neq \frac{1}{2}$ when $\alpha_1 \neq \alpha_2$; in this case, different arrival rates show up in terms of different bargaining power and the same threat point. Hence, different arrival rates can be captured either by different bargaining power or by different threat points, at least if we are only concerned with steady states.

Steady states of such a model have already been analyzed in Shi (1995a) and Trejos and Wright (1995a), and it is known that some substantive results can change due to the possibility of breakdowns. Using (24), one can also study dynamic equilibria in this version of the model (results are available upon request).

6 Conclusion

This paper has analyzed models of search and bargaining in potentially non-stationary environments. Although we concentrated exclusively on equilibria with immediate trade, dynamics are important in that the terms of settlement depend in general on when buyers and sellers meet. As a function of time, the terms of trade can be characterized by a relatively simple dynamical system, which may be useful for applications in dynamic economics in much the same way that the Nash solution is useful in static or steady state analyses. In the context of the search-theoretic model of fiat money, it was demonstrated that our solution does not give the same answer as the myopic Nash solution except in steady state. In particular, we constructed an example with price cycles in the forward looking model, something that cannot happen in the model with myopic bargaining solution.

Appendix

Proof of Lemma 1: By always rejecting offers and making offers $q_i(t) \in \mathcal{A}(t)$, agent i can always guarantee a non-negative payoff. Hence an equilibrium offer must lie in $\mathcal{A}(t)$. Then $q_i(t)$ must be bounded because $\mathcal{A}(t)$ is uniformly bounded. $\square$

Proof of Lemma 2: We must show that for all t, for small Δ, $q_2(t) - q_1(t) = O(\Delta^a)$ where $a \geq 1$. By way of contradiction, suppose that at some t we have $q_2(t) - q_1(t) = O(\Delta^a)$ with $a < 1$. Notice that ITE requires $q_2(t) > q_1(t)$, while Lemma 1 requires $a \geq 0$. Now let $h = h_0 \Delta^b$, where $h_0 > 0$ and $a < b < 1$, and consider the time interval $T_h = [t, t+h]$. By construction, $h \to 0$ as $\Delta \to 0$. Also, if N denotes the number of Δ time periods in T_h then $N \to \infty$ as $\Delta \to 0$.

The following result sets up the required contradiction.

Claim: Fix $\Delta > 0$ and $k > 0$. Let $n = 1, 2, ...$, and let M be the number of time periods in an ITE where $t + n\Delta \in T_h$ and

$$\frac{u_1[q_2(t+n\Delta), t+n\Delta]}{(1+r\Delta)^n} > u_1[q_1(t), t] + k\Delta. \tag{32}$$

Then, as $\Delta \to 0$, $\frac{M}{N} \to 0$.

Proof: Let $P_1(t)$ be the expected payoff to player 1 at t if agreement is not reached at t. Player 1 can always use the following strategy in the subgame:

1. Always reject player 2's offer;

2. In period $t + n\Delta$, propose $q > q_2(t+n\Delta)$ if (32) does not hold;

3. In period $t + n\Delta$, propose $q = q_2(t+n\Delta)$ if (32) holds.

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Given player 2's strategy in ITE, this strategy implies

$$P_1(t) \geq \{u_1[q_1(t), t] + k\Delta\} \left[1 - \left(\tfrac{1}{2}\right)^M\right]. \qquad (33)$$

Settlement occurs in the third contingency in the above list; the probability that this never occurs is $(1/2)^M$, in which case $u_1 \geq 0$. Now ITE requires $P_1(t) \leq u_1[q_1(t), t]$. This and (33) imply

$$\left(\frac{1}{2}\right)^M \geq \frac{k\Delta}{u_1[q_1(t), t] + k\Delta},$$

or, equivalently,

$$M \leq \frac{\log(u_1 + k\Delta) - \log(k\Delta)}{\log(2)}.$$

Now consider the limit as $\Delta \to 0$. If $u_1 = 0$ then $M = 0$. If $u_1 > 0$ (but bounded) then, noting that $1/N = O(\Delta^{1-b})$, we have

$$\frac{M}{N} \leq O(-\Delta^{1-b} \log \Delta).$$

Hence, $\frac{M}{N} \to 0$. This proves the claim.

