Exact ratchet description of Parrondo's games with self-transitions

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ABSTRACT

We extend a recently developed relation between the master equation describing the Parrondo's games and the formalism of the Fokker–Planck equation to the case in which the games are modified with the introduction of "self–transition probabilities". This accounts for the possibility that the capital can neither increase nor decrease during a game. Using this exact relation, we obtain expressions for the stationary probability and current (games gain) in terms of an effective potential. We also demonstrate that the expressions obtained are nothing but a discretised version of the equivalent expressions in terms of the solution of the Fokker–Planck equation with multiplicative noise.

Keywords: Master and Fokker–Planck equations, Parrondo's games, multiplicative noise

1. INTRODUCTION

Parrondo's paradox is a counter–intuitive effect that has attracted much attention in the last years.[1] It is based upon the ratchet effect[2]: the combination of two dynamics, both negatively biased, leading to a positively biased dynamics. Briefly speaking, the Parrondo's paradox combines simple coin-tossing losing games to produce a winning game. This paradoxical result has been widely studied in its fundamental aspects[3–9] and also in relation to some sociological applications,[10,11] control theory,[12,13] pattern formation,[14–16] molecular motors[17,18] and economics.[19]

In previous work[20,21] (see also reference[22]) we established a closed relation between the parameters defining a Parrondo game and its physical Brownian ratchet. We were able to obtain a set of equations relating the drift $F(x)$ and diffusion $D(x)$ characterizing a Brownian ratchet with the probabilities p_i defining a Parrondo game. A simplifying feature used in that work is that we consider only games whose output is a win or a lose, i.e. we did not accept draw as a result. This fact translates into a constant diffusion function $D(x) = D$. In the language of stochastic processes,[23] this implies that the noise term in the Langevin equation describing these games is additive.

Recently a new set of Parrondo games have been presented.[24] These games introduce a *self–transition* probability which denotes the probability of the player to remain with the same amount of capital after a round is played. We will show that the existence of this so-called *self–transition* probability allows the diffusion term $D(x)$ to vary in space, and so the Fokker–Planck equation corresponding to these games corresponds to a Langevin equation with a multiplicative noise term.

The outline of the paper is as follows: In Section 2 we review the main results concerning the *additive noise* approach. In Section 3 we show how the expressions obtained in the previous section vary due to the inclusion of the *self–transition* probability and both approaches are compared in Section 4. Finally, in Section 5 we present our main conclusions.

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2. ADDITIVE NOISE

A master equation can be written for a game if we consider it as a discrete process, in which both the time and "space" (the capital owned by the player) are both discretised with a step size $\Delta = 1$. If τ denotes the discretised time and $P_i(\tau)$ the probability that at time τ the capital is i, then we have the following master equation

$$P_i(\tau+1) = a^i_{-1}P_{i-1}(\tau) + a^i_0 P_i(\tau) + a^i_1 P_{i+1}(\tau) \tag{1}$$

where $a^i_{-1} \equiv p_{i-1}$ is the probability of winning when the capital is $i-1$, $a^i_0 \equiv r_i$ is the probability of remaining with the same capital when the capital is i and $a^i_1 \equiv q_{i+1}$ is the probability of losing when the capital is $i+1$. These terms must fulfill the normalization condition: $a^{i+1}_{-1} + a^i_0 + a^{i-1}_1 = 1$ or $p_i + r_i + q_i = 1$.

An straightforward algebra shows that this equation can be rewritten as a continuity equation

$$P_i(\tau+1) - P_i(\tau) = -\left[J_{i+1}(\tau) - J_i(\tau)\right] \tag{2}$$

where

$$J_i(\tau) = \frac{1}{2}[F_i P_i(\tau) + F_{i-1}P_{i-1}(\tau)] - [D_i P_i(\tau) - D_{i-1}P_{i-1}(\tau)] \tag{3}$$

and

$$D_i = \frac{1}{2}(a^{i+1}_{-1} + a^{i-1}_1) = \frac{1}{2}(1 - a^i_0) \qquad F_i = a^{i+1}_{-1} - a^{i-1}_1. \tag{4}$$

Eq.(1) can be compared with a consistent discretization of the Fokker–Planck equation of general form[23]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \tag{5}$$

where

$$J(x,t) = F(x)P(x,t) - \frac{\partial D(x)P(x,t)}{\partial x} \tag{6}$$

is the probability current with drift $F(x)$ and diffusion $D(x)$. When the diffusion term is constant, $D(x) = D_0$, it is known that the equivalent Langevin equation has an additive noise term, otherwise noise appears multiplicatively.

