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Master Thesis in Mathematics

Flow analysis and first integrals of a family of 3D Lotka-Volterra Systems

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CERTIFIQUEN:

que el contingut del present Treball de Final de Máster titulat "Flow Analysis and First Integrals of a Family of 3D Lotka–Volterra Systems" ès material original que ha estat realitzat pel Sr. Adrian C. Murza sota la nostra direcció.

Palma, a 21 de Setembre de 2009

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A mi querida Dolores

A mi querida Dolores

"By a small sample we may judge of the whole piece"

– Miguel de Cervantes, *Don Quixote*–

Contents

Chapter 1

Introduction

> *"There is nothing mysterious, as some have tried to maintain, about the applicability of mathematics. What we get by abstraction from something can be returned."*
>
> – Raymond Louis Wilder, *Introduction to the Foundations of Mathematics–*

Differential equations appear in many interdisciplinary areas, at the interface between mathematics and a variety of fields, from biology and chemistry to economy and physics. They are mathematically studied from several different perspectives, mostly concerned with their solutions, functions that make the equation hold true. In general, not all differential equations admit solutions in terms of elementary functions. Computer simulations and/or the qualitative theory of differential equations are being used to the analyze differential equations whose explicit solutions are hard to find.

1.1 Qualitative theory of differential equation vs. numerical simulations of individual solutions

Numerical methods simplify the task of determining the solutions of a differential equation of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{s}, \mathbf{x})$ subjected to the initial condition $\mathbf{x}(\mathbf{t_0}) = \mathbf{x_0}$. This simplification may result useful when one is interested in calculating some specific parameters of the equation. However, numerical methods have two important limitations:

(a) On the one hand, they introduce two kinds of approximations.

   (i) roundoff error provoked by the finite computers arithmetic
   (ii) truncation error associated to the calculation procedure (see for example Newton or Runge-Kutta methods)

   If the dynamical system is stable, these kinds of errors do not qualitatively influence the result, because in the long asymptotic limit the approximated solution converges to the real orbit. However, when the system is unstable, the errors introduced by the numerical methods are dramatically increased, such that the predicted trajectory is far from the true solution.

(b) On the other hand, even when dealing with a stable system, numerical methods might be useful when one searches for a specific feature of the system like a particular trajectory or the flying time of a specific orbit, but they are insufficient when the interest is focused on the global dynamics and its changes.

By the contrary, while not using these limiting approximations, the qualitative theory of differential equations captures the essential features of the dynamical behavior. For these reasons, in Chapter 2, we will introduce

elements of the qualitative theory to study the behavior of a chosen family of polynomial differential systems. The polynomial systems are of Lotka-Volterra type (LVS).

1.2   Lotka-Volterra or predator-prey interactions

The Lotka-Volterra Equations, also known as the predator-prey equations, are a pair of first-order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, the predator and its prey. The system is generically described by

$$
\begin{aligned}
\frac{dx}{ds} &= x(\alpha - \beta y), \\
\frac{dy}{ds} &= -y(\gamma - \delta x),
\end{aligned}
\tag{1.2.1}
$$

where $y \geq 0$ is the number of some predator (for example, wolves); $x \geq$ is the number of its prey (for example, rabbits); dy/ds and dx/ds represents the growth of the two populations over the time s, while $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $\delta \geq 0$ are parameters representing the interaction of the two species. Note that we use the variable s to represent time, instead of the standard variable t, that will be used for other purposes. This notation applies to all chapters of this Thesis. The model was proposed independently by Alfred J. Lotka in 1925 [39] and Vito Volterra in 1926 [38]. Let us briefly explain how it was constructed.

The prey x is assumed to have an unlimited food supply, and to reproduce exponentially unless subject to predation; this exponential growth is represented in the equation above by the term αx. The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet; this is represented above by βxy. If either x or y is zero then there can be no predation. With these two terms the equation for the prey variation over time, dx/ds, can be written as the first equation in System (1.2.1) which means that the change in the prey's number is given by its own growth minus the rate at which it is preyed upon. Since in this equation $\alpha - \beta y$ multiplies x, it makes sense to call the term $\alpha - \beta y$, the *global growth rate of x* in System (1.2.1).

