Entanglement Distribution and Entangling Power of Quantum Gates

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Abstract

Quantum gates, that play a fundamental role in quantum computation and other quantum information processes, are unitary evolution operators $\hat{U}$ that act on a composite system changing its entanglement. In the present contribution we study some aspects of these entanglement changes. By recourse of a Monte Carlo procedure, we compute the so called "entangling power" for several paradigmatic quantum gates and discuss results concerning the action of the CNOT gate. We pay special attention to the distribution of entanglement among the several parties involved.

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I. INTRODUCTION

Entanglement is one of the most fundamental aspects of quantum physics [1] and can be regarded as a physical resource associated with peculiar non-classical correlations between separated quantum systems. Entanglement lies at the basis of important quantum information processes [2–5] such as quantum cryptographic key distribution, quantum teleportation, superdense coding and quantum computation [6–8].

Quantum gates, the quantum generalization of the so-called standard logical gates, play a fundamental role in quantum computation and other quantum information processes, being described by unitary transformations $\hat{U}$ acting on the relevant Hilbert space (usually, that for a multi-qubit system). In general, a quantum gate acting on a composite system changes the entanglement of the system's concomitant quantum state. It is then a matter of interest to obtain a detailed characterization of the aforementioned entanglement changes ([9–12]). To such an end, the study is greatly simplified if one is able to conveniently parameterize the gate's non-local features. We need $N^2 - 1$ parameters to describe a unitary transformation $U(N)$ in a system of $N = N_A \times N_B$ dimensions. In the case of two qubits ($N = 2 \times 2$) one needs just three parameters $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3)$ ([10]), since any two-qubits quantum gate can be decomposed in the form of a product of local unitarities, acting on both parties, and a "nuclear" part $\tilde{U}$, which is completely non-local. Given a quantum gate U, the concomitant distribution of entanglement changes is equivalent, on average, to the one produced by $\tilde{U}$, and we need to know the vector λ.

In addition to studying changes in the entanglement of a given state produced by quantum gates, we would like to ascertain entangling capabilities of unitary operations or evolutions. In point of fact, the latter enterprise complements the former. By looking at the distribution of entanglement changes induced by several quantum gates, one can deduce a special formula that quantifies the "entangling power". To such an end we use the definition introduced by Zanardi *et al.* [13], and introduce a new one as well, based exclusively on the shape of a particular probability (density) distribution: that for finding a state with a given entanglement change ΔE, measured in terms of the so called entanglement of formation [14]. We will see that the distribution obtained by randomly picking up two states measuring their relative entanglement change is optimal in the context of our new measure. Moreover, the two-qubits instance will be seen to be rather peculiar in comparison with its counterpart for larger dimensions (bipartite systems, like two-qudits $N_A \times N_A$, for N_A = 3,4,5 and 6).

Extending the above considerations to mixed states requires the introduction of a measure for the simplex of eigenvalues of the matrix $\hat{\rho}$ instead of dealing with pure states distributed according to the invariant Haar measure. Rather than mimicking the aforementioned evaluation, which could be easily achieved by introducing a proper measure for the generation of mixed states, we will generate them in the fashion of Refs. ([15–21]). In such a connection we discuss the action of the exclusive-OR or controlled-NOT gate (CNOT in what follows) in the 15-dimensional space $\mathcal{S}$ of mixed states and compare our results with those obtained using the well known Hilbert-Schmidt and Bures metrics [22].

Also we study numerically how the entanglement is distributed when more than two parties are involved (multipartite entanglement). By applying locally the CNOT gate to a given pair of two-qubits in a system of pure states composed by three or four qubits, we shall study the concomitant distributions of entanglement changes among different qubits, pointing out the differences between them. Great entanglement changes are appreciated as we increase the relevant number of qubits.

The paper is organized as follows: in section II we summarize the optimal parametrization of two-qubits gates, which will be useful in order to discuss, in section III, the computation of the so called "entangling power" [13] for several quantum gates. In this regard we will introduce a new measure for general bipartite states (two-qubits and two-qudits). The extension to mixed states is dealt with in section IV by recourse to an appropriate heuristic approach that uses the Hilbert-Schmidt and Bures metrics. In section V we study some basic properties of the distribution of entanglement in multipartite systems and the effects produced by two-qubits gates acting upon them. Finally, some conclusions are drawn in section VI.

II. OPTIMAL PARAMETRIZATION OF QUANTUM GATES FOR TWO-QUBITS SYSTEMS

Two-qubits systems are the simplest quantum ones exhibiting the entanglement phenomenon. They play a fundamental role in quantum information theory. There remain still some features of these systems, related to the phenomenon of entanglement, that have not yet been characterized in enough detail, as for instance, the manner in which $P(\Delta E)$, the probability of generating a change ΔE associated to the action of these operators, is distributed under the action of certain quantum gates. In this vein it is also of interest to express the general quantum two-qubits gate in a way as compact as possible, i.e., to find an optimal parametrization.

Since any quantum logical gate acting on a two-qubits system can be expressed in the form [23],

$$(v_1 \otimes v_2)\, \exp\left[-i \sum_{i=1}^{3} \lambda_k \sigma_k \otimes \sigma_k\right] (w_1 \otimes w_2)\,, \tag{1}$$

where the transformations $v_{1,2}$ and $w_{1,2}$ act only on one of the two qubits, and σ_k are the Pauli matrices. Note that it is always possible to chose the λ-parameters in such a way that

$$\begin{aligned} \lambda_1 \geq \lambda_2 \geq |\lambda_3|, \\ \lambda_1, \lambda_2 \in [0, \pi/4], \\ \lambda_3 \in (-\pi/4, \pi/4], \end{aligned} \tag{2}$$

and consider the parameterized unitary transformation

$$\tilde{U}_{(\lambda_1,\lambda_2,\lambda_3)} \;=\; \exp\left[-i \sum_{i=1}^{3} \lambda_k \sigma_k \otimes \sigma_k\right]. \tag{3}$$

From previous work [23,24] we know that the unitary transformations (1) and (3) share the same probability distribution $P(\Delta E)$ of entanglement changes. Consequently, the $P(\Delta E)$-distribution generated by any quantum logical gate acting on a two-qubits system coincides, for appropriate values of the λ-parameters, with the distribution of entanglement changes associated with a unitary transformation of the form (3). This means that the set of all possible $P(\Delta E)$-distributions for two-qubits gates constitutes, in principle, a three-parameter family of distributions.

As an example of this equivalence between gates, let us consider the CNOT and $\hat{U}_\theta$ gates

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, U_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{4}$$

It can be shown [24] that, defining the auxiliary gates

$$U_{LA} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \otimes \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}, U_{LB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & 1 \end{pmatrix}, \tag{5}$$

the formal relation $U_{\pi/2} = U_{LA}\, \mathrm{CNOT}\, U_{LB}$ holds, implying that both CNOT and $U_{\pi/2}$ share the same $P(\Delta E)$-distribution.