By symmetry, the same result holds for player 2. Hence, as $\Delta \to 0$, most time periods $t + n\Delta \in T_h$ are characterized by

$$\frac{u_1[q_2(t + n\Delta), t + n\Delta]}{(1 + r\Delta)^n} \leq u_1[q_1(t), t] + k\Delta \qquad (34)$$

$$\frac{u_2[q_1(t + n\Delta), t + n\Delta]}{(1 + r\Delta)^n} \leq u_2[q_2(t), t] + k\Delta \qquad (35)$$

By concavity, (34) implies

$$\frac{u_1[q_2(t + n\Delta), t + n\Delta]}{(1 + r\Delta)^n} \leq u_1[q_2(t + n\Delta), t] \\ + [q_1(t) - q_2(t + n\Delta)]\frac{\partial u_1[q_2(t + n\Delta), t]}{\partial q} + k\Delta.$$

This can be rewritten

$$q_2(t + n\Delta) \le q_1(t) + R_1(t, t + n\Delta, \Delta),$$

where $R_1(t, t + n\Delta, \Delta)$ is defined to make the statements equivalent.

We know $q_2(t+n\Delta)$ is bounded and u_1 is continuous with a bounded time derivative. As $n\Delta < h$, it follows that $|R_1(t, t + n\Delta, \Delta)| = O(\Delta^b)$. Similarly,

$$q_1(t + n\Delta) \ge q_2(t) + R_2(t, t + n\Delta, \Delta)$$

where $|R_2(t, t+n\Delta, \Delta)| = O(\Delta^b)$ (note that the inequality is reversed because $\partial u_2/\partial q < 0 < \partial u_1/\partial q$). Subtracting,

$$q_2(t+n\Delta) - q_1(t+n\Delta) \le -[q_2(t) - q_1(t)] + R_1(t, t+n\Delta, \Delta) - R_2(t, t+n\Delta, \Delta).$$

But $q_2(t) - q_1(t) > 0$ and is $O(\Delta^a)$, where $a < b$. Hence, as $\Delta \to 0$, there must exist many time periods $t+n\Delta \in T_h$ where $q_2(t+n\Delta) - q_1(t+n\Delta) < 0$, which is a contradiction. This completes the proof. $\square$

Proof of Theorem 4: The generalized versions of (8) and (9) are

$$u_1[q_1(t), t] = \frac{1 - \lambda_1\Delta}{1 + r_1\Delta} \{\pi_1 u_1[q_2(t + \Delta), t + \Delta] + \pi_2 u_1[q_1(t + \Delta), t + \Delta]\}$$

$$+ \frac{\gamma_1\Delta + \lambda_1\Delta b_1}{1 + r_1\Delta} + o(\Delta)$$

$$u_2[q_2(t), t] = \frac{1 - \lambda_2\Delta}{1 + r_2\Delta} \{\pi_1 u_1[q_2(t + \Delta), t + \Delta] + \pi_2 u_1[q_1(t + \Delta), t + \Delta]\}$$

$$+ \frac{\gamma_2\Delta + \lambda_2\Delta b_2}{1 + r_2\Delta} + o(\Delta),$$

33

where $o(\Delta)$ appears because $\lambda_i \Delta + o(\Delta)$ is the probability of a breakdown, by the Poisson assumption. For any t, let $q = \pi_1 q_1 + \pi_2 q_2$ and $\varepsilon = q_1 - q_2$. Notice that $q_1 - q = \pi_1 \varepsilon$ and $q_2 - q = -\pi_2 \varepsilon$. Then, as in the proof of Theorem 1, approximate the above equations around q:

