Global dynamics of oscillator populations under common noise

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Abstract – Common noise acting on a population of identical oscillators can synchronize them. We develop a description of this process which is not limited to the states close to synchrony, but provides a global picture of the evolution of the ensembles. The theory is based on the Watanabe-Strogatz transformation, allowing us to obtain closed stochastic equations for the global variables. We show that on the initial stage, the order parameter grows linearly in time, while on the late stages the convergence to synchrony is exponentially fast. Furthermore, we extend the theory to nonidentical ensembles with the Lorentzian distribution of natural frequencies and determine the stationary values of the order parameter in dependence on driving noise and mismatch.

Introduction. – Synchronization of oscillations by a periodic forcing is a general phenomenon observed in numerous experiments. In this setup the system follows the driving and has, in particular, the same frequency, so one often speaks on frequency entrainment. Much more non-trivial is the effect of synchronization by an external noise. Here one can also distinguish between the cases when the driven system is entrained by the noise (synchrony) and the situations when the noise is not followed (asynchrony). While the difference between these regimes can be hardly seen by observing just one responding oscillator, it becomes evident if an ensemble of identical systems driven by the same noise is observed: in the case of synchronization all the oscillators in the ensemble follow the forcing and their states thus coincide, while in the asynchronous state the states of systems remain different. This effect is therefore called synchronization by common noise [1–6]. An interesting realization of this type of synchronization is the effect of reliability of neurons [7]. Here one does not use an ensemble of identical neurons, but takes one neuron and applies the same pre-recorded noise to it several times. The synchronous case appears as a reliable respond to the forcing where all the noise-induced spikes are at the same position at all runs, while for asynchrony (antireliability) the same noise produces different spike patterns [8,9]. Synchronization by common noise was also observed in physical experiments with phase-locked loop [10]

and noise-driven lasers [11].

Synchronization by common noise can be characterized by the largest Lyapunov exponent of the noise-driven dynamics. This exponent governs the growth/decay of small perturbations of a synchronous state; a negative exponent corresponds to synchrony while a positive one to asynchrony [1–6] (notice that here the largest Lyapunov exponent can be interpreted as a "transverse" one, determining the growth/decay of the difference between oscillators in the ensemble). For periodic oscillators, which in the noise-free case have a zero maximal Lyapunov exponent, small noise generally leads to a negative exponent (while large noise can desynchronize); for chaotic systems with a positive Lyapunov exponent, strong noise can synchronize (see examples of the synchronization-desynchronization transition in [1,2,12–14]). Calculation of the Lyapunov exponent is a relatively easy numerical task, and in many cases it can be obtained analytically [4–6]. This theory is, however, restricted to the linear analysis of a stability of the synchronous state, and does not allow one to follow the evolution starting from a broad distribution of the phases.

The goal of this letter is to present a *global* analytic theory of the synchronization by common noise, i.e. a theory describing the evolution toward synchrony of the population starting from the distributed, asynchronous state. Our theory is based on the Watanabe-Strogatz ansatz [15] that allows one to describe a population of phase oscil-

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lators under common forcing via closed equations for the macroscopic, global variables. We will show that the resulting equations can be written as a noise-driven Hamiltonian system, and will analyze the evolution of ensembles close to synchrony (where the results of the Lyapunov analysis will be recovered) as well as the evolution starting from the maximally asynchronous state.

Global variables description of ensembles. – Our goal is to describe an ensemble of identical (phase) oscillators, and one can show that the following model is general enough to describe all interesting cases:

$$\dot\varphi_k = \Omega(t) + \mathrm{Im}(F(t)e^{-i\varphi_k}), \quad k = 1,\ldots,N . \quad (1)$$

Here oscillators are described by their phases φ_k, and $\Omega(t)$ and $F(t)$ are time-dependent common forces (without loss of generality we can assume that $F(t)$ is real, otherwise one shifts the phases φ by the argument of complex F and correspondingly redefines Ω). We want to characterize the evolution of the ensemble especially for noisy forces.