We have explored the two-qubits space by means of a Monte Carlo simulation [15,16,25] and in Fig. 1 we depict the action of several gates acting on two-qubits pure states, as described by different values of the vector $(\lambda_1, \lambda_2, \lambda_3)$. We see how different the associated entanglement probability distributions are. In point of fact, the CNOT gate (solid line) is equivalent (on average) to $(\pi/4, 0, 0)$. Curve 1 corresponds to $\lambda = (\pi/4,\ \pi/8,\ 0)$, curve 2 to ($\pi/4$, $\pi/8$, $\pi/16$), curve 3 to ($\pi/4$, 0, 0), curve 4 to ($\pi/4$, $\pi/8$, -$\pi/8$), and curve 5 to ($\pi/8$, $\pi/8$, $\pi/8$). All these gates have the common property that they reach the extremum $|\Delta E| = 1$ change if have given the appropriate λ vectors. This is not the case for other gates like the $U_{\pi/4}$ one [12]. The vertical dashed line represents any gate that can be mapped to the identity $\hat{I}$, so that no change in the entanglement occurs (we get a delta function $\delta(\Delta E)$).

III. QUANTUM GATES' ENTANGLING POWER: QUBITS AND QUDITS

As stated, a quantum gate (QG), represented by a unitary transformation $\hat{U}$, changes the entanglement of a given state. As a matter of fact, we may think of the QG as an "entangler". This particular transformation represents the abstraction of some physical interaction taking place between the different degrees of freedom of the pertinent system. A natural question then arises: how good a quantum gate is as an entangler?, or in other words, can we quantify the set of quantum gates in terms of a certain "entanglement capacity"? The question is of some relevance in Quantum Information. A quantum gate robust against environmental influence becomes specially suitable in the case of networks of quantum gates (quantum circuits, quantum computer, etc) as described by Zanardi *et al.* [13], where the so called "entangling power" $\epsilon_P(\hat{U})$ of a quantum gate $\hat{U}$ is defined as follows

$$\epsilon_P(\hat{U}) \equiv \overline{E((\rho_A \otimes \rho_B)\hat{U}(\rho_A \otimes \rho_B)^\dagger)}^{\rho_A,\rho_B}, \qquad (6)$$

where the bar indicates averaging over all (pure) product states in a bipartite quantum state described by $\rho_{AB} = \rho_A \otimes \rho_B \in \mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ and E represents a certain measure of entanglement, in our case the entanglement of formation, that, in the case of pure states becomes just the binary von Neumann entropy of either reduced state $E(\rho_{AB}) = -\mathrm{Tr}(\rho_A \log_2 \rho_A) = -\mathrm{Tr}(\rho_B \log_2 \rho_B)$. It greatly simplifies the numerics of our study to assume that the separable states ρ_{AB} are all equally likely. The corresponding (special) form of (6) exhibits the advantage that it can be generalized to any dimension for a bipartite system. In our case, we are mostly interested in two-qubits systems (the 2×2 case). In [13] the concept of *optimal* gate is introduced, where by *optimal* one thinks of a gate that makes (6) maximal. It is shown there that the CNOT gate is an optimal gate.

Let us suppose now that we make use of the special parameterization $\mathcal{P}$ given in section II for the unitary transformations $U(N)$. In the case of the CNOT gate, it was clear that $\mathcal{P}$ is equivalent (on average) to the $(\pi/4, 0, 0)$ gate. This fact allow us to see how the entangling power (6) evolves when we perturb the CNOT gate in the form $(\pi/4, x, x)$, x

being a continuous parameter. To such an end we numerically generate *separable* [26] states $\rho_A \otimes \rho_B$ according to the Haar measure on the group of unitary matrices $U(N)$ that induces a unique and uniform measure ν on the set of pure states of two-qubits ($N = 4$) [15,16,25]. The corresponding results are shown in Fig. 2. Every point has been obtained averaging a sampling of 10^9 states, so that the associated error is of the order of the size of the symbol. It is clear from the plot that large deviations imply a smaller entangling power $\epsilon_P(CNOT_{pert.})$. Notice that a small perturbation around the origin (CNOT gate) *increases* the entangling power. This fact leads us to conclude that, in the space of quantum gates, and in the vicinity of an optimal gate, there exists an infinite number of optimal gates. On the other hand, if we perturb a quantum gate which is not optimal, like $(\pi/8, x, x)$, any deviation, no matter how small, will lead to an increasing amount of the entangling power ϵ_P. This latter case is depicted in the inset of Fig. 2.

It is argued in [13] that the two-qubits case presents special statistical features, as far as the entangling power is concerned, when compared to $N_A \times N_A$ systems (two-qudits). We investigate this point next, not by making use of any quantum gate, or by recourse to Eq. (6). What we do instead might be regarded a "no gate action": we look at the probability (density) distribution P_R obtained by randomly picking up two pure states generated according to the Haar measure in $N_A \times N_A$ dimensions, and determine then the relative entanglement change ΔE in passing form one of these states to the other. The distribution P_R is [24]

$$P_R(\Delta E) \; = \; \int_0^{1-|\Delta E|} dE \, P(E) \, P(E + |\Delta E|). \tag{7}$$

The distribution $P_R(\Delta E)$ is thus related to the probability density $P(E)$ of finding a quantum state with entanglement E. Notice that the above expression holds for any states space measure invariant under unitary transformations and for any bipartite quantum system consisting of two subsystems described by Hilbert spaces of the same dimensionality. We must point out that the entanglement is measured for every two-qudits in terms of $E = S(\rho_A)/\log(N_A)$, where S is the von Neumann entropy, so that it ranges from 0 to 1

(N_A is the dimension of subsystem A). The resulting distributions are depicted in Fig. 3. The five curves represent the $2 \times 2, 3 \times 3, 4 \times 4, 5 \times 5$ and 6×6 systems. A first glance at the corresponding plot indicates a sudden change in the available range of ΔE. The width of our probability distribution is rather large for two-qubits and it becomes narrower as we increase the dimensionality of the system. With this fact in mind, one may propose the *natural width* of these distributions as some measure of its entangling power. We choose the maximum spread of the distribution in ΔE at half its maximum height $P(0)$. If we use this definition of entangling power $W_{\Delta E}$, Fig. 3 provides numerical evidence for the peculiarity of the two-qubits instance. One may dare to conjecture, from inspection, that for large N_A, $W_{\Delta E}$ decays following a power law: $W_{\Delta E} \sim 1/N_A^{\alpha}$.

IV. TWO-QUBITS SPACE METRICS AND THE ENTANGLING POWER OF A QUANTUM GATE

So far we have considered the QG "entangling power" as applied to the case of pure states of two-qubits. In order to do so, it has been sufficient to generate pure states according to the invariant Haar measure. In passing to mixed two-qubits states, the situation becomes more involved. Mixed states appear naturally when we consider a pure state that is decomposed into an statistical mixture of different possible states by environmental influence (a common occurrence). It may seem somewhat obvious to extend to mixed states the previous study of the entangling power of a certain quantum gate by following the steps given by formula (6). Instead, we will consider a heuristic approach to the problem.

