Shear Diversity Prevents Collective Synchronization

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Large ensembles of heterogeneous oscillators often exhibit collective synchronization as a result of mutual interactions. If the oscillators have distributed natural frequencies and common shear (or nonisochronicity), the transition from incoherence to collective synchronization is known to occur at large enough values of the coupling strength. However, here we demonstrate that shear diversity cannot be counterbalanced by diffusive coupling leading to synchronization. We present the first analytical results for the Kuramoto model with distributed shear and show that the onset of collective synchronization is impossible if the width of the shear distribution exceeds a precise threshold.

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Collective synchronization is a form of self-organization in time that results from the interactions among a large heterogeneous population of self-sustained oscillators [1–5]. This phenomenon is observed in a large variety of systems that range from biology to chemistry, physics, and engineering (see, e.g., [6]). For the sake of mathematical simplicity, most theoretical advances in this field consider oscillators with different natural frequencies. Nevertheless, it is of great importance to know how other sources of heterogeneity influence collective synchronization. This question has been addressed in the cases of heterogeneous patterns of connectivity [7] and interaction delays [8].

The so-called shear (or nonisochronicity) is a crucial nonlinear ingredient for the formation of patterns in oscillatory extended media [2,9], as well as for the onset of complex behaviors in ensembles of identical limit-cycle oscillators [10]. Although shear is a generic feature of oscillators, studies considering shear diversity in ensembles of oscillators are very scarce and focus on a regime far from synchronization [11]. Here we will show that distributed shear plays a key role in collective synchronization and may even prevent its onset.

Mathematical formulations of collective synchronization usually consider as elementary oscillatory unit a normal form describing a system near the onset of oscillations, the so-called Stuart-Landau (SL) oscillator [2]:

$$\dot{\varrho} = \varrho(1-\varrho^2), \qquad \dot{\theta} = \omega + q(1-\varrho^2). \tag{1}$$

Here the natural frequency ω determines the frequency of rotation on the attractor of radius $\varrho(t) = 1$. Additionally, q quantifies the shear of the flow, i.e., how much perturbations off the limit cycle modify the angular frequency $\dot{\theta}$. It is then usual to consider an ensemble of $N \gg 1$ globally coupled SL oscillators (1), a mean-field version of the complex Ginzburg-Landau equation with disorder [12]:

$$\dot{z}_j = z_j[1 + i(\omega_j + q_j) - (1 + iq_j)|z_j|^2] + K(Z - z_j), \tag{2}$$

where $z_j = \varrho_j e^{i\theta_j}$, $j = 1, \ldots, N$, and $Z = N^{-1}\sum_{k=1}^{N} z_k$. We assume a purely dissipative coupling, $K \in \mathbb{R}$.

Previous works studying model (2) have adopted the simplifying assumption that diversity is only present in the natural frequencies ω_j, and the shear is either absent $q_j = 0$ [12,13] or constant $q_j = q$ [14]. However, in a heterogeneous ensemble, either the inherent (say, genetic variability for living organisms or tolerances for electronic circuits) or the imposed (e.g., experiments using coupled chemical reactors [15]) disorder will generically be reflected in both natural frequency and shear terms.

The aim of this Letter is to analyze the genuine problem of collective synchronization in a large ensemble of SL oscillators (2) with ω_j and q_j distributed. In our mathematical analysis we will assume that the oscillators are weakly coupled; i.e., $|K|$ is small. In this case, the dynamics of Eq. (2) is well described by the phases only [2],

$$\dot{\theta}_j = \omega_j + Kq_j + KR[\sin(\Psi - \theta_j) - q_j\cos(\Psi - \theta_j)]. \tag{3}$$

Here, the complex order parameter $r = Re^{i\Psi} = N^{-1}\sum_{k=1}^{N} e^{i\theta_k}$ is a mean field and measures the degree of synchronization in the population. The well-known Kuramoto model is recovered in the fully isochronous case, $q_j = 0$ [2,5,12,16], whereas the nonisochronous case without disorder, $q_j = q$, corresponds to the so-called Sakaguchi-Kuramoto model [17,18].