Similar arguments can be used to derive an equation for the predator y variation over time, dy/ds. In this case, we will use δxy to represent the growth of the predator population (a different constant, δ, is used as the rate at which the predator population grows is not necessarily equal to the rate at which it consumes the prey). The term $-\gamma y$ represents the natural death of the predators; it is an exponential decay. Hence the equation represents the change in the predator population as the growth of the predator population, minus natural death. The predator differential equation can be expressed as the second equation in System (1.2.1). Taken together, these differential equations form the predator-prey LVS. Since in this equation $\gamma - \delta x$ multiplies y, we call the function $\gamma - \delta x$, the *global growth rate of y* in System (1.2.1). Therefore, both global growth rates $\alpha - \beta y$ and $\gamma - \delta x$, are in this simplest case polynomials of degree one.

The predator-prey system admits a periodic solution, which do not have a simple expression in terms of the usual trigonometric functions. There is however, an easy way to find an essential argument that suggests periodic solutions. In the neighborhood of the points in the phase space of System (1.2.1) where $\frac{dx}{ds} \neq 0$, based on the Implicit Function Theorem, we can express y as a function of x by using the equation $(\alpha - \beta y)\frac{dy}{dx} = (\delta x - \gamma)\frac{y}{x}$. After integration, one arrives to the function

$$
\ln\left(x^{\gamma} y^{\alpha}\right) - (\delta x + \beta y) = c,
\tag{1.2.2}
$$

of the predator-prey System (1.2.1). Trajectories which are solutions of the form $(x(s), y(s))$ of System (1.2.1), are contained in the level curves (1.2.2). This way, functions $H(x, y)$ whose level curves contain the solutions of system, are called first integrals of System (1.2.1). They are key elements of the present Master Thesis, because, as it will be discussed later, first integrals are crucial in proving the existence of periodic solutions of differential systems.

On the other hand, it should be mentioned here that the form of the predator-prey systems can be easily extended to a more general class of 2D LVS, by considering global growth rates $f(x,y)$ and $g(x,y)$, in the form of polynomials of degree higher than one,

$$\begin{aligned} \dot{x} &= xf(x,y) \\ \dot{y} &= yg(x,y), \end{aligned} \qquad (1.2.3)$$

where $\dot{x}$ and $\dot{y}$ represent derivatives with respect to time, of x and y, respectively.

The general feature of LVS is that the time-derivative of each variable of the system is expressed as the product of that variable and the corresponding global growth rate. While conserving this general feature, the initial 2D LVS have been extended in two directions:

(a) increasing the degree of the polynomials which form the global growth rates

(b) increasing the number of variables

To this last point belongs the case of our family of 3D LVS, which will be introduced in the next section.

1.3   The 3D family of polynomial differential systems of Lotka-Volterra type

Initially proposed for modeling the interaction between biological species, they have found a large variety of applications from the genetic circuits [35] to the market economy [36]. It is however, the interdisciplinary field of biochemical interactions where the LVS have known one of their most fruitful and promising applications. For instance, enzyme kinetics [34] , circadian clocks [25] and genetic networks [27,28] often produce sustained oscillations modeled with LVS.

In this Thesis, we are interested in a family of 3D LVS that models a closed sequence of chemical equilibria. But what does such a chemical system look like? First of all, the chemical system is assumed to be closed, which means that the mass of the system is constant in time. Second, it is composed by four coexisting macromolecular species denoted by X, Y, Z and V. The interaction between them depends of course, on their chemical structure; however, the interaction between same macromolecules has been seen in different ways within the last thirty years.

Chemical equilibrium is the process related to most part of macromolecular interactions. However, the concept of chemical equilibrium has suffered changes over the years. Initially, it has been proposed as the state in a chemical process, in which the chemical activities or concentrations of the reactants and products show no net change over time. Usually, this would be the state that results when the forward chemical process proceeds at the same rate as their reverse reaction. The reaction rates of the forward and reverse reactions are generally not zero but, being equal, there are no net changes in any of the reactant or product concentrations. This process is called dynamic equilibrium.

In his seminal paper [34], Wyman suggested however, that macromolecular interactions can be understood as a reaction network whose state is somewhat different than the dynamic equilibrium. He compared such a reaction network with a "turning wheel" of one–step transitions of the macromolecule, which circulates in a closed reaction path involving the possible states (four in our case). The turning wheels have been used by Di Cera et al. in [23] as a generic model for macromolecular unidirectional autocatalytic interactions. However, the concept of turning wheels is not so far from the dynamic equilibrium. When in a chemical reaction at equilibrium, certain parameters such as temperature and pressure are changed or catalyzers are added, one of the two reactions is favored with respect to its opposite and consequently, its rate increases. This leads to a shifted chemical equilibrium. Macromolecular reactions frequently involve enzymes which are biochemical catalyzers, or show autocatalyzed processes (such as helix-coil intramolecular reactions in protein folding). The subtleties of the shifted chemical equilibrium have actually been discussed and analogies with turning wheels have been introduced in Wyman's paper [34].