Now we present two particularly important applications, which can be described using Eqs. (1). One is an ensemble of self-sustained oscillators under common external force. Unforced oscillators are described by $\dot\varphi_k = \omega$, and a noisy forcing is typically described by

$$\dot\varphi_k = \omega - \sigma\xi(t)\sin\varphi_k . \quad (2)$$

This system, previously considered in [4,6], reduces to Eq. (1) with constant $\Omega = \omega$ and $F(t) = \sigma\xi(t)$. Another relevant physical setup is a sequential array of shunted Josephson junctions subject to a common current $I(t)$:

$$\frac{\hbar}{2eR}\frac{d\varphi_k}{dt} + I_c\sin\varphi_k = I(t) . \quad (3)$$

Here φ_k is the junction's phase (difference of the phases of the macroscopic wave functions in superconductors constituting the junction), R is the resistance of the shunt, I_c is the critical current. Supposing that the current $I(t)$ has a constant and noisy component $I = I_0 + I_1(t)$, then by rescaling time $t \to \frac{I_c 2eR}{\hbar}t$, we can write system (3) as

$$\dot\varphi_k = \omega + \sigma\xi(t) - \sin\varphi_k , \quad (4)$$

which is also Eq. (1) with constant $F = 1$ and time-dependent $\Omega(t) = (I_0 + I_1(t))(I_c)^{-1} = \omega + \sigma\xi(t)$.

In the seminal work [15] Watanabe and Strogatz (WS) demonstrated that the ensemble (1), for any $N > 3$, can be fully described with three global variables and $N-3$ constants of motion. We will use here the formulation of the WS theory given in [16]. The transformation to the global variables ρ, Φ, Ψ and constants ψ_k is performed according to

$$e^{i\varphi_k} = e^{i\Phi}\frac{\rho + e^{i(\psi_k - \Psi)}}{\rho e^{i(\psi_k - \Psi)} + 1} , \quad (5)$$

with an additional condition $\sum_k e^{i\psi_k} = 0$. The closed system of equations for ρ, Φ reads (as Ψ does not enter in the dynamical equations for $\dot\rho$ and $\dot\Phi$, it does not to be taken into account.)

$$\dot\rho = \frac{1 - \rho^2}{2}\mathrm{Re}(F(t)e^{-i\Phi}) ,$$
$$\dot\Phi = \Omega(t) + \frac{1 + \rho^2}{2\rho}\mathrm{Im}(F(t)e^{-i\Phi}) . \quad (6)$$

The physical meaning of the global variables ρ, Φ is clear from their definition (5). The case of uniformly spread constants of motion ψ_k is easiest to interpret, because as it has been shown in [16,17], for a uniform distribution of constants ψ_k one has $\rho\exp(i\Phi) = N^{-1}\sum_k \exp(i\varphi_k)$. This means that $z = \rho e^{i\Phi}$ is the complex Kuramoto order parameter widely used for characterizing synchrony in the ensemble. Thus, ρ is roughly proportional to the mean field amplitude: for $\rho = 0$ the phases φ_k are uniformly spread while for $\rho = 1$ they form a cluster (from which at most one oscillator with $\psi_k - \Psi = \pi$ may deviate) of perfect synchrony. The variable Φ, being the phase of the mean field, characterizes the position of the maximum in the distribution of phases. Finally, the variable Ψ characterizes the offset of the phases of individual oscillators with respect to Φ. For a non-uniform distribution of the constants ψ_k, the complex variable z does not coincide with the Kuramoto order parameter; nevertheless, the limits $\rho \to 0$ and $\rho \to 1$ correspond to fully asynchronous and fully synchronous cases, so ρ yields a proper characterization of synchrony.

Hamiltonian formulation. – Our goal in this paper is to describe the evolution of the ensemble (1) for noisy $\Omega(t), F(t)$, by virtue of the global variables dynamics (6). Remarkably, one can reformulate the basic equations (6) as a *Hamiltonian* system [18]. Indeed, in variables

$$q = \frac{\rho\cos\Phi}{\sqrt{1 - \rho^2}} , \qquad p = -\frac{\rho\sin\Phi}{\sqrt{1 - \rho^2}} , \quad (7)$$

the equations read

$$\dot q = \frac{\partial H}{\partial p} = \Omega(t)p + F(t)\frac{1 + q^2 + 2p^2}{2\sqrt{1 + p^2 + q^2}} ,$$
$$\dot p = -\frac{\partial H}{\partial q} = -\Omega(t)q - F(t)\frac{qp}{2\sqrt{1 + p^2 + q^2}} , \quad (8)$$

with Hamiltonian

$$H(q,p,t) = \Omega(t)\frac{p^2 + q^2}{2} + F(t)\frac{p\sqrt{1 + p^2 + q^2}}{2} . \quad (9)$$