$$u_1[q(t), t] + \pi_1 \varepsilon \frac{\partial u_1[q(t), t]}{\partial q} = \frac{1 - \lambda_1 \Delta}{1 + r_1 \Delta} u_1[q(t+1), t+1]$$

$$+ \frac{\gamma_1 \Delta + \lambda_1 \Delta b_1}{1 + r_1 \Delta} + o(\Delta)$$

$$u_2[q(t), t] - \pi_2 \varepsilon \frac{\partial u_2[q(t), t]}{\partial q} = \frac{1 - \lambda_2 \Delta}{1 + r_2 \Delta} u_2[q(t+1), t+1]$$

$$+ \frac{\gamma_2 \Delta + \lambda_2 \Delta b_2}{1 + r_2 \Delta} + o(\Delta).$$

Multiply the first by $\pi_2 \partial u_2[q(t), t]/\partial q$ and the second by $\pi_1 \partial u_1[q(t), t]/\partial q$, then add to get the following generalization of (13):

$$\pi_2 \left\{ u_1[q(t), t] - \frac{\gamma_1 \Delta + \lambda_1 \Delta b_1}{1 + r_1 \Delta} - \frac{1 - \lambda_1 \Delta}{1 + r_1 \Delta} u_1[q(t+\Delta), t+\Delta] \right\} \frac{\partial u_2[q(t), t]}{\partial q}$$

$$+ \pi_1 \left\{ u_2[q(t), t] - \frac{\gamma_2 \Delta + \lambda_2 \Delta b_2}{1 + r_2 \Delta} - \frac{1 - \lambda_2 \Delta}{1 + r_2 \Delta} u_2[q(t+\Delta), t+\Delta] \right\} \frac{\partial u_1[q(t), t]}{\partial q}$$

$$= o(\Delta). \tag{36}$$

Finally, multiply (36) by $(1 + r_1 \Delta)(1 + r_2 \Delta)$, divide by Δ, let $\Delta \to 0$, and simplify to arrive at the differential equation (24). $\square$

Proof of Theorem 5: In a steady state where q is constant, (36) implies

:

$$\pi_2(r_1 + \lambda_1)\left(\bar{u}_1 - \frac{\gamma_1 + \lambda_1 b_1}{r_1 + \lambda_1}\right)\bar{u}_2' + \pi_1(r_2 + \lambda_2)\left(\bar{u}_2 - \frac{\gamma_2 + \lambda_2 b_2}{r_2 + \lambda_2}\right)\bar{u}_1' = 0 \quad (37)$$

Comparing this with (25), the desired result follows immediately. □

Proof of Theorem 6: For these preferences (14) implies

$$q = \theta(\varphi_2 - T_2) - (1 - \theta)(\varphi_1 - T_1), \tag{38}$$

while (24) implies

$$
\begin{aligned}
\dot{q} - [\pi_2(r_1 + \lambda_1) + \pi_1(r_2 + \lambda_2)]q &= \pi_2[(r_1 + \lambda_1)\varphi_1 - \dot{\varphi}_1] \\
-\pi_1[(r_2 + \lambda_2)\varphi_2 - \dot{\varphi}_2] &- \pi_2[\gamma_1 + \lambda_1 b_1] + \pi_1[\gamma_2 + \lambda_2 b_2].
\end{aligned}
\tag{39}
$$

We want to know when we can find a solution to (39) of the form (38).