In the remainder of this section, we impose $a^i_0 = 0$, that is, we can only win or lose when playing the games. This condition implies a constant diffusion term $D_i = D = \frac{1}{2}$ and a drift given by $F_i = 2p_i - 1$. It is typical of the games that the probabilities a^i_j $(j = -1, 0, 1)$ are defined as a function of the capital i modulus a number L, i.e. $a^i_j = a^{i \bmod L}_j$.

For a stationary state and constant current, i.e., $P_i(\tau) = P_i$ and $J_i = J$, the following expressions can be obtained

$$P^{st}_i = \mathrm{e}^{-\frac{V_i}{D}}\left[P^{st}_0 - \frac{J}{D}\sum_{j=1}^{i}\frac{\mathrm{e}^{\frac{V_j}{D}}}{1-F_j}\right] \tag{7}$$

$$J = D \cdot P^{st}_0 \frac{1 - \mathrm{e}^{\frac{V_L}{D}}}{\sum_{j=1}^{L}\frac{\mathrm{e}^{\frac{V_j}{D}}}{1-F_j}} \tag{8}$$

here the current J is obtained through periodicity conditions $P_0 = P_L$.

In the previous equations we have defined a potential in terms of the game probabilities as

$$V_i = -D\sum_{k=1}^{i}\ln\left[\frac{1+F_{k-1}}{1-F_k}\right] = -\frac{1}{2}\sum_{k=1}^{i}\ln\left[\frac{p_{k-1}}{1-p_k}\right] \tag{9}$$

This definition assures periodicity of the potential when the set of probabilities $\{p_i, q_i\}$ define a fair game. Equivalently, imposing $V_0 = V_L$ in Eq.(9) what we obtain is the well–known fairness condition[1] $\prod_{k=1}^{L-1} p_i = \prod_{k=1}^{L-1}(1-p_i)$.

It can be checked that this discretised potential V_i is equivalent to a discretization of the *physical* potential $V(x) = -\int F(x)dx$. For this purpose consider our discrete potential for a step size $\Delta \neq 1$. After some manipulation and expanding up to first order in Δ we obtain

$$\begin{aligned} V_i &= -\frac{1}{2}\sum_{k=1}^{i} \ln\left[\frac{1+\Delta\cdot F_{k-1}}{1-\Delta\cdot F_k}\right] \\ &\approx -\Delta\cdot\left(\frac{1}{2}F_0 + \sum_{k=1}^{i-1} F_k + \frac{1}{2}F_i\right) \end{aligned} \tag{10}$$

which is nothing but a discrete integral of the drift obtained with Simpson's trapezoidal method.

We can also compare the discretised solutions obtained before (7) and (8) with the continuous solutions for the Langevin equation with additive noise[23] $\dot{x} = F[x(t), t] + D\xi(t)$,

$$P(x) = \mathrm{e}^{-\frac{V(x)}{D}}\left[P(0) - \frac{J}{D}\cdot\int_0^x \mathrm{e}^{-\frac{V(x')}{D}}dx'\right] \tag{11}$$

$$J = \frac{P(0)\cdot D\cdot\left[1-\mathrm{e}^{\frac{V(L)}{D}}\right]}{\int_0^L \mathrm{e}^{-\frac{V(x')}{D}}dx'} \tag{12}$$

It is clear that both sets of equations (7–8) and (11–12) are equivalent. There is only one term that needs to be analyzed in some detail, namely:

$$\sum_{j=1}^{i}\frac{\mathrm{e}^{\frac{V_j}{D}}}{1-F_j} = \sum_{j=1}^{i}\mathrm{e}^{-\sum_{k=1}^{j}\ln\left[\frac{1+F_{k-1}}{1-F_k}\right]-\ln(1-F_j)} \approx \sum_{j=1}^{i}\mathrm{e}^{-\frac{\frac{F_0}{2}+\sum_{k=1}^{i}F_k+\frac{F_j}{2}}{D}+F_j}. \tag{13}$$

This equation is the numerical approximation using Simpson's rule to the integral $\int_0^x \mathrm{e}^{-\frac{V(x')}{D}}dx'$, except by the presence of an extra term F_j. However, this latter term would tend to zero when $\Delta \to 0$ (remember that we have set $\Delta = 1$ in the previous equations) and Eq. (13) can be considered a consistent numerical discretization of the corresponding integral in Eq.(11).