One can also formulate the dynamics in "action-angle" variables [19], where the angle is the WS variable Φ and the action is defined according to

$$J = \frac{q^2 + p^2}{2} = \frac{\rho^2}{2(1 - \rho^2)} . \quad (10)$$

The Hamiltonian equations now read

$$\dot{J} = F(t)\frac{\sqrt{2J(2J+1)}}{2}\cos(\Phi) = -\frac{\partial H}{\partial \Phi} ,$$

$$\dot{\Phi} = \Omega(t) - F(t)\frac{4J+1}{2\sqrt{2J(2J+1)}}\sin(\Phi) = \frac{\partial H}{\partial J} , \quad (11)$$

with Hamiltonian,

$$H(J,\phi,t) = \Omega(t)J - F(t)\frac{\sqrt{2J(2J+1)}}{2}\sin\Phi . \quad (12)$$

The action J yields, according to its relation to the Kuramoto order parameter ρ, a natural characterization of synchrony in the population of oscillators: $J \to 0$ corresponds to a maximally asynchronous, uniformly spread state, while $J \to \infty$ corresponds to a perfect synchrony where all oscillators cluster in a single point in phase space.

The Hamiltonian formulation allows us to give a general qualitative description of the dynamics. Notice that since the Hamiltonian is time-dependent, the energy in eqs. (8,11) is not conserved. Typically, noise leads to a growth of energy, either diffusive or exponential (in exceptional cases, e.g. if in (12) $F(t)$ vanishes, the system possesses an integral and no growth of energy is observed). Thus, the action variable J grows and the system tends to synchrony. We analyze below this growth for large and small values of J, i.e. close to synchrony and close to asynchrony, using $\langle J \rangle$ as the order parameter characterizing the level of synchrony.

Dynamics close to synchrony. – Close to synchrony, i.e. for $J \gg 1$, we can approximate the Hamiltonian (12) as $H(J,\phi,t) = (\Omega(t) - F(t)\sin\Phi)J$ which leads to a skew system where the dynamics of Φ does not depend on J:

$$\dot{J} = F(t)J\cos(\Phi) , \quad (13)$$

$$\dot{\Phi} = \Omega(t) - F(t)\sin(\Phi) . \quad (14)$$

This yields $\ln J(t) = \ln J(0) + \int_0^t F(t')\cos\Phi(t')dt'$ and $\Phi(t)$ is a solution of (14). On average, $\ln J$ grows linearly in time with the rate given by the Lyapunov exponent

$$\left\langle \frac{d}{dt}\ln J \right\rangle = -\lambda = \langle F(t)\cos(\Phi) \rangle . \quad (15)$$

The same Lyapunov exponent appears when one directly analyses stability of the cluster $\varphi_1 = \varphi_2 = \ldots = \varphi_N = \Phi$ in system (1). The phase of the cluster obeys (14) and the small deviation of one of the phases $\delta\varphi$ obeys

$$\frac{d}{dt}\delta\varphi = -F(t)\cos(\Phi)\delta\varphi ,$$

what means that $\delta\varphi \sim \exp(\lambda t)$ with the Lyapunov exponent defined in (15). For both applications (2) and (4), the calculations of the Lyapunov exponent for a white Gaussian noise $\xi(t)$ have been already reported in the literature (see Fig. 9.4 in book [3] for (4) and refs. [4–6] for (2)).
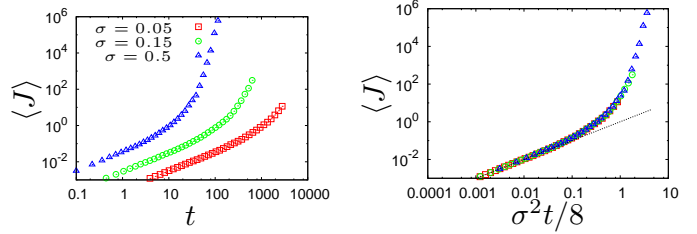

Fig. 1: (Color online) Evolution of $\langle J(t) \rangle$ for noise-driven oscillators for different σ (right panel: the same in a rescaled time), for $\omega = 2$ and different noise intensities σ. Dashed line corresponds to relation (19).