The space of mixed states $\mathcal{S}$ of two-qubits is 15-dimensional, which implies that it clearly possesses non-trivial properties, a systematic survey of which can be found in [15,16,29,30]. In general, the space $\mathcal{S}$ of all (pure and mixed) states $\hat{\rho}$ of a quantum system described by an N-dimensional Hilbert space can be regarded as a product space $\mathcal{S} = \mathcal{P} \times \Delta$ [15,16]. Here $\mathcal{P}$ stands for the family of all complete sets of orthonormal projectors $\{\hat{P}_i\}_{i=1}^N$, $\sum_i \hat{P}_i = I$ (I being the identity matrix). Δ is the set of all real N-uples $\{\lambda_1, \ldots, \lambda_N\}$, with $0 \leq \lambda_i \leq 1$, and $\sum_i \lambda_i = 1$. The general state in $\mathcal{S}$ is of the form $\hat{\rho} = \sum_i \lambda_i P_i$. The Haar measure on the group of unitary matrices $U(N)$ induces a unique, uniform measure ν on the set $\mathcal{P}$ [15,16,25]. On the other hand, since the simplex Δ is a subset of a $(N-1)$-dimensional hyperplane of $\mathcal{R}^N$, the standard normalized Lebesgue measure $\mathcal{L}_{N-1}$ on $\mathcal{R}^{N-1}$ provides a natural measure for Δ. The aforementioned measures on $\mathcal{P}$ and Δ lead to a natural measure μ on the set $\mathcal{S}$ of quantum states [15,16],

$$\mu = \nu \times \mathcal{L}_{N-1}. \tag{8}$$

Since we consider the set of states of a two-qubits system, our system will have $N = 4$. All present considerations are based on the assumption that the uniform distribution of states of a two-qubit system is the one determined by the measure (8). Thus, in our numerical computations we are going to randomly generate states of a two-qubits system according to the measure (8). At the same time, we compute distances between states, which can be evaluated by certain measures [22]. The ones that are considered here are the Bures distance

$$d_{Bures}(\hat{\rho}_1, \hat{\rho}_2) \;=\; \left(2 - 2\,Tr\sqrt{(\sqrt{\hat{\rho}_2}\hat{\rho}_1\sqrt{\hat{\rho}_2})}\right)^{\frac{1}{2}}, \tag{9}$$

and the Hilbert-Schmidt distance

$$d_{HS}(\hat{\rho}_1, \hat{\rho}_2) \;=\; \sqrt{|Tr[\hat{\rho}_1 - \hat{\rho}_2]^2|}. \tag{10}$$

The goal is to generate unentangled states $\hat{\rho}$ (according to (8)) of two-qubits and to compute by means of measures (9,10) the average distance reached in $\mathcal{S}$ by a final state $\hat{\rho}'$, once the CNOT gate (4) is applied. In other words, we quantify the action of the CNOT gate acting on the set $\mathcal{S}'$ of completely separable states. The several distances between final (after CNOT) and initial states are computed, and a probability (density) distribution is then obtained.

The probability distributions for the Bures and Hilbert-Schmidt distances are depicted in Fig. 4 and Fig. 5, respectively. However, one has to bear in mind that these absolute distances between states do not take into account the fact that the set $\mathcal{S}'$ may have (and indeed such is the case) a certain non-trivial geometry, which makes the shape of the convex set of separable states $\mathcal{S}'$ highly anisotropic [31]. Therefore, in order to clarify the action of the CNOT gate, we separate the set $\mathcal{S}'$ into two parts: I) $\mathcal{S}'_I$, which is the set of unentangled states inside the minimal separable ball around $\frac{1}{4}\hat{I}$ of radius d_{min}, as measured with either (9) or (10), and II) $\mathcal{S}'_{II}$, which is nothing but $\mathcal{S}' - \mathcal{S}'_I$. In point of fact, d_{min} corresponds to the radius of a hypersphere in 15 dimensions whose interior points have Tr($\hat{\rho}^2$)$\leq 1/3$ ([21]). As seen from Figs. 4 or 5, the first case exhibits a well defined range. This is due to the fact that any unitary evolution (CNOT in our case) does not change Tr($\hat{\rho}^2$), so that the CNOT gate cannot produce entanglement at all or, in other words, cannot "move" to any extent a state $\hat{\rho}$ out of $\mathcal{S}'_I$. On the other hand, CNOT may entangle in $\mathcal{S}'_{II}$ and displace the whole distribution to the right. Indeed, if we consider for both graphs the total set $\mathcal{S}'$, the concomitant distributions look rather alike. The crossing point of the three curves in Figs. 4 and 5 corresponds to the border defined by d_{min}^{Bures} and d_{min}^{HS}, respectively.

In view of these results, one may call a QG "strong" if its entangling power, in acting on a separable state, is great. Thus a semi-quantitative strength-measure could be the average value of the distance $\overline{d}^{\mathcal{S}'}$ over the whole set of separable states. However, it should be pointed out that any definition of entangling power for mixed states would turn out to be metric-dependent, i.e., it depends on the set of eigenvalues Δ wherefrom $\hat{\rho}$ is generated.

V. ENTANGLEMENT DISTRIBUTION IN MULTIPLE QUBITS SYSTEMS

So far we considered logical QGs acting on two-qubits systems. We pass now to multi-partite ones (nothing strange: the environment can be regarded as a third party), composed of many subsystems. We thus deal with a network of qubits, interacting with each other, and with a given configuration. More specifically, one could consider the set $\mathcal{S}$ of pure states $\hat{\rho} = |\Psi\rangle\langle\Psi|$ "living" in a Hilbert space of n parties (qubits) $\mathcal{H} = \otimes_{i=1}^{n}\mathcal{H}_i$.

The usual three party, physically-motivated case, is the two-qubits system interacting with an environment which, as a first approximation, could be treated roughly as a qubit (two-level system). In any case, the issue of how the entanglement present in a given system is *distributed* among its parties is interesting in its own right. Therefore, it should be of general interest to study the general case of multipartite networks of qubits on the one hand, while discussing, on the other one, how the dimensionality (the number of qubits) affects the distribution of the bipartite entanglement between pairs when we apply, locally, a certain quantum gate.