The inverse process can also be performed, i.e., for a given potential we can obtain its corresponding probabilities defining a Parrondo game. This set of probabilities $\{p_i, q_i\}$ for a given potential V_i can be obtained inverting Eq.(9) for $\{p_i\}$. The resulting expression is

$$F_i = (-1)^i \mathrm{e}^{V_i/D}\left[\frac{\sum_{j=1}^{L}(-1)^j[\mathrm{e}^{-V_j/D}-\mathrm{e}^{-V_{j-1}/D}]}{(-1)^L\mathrm{e}^{(V_0-V_L)/D}-1} + \sum_{j=1}^{i}(-1)^j[\mathrm{e}^{-V_j/D}-\mathrm{e}^{-V_{j-1}/D}]\right] \tag{14}$$

which together with $F_i = 2p_i - 1$ can be used for determining the probabilities p_i.

In summary, we have developed a method for obtaining the stationary probability, current and potential that correspond to a given set of probabilities defining a game as well as the inverse process of obtaining the game probabilities $\{p_i, q_i\}$ for a given potential V_i. This method is limited to the case $a_0^i = 0$, leading to a constant diffusion coefficient, or an additive noise.

3. MULTIPLICATIVE NOISE

We go now a step forward, and calculate how these previous expressions obtained for the stationary probability, current and the defined potential vary when we consider the case $a_0^i \neq 0$ (which is equivalent to $r_i \neq 0$). As we stated before, considering this term implies that the player has now a certain probability of remaining with the same capital after a round played.

The drift and diffusion terms now read

$$F_i = a_{-1}^{i+1} - a_1^{i-1} = 2p_i + r_i - 1 \tag{15}$$

$$D_i = \frac{1}{2}(1 - a_0^i) = \frac{1}{2}(1 - r_i) \tag{16}$$

It can be appreciated that now both terms, the diffusion D_i as well as the drift F_i, may vary on every site. Using Eq.(3) and considering the stationary case $P_i(\tau) = P_i$ together with a constant current $J_i = J$, we get

$$P_i^{st} = \frac{J}{\frac{1}{2}F_i - D_i} - \left(\frac{\frac{1}{2}F_{i-1} + D_{i-1}}{\frac{1}{2}F_i - D_i} \right) \cdot P_{i-1}^{st}. \tag{17}$$

The previous equation has a general form $x_i = a_i + b_i x_{i-1}$, from which a solution can be derived as $x_n = \left[\prod_{k=1}^n b_k\right] \cdot x_0 + \sum_{j=1}^n a_j \cdot \left[\prod_{k=j+1}^n b_k\right]$. Applying the latter result to the stationary probability we have

$$P_n^{st} = \left[\prod_{k=1}^n \frac{D_{k-1} + \frac{1}{2}F_{k-1}}{D_k - \frac{1}{2}F_k} \right] \cdot P_0^{st} - J \sum_{j=1}^n \frac{1}{D_j - \frac{1}{2}F_j} \left[\prod_{k=j+1}^n \frac{D_{k-1} + \frac{1}{2}F_{k-1}}{D_k - \frac{1}{2}F_k} \right] \tag{18}$$

We can solve for the current J using Eq.(17) together with the periodic boundary condition $P_L^{st} = P_0^{st}$

$$J = \frac{P_0^{st} \cdot \left(\prod_{k=1}^L \left[\frac{\frac{1}{2}F_{k-1} + D_{k-1}}{D_k - \frac{1}{2}F_k} \right] - 1 \right)}{\sum_{j=1}^L \frac{1}{D_j - \frac{1}{2}F_j} \prod_{k=j+1}^L \left[\frac{\frac{1}{2}F_{k-1} + D_{k-1}}{D_k - \frac{1}{2}F_k} \right]} \tag{19}$$

An *effective potential* can be defined in a similar way to its continuous analog as

$$V_i = -\sum_{j=1}^i \ln \left(\frac{1 + \frac{1}{2}\frac{F_{j-1}}{D_{j-1}}}{1 - \frac{1}{2}\frac{F_j}{D_j}} \right) = -\sum_{j=1}^i \ln \left(\frac{\frac{p_{j-1}}{1-r_{j-1}}}{\frac{1-p_j-r_j}{1-r_j}} \right). \tag{20}$$