Notice (39) is linear and first order, and can be integrated using an integrating factor. In general, such a solution will be a complicated integral of future values of φ_i and $\dot{\varphi}_i$. However a special case arises when $\dot{q}$, $\dot{\varphi}_1$ and $\dot{\varphi}_2$ have the same integrating factor, which requires $r_1 + \lambda_1 = r_2 + \lambda_2 = r + \lambda$. In that case, (39) simplifies to

$$
\begin{aligned}
\dot{q} - (r + \lambda)q &= \pi_2[(r + \lambda)\varphi_1 - \dot{\varphi}_1] - \pi_1[(r + \lambda)\varphi_2 - \dot{\varphi}_2] \\
&- \pi_2[\gamma_1 + \lambda_1 b_1] + \pi_1[\gamma_2 + \lambda_2 b_2].
\end{aligned}
$$

Integrating, applying Lemma 1, and assuming that π_i does not depend on t yields

$$q = \pi_1\varphi_2 - \pi_2\varphi_1 + \int_t^\infty e^{-R(t,\tau)}\left[\pi_2(\gamma_1 + \lambda_1 b_1) - \pi_1(\gamma_2 + \lambda_2 b_2)\right]d\tau,$$

where $R(t, \tau)$ is defined in the statement of the theorem. Comparing this with (38) yields the desired threat points and bargaining power. The special case where the parameters are time-invariant follows from simplification. □

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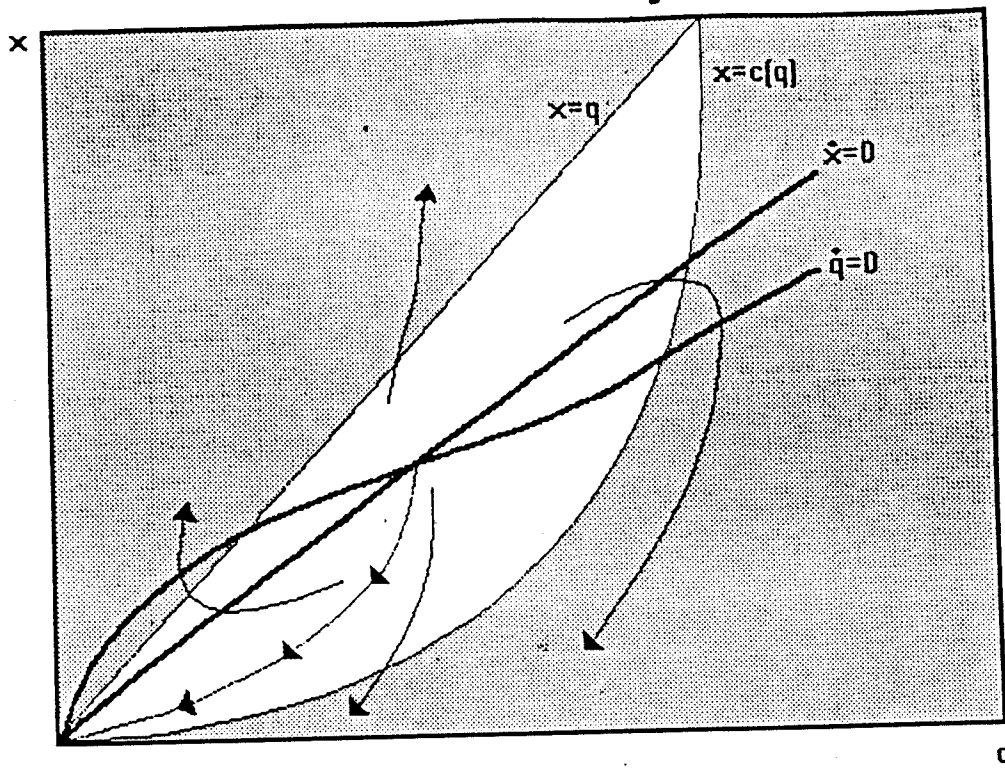
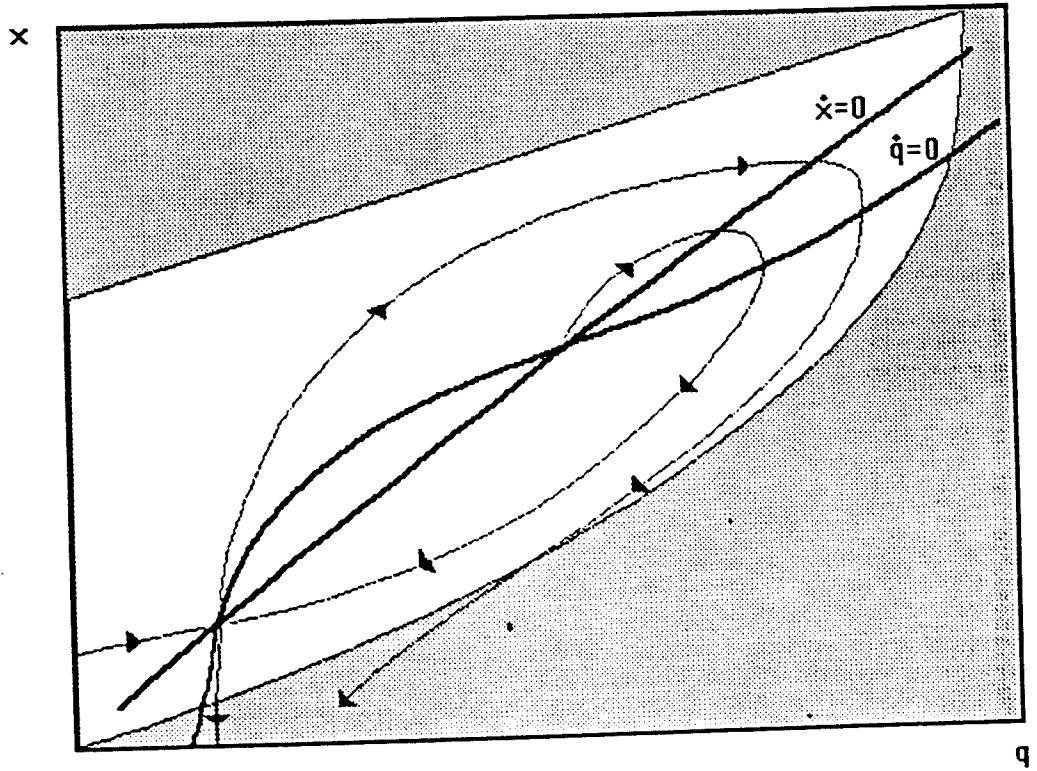Figure 1: Monetary and nonmonetary steady states (source and saddle).