To analyze model (3) we adopt its thermodynamic limit $N \to \infty$. Thus we drop the indices and introduce the probability density for the phases $f(\theta, \omega, q, t)$ [19]. Then, the quantity $f(\theta, \omega, q, t)d\theta d\omega dq$ represents the ratio of oscillators with phases between θ and $\theta + d\theta$, natural frequencies between ω and $\omega + d\omega$, and shear between q and $q + dq$. The density f obeys the continuity equation

$$\partial_t f = -\partial_\theta\left(\left\{\omega + Kq + \frac{K}{2i}[re^{-i\theta}(1 - iq) - \text{c.c.}]\right\}f\right),$$

where c.c. stands for complex conjugate of the preceding term, and the complex order parameter is

$$r = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{2\pi} e^{i\theta} f(\theta, \omega, q, t) d\theta d\omega dq. \quad (4)$$

If the phases are uniformly distributed r vanishes. This state is customarily referred to as incoherence. Since $f(\theta, \omega, q, t)$ is real and 2π periodic in the θ variable, it admits the Fourier expansion

$$f(\theta, \omega, q, t) = \frac{p(\omega, q)}{2\pi} \sum_{l=-\infty}^{\infty} f_l(\omega, q, t) e^{il\theta}, \quad (5)$$

where $f_l = f_{-l}^*$, $f_0 = 1$, and $p(\omega, q)$ is the joint probability density function (PDF) of ω and q. The first Fourier mode is important because it determines the order parameter (4):

$$r^*(t) = \iint_{-\infty}^{\infty} p(\omega, q) f_1(\omega, q, t) d\omega dq. \quad (6)$$

Inserting the Fourier series (5) into the continuity equation we obtain the following set of integro-differential equations for the Fourier modes

$$\partial_t f_l = -il(\omega + Kq) f_l + \frac{Kl}{2}[r^*(1 + iq) f_{l-1} - r(1 - iq) f_{l+1}]. \quad (7)$$

Recently Ott and Antonsen found that the ansatz [20,21]

$$f_l(\omega, q, t) = \alpha(\omega, q, t)^l \quad (8)$$

is a particular solution of the Kuramoto model and related systems with distributed natural frequencies. Here we also resort to (8) as it turns out to be a solution in our case if α obeys

$$\partial_t \alpha = -i(\omega + Kq)\alpha + \frac{K}{2}[r^*(1 + iq) - r(1 - iq)\alpha^2]. \quad (9)$$

The idea behind the approach of Ott and Antonsen is to simplify an infinite set of equations—like Eq. (9)—using distributions that can be inserted in Eq. (6) and integrated via Cauchy's residue theorem (see below).

In this Letter we assume that ω and q are independent random variables, $p(\omega, q) = g(\omega)h(q)$. Moreover, we restrict the analysis to symmetric unimodal PDFs $g(\omega)$ and $h(q)$ centered at ω_0 and q_0, respectively. We start choosing g and h to be Lorentzian (Cauchy) PDFs,

$$g(\omega) = \frac{\delta/\pi}{(\omega - \omega_0)^2 + \delta^2}; \qquad h(q) = \frac{\gamma/\pi}{(q - q_0)^2 + \gamma^2}.$$

The integrals in (6) can be evaluated by means of the residue theorem with the contour closings at infinity in the lower or the upper half-plane of $\mathbb{C}$, granted $\alpha = f_1$ can be continued from real ω and q into complex $\omega = \omega_r + i\omega_i$ and $q = q_r + iq_i$.

Regarding variable ω, analyticity of α holds in the lower half complex ω plane (see [20]). As $g(\omega) = (2\pi i)^{-1}[(\omega - \omega_0 - i\delta)^{-1} - (\omega - \omega_0 + i\delta)^{-1}]$ has only one simple pole $\omega^p = \omega_0 - i\delta$ inside this integration contour, only the value of $\alpha = f_1$ at $\omega = \omega^p$ counts in the integral over ω in Eq. (6).