Dynamics close to asynchrony. – Here we describe the dynamics for small values of J, i.e. close to the asynchronous regime with nearly uniform distribution of the phases φ_k. We perform the analysis separately for the noise-driven oscillators (2) and noise-driven Josephson junctions (4).

An ensemble of noise-driven oscillators (2) is described by Hamiltonian (9) with $F(t) = \sigma\xi(t)$ and $\Omega = \omega = const$. For small $q, p \ll 1$ we can approximate the Hamiltonian as,

$$H(q,p,t) = \omega\frac{p^2+q^2}{2} + \sigma\xi(t)\frac{p}{2} . \quad (16)$$

Assuming for simplicity of presentation that we start from the vanishing order parameter, i.e. $q(0) = p(0) = 0$, we can easily solve the resulting linear equations:

$$q(t) + ip(t) = \frac{\sigma}{2}\int_0^t \exp[i\omega(t'-t)]\xi(t')dt', \quad (17)$$

which yields for the action J, after averaging,

$$\langle J(t) \rangle = \tfrac{1}{2}\left\langle q^2 + p^2 \right\rangle =$$

$$= \tfrac{\sigma^2}{8}\int_0^t \int_0^t e^{i\omega(t'-t'')}\langle \xi(t')\xi(t'') \rangle\, dt'dt''$$

Introducing $\tau = t' - t''$ and assuming that $\langle \xi(t')\xi(t'') \rangle$ depends only on τ one can integrate over t' so that

$$\langle J(t) \rangle = t\frac{\sigma^2}{8}\int_{-t}^t \langle \xi(0)\xi(\tau) \rangle \cos\omega\tau(1 - \frac{|\tau|}{t})d\tau . \quad (18)$$

Asymptotically, for large t this describes a diffusive linear growth of J (provided the integral converges, i.e. the correlation function of noise decays fast enough); in the case of white noise, when $\langle \xi(0)\xi(t') \rangle = \delta(t')$, we get

$$\langle J(t) \rangle = \frac{\sigma^2}{8}t . \quad (19)$$

In Fig. 1 we compare for model (2) the theoretical results with the numerical ones (obtained from eqs. (8),(11)).

In the case of noise-driven Josephson junctions (4) we need to analyze the Hamiltonian

$$H(J,\Phi,t) = (\omega + \sigma\xi(t))J - \frac{\sqrt{2J(2J+1)}}{2}\sin\Phi . \quad (20)$$
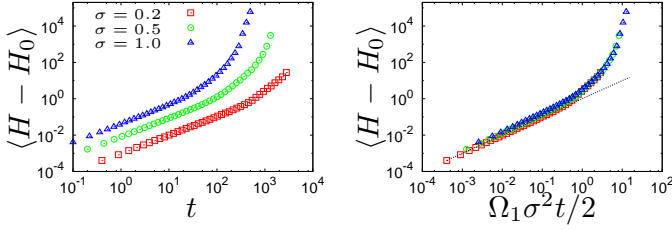
Fig. 2: (Color online) Evolution of $\langle H - H_0 \rangle$ for the ensemble of Josephson junctions (4) for $\omega = 5$ and different σ (right panel: in a rescaled time). The dashed line corresponds to formula (22).

We restrict our attention to the nontrivial case $\omega > 1$, so that the Hamiltonian is bounded from below. The "lowest-energy" state here for $\sigma = 0$ is the steady state with $\Phi_0 = \pi/2$, $J_0 = (\omega - \sqrt{\omega^2 - 1})/(4\sqrt{\omega^2 - 1})$. This steady state does not correspond to full asynchrony (which is not a stationary solution of the equations, because the phases rotate non-uniformly), but to a stationary distribution of the phases of the ensemble (4). Close to this equilibrium, we can linearize the equations of motion, so that the Hamiltonian in the vicinity of J_0, Φ_0 reads

$$H(\delta J, \delta \Phi, t) = H_0 + \sigma \xi(t) \delta J + \frac{\omega_1 (\delta \Phi)^2 + \omega_2 (\delta J)^2}{2} , \quad (21)$$

with $\omega_1 = 0.5\sqrt{2J_0(2J_0 + 1)}$, $\omega_2 = 2(\omega^2 - 1)(2J_0(2J_0 + 1))^{-1/2}$. The solution of the linear equations of motion (starting from the equilibrium point) is, similar to (17),

$$\delta \Phi + i\frac{\omega_2}{\kappa}\delta J = \sigma \int_0^t \exp[i\kappa(t' - t)]\xi(t')dt' ,$$

where $\kappa^2 = \omega^2 - 1$. After averaging we obtain

$$\left\langle (\delta \Phi)^2 + \frac{\omega_2^2}{\kappa^2}(\delta J)^2 \right\rangle =$$

$$= t\sigma^2 \int_{-t}^t \langle \xi(0)\xi(\tau) \rangle \cos \kappa\tau (1 - \frac{|\tau|}{t})d\tau .$$