In what follows we consider the Coffman *et al.*−approach of [32] and consider firstly the case of three qubits in a pure state $\hat{\rho}_{ABC}$. An important inequality exists that refers to how the entanglement between qubits is pairwise distributed. The entanglement is measured by the concurrence squared C^2. Even though we handle pure states, once we have traced over the rest of qubits we end up with mixed states of two qubits, so that a measure for mixed states is needed. C^2 is related to the entanglement of formation [14]. It ranges from 0 to 1. The concurrence is given by $C = max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$, λ_i, $(i = 1, \ldots 4)$ being the square roots, in decreasing order, of the eigenvalues of the matrix $\rho\tilde{\rho}$, with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. The latter expression has to be evaluated by recourse to the matrix elements of ρ computed with respect to the product basis. Considering the reduced density matrices $\hat{\rho}_A = Tr_{BC}\,(\hat{\rho}_{ABC})$, $\hat{\rho}_{AB} = Tr_C\,(\hat{\rho}_{ABC})$ and $\hat{\rho}_{AC} = Tr_B\,(\hat{\rho}_{ABC})$, the following elegant relation is derived:

$$C^2_{AB} + C^2_{AC} \leq 4\,det\hat{\rho}_A\,(\equiv C^2_{A(BC)}), \tag{11}$$

where $C^2_{A(BC)}$ shall be regarded as the entanglement of qubit A with the rest of the system. In fact, we are more concerned in quantifying (do not be confused with distances of the previous section) $d_W \equiv C^2_{A(BC)} - C^2_{AB} - C^2_{AC}$. From inspection, d_W ranges from 0 to 1 and can be regarded as a legitimate multipartite entanglement measure, endowed with certain properties [32].

In Fig. 6 the probability (density) function $P(d_W)$ is obtained by generating a sample of pure states of three qubits according to the invariant Haar measure, as we did for $n = 2$ in sections II-III. It is interesting to notice the bias of the distribution, and the remarkable fact that numerical evaluation indicates that $\overline{d_W} \simeq 1/3$.

Now, suppose that we apply the CNOT gate to the pair of qubits AB. This means that the unitarity acting on the state $\hat{\rho}_{ABC}$ is described by $\hat{U}_{AB}^{CNOT} \otimes \hat{I}_C$, where $\hat{I}_X$ is the identity acting on qubit X. Making then a numerical survey of the action of this operator on the evolution of the system we show the concomitant, pairwise entanglement-change ΔE as the probability distributions plotted in Fig. 7a (as measured by the entanglement of formation E). Two types of entanglement are present in the system, namely, the one between the pair AB, where the gate is applied, and the remaining possibilities AC and BC, symmetric on average. The solid thick line depicts the first kind AB, while the second type AC, BC exhibits a sharper distribution (dashed line). One is to compare this distributions to the one obtained by picking up two states at random (solid thin line), which resembles the case of Fig. 3. Again, the random case exhibits a larger width for the distribution. When compared to the two-qubits CNOT case (thin dot-dashed line), we may think of the existence of a third party as a rough "thermal bath" that somehow dilutes the entanglement available to the pair AB, as prescribed by the relation (11). This is why the CNOT distribution for $n = 3$ seems "sharper" than that for $n = 2$. As a matter of fact, if we continue increasing the number of qubits present in the system, we can numerically check that the generalization of (11) still holds. In such a (new) instance, the action of the CNOT gate is equivalent to the evolution governed by $\hat{U}_{AB}^{CNOT} \otimes \hat{I}_C \otimes \hat{I}_D$. As it is shown in Fig. 7b, the new distribution of the entanglement changes for $n = 4$ in the AB pair (dashed line, out of scale) and, as expected, is more peaked than for the $n = 2$ (dot-dashed line) and, $n = 3$ (solid line) cases, reinforcing our thermodynamical analogy [24]. If we compute their entangling power (EP) with $W_{\Delta E}$, our new measure defined in section III, we could conjecture that the EP decreases exponentially with the number of qubits n ($W_{\Delta E}^{n=2} \simeq 0.437, W_{\Delta E}^{n=3} \simeq 0.196, W_{\Delta E}^{n=4} \simeq 0.002$).

VI. CONCLUSIONS

In the present work we have focused attention upon the action of quantum gates as applied to multipartite quantum systems and presented the results of a systematic numerical survey. In particular, we investigated aspects of the quantum gate or unitary operation (acting on two-qubits states) as conveniently represented by a vector $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3)$, visualizing the "entangling power" of unitary quantum evolution from two different perspectives.

- The first one refers to pure states of a bipartite system. One has here a well defined formula that quantifies the ability of a given transformation $\hat{U}$ to entangle, on average, a given state that pertains to the set $\mathcal{S}'$ of unentangled pure states. We have seen that the collective of all possible quantum gates, as defined by the vector λ, posseses the following property: in the vicinity of an optimal gate there are infinite quantum gates which are optimal as well. In addition, we introduced a measure of the entangling power above referered to: $W_{\Delta E}$, on the basis of the probability (density) distribution (associated with a quantum gate) of finding a state that experiences a given change ΔE in its entanglement E. A power-law decay is conjectured: $W_{\Delta E} \sim 1/N_A^{\alpha}$, N_A being the dimension of the subsystem ($N = N_A \times N_A$).

- The second instance deals with mixed states and the metrics of the 15-dimensional space $\mathcal{S}$ of mixed states of two-qubits. We introduce an heuristic measure based on an average distance $\overline{d}$ obtaind from the distribution of distances between states in $\mathcal{S}$, as defined by the action of a definite quantum gate acting (again) on the set of unentangled states $\mathcal{S}'$.

Finally, we have studied i) some basic properties of the distribution of entanglement in multipartite systems (MS) (network of qubits) and ii) the effects produced by two-qubits gates acting upon MS. The fact that the entanglement between pairs becomes diluted by the presence of third or fourth parties becomes apparent from the concomitant distribution of entanglement changes. Their natural width $W_{\Delta E}$ decreases with the number of parties n, in what seems to be an exponential fashion.

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FIGURE CAPTIONS

Fig.1- $P(\Delta E)$-distributions generated by the two-qubits quantum gates, parametrized in an optimal way. Curve 1 corresponds to $\lambda = (\pi/4,\ \pi/8,\ 0)$, curve 2 to $(\pi/4,\ \pi/8,\ \pi/16)$, curve 3 to $(\pi/4,\ 0,\ 0)$ (or equivalently to the CNOT gate), curve 4 to $(\pi/4,\ \pi/8,\ -\pi/8)$ and curve 5 to $(\pi/8,\ \pi/8,\ \pi/8)$. The vertical line represents any gate that can be mapped to the identity $\hat{I}$. All depicted quantities are dimensionless.

Fig.2- Entangling power ϵ_P of the perturbed CNOT gate, expressed in the form of $(\pi/4, x, x)$. Small perturbations around this optimal gate ($x = 0$) find gates which are also optimal (greater ϵ_P). Large deviations diminish the concomitant ϵ_P. A perturbed non-optimal gate, like $(\pi/8, x, x)$ shown in the inset, increases its ϵ_P. See text for details. All depicted quantities are dimensionless.