In consequence, the unidirectional autocatalytic reactions introduced by Di Cera et al. in [23], can receive a more general treatment when they form part of shifted chemical equilibria. The modeling of such a chemical

system was carried out by Murza et al. in [24], who considered a closed sequence of three chemical equilibria, which translated into a planar polynomial LVS with very simple dynamics. In this Thesis, we extend that approach, by considering the chemical shifted equilibria between four chemical species.

Another difference with respect to [23], is that while they used first order reactions of constant reaction rates, in [24] pseudo-first order reactions were considered, and reaction rates were defined as functions of the time dependent product concentrations, multiplied by their reaction rate constants, in agreement with [34].

The chemical system to be modeled in this Thesis, is represented by the following scheme of autocatalytic chemical reactions between the chemical species X, Y, Z and V.
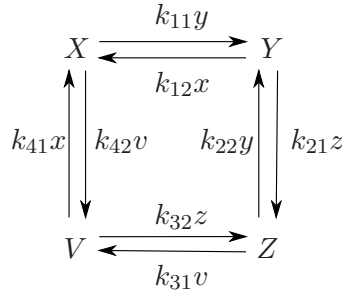

Figure 1.3.1. Closed sequence of chemical equilibria between X, Y, Z and V.

Now given the scheme above, the first question we ask is, how can we translate this closed path between chemical equilibria into a system of differential equations? To do that, we shall first take into account that functions $x(s), y(s), z(s)$ and $v(s)$ are concentrations at time s of the chemical species X, Y, Z, V respectively. The parameters $k_i = k_{i2} - k_{i1}$ for $i = 1, 2, 3, 4$ are differences of pairs of reaction rate constants corresponding to each chemical equilibrium, while $k_{ij} \geq 0$, for $j = 1, 2$ and $i = 1, 2, 3, 4$. Moreover, the reaction rates are not constant; they are proportional to the time-dependent concentration of the reaction product, in each reaction (same feature as in System (1.2.3)). This kind of processes whose reaction rates are increased by the reaction product, are called autocatalytic reactions.

Once all variables and reaction rates are defined, we are prepared to describe the rule by which we constructed the time derivatives of each and every chemical species. We considered that the velocity of consuming an arbitrary chemical species is proportional to its concentration at every moment, and also proportional to the concentration of their reaction products, as mentioned above. For example, let's take two generic chemical species in equilibrium, ϵ and ψ. The reaction rates are $w_1\psi$ from ϵ to ψ and $w_2\epsilon$ from ψ to ϵ; by applying this kinetic ansatz, the time derivative of ϵ is $\dot{\epsilon} = w_2\psi\epsilon - w_1\epsilon\psi = \epsilon\psi(w_2 - w_1)$. By proceeding in a similar way with the four chemical species joined through the closed chemical path, it is easy to write the 4–parameter family of nonlinear differential equations

$$\begin{aligned}
\dot{x} &= x(k_1 y - k_4 v), \\
\dot{y} &= y(k_2 z - k_1 x), \\
\dot{z} &= z(k_3 v - k_2 y), \\
\dot{v} &= v(k_4 x - k_3 z).
\end{aligned} \tag{1.3.1}$$

In this system, the time-derivative of each variable is expressed as the product of that variable and the corresponding global growth rate, which -as explained in the previous section- indicate that System (1.3.1) is a LVS.

At this point, it is useful to remember the initial assumption of a closed chemical system. This means that the mass of the system does not change with time, is constant. Let us assume that time-dependent mass-concentrations of the four chemical species in System (1.3.1) have been obtained after normalization with respect to the mass of the chemical system. In this case, the sum of these mass-concentration of the chemical species equals 1, i.e. $x + y + z + v = 1$. As a first consequence, the behavior of the LVS System (1.3.1) which

is relevant in our study, is restricted to the lineal manifold $x + y + z + v = 1$. Moreover, we can use this linear manifold to express one of the four time-dependent variables, x, y, z or v as a linear combination of the other three. We choose v, so that we have $v = 1 - x - y - z$. If we replace v in System (1.3.1) by $v = 1 - x - y - z$, then we arrive to the three–dimensional polynomial differential system

$$\begin{cases} \dot{x} = x(k_1 y - k_4(1 - x - y - z)), \\ \dot{y} = y(k_2 z - k_1 x), \\ \dot{z} = z(-k_2 y + k_3(1 - x - y - z)). \end{cases} \qquad (1.3.2)$$

It is worth emphasizing that by applying the law of mass conservation to the System (1.3.1), we reduced the system dimension by one, so that the object of our study in this Thesis is the analysis in $\mathbb{R}^3$ of the LVS (1.3.2). Moreover, because x, y, and z are mass-concentrations, they are non-negative real numbers defined over $[0, 1]$. These two reasons taken together, determine that the analysis of our System (1.3.2), is restricted to the bounded region $\mathcal{T}$

$$\mathcal{T} = \{x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}. \qquad (1.3.3)$$

in $\mathbb{R}^3$.