The integration over q in Eq. (6) is more intricate. We have to choose an integration contour such that, if α is analytic and $|\alpha| \leq 1$ everywhere inside the contour at $t = 0$, this will hold for all $t > 0$. As α is a solution of Eq. (9), the analyticity of α at $t = 0$ is preserved as t grows if α remains finite [22]. Moreover, if α is analytic, the Cauchy-Riemann conditions imply $\partial_{q_r}|\alpha| + \partial_{q_i}|\alpha| \geq 0$, and the maximum of $|\alpha|$ is necessarily located on the boundary (namely, on the integration contour). First of all, setting $\alpha = |\alpha|e^{-i\psi}$ in Eq. (9), we obtain that, on the real q axis, $|\alpha|$ is governed by

$$\partial_t|\alpha| = -\delta|\alpha| + \frac{K}{2}\mathrm{Re}[r^* e^{i\psi}(1 + iq)](1 - |\alpha|^2). \quad (10)$$

The fact that $\partial_t|\alpha| = -\delta < 0$ at $|\alpha| = 1$ guarantees that, if an initial condition satisfies $|\alpha(\omega, q, t = 0)| < 1$, this will hold for all $t > 0$. Consequently, the series in Eq. (5) remains convergent. Regarding the semicircular path $q = |q|e^{i\vartheta}$ with $|q| \to \infty$, at $|\alpha| = 1$ Eq. (9) yields

$$\partial_t|\alpha| = (1 - R\cos\chi)K|q|\sin\vartheta, \quad (11)$$

where $\chi(\omega, q, t) = \psi(\omega, q, t) - \Psi(t)$. In this equation the desired relation, $\partial_t|\alpha| \leq 0$, is fulfilled in the lower half complex q plane $\vartheta \in (-\pi, 0)$ only if $K > 0$, and in the upper half-plane $\vartheta \in (0, \pi)$ for $K < 0$.

The integral over q in (6) can now be conveniently evaluated, and yields

$$r^*(t) = \alpha(\omega = \omega^p, q = q^p, t) = a(t), \quad (12)$$

with $q^p = q_0 - i\gamma$ for $K > 0$, and $q^p = q_0 + i\gamma$ for $K < 0$. Thus, among the infinite set of equations, Eq. (9), only the one at $(\omega, q) = (\omega^p, q^p)$ is needed:

$$\dot{a} = -i\omega^p a + \frac{K}{2}(1 - iq^p)(1 - |a|^2)a. \quad (13)$$

The dynamics of the radial component $|a| = R$ obeys

$$\dot{R} = \left[-\delta + \frac{K}{2}(1 \mp \gamma)(1 - R^2)\right]R, \quad (14)$$

where "$\mp$" stands for "$-$" if $K > 0$ and "$+$" if $K < 0$. The incoherent state, $R = 0$, is always stable except above

$$K_c = \frac{2\delta}{1 - \gamma} \quad \text{if } \gamma < 1. \quad (15)$$

At K_c a stable nontrivial solution, corresponding to a partially synchronized state, appears with $R^2 = (K - K_c)/K$. Equation (15) is depicted in Fig. 1(a) and compared with numerical simulations of Eq. (2). In the Kuramoto model—recovered when $\gamma = 0$—a large enough coupling strength K *always* results in partial synchronization of the population above K_c, for any width δ of the frequency distribution $g(\omega)$. However, here we find that the width γ of the shear distribution $h(q)$ has a more severe effect on the synchronization transition. If $\gamma \geq \gamma_d = 1$, synchrony disappears for all K and δ, and incoherence becomes the only stable state. It is noteworthy that this is a collective phenomenon caused by the presence of distributed shear, and it has no counterpart in the case of two coupled SL oscillators [11,23,24].

To confirm the generality of these findings for other PDFs [25], next we follow [19] and perform the linear stability analysis of Eq. (7) about the incoherent state $f_{l\neq 0} = 0$. We find that the only potentially unstable modes are $l = \pm 1$. Inserting $f_1(\omega, q, t) = b(\omega, q)e^{\lambda t}$ into Eq. (7), the discrete spectrum of eigenvalues λ can be obtained by virtue of a self-consistency argument. This yields the integral equation

$$\frac{2}{K} = \iint_{-\infty}^{\infty} \frac{1 + iq}{\lambda + i(\omega + qK)} g(\omega)h(q)d\omega dq. \tag{16}$$