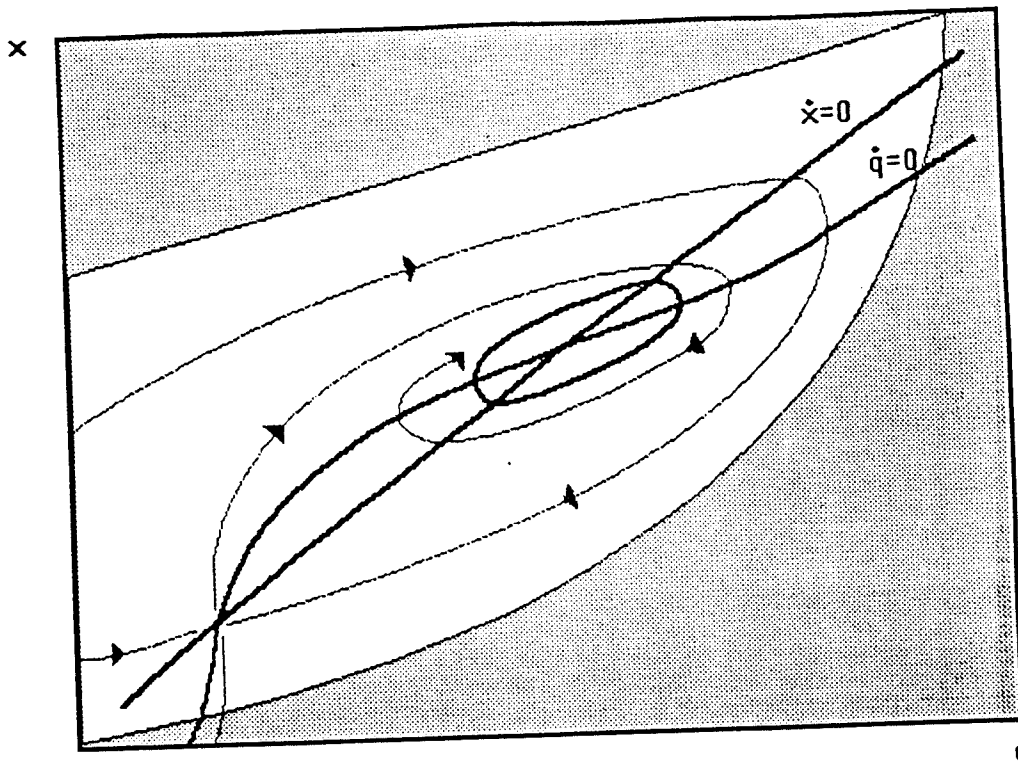Figure 2: Two monetary steady states, one source and one saddle.