It is important to note that, as in the previous case $a_0^i = 0$, the potential must verify periodic conditions $V_0 = V_L$ when the set of probabilities define a fair game. It is an easy task to check that using Eq.(20) together with the periodic boundary condition, what we obtain is the fairness condition for a given set of probabilities defining a game with *self–transition*,[24] that is

$$\prod_{k=1}^{L-1} p_i = \prod_{k=1}^{L-1} q_i = \prod_{k=1}^{L-1} (1 - p_i - r_i) \tag{21}$$

By means of Eq.(20) we can obtain the stationary probability (18) and current (19) in terms of the defined potential as

$$P_n^{st} = \mathrm{e}^{-V_n} \left(\frac{D_0 \cdot P_0^{st}}{D_n} - J \sum_{j=1}^n \frac{\mathrm{e}^{V_j}}{D_n \left(1 - \frac{1}{2}\frac{F_j}{D_j}\right)} \right) \tag{22}$$

$$J = \frac{P_0^{st}\left[D_0 - D_L\cdot \mathrm{e}^{V_L}\right]}{\sum_{j=1}^{L}\frac{\mathrm{e}^{V_j}}{\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)}} \tag{23}$$

These are the new expressions which, together with Eqs.(15) and (16) allow us to obtain the potential, current and stationary probability for a given set of probabilities $\{p_i, r_i, q_i\}$ defining a Parrondo game with *self–transition*. We will now show that the set of Eqs.(20),(22),(23) can be related in a consistent form with the continuous solutions corresponding to the Fokker–Planck equation of a process with multiplicative noise.[23]

Given a Langevin equation with multiplicative noise

$$\dot{x} = F[x(t),t] + \sqrt{B[x(t),t]}\cdot\xi(t) \tag{24}$$

interpreted in the sense of Ito, we can obtain its associated Fokker–Planck equation given by Eq.(5) recalling that $D(x,t) = \frac{1}{2}B(x,t)$. The general solution for the stationary probability density function $P(x,t)$ is given by

$$P^{st}(x) = \frac{\mathrm{e}^{\int^x \Psi(x)dx}}{D(x)}\cdot\left[\mathcal{N} - J\int^x \mathrm{e}^{-\int^{x'}\Psi(x'')dx''}dx'\right] \tag{25}$$

where $\mathcal{N}$ is a normalization constant and $\Psi(x) = \frac{F(x)}{D(x)}$. Making use of the periodicity and the normalization condition$P(0) = P(L)$ and $\int_0^x P(x)dx = 1$ we obtain the following expressions for $\mathcal{N}$ and J

$$\mathcal{N} = P(0)\cdot D(0) \qquad J = \frac{P(0)\cdot\left(D(0) - D(L)\mathrm{e}^{\int_0^L\Psi(x)dx}\right)}{\int_0^L \mathrm{e}^{-\int_0^{x'}\Psi(x'')dx''}dx'} \tag{26}$$

Comparing the discrete equations for the current and stationary probability (22-23) with the continuous solutions (25-26) we have the following equivalences

$$P_0^{st}\cdot D_0 \equiv P(0)\cdot D(0) \tag{27}$$

$$D_j \equiv D(x) \tag{28}$$

$$\mathrm{e}^{V_n} \equiv \mathrm{e}^{\int^x\Psi(x)dx} \tag{29}$$

$$\sum_{j=1}^{n}\frac{\mathrm{e}^{V_j}}{\left(1-\frac{1}{2}\frac{F_j}{D_j}\right)} \equiv \int^x \mathrm{e}^{-\int^{x'}\Psi(x'')dx''}dx' \tag{30}$$

It is clear the identification of the terms in Eqs.(27) and (28). Now we need to demonstrate the equivalence given by Eqs.(29) and (30). If we define a *discretised function* as $\psi_j = \frac{F_{j-1}}{D_{j-1}}$ and we use the Taylor expansion up to first order of the logarithm $\ln(1+x)\approx x$ already used in the previous section we get

$$V_n = -\sum_{j=1}^{n}\ln\left(\frac{1+\frac{1}{2}\psi_{j-1}}{1-\frac{1}{2}\psi_j}\right) \approx -\frac{1}{2}\sum_{j=1}^{n}(\psi_{j-1}+\psi_j) = -\left(\frac{1}{2}\psi_0 + \sum_{k=1}^{n-1}\psi_k + \frac{1}{2}\psi_n\right) \tag{31}$$