This relation means that, asymptotically, the "energy" defined as $H - H_0 = 0.5(\omega_1(\delta \Phi)^2 + \omega_2(\delta J)^2)$ grows linearly in time, for the white noise we get

$$\langle H - H_0 \rangle = \frac{\omega_1 \sigma^2}{2}t . \quad (22)$$

This relation is checked in in Fig. 2, where numerical simulations of eqs. (8),(11) are presented.

The results above show that there are two stages for the transition to synchrony in ensembles of oscillators driven by common noise. As the system is described by a noise-driven Hamiltonian, it is natural to characterize the evolution through the "energy" – the value of the noise-independent part of the Hamilton function. At the initial stage, close to asynchrony, the growth of the energy is diffusive, its averaged square grows linearly in time according

to (19) and (22). When the energy reaches a level of order one, a crossover to the other type of behavior, namely to an exponential growth of energy, is observed from numerical simulation (cf. Figs. 1,2). This latter stage means that the final convergence of the ensemble of oscillators to a synchronous cluster is exponentially fast.

Nonidentical oscillations. – Here we extend the theory to the case of nonidentical oscillators (2) having a distribution of frequencies $g(\omega)$. In this case one first generalizes the WS description by assuming the frequency dependence of the variables $\rho(\omega), \Phi(\omega)$ [16]. It is convenient to introduce one complex variable $z(\omega) = \rho e^{i\Phi}$ that obeys

$$\dot{z} = i\omega z + \frac{\sigma\xi(t)(1 - z^2)}{2} . \quad (23)$$

The distribution of the phases is now characterized by the global order parameter $Z = \int z(\omega)g(\omega)d\omega$. Following the approach of Ott and Antonsen [20], it is possible to obtain a closed equation for this order parameter in the case of a Lorentzian distribution $g(\omega) = \pi^{-1}\gamma(\gamma^2 + (\omega - \omega_0)^2)^{-1}$. Then, assuming that $z(\omega)$ as function of complex ω does not have singularities in the upper half-plane, one can perform the integration to get $Z = z(\omega_0 + i\gamma)$. Thus, the equation for Z follows from (23) with $\omega \to \omega_0 + i\gamma$:

$$\dot{Z} = i\omega_0 Z - \gamma Z + \frac{\sigma\xi(t)(1 - Z^2)}{2} . \quad (24)$$

Transforming from Z to the canonical variables as in (7),(10), we obtain the same equations as (8) and (11) but with additional non-Hamiltonian, damping terms:

$$\dot{q} = \omega_0 p + \sigma\xi(t)\frac{1 + q^2 + 2p^2}{2\sqrt{1 + p^2 + q^2}} - \gamma q(1 + p^2 + q^2) ,$$

$$\dot{p} = -\omega_0 q - \sigma\xi(t)\frac{qp}{2\sqrt{1 + p^2 + q^2}} - \gamma p(1 + p^2 + q^2) ,$$

$$(25)$$

and

$$\dot{J} = \sigma\xi(t)\frac{\sqrt{2J(2J + 1)}}{2}\cos(\Phi) - 2\gamma J(1 + 2J) ,$$

$$\dot{\Phi} = \omega_0 - \sigma\xi(t)\frac{4J + 1}{2\sqrt{2J(2J + 1)}}\sin(\Phi) . \quad (26)$$

With damping terms, the energy does not grow indefinitely, but saturates as shown in Fig. 3. The saturation level corresponds to a bunch of oscillators that do not form a perfect cluster, but have a finite spread.