Fig.3- $P(\Delta E)$-distributions generated (ΔE being the change in the entanglement of formation) by randomly choosing the initial and final pure two-qubits states (2×2), and several two-qudits states ($N_A \times N_A$, for $N_A = 3, 4, 5, 6$). The two-qubits instance appears to be a peculiar case. All depicted quantities are dimensionless.

Fig.4- Probability (density) distributions of finding a state of two-qubits (pure or mixed) being sent a distance d_{Bures} away from the original state $\hat{\rho}$, after the action of the CNOT gate. All initial states belong to the set $\mathcal{S}'$ of separable states. Two regions are defined. See text for details. All depicted quantities are dimensionless.

Fig.5- Same as in Fig. 4, using the Hilbert-Schmidt distance d_{HS} between states. Both figures show similar qualitative features. See text for details. All depicted quantities are dimensionless.

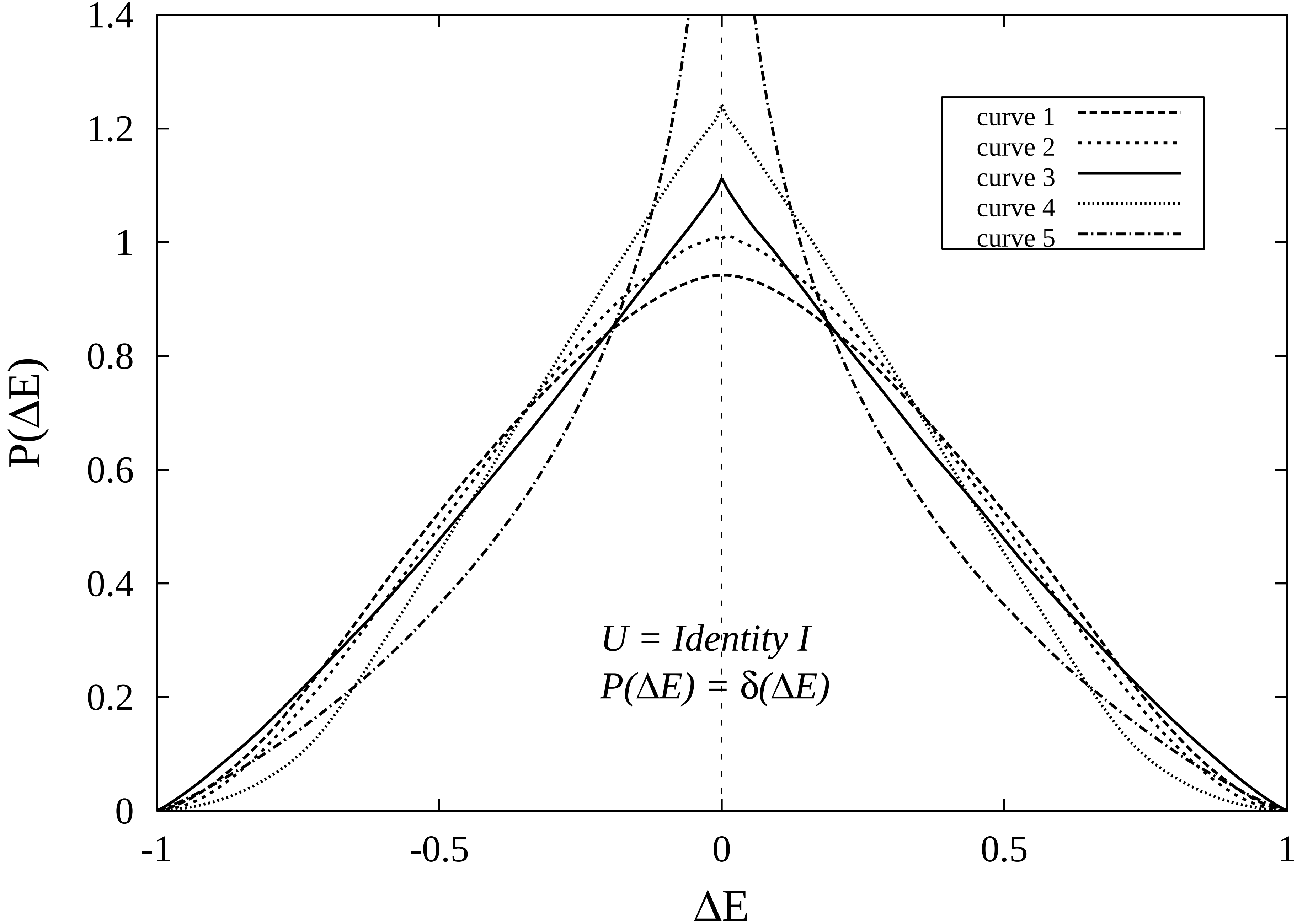
Fig.6- Probability (density) distribution of finding a pure state of three-qubits with a given value of d_W (11), a measure of the distribution of the pairwise entanglement in the

system. The curve is biased to low values of d_W, and $\overline{d_W} \simeq 1/3$. All depicted quantities are dimensionless.

Fig.7a- $P(\Delta E)$-distributions generated by the CNOT quantum gate $\hat{U}_{AB}^{CNOT} \otimes \hat{I}_C$, acting on the pair AB of a pure state of three-qubits. The resulting distribution (solid thick line) is to be compared with the one of the pairs AC, BC, equal on average (dashed line), the random case where no gate is applied (solid thin line) and the case of solely two-qubits CNOT gate $P(\Delta E)$ distribution (thin dot-dashed line). As compared to the three-qubit random instance, it possesses a width slightly inferior, being much narrower than in the two-qubits case. This fact indicates that the entanglement available to the pair AB is diluted by the presence of a third party. All depicted quantities are dimensionless.