When comparing System (1.3.2) with the one obtained from Di Cera's model restricted to $n = 4$, (i.e. Equation (7) in [23]), we see that they are identical. Apparently this is surprising, because, while both chemical systems (our and their) are formed by the same chemical species, our system consists of chemical equilibria, while in Di Cera's model reactions are unidirectional. The explanation lies on the fact that, in [23] the unidirectional chemical reactions were modeled with constants rates, while in this work we consider non-constant reaction rates; these are proportional to the reaction product concentrations, as suggested by Wyman in [34]. These non-constant reaction rates can be determined in many cases by changing the physical/chemical conditions under which the reaction takes place. There should be no surprise to find that the reaction mechanism between same chemical species under different conditions of pressure, temperature, pH or catalyzers, may be different. And in many cases, the same dynamical system is used to model the kinetics given rise by different reaction mechanisms.

There is an interesting affirmation in [23], regarding the region in the parameter space which corresponds to periodic orbits of System (1.3.2). But let us first briefly explain what a periodic orbit is. Roughly speaking, an orbit is a curve of the dynamical system, i.e. a collection of points related by the evolution function of the dynamical system. It represents a one-dimensional manifold of the phase space of the system. A non-constant curve in the phase space, $\phi(s)$ is a periodic orbit if there exists a constant T called period, such that $\phi(s) = \phi(s + T)$. We can now enunciate Di Cera's affirmation in [23]. In that paper they claim that *System (1.3.2) exhibits periodic orbits only when the parameter* $\mathbf{k} = (k_1, k_2, k_3, k_4)$ *is in the three dimensional manifold* $\mathcal{S} = \{\mathbf{k} \in \mathbb{R}^4 \setminus \{\mathbf{0}\} : k_1 k_3 - k_2 k_4 = 0\}$. In Chapter 4 while we show that this is a necessary condition, we prove that for periodic orbits, in addition reaction constants k_i have to be of the same sign, and this is expressed by $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$, where $\mathcal{PS} = \{\mathbf{k} \in \mathbb{R}^4 : k_1 k_2 > 0, k_1 k_3 > 0, k_1 k_4 > 0\}$. In addition, we prove that the converse also hold true: if the orbits are periodic, then $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$. In short, we prove that System (1.3.2) exhibits periodic orbits if and only if $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$.

The main goal of this Thesis is the global analysis, under the point of view of the qualitative theory, of the flow of System (1.3.2) restricted to the region $\mathcal{T}$, by using methods of the qualitative theory of differential equations.

The qualitative theory offers two kinds of tools which enable the analysis of a dynamical system. On the one hand there are tools of local character, that will be introduced in Chapter 2. Some of these tools such as the Hartman–Grobman Theorem, enable characterizing the singular points of the dynamical system; other techniques are used to the analysis of the flow in the neighborhood of localized structures such as singular points or periodic orbits. We should also mention the Poincaré–Bendixson Theorem, which allows the analysis of the α and $\omega-$ limit sets in planar dynamical systems, i.e. the values to which the orbits of the dynamical system tend, as time approaches the extreme values in the interval of definition.

In addition, the qualitative theory contains tools of global character, such as the study of algebraic manifolds

which are invariant by the flow of the differential system. The invariant algebraic manifolds are relevant in our analysis of System (1.3.2), because the calculation of a sufficient number of them enables the calculation of first integrals, i.e. functions which are constant along trajectories in the phase space.

Along the centuries it was very difficult for mathematicians to discover invariants of motion, which are important in describing mechanical systems (e.g. energy). This motivated Darboux, who devoted most part of his life to the study of these mathematical objects. In 1878, Darboux published a theory to find the explicit expression of first integrals, in the case of polynomial 2D systems. We invite the reader to take a look at the Chapter 3 of this Thesis, where details of their definition and construction are presented.