$$\sum_{j=1}^{n}\frac{\mathrm{e}^{V_j}}{1-\frac{1}{2}\psi_j} = \sum_{j=1}^{n}\mathrm{e}^{V_j - \ln\left(1-\frac{1}{2}\psi_j\right)} \approx \sum_{j=1}^{n}\mathrm{e}^{-\frac{1}{2}\left(\sum_{k=1}^{j}[\psi_{k-1}+\psi_k]-\psi_j\right)} = \sum_{j=1}^{n}\mathrm{e}^{-\left(\frac{1}{2}\psi_0 + \sum_{k=1}^{j}\psi_k + \frac{1}{2}\psi_j\right)+\frac{1}{2}\psi_j} \tag{32}$$

It can be clearly seen that Eq.(31) corresponds to the numerical integration of the function $\Psi(x)$ defined previously, but with a $\Delta = 1$ (the difference in the sign is due to the way we have defined our potential). It can be demonstrated that when $\Delta \neq 1$ both expressions agree up to first order in Δ,

$$V_{n\Delta} = -\Delta \left(\frac{1}{2}\psi_0 + \sum_{k=1}^{n-1} \psi_k + \frac{1}{2}\psi_n \right) \tag{33}$$

In the case of Eq.(32) what we obtain is nearly the Simpson's numerical integration method but for an extra term. As in the previous case, when $\Delta \neq 1$ then we have up to a first order an extra Δ term,

$$\sum_{j=1}^{n} \frac{\mathrm{e}^{V_{j\Delta}}}{1 - \frac{1}{2}\psi_{j\Delta}} \approx \Delta \cdot \sum_{j=1}^{n} \mathrm{e}^{-\Delta\left(\frac{1}{2}\psi_0 + \sum_{k=1}^{j} \psi_{k\Delta} + \frac{1}{2}\psi_{j\Delta}\right) + \frac{1}{2}\Delta\psi_{j\Delta}} \tag{34}$$

So when $\Delta \to 0$ the contribution of the *extra* term can be neglected as compared to that of the sum.

We can also perform the inverse process, that is, to obtain the set of probabilities $\{p_i, r_i, q_i\}$ for a given potential V_i. If we call $A_n = \frac{p_n - q_n}{p_n + q_n}$, we need only to solve Eq.(20) for A_n obtaining

$$A_n = (-1)^n \cdot \mathrm{e}^{V_n} \left[\frac{\sum_{j=1}^{L} (-1)^j (\mathrm{e}^{-V_j} - \mathrm{e}^{-V_{j-1}})}{(-1)^L \cdot \mathrm{e}^{V_0 - V_L} - 1} + \sum_{j=1}^{n} (-1)^j \cdot \left(\mathrm{e}^{-V_j} - \mathrm{e}^{-V_{j-1}}\right) \right] \tag{35}$$

Once these values are obtained, we must solve for the probabilities together with the normalization condition $p_i + r_i + q_i = 1$. As we have a free parameter in the set of solutions, we can fix the r_i values on every site and the rest of parameters can be obtained through

$$p_i = \frac{1}{2}(1 + A_i)(1 - r_i) \tag{36}$$
$$q_i = \frac{1}{2}(1 - A_i)(1 - r_i). \tag{37}$$

In this way what we have is a method for inverting an *effective* potential, fixing a parameter that in our case is the diffusion in every site (remember that the parameter r_i is related to the diffusion coefficient by Eq.(16)) or equivalently the temperature.

4. COMPARISON BETWEEN BOTH SYSTEMS

We have presented two methods for obtaining the potential defined by a set of game probabilities or vice versa. If we suppose an additive noise what we get is a set of probabilities $\{p_i, q_i\}$, whereas if the noise is multiplicative what we get is $\{p_i, r_i, q_i\}$. We want to investigate which relation may exist between these two sets of probabilities.