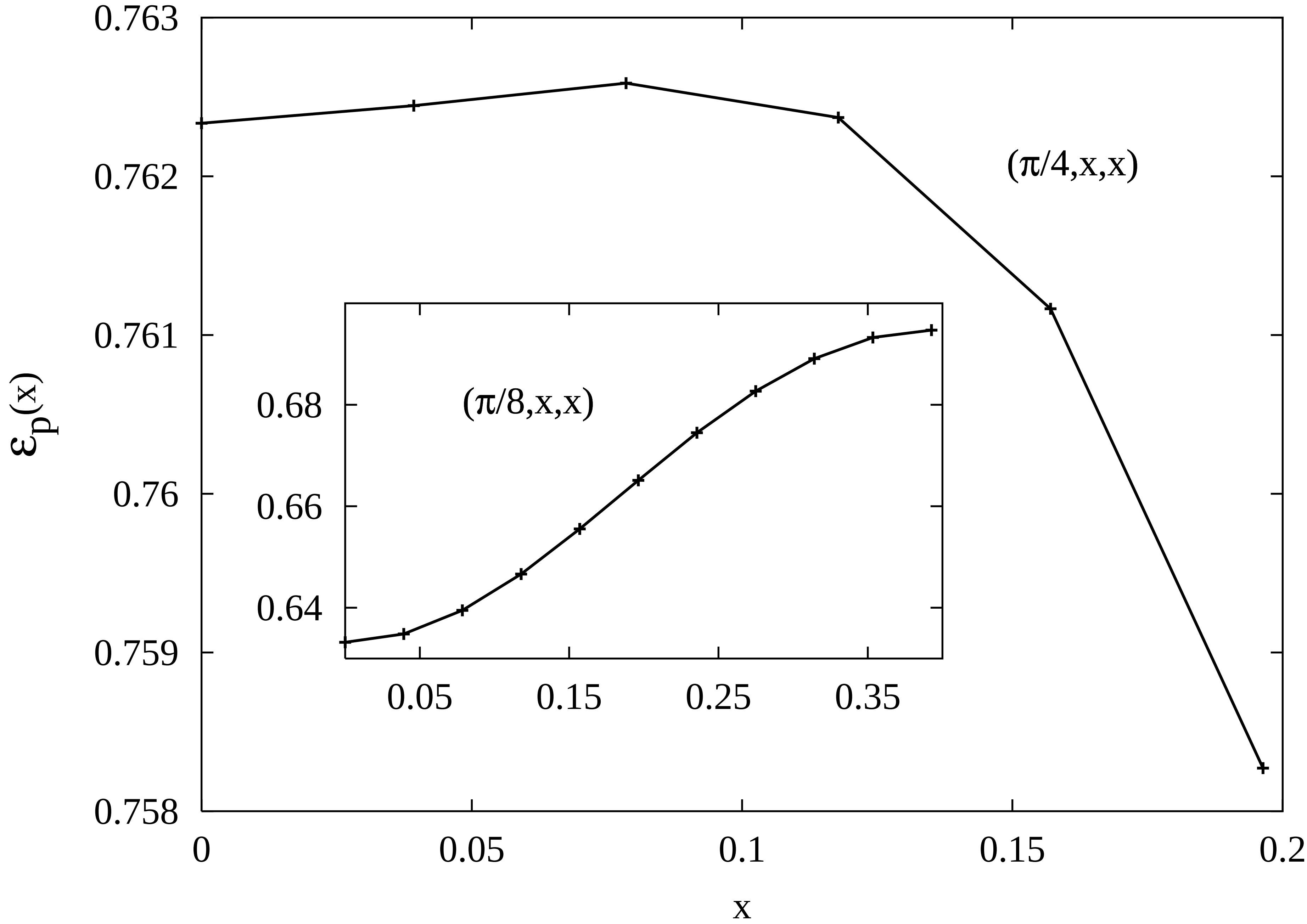
Fig.7b- These distributions result from the action of the CNOT gate $\hat{U}_{AB}^{CNOT}$ on two-qubits ($n = 2$, dot-dashed line), $\hat{U}_{AB}^{CNOT} \otimes \hat{I}_C$ on three qubits ($n = 3$, solid line), and $\hat{U}_{AB}^{CNOT} \otimes \hat{I}_C \otimes \hat{I}_D$ on four qubits ($n = 4$, dashed line) pure states. The width of these distributions, or entangling power $W_{\Delta E}$ (see text), decreases exponentially as the number of qubits is increased. All depicted quantities are dimensionless.

fig. 1



1.4
1.2
1
0.8
0.6
0.4
0.2
0
P(ΔE)
-1
-0.5
0
0.5
1
ΔE
curve 1
curve 2
curve 3
curve 4
curve 5
U = Identity I
P(ΔE) = δ(ΔE)

fig. 2



0.763
0.762
0.761
0.76
0.759
0.758
0
0.05
0.1
0.15
0.2
x
ε_p(x)
(π/4,x,x)
(π/8,x,x)
0.68
0.66
0.64
0.05
0.15
0.25
0.35