While the original Darboux theory of integrability concerned 2D polynomial differential systems, Jouanoulou, [16], Christopher [19], [20], [21], Rodriguez and Llibre [26] and Cairo and Llibre [17], among others, extended it in different directions; for instance, to polynomial systems in $\mathbb{R}^3$; in Chapter 4, following their work, we apply Darboux's theory of integrability to analyze System (1.3.2). The calculation of two independent first integrals proves that the system is integrable and we determine the exact integrability conditions in the parameter space. Under these conditions, we will see that the orbits in the interior of $\mathcal{T}$ are periodic orbits.

In addition, we use Darboux functions which are not first integrals, to completely characterize the direction of the flow. By using these tools, we characterize the asymptotic behavior of the orbits and the changes taking place, which allow the search for bifurcation manifolds in the parameter space. By bifurcation we mean a change in the qualitative or topological structure of a dynamical system. Most commonly, a bifurcation is observed in the change of certain family of orbits or the flow of the vector field of the dynamical system, when one or more parameters (called bifurcation parameters) are changed. In Chapter 4 we will characterize the bifurcation undergone by System (1.3.2).

The work carried out in this Master Thesis gave rise to the manuscript 'Global dynamics of a family of 3D Lotka-Volterra Systems', submitted to *Dynamical Systems: An International Journal*, and currently it is under refereed review.

Chapter 2

Qualitative theory of differential equations

"I have tried to avoid long numerical computations, thereby following Riemann's postulate that proofs should be given through ideas and not voluminous computations."

– David Hilbert, *Report on Number Theory*–

In this chapter we present some techniques of the qualitative theory used to the study of nonlinear systems of differential equations. We have selected those techniques which will be used in Chapter 4 to analyze the flow of the vector field generated by System (1.3.2). In particular, System (1.3.2) belongs to the vast majority of differential systems whose exact analytical solution cannot be determined in terms of elementary functions. In general, even if such analytical expressions are available, their functional forms may be so complicated that they may not provide much information on the details of the flow behavior. By the contrary, it is the qualitative theory responsible for describing, in most cases, the set of the solutions (flow) of the vector field generated by these systems.

An important part of the qualitative theory deals with asymptotic behavior of solutions. It searches for analytical methods to answer questions like: -what happens with the solutions after a large time? -what are the sets the solution of the differential system converges to, after a large time? -do these solutions experiment qualitative changes over time? Part of the answer to these questions, involves the analysis of singular points and periodic orbits.

Along this work, our attempt is to describe all results in the general $n-$dimensional case. One can notice however, that we have also included results which are only available in 2D. An example is the Poincaré–Bendixson Theorem, which is useful in analyzing the flow in the region $\mathcal{T}$, restricted to a two-dimensional surface, as it will be shown in Chapter 4. We emphasize theorems such as Hartman–Grobman whose importance far overpasses the environment of our work; it allows the qualitative description of any dynamical system in the neighborhood of a hyperbolic singular point; or the above mentioned Poincaré–Bendixson Theorem, which characterizes the type of limit sets in the flows defined over planar surfaces.

The techniques introduced in this chapter have a local character, i.e. they allow the analysis of the flow in the neighborhood of localized structures like singular points and periodic orbits, while techniques of more global character are presented in Chapter 3.

2.1 Solutions of differential equations

Let U be a subset of $\mathbb{R}^n$ and W an open subset of U, we say that a function $\mathbf{f} : U \to \mathbb{R}^n$ is *Lipschitz on W* if there exists a constant $L \in \mathbb{R}$ such that for every $\mathbf{x}, \mathbf{y} \in W$

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \le L \|\mathbf{x} - \mathbf{y}\|.$$

The constant L is called a *Lipschitz constant for* $\mathbf{f}$. Here and subsequently, $\|\ \|$ denotes the Euclidean norm in $\mathbb{R}^n$. As $\mathbb{R}^n$ is a finite–dimensional vector space, if $\mathbf{f}$ is Lipschitz with respect to a norm of $\mathbb{R}^n$, then $\mathbf{f}$ is Lipschitz with respect to any other norm of $\mathbb{R}^n$. Hence, the definition of Lipschitz function doesn't depend on the chosen norm.

In the particular case of $\mathbf{f}$ Lipschitz on whole U, we call $\mathbf{f}$ *globally Lipschitz*. On the other hand, if for every $\mathbf{x}_0 \in U$ there exists a neighborhood W of $\mathbf{x}_0$ in U such that $\mathbf{f}$ is Lipschitz on W, then we call $\mathbf{f}$ *locally Lipschitz* on U.