Starting with the sets of probabilities of the original Parrondo's games with periodicity $L = 3$, we will modify them in order to introduce the new parameter r as follows

$$\left\{ \begin{array}{rcl} p' &=& p \cdot (1-r) = \frac{1}{2} \cdot (1-r) \\ p'_1 &=& p_1 \cdot (1-r) = \frac{1}{10} \cdot (1-r) \\ p'_2 &=& p_2 \cdot (1-r) = \frac{3}{4} \cdot (1-r) \end{array} \right. \tag{38}$$

In Fig.1 we plot the current J vs r obtained varying the game probabilities in three different ways. In Fig.(1)a we find the current when only the probabilities of game A are varied, keeping the probabilities of game B fixed. In Fig.(1)b the probabilities of game B are varied instead of game A. Finally in Fig.(1)c both sets of probabilities are changed using Eq.(38). In all three cases the current diminishes gradually with increasing r, although in the first case the curve appears to be more convex, and in the second and third case the trend is more linear.

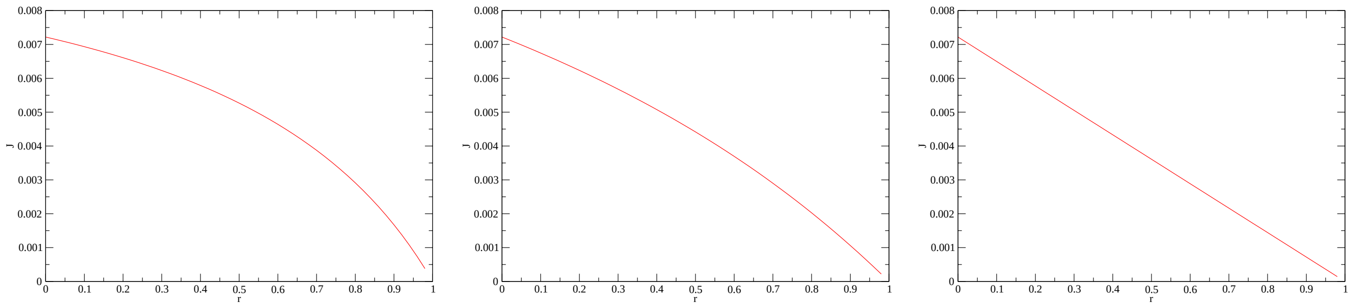

Figure 1. The current is plotted for three different cases: a) The probabilities of game A are varied, keeping game B probabilities fixed; b) now only game B probabilities are varied; c) both sets of probabilities are varied.

As noted in,[24] another way of modifying the probabilities, where the winning probability is kept fixed and varying the losing probability q (it can be considered as if q split into the new probability r and a new q value) is the following

$$\left\{ \begin{array}{rcl} p' & = & p, \quad r' = r, \quad q' = q - r \\ p'_1 & = & p_1, \quad r'_1 = r, \quad q'_1 = q_1 - r \\ p'_2 & = & p_2, \quad r'_2 = r, \quad q'_2 = q_2 - r \end{array} \right. \tag{39}$$

This alternative way of modifying the probabilities leads to a variation of the current presented in Fig.(2). In all cases the current J increases because the losing probability diminishes with r, and so the winning rate at a constant p_i increases with r.

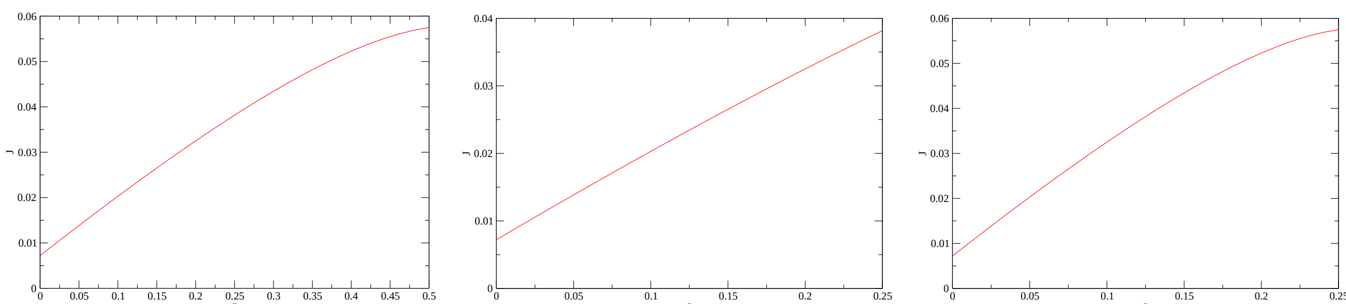

Figure 2. As in the previous figure, we plot the current versus the *self–transition* probability r for three different cases: a) we vary only the probabilities of game A; b) we only vary the probabilities of game B; c) we vary the probabilities of both games using (39).