fig. 3

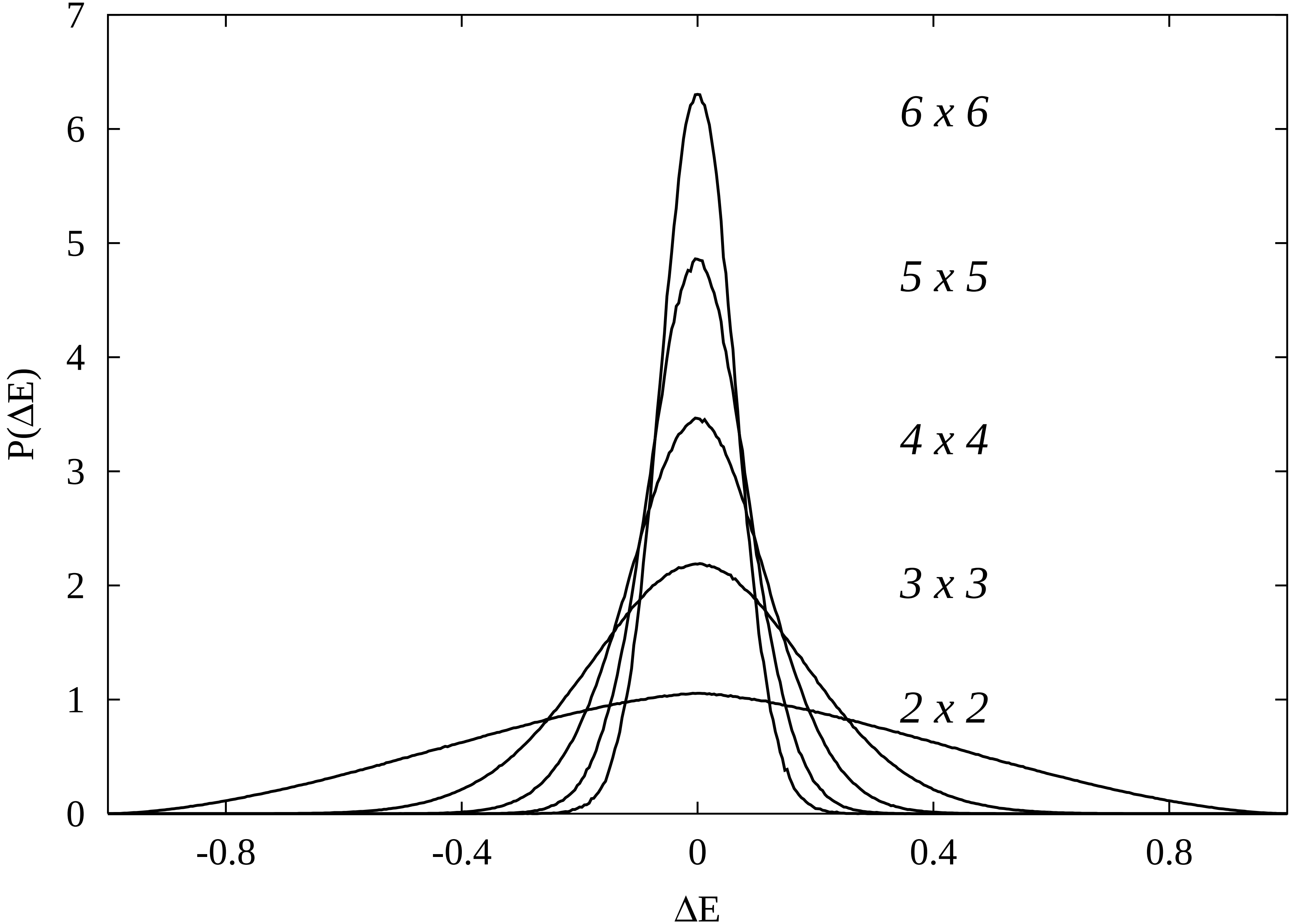
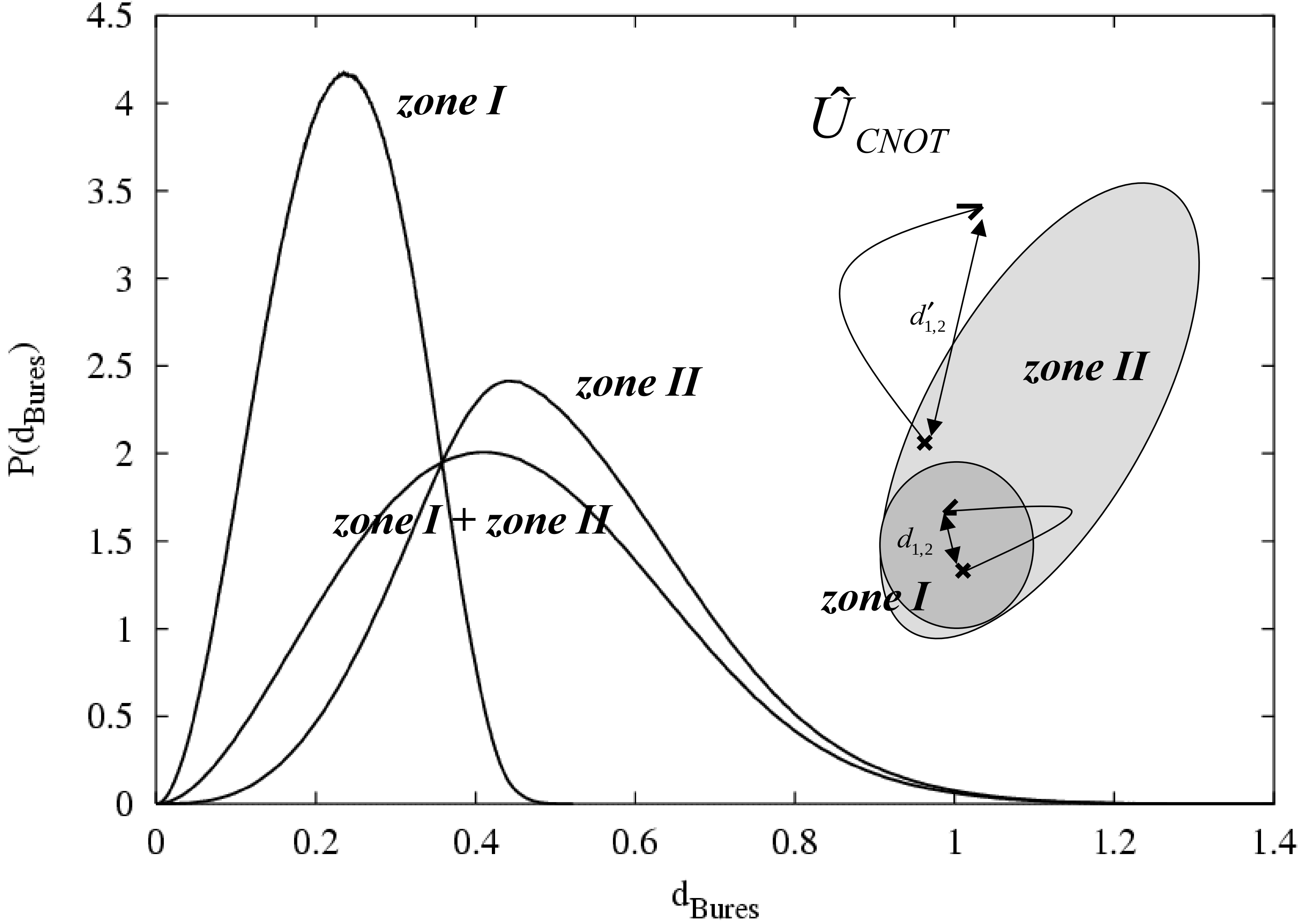
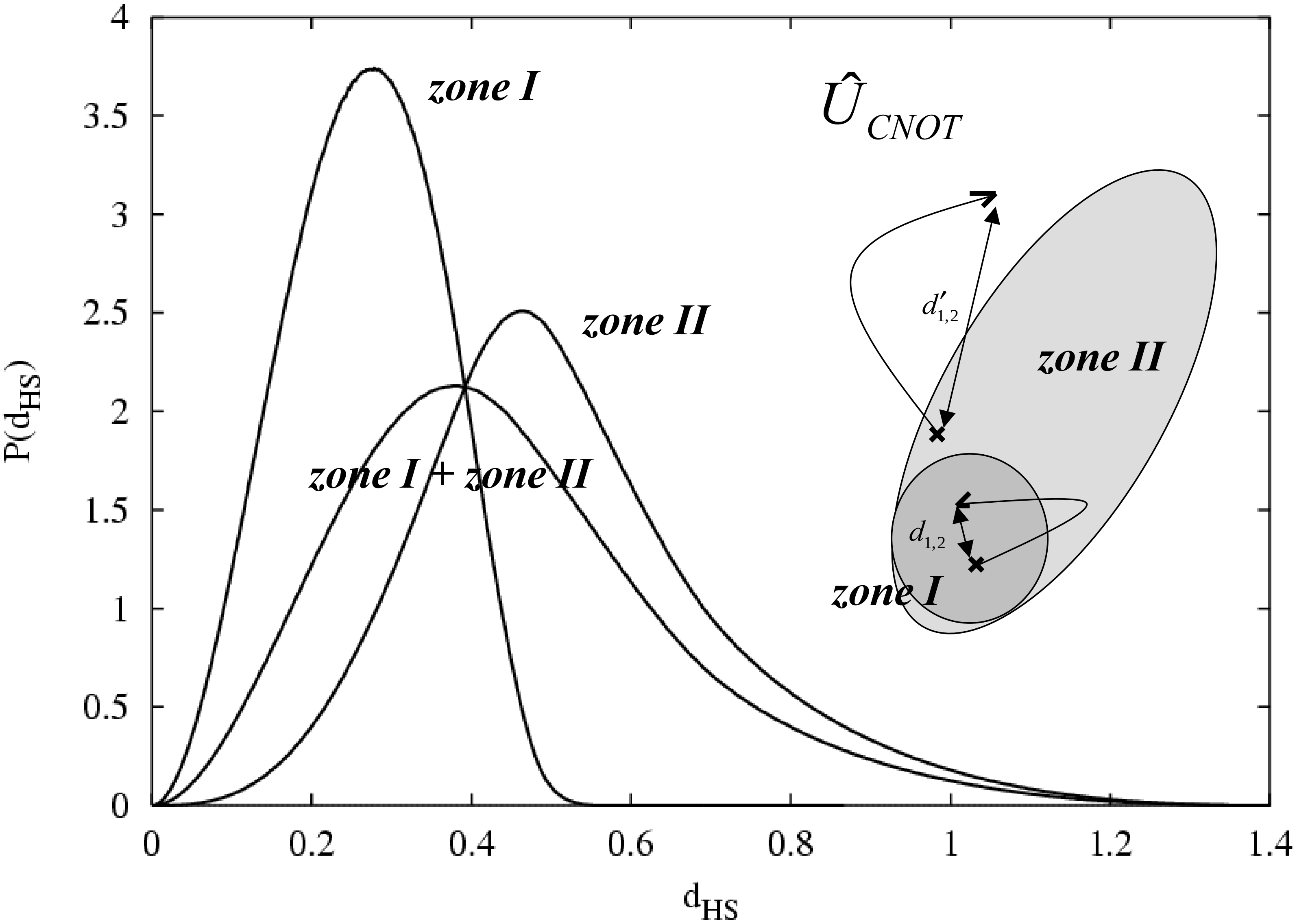