The border of unstable incoherence K_c is found imposing $\mathrm{Re}(\lambda) \to 0^+$. If $g(\omega)$ and $h(q)$ are Gaussian functions with variances σ^2 and ν^2, respectively, and $h(q)$ has zero mean ($q_0 = 0$), the critical coupling can be explicitly obtained:

$$K_c = \sigma\sqrt{\frac{-\pi + 8\nu^2 + \sqrt{\pi(\pi + 16\nu^2)}}{2\nu^2(\pi - 2\nu^2)}}. \tag{17}$$

This function is plotted in Fig. 1(b) and compared with the results of numerical simulations. Remarkably, we find again a threshold for the dispersion of $h(q)$, $\nu_d = \sqrt{\pi/2} = 1.253\ldots$, above which incoherence is stable for all K.

We have found that the divergence of K_c does not only exist for Lorentzian and Gaussian distributions, but for any symmetric unimodal distribution. This divergence occurs at a shear diversity that is conveniently expressed in terms of the peak value $h(q_0)$. If $q_0 = 0$, the divergence occurs at

$$h(0) = \pi^{-1}. \tag{18}$$

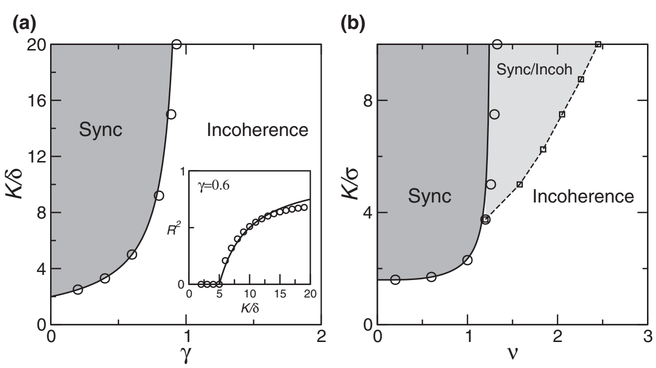

FIG. 1. Phase diagram for (a) Lorentzian and (b) Gaussian PDFs, $g(\omega)$ and $h(q)$. (a) Solid line: synchronization critical coupling, Eq. (15). Inset: R^2 as a function of K; the solid line is $R^2 = (K - K_c)/K$. (b) Solid line: critical coupling given by Eq. (17). In both (a) and (b) the symbols correspond to numerical results obtained using an ensemble of SL oscillators, Eq. (2), with (a) $N = 40000$, $\delta = 0.01$, $\omega_0 = q_0 = \frac{1}{2}$ and (b) $N = 22500$, $\sigma = 0.02$, $\omega_0 = q_0 = 0$. Parameters ω_j and q_j were deterministically selected to represent $p(\omega, q)$, and averages were done over time.

Otherwise, if $h(q)$ is not centered at zero, K_c also diverges at a certain value of $h(q_0) = h_d$, but a simple distribution-independent formula like (18) does not exist.

Our numerical calculations of Eq. (2) using Gaussian PDFs also reveal that incoherence and synchronization coexist in a region with large K/σ [light shaded region, Fig. 1(b)]. Note that this region is not present in the Lorentzian case [Fig. 1(a)], in spite of the similar bell-shaped form of these two distributions. Figure 1(b) suggests that the destabilization of incoherence may also occur through a subcritical bifurcation for certain PDFs. To elucidate the supercritical or subcritical character of the synchronization transition, we carry out a self-consistency analysis in the manner of Kuramoto [2,12,16,17] in the limit of large coupling and/or very small frequency dispersion, i.e., $Kg(\omega_0) \gg 1$. After going into a rotating framework $\theta \to \theta + \omega_0 t$, rescaling time $t \to K^{-1}t$, and neglecting the ω_j/K term, we approximate Eq. (3) by

$$\dot{\theta}_j = q_j + R[\sin(\Psi - \theta_j) - q_j \cos(\Psi - \theta_j)]. \tag{19}$$