Another aspect to consider is the possibility of obtaining two different sets of probabilities that describe the same discretised potential V_i. One way is through Eq.(14). What we have then is a set of probabilities $\{p_i, q_i\}$. The second possibility involves using Eq.(35) together with (36) and (37) for obtaining the probabilities $\{p_i, r_i, q_i\}$. The potential used for finding these probabilities is asymmetric with periodicity L and amplitude A,

$$V(x) = A\left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{4}\sin\left(\frac{4\pi x}{L}\right)\right] \tag{40}$$

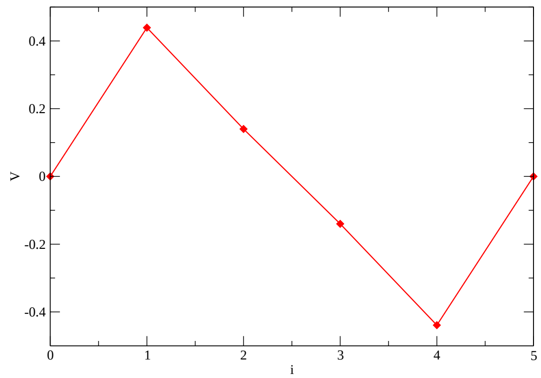

Figure 3. Potential defined by Eq.(40) with $A = 0.4$ and $L = 5$.

This potential has been used widely in the literature as a *ratchet potential*.[2] In Fig.(3) we have plotted the potential with the parameter values $L = 5$ and $A = 0.4$ used for obtaining the following probability sets

$$\left\{ \begin{array}{lcll} p_0 & = & 0.187, & q_0 = 0.81299 \\ p_1 & = & 0.70986, & q_1 = 0.29014 \\ p_2 & = & 0.473696, & q_2 = 0.52630 \\ p_3 & = & 0.6419944, & q_3 = 0.35801 \\ p_4 & = & 0.5240148, & q_4 = 0.47599 \end{array} \right. \longrightarrow J = 1.1288 \cdot 10^{-3} \tag{41}$$

$$\left\{ \begin{array}{llll} p_0 & = 0.07480, & r_0 = 0.60, & q_0 = 0.3252 \\ p_1 & = 0.28394, & r_1 = 0.60, & q_1 = 0.1161 \\ p_2 & = 0.18948, & r_2 = 0.60, & q_2 = 0.2105 \\ p_3 & = 0.25680, & r_3 = 0.60, & q_3 = 0.1432 \\ p_4 & = 0.20961, & r_4 = 0.60, & q_4 = 0.1904 \end{array} \right. \longrightarrow J = 5.7677 \cdot 10^{-4} \tag{42}$$

As one of the three probabilities must be fixed (because there is a free parameter for every site i), we decide to fix the $r_i = 0.6\,\forall i$ in the sets of probabilities given by (42). The current J is obtained alternating with a game A with probabilities $p = \frac{1}{2}$, $q = \frac{1}{2}$ in the former case and $p = \frac{1}{4}$, $r = \frac{1}{2}$ and $q = \frac{1}{4}$ in the latter.

The fact that we can obtain different sets of probabilities, both describing different dynamics but coming from the same potential $V(x)$, it is not surprising if we take into account that a system with multiplicative noise is equivalent, in the sense that both have the same stationary probability distribution, to another system with additive noise

$$\dot{x} = F(x) + D(x) \cdot \xi(t) \longrightarrow \dot{x} = \bar{F}(x) + \xi(t) \tag{43}$$

but with a renormalized drift term $\bar{F}(x)$ given by $\bar{F}(x) = -\frac{\partial \bar{V}}{\partial x}$ where $F(x) = -\frac{\partial V}{\partial x}$ and $\bar{V} = \int \frac{F(x)}{D(x)} dx + \ln D(x)$.

5. CONCLUSION

We have presented a consistent way of relating the master equation for the Parrondo games with the formalism of the Fokker–Planck equation describing Brownian ratchets. This relation works in two ways: we can obtain the physical potential corresponding to a set of probabilities defining a Parrondo game, as well as the current and its stationary probability distribution. Inversely, we can also obtain the probabilities corresponding to a given physical potential. Our relations work both in the cases of additive noise or multiplicative noise. With the new relations introduced for the case of multiplicative noise in this paper, we have now a precise and of general validity connection between individual Brownian ratchets and single Parrondo's games. We have also presented a comparison of the efficiency of the games with and without self–transitions.

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