The stationary level of the order parameter can be estimated for a state close to synchrony and for a wide distribution (asynchrony). Close to synchrony, i.e. for $J \gg 1$, we have instead of (15)

$$\left\langle \frac{d}{dt}\ln J \right\rangle = \sigma\xi(t)\cos \Phi - 2\gamma J . \quad (27)$$

If we neglect the fluctuations of the growth rate and assume $\sigma\xi(t)\cos \Phi = \langle \sigma\xi(t)\cos \Phi \rangle = -\lambda$, then the stationary value of J is $J_{st} = |\lambda|/(2\gamma)$. Close to asynchrony we
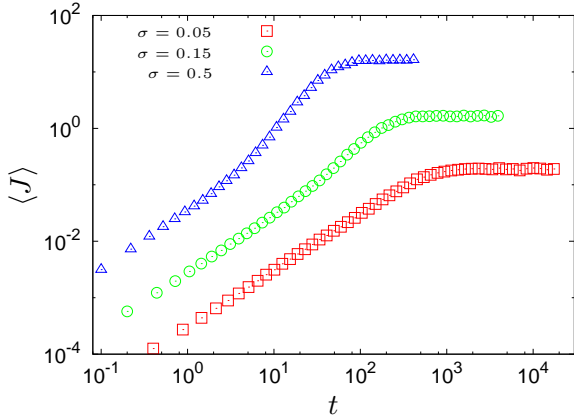
Fig. 3: (Color online) Evolution of the order parameter according to Eqs. (25),(26) for $\omega_0 = 2$, $\gamma = 10^{-3}$ and different noise intensities σ.

can use the same approximation as in (16), but now the equations read

$$\dot{q} = \omega_0 p - \gamma q + 0.5\sigma\xi(t) , \quad \dot{p} = -\omega_0 q - \gamma p ,$$

with the average stationary energy

$$\left\langle q^2 + p^2 \right\rangle = \frac{\sigma^2}{8\gamma} \int_{-\infty}^{\infty} \langle \xi(0)\xi(t) \rangle \cos\omega_0 t e^{-\gamma|t|} dt ,$$

which in the case of the white noise yields $\langle J \rangle = \sigma^2/(16\gamma)$.

Remarkably, in both limits the average value of J scales as γ^{-1}. This is confirmed by numerics presented in Fig. 4. In this figure we present also simulations of the oscillator populations, which fit nicely the results from the modeling of the WS variables.

Discussion. – In this letter we have developed a global theory of synchronization of oscillator populations by common noise. Our analysis is based on the Watanabe-Strogatz ansatz [15], which is not restricted by a number of elements in the populations and results in an explicit time-dependence of the global variables on the common forcing terms. These variables can be interpreted as order parameters characterizing the population of identical oscillators. For noisy forcing we thus obtained a closed set of stochastic differential equations for the global variables. An important step in our consideration is a representation of the WS equations as a nonautonomous Hamiltonian system; transition to synchrony then appears as the growth of energy due to the noisy driving. While for the situations close to synchrony the results are essentially the same as the previous ones derived from the linear perturbation approach, we have demonstrated that when starting from a broad initial distribution, the energy first grows linearly, and only after a formation of a concentrated cluster the exponential convergence to synchrony sets on. Furthermore, by virtue of the Ott-Antonsen theory [20] we have extended the analysis to populations of non-identical oscillators with a Lorentzian distribution of natural frequen-
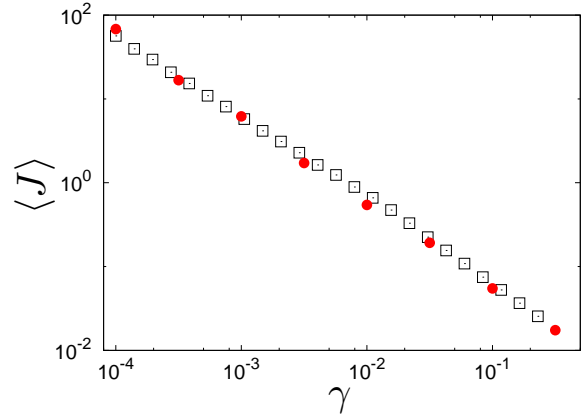

Fig. 4: (Color online) The stationary level of the order parameter for an ensemble of the oscillators with a Lorentzian distribution of frequencies, vs. the distribution width. Parameters: $\omega_0 = 2$, $\sigma = 0.3$. Squares: modeling eqs. (25),(26); filled circles: modeling the ensemble of $N = 5000$ oscillators for the same parameters.

cies. Here the theory is valid in the thermodynamic limit of very large ensembles only.

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