Fig. 4

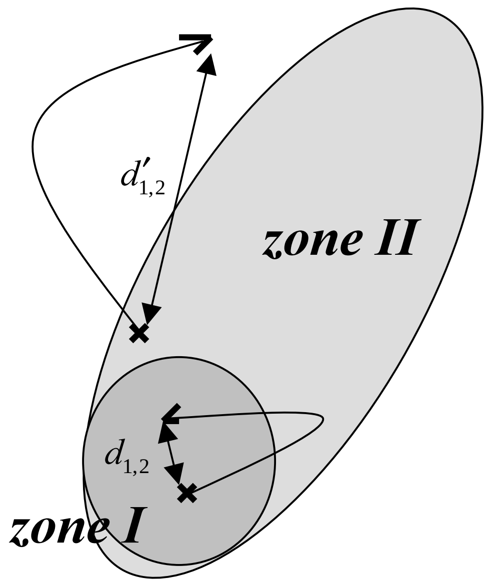


4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
P(d_Bures)
d_Bures
zone I
zone II
zone I + zone II
Û CNOT
d'1,2
d1,2
zone II
zone I

Fig. 5



4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
P(d_HS)
d_HS
zone I
zone II
zone I + zone II
Û_CNOT
d'_{1,2}
zone II
d_{1,2}
zone I



$d'_{1,2}$
zone II
$d_{1,2}$
zone I

Fig. 6

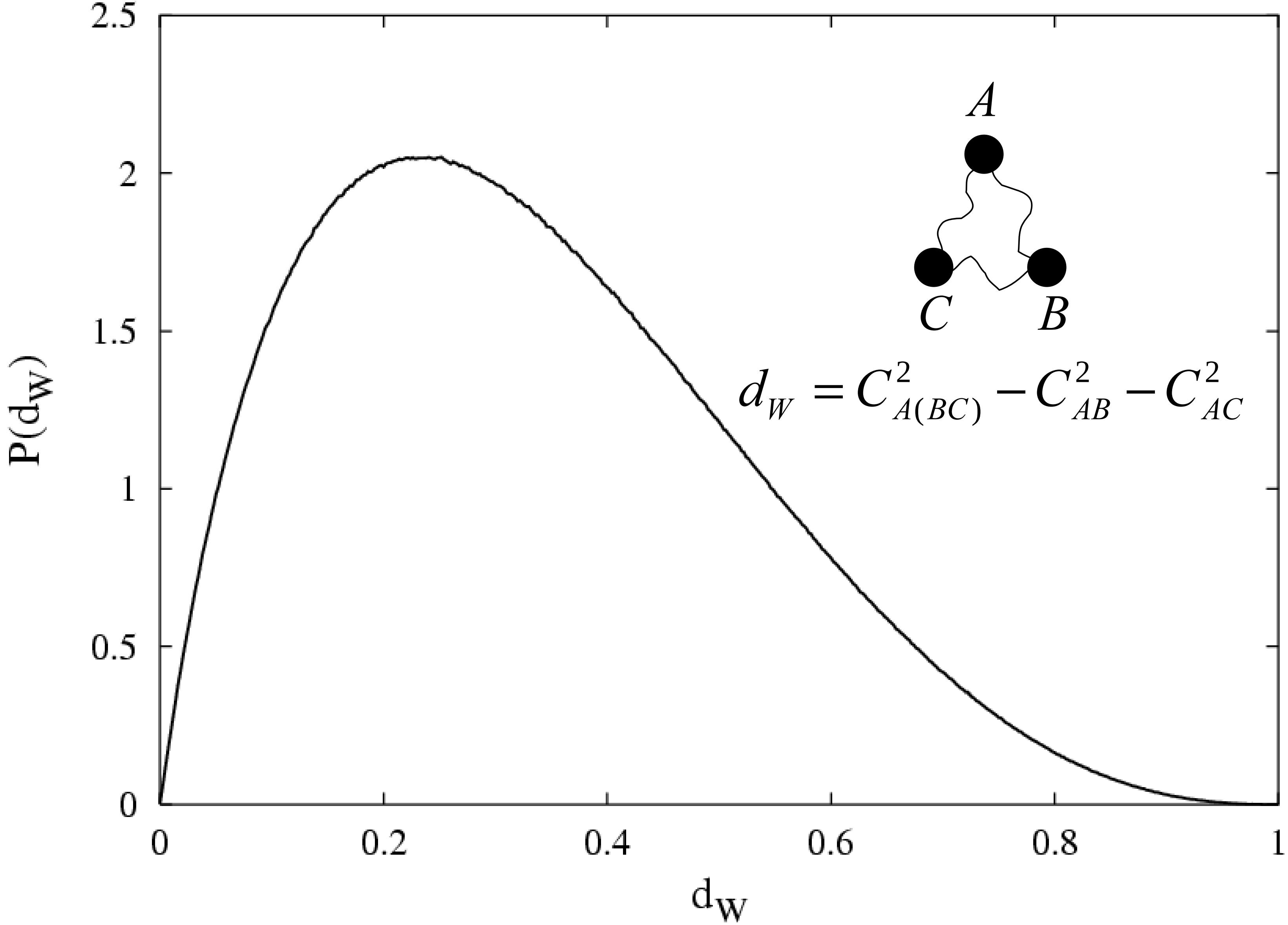


fig. 7