For the purposes of this work it is enough to consider a *differential equation* or a *system of ordinary differential equations* as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.1.1}$$

where $\mathbf{x} = \mathbf{x}(s) \in U$, U is an open subset of $\mathbb{R}^n$ and $\mathbf{f} : U \to \mathbb{R}^n$ is a locally Lipschitz function on U. From now on, $\dot{\mathbf{x}}$ denotes the derivative of $\mathbf{x}(s)$ respect to $s \in \mathbb{R}$. As usual, the domain of $\mathbf{f}$, i.e. the set U, is called the *phase space*, the variable $\mathbf{x}$ is called the *dependent variable*, and s is called the *independent variable* or *time*.

In a more general context, Equation (2.1.1) is known as an *autonomous ordinary differential equation* (opposite to *non–autonomous differential equations*) where the function $\mathbf{f}$ doesn't depend, explicitly, on the independent variable s.

A smooth function $\phi : I \to U$, where I is an open interval of $\mathbb{R}$, is said to be a *solution* of the differential Equation (2.1.1) if $\dot{\phi}(s) = \mathbf{f}(\phi(s))$ for every $s \in I$.

Geometrically, a differential Equation (2.1.1) assigns to every point $\mathbf{x}$ in the phase space U, a vector $\mathbf{f}(\mathbf{x})$ of the tangent space to U at $\mathbf{x}$. Then, a solution of the differential equation is a curve $\phi : I \to U$ whose tangent vector at any value, $\dot{\phi}(s)$, coincides with the vector $\mathbf{f}(\phi(s))$. From this we call *vector field* associated with the differential Equation (2.1.1), the function $\mathbf{f}$.

In some situations, for compactness and simplicity of notation, it is usual to express the vector field in the form of a differential operator

$$X = f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n}, \tag{2.1.2}$$

where $\mathbf{f}$ is the vector function $\mathbf{f} = (f_1, \ldots, f_n)$, and $\frac{\partial}{\partial x_i}$ for any $1 \le i \le n$, represents the vector field which in standard coordinates corresponds to the system of differential equations

$$\dot{x}_i = 1,$$
$$\dot{x}_j = 0,$$

for $j \ne i$.

Existence of solutions of differential Equation (2.1.1) is not obvious and it depends on some properties of the vector field $\mathbf{f}$. The same happens with uniqueness of solutions satisfying *initial conditions* $(s_0, \mathbf{x}_0)$; i.e. such that $\phi(s_0) = \mathbf{x}_0$. The basic result in this direction is Picard's Theorem; a proof of it can be found in [14, pp. 13–14] and [6], for instance.

Theorem 2.1.1 (Existence and uniqueness) *Let U be an open subset of $\mathbb{R}^n$, $\mathbf{f} : U \to \mathbb{R}^n$ be a locally Lipschitz function on U, $s_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in U$. There exists a constant $c > 0$ and a unique solution $\phi : (s_0 - c, s_0 + c) \to U$ of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ such that $\phi(s_0) = \mathbf{x}_0$.*

To emphasize the dependence of solutions with respect to initial conditions we denote by $\phi(s; s_0, \mathbf{x}_0)$ the

solution of the differential Equation (2.1.1) passing through $\mathbf{x}_0$ at time $s = s_0$ i.e. such that $\phi(s_0; s_0, \mathbf{x}_0) = \mathbf{x}_0$.

Remark 2.1.1 *If* $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^n)$, *it is easy to conclude that* $\mathbf{f}$ *is locally Lipschitz. In particular, given* $\mathbf{x}_0 \in \mathbb{R}^n$, *and considering the closed unit ball with center at* $\mathbf{x}_0$, $\bar{\mathcal{B}}_1(\mathbf{x}_0)$, *it is easy to conclude that*

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| \leq L \, ||\mathbf{x} - \mathbf{y}||,$$

where $L = \max\limits_{\xi \in \bar{\mathcal{B}}_1(\mathbf{x}_0)} ||D\mathbf{f}(\xi)||$.

Hence, we have the following corollary.

Corollary 2.1.1 *The initial conditions problem* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *where* $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}^n)$, *and* $\mathbf{x}(\mathbf{s}_0) = \mathbf{x}_0$, *has a unique solution.*

Remark 2.1.2 *Since a polynomial system is a differential system* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, *and* f_i *is a polynomial function in* n *variables then* f *is* $\mathcal{C}^1$. *Therefore initial condition problem associated to System* (1.3.2)*, it has a unique solution.*

2.1.1 Prolongability of solutions

From the existence and uniqueness Theorem we obtain conditions on the vector field $\mathbf{f}(\mathbf{x})$ to have exactly one solution passing through a prefixed point. This solution is defined in an open interval of the variable s sufficiently small. In next result we maximize the interval where such solution is defined. First we need to introduce the following definitions.