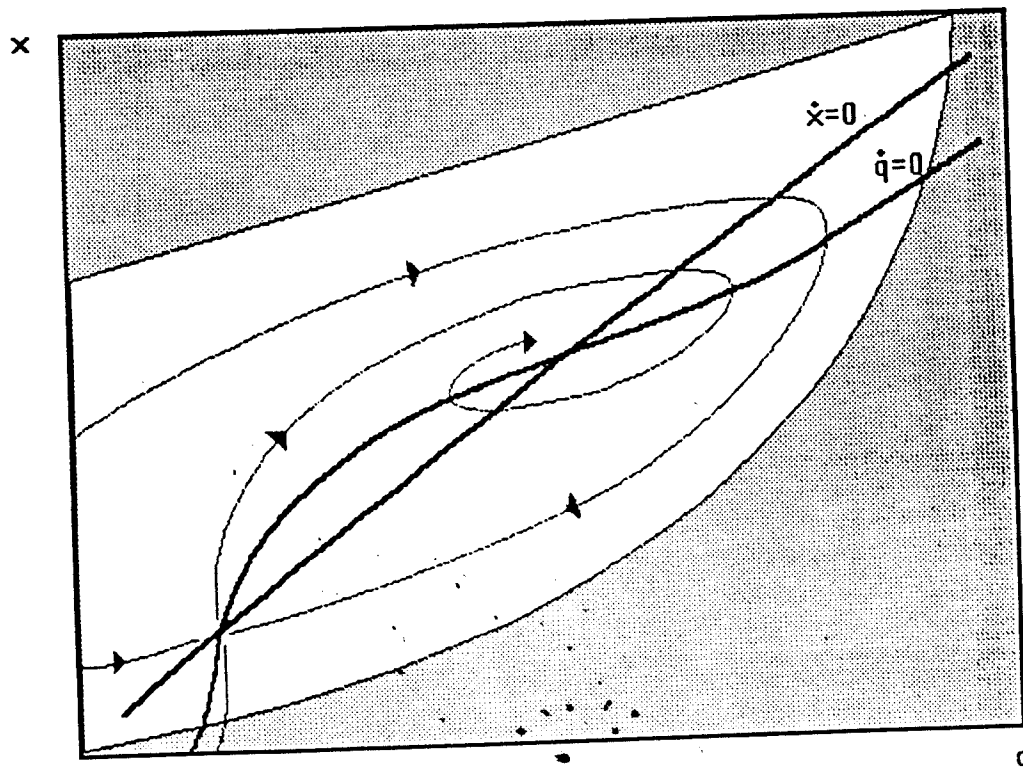Figure 3: Stable limit cycle around the steady state which is a source.

Figure 4: Cycle collapses into steady state, which becomes a sink.