Hereafter we assume $q_0 = 0$. In a partially synchronized state the population splits into two groups, the synchronized (or locked) subpopulation with $|q| \leq q_{\max} = R/\sqrt{1 - R^2}$ and the desynchronized (or drifting) one with $|q| > q_{\max}$. Both subpopulations contribute to the order parameter $R = \langle\cos\theta\rangle_s + \langle\cos\theta\rangle_{ds}$, where we have chosen a reference frame where $\Psi = 0$ and the brackets denote averages over each subpopulation. We can now make an expansion in powers of R for each contribution. Up to cubic order we obtain:

$$\langle\cos\theta\rangle_s \simeq R\frac{\pi}{2}h(0) + R^2\frac{2}{3}h(0) + R^3\frac{\pi}{8}\left[h(0) + \frac{h''(0)}{2}\right],$$

$$\langle\cos\theta\rangle_{ds} \simeq \frac{R}{2} - R^2\frac{2}{3}h(0) + \frac{R^3}{4}\left[\frac{1}{2} + \int_0^\infty \frac{h'(q)}{q}dq\right],$$

where we have assumed $h(q)$ is twice differentiable at the origin. With these expansions, and discarding the trivial solution $R = 0$, we find that R follows asymptotically a square-root dependence of the form:

$$R = \sqrt{\frac{1 - \pi h(0)}{J}}, \tag{20}$$

where

$$J = \frac{1}{2}\left[1 + \frac{\pi h''(0)}{4} + \int_0^\infty \frac{h'(q)}{q}dq\right] \tag{21}$$

is evaluated at $h(0) = \pi^{-1}$. The sign of J determines the orientation of the bifurcating branch in Eq. (20), as shown in the conjectured bifurcation scenarios in Figs. 2(a) and 2(b). When $J > 0$, as for the Gaussian distribution ($J = \frac{1}{2} - \frac{3}{4\pi}$), a partially synchronized solution branches off from incoherence subcritically. This is in concordance with the numerical results in Fig. 1(b). The scenario for $J < 0$, Fig. 2(b), is also followed by PDFs with

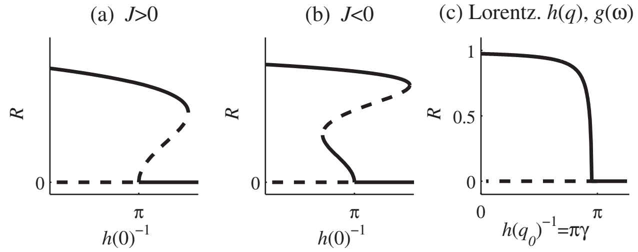

FIG. 2. (a),(b) Sketch of the possible bifurcation scenarios for large K values, depending on the sign of J, Eq. (21). Solid (dashed) lines indicate stable (unstable) solutions. (c) Bifurcation diagram for Lorentzian PDFs at $K/\delta = 40$, obtained from Eq. (14).

a nondifferentiable maximum of type $h(q) = h(0) + h'(0^+)|q| + \cdots$, like the Laplace distribution (formally $J = -\infty$). It is important to note that bistability between incoherence and synchronization is found irrespective of the sign of J, because Eq. (19) has a stable fixed point at $\theta_j = \Psi$ ($R = 1$) that persists as a solution with R near 1, provided $g(\omega)$ has a small dispersion (or K is large enough).

The case of Lorentzian $h(q)$ is quite peculiar. On the one hand, J vanishes for this PDF, which is consistent with the infinitely abrupt transition predicted by the Ott and Antonsen ansatz in the limit $K \to \infty$; see Fig. 2(c). On the other hand, according to our numerical simulations, the synchronized solution $R = 1$ of Eq. (19) does not persist for $\gamma > 1$ if $g(\omega)$ is not a delta function. This indicates that, for heavy-tailed $h(q)$, the term ω_j/K neglected in Eq. (19) may become relevant and destroy the synchronized solution beyond a certain critical value of $h(q_0)^{-1}$.

In summary, this Letter uncovers the effect of shear diversity on the collective synchronization of globally coupled oscillators. We have obtained the first analytical results for the Kuramoto model with distributed shear (3). If shear is widely distributed, incoherence is always stable and for some distributions —such as the Lorentzian one— synchronization is impossible. The techniques used here can be readily applied to a number of extensions of the model (3), such as considering other distributions $p(\omega, q)$, periodic or stochastic driving, time delays, or networks and communities of oscillator populations.

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