We say that $\phi : I \to U$, with $\phi = \phi(s; s_0, \mathbf{x}_0)$, is a *maximal solution* of Equation (2.1.1), if for every solution $\psi : J \to U$, with $\psi = \psi(s; s_0, \mathbf{x}_0)$, we have $J \subseteq I$. We call *maximal interval of definition* to the interval of definition of the maximal solution $\phi(s; s_0, \mathbf{x}_0)$ and we denote it by $I_{(s_0, \mathbf{x}_0)}$. *From now on we will only consider maximal solutions.*

In general, the maximal interval of definition of solutions is not whole $\mathbb{R}$. Nevertheless, under restricted conditions on the vector field (see [14]), it can be concluded that $I_{(s_0, \mathbf{x}_0)} = \mathbb{R}$.

Proposition 2.1.1.1 *If function* $\mathbf{f}$ *is globally Lipschitz in* $\mathbb{R}^n$, *then* $I_{(s_0, \mathbf{x}_0)} = \mathbb{R}$.

For a proof see [14, pp.15, Prop. 4].

But these restricted conditions are not satisfied by polynomial systems. In the following result we analyze the maximal interval of definition for a particular set of solutions, contained in a compact.

Theorem 2.1.1.1 *Let* $\mathbf{f} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a locally Lipschitz function, let* $\mathbf{x}_0$ *be a point in* U *and* $I_{(s_0, \mathbf{x}_0)} = (\omega_-(x_0), \omega_+(x_0))$. *If* $\omega_+(x_0) < +\infty$ *(respectively* $\omega_-(x_0) < -\infty$*), then* $\phi(s; s_0, \mathbf{x}_0)$ *tends to* ∂U *(boundary of* U*), as* s *tends to* $\omega_+(x_0)$ *(respectively* s *tends to* $\omega_-(x_0)$*).*

For a proof of Theorem 2.1.1.1, we refer the reader to [40, pp. 3, Th. 1.2].

Remark 2.1.1.1 *In Proposition 3.3.1, we show that the faces of the compact set* $\mathcal{T}$ *are invariant by the flow defined by System* (1.3.2)*. Thus, every orbit of System* (1.3.2) *intersecting* $\mathcal{T}$, *remains contained in* $\mathcal{T}$ *for every* $s \in I_{(s_0, \mathbf{x}_0)}$. *From Theorem 2.1.1.1, it follows that* $I_{(s_0, \mathbf{x}_0)} = \mathbb{R}$.

2.1.2   Dependence of solutions on initial conditions and parameters

Consider the family of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda),$$

where $\mathbf{f} : U \times V \to \mathbb{R}^n$, U is an open subset of $\mathbb{R}^n$, and V is an open subset of $\mathbb{R}^p$. The set V is called the *parameter space* of the differential equation.

Assuming that $\lambda_0 \in V$, $s_0 \in \mathbb{R}$, $\mathbf{f}$ is Lipschitz and $\mathbf{x}_0 \in \mathbb{R}^n$, there exists exactly one solution of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda_0)$ passing through $\mathbf{x}_0$ at time s_0. We denote such solution by $\phi(s; s_0, \mathbf{x}_0, \lambda_0)$. In next theorem we summarize the behavior of the solution $\phi(s; s_0, \mathbf{x}_0, \lambda_0)$ when we vary any of the initial conditions s_0, $\mathbf{x}_0$ or parameters λ_0. First we introduce some additional definitions.

Let W be an open subset of U. The function $\mathbf{f}(\mathbf{x}, \lambda)$ is said to be *Lipschitz with respect to the first variable in W* if $\lambda \in V$, there exists a positive constant $L \in \mathbb{R}$ such that for every $\mathbf{x}, \mathbf{y} \in W$ and

$$\|\mathbf{f}(\mathbf{x}, \lambda) - \mathbf{f}(\mathbf{y}, \lambda)\| \le L \|\mathbf{x} - \mathbf{y}\|.$$

In particular, if $\mathbf{f}$ is Lipschitz with respect to the first variable in U, then we say that $\mathbf{f}$ is *globally Lipschitz with respect to the first variable*. The function $\mathbf{f}$ is said to be *locally Lipschitz with respect to the first variable* if for every $\mathbf{x}_0 \in U$ there exists a neighborhood W of $\mathbf{x}_0$ in U in such a way that $\mathbf{f}$ is Lipschitz respect to the first variable in W. To abbreviate we will call $\mathbf{f}$ either globally or locally Lipschitz when no confusion can arise.

A proof of the following theorem can be found in Hartman [4, pp. 93–96] and Lefschetz [8, pp. 36–43].

Theorem 2.1.2.1 (Dependence on initial conditions and parameters) *Let U be an open subset of $\mathbb{R}^n$ and let V be an open subset of $\mathbb{R}^p$. If $\mathbf{f} : U \times V \to \mathbb{R}^n$ is a locally Lipschitz function with respect to the first variable in U and $\mathbf{f} \in \mathcal{C}^r(U \times V)$, with $r \ge 0$, then for every $(s_0, \mathbf{x}_0, \lambda_0) \in \mathbb{R} \times U \times V$, the solution $\phi(s; s_0, \mathbf{x}_0, \lambda_0)$ of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda_0)$ is r times continuously differentiable with respect to $\mathbf{x}_0$ and λ_0 and $r + 1$ times continuously differentiable with respect to s.*

2.1.3   Other properties

We introduce now some other properties of the solutions of a differential equation. We say that $\phi : \mathbb{R} \to \mathbb{R}^n$ is a *periodic function* if there exists a positive constant T such that $\phi(s + T) = \phi(s)$ for every $s \in \mathbb{R}$. The minimum value of T satisfying previous property is called the *period* of the function ϕ.

For a proof of the following result we refer the reader to [15, pp. 8–9]. Note that in this reference the author suppose that the vector field is differentiable, but it is easy to check that such hypothesis can be substituted by the uniqueness of the solutions.

Proposition 2.1.3.1 *Consider the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f} : U \to \mathbb{R}^n$ such that $\mathbf{f}$ is locally Lipschitz and $I_{(s_0, \mathbf{x}_0)} = \mathbb{R}$.*

(a) If $\phi(s; s_0, \mathbf{x}_0)$ is a solution, then $\phi(s + \tau, s_0, \mathbf{x}_0)$ is also a solution for every $\tau \in \mathbb{R}$.

(b) Let $\phi(s; s_1, \mathbf{x}_1)$ and $\phi(s; s_2, \mathbf{x}_2)$ be two solutions satisfying $\phi(\tau_1; s_1, \mathbf{x}_1) = \phi(\tau_2; s_2, \mathbf{x}_2)$ for fixed $\tau_1, \tau_2 \in \mathbb{R}$. Then, $\phi(s - (\tau_2 - \tau_1); s_1, \mathbf{x}_1) = \phi(s; s_2, \mathbf{x}_2)$ for every $s \in \mathbb{R}$.

(c) Let $\phi(s; s_0, \mathbf{x}_0)$ be a solution and suppose that there exist $\tau_1, \tau_2 \in \mathbb{R}$, $\tau_1 < \tau_2$, such that $\phi(\tau_1; s_0, \mathbf{x}_0) = \phi(\tau_2; s_0, \mathbf{x}_0)$. Then $\phi(s; s_0, \mathbf{x}_0)$ is a periodic function whose period is a multiple of $\tau = \tau_2 - \tau_1$.

2.2   Orbits

At this moment we introduce some dynamical aspects around the solutions of a differential equation. Take $s_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in U$, and let $\phi(s; s_0, \mathbf{x}_0)$ be a maximal solution of the differential Equation (2.1.1). We call the

set
$$\gamma\left(s_0, \mathbf{x}_0\right) := \left\{\mathbf{x} \in U : \mathbf{x} = \phi\left(s, s_0, \mathbf{x}_0\right) \text{ and } s \in I_{\left(s_0, \mathbf{x}_0\right)}\right\},$$

orbit of the solution $\phi\left(s; s_0, \mathbf{x}_0\right)$.

If the phase space is $\mathbb{R}^n$ and $I_{\left(s_0, \mathbf{x}_0\right)} = \mathbb{R}$, see Proposition 2.1.1.1, then $\gamma\left(s_0, \mathbf{x}_0\right) = \gamma\left(s_0 + \tau, \mathbf{x}_0\right)$ for every $\tau \in \mathbb{R}$, see Proposition 2.1.3.1(a). Hence, we will use $\gamma\left(\mathbf{x}_0\right)$ to denote the orbit through $\mathbf{x}_0$. Moreover, if $\mathbf{x}_1 \in \gamma\left(\mathbf{x}_0\right)$, then there exists $s_1 \in \mathbb{R}$ such that $\mathbf{x}_1 = \phi\left(s_1; s_0, \mathbf{x}_0\right)$. Applying Proposition 2.1.3.1(b) to the solutions $\phi\left(s; s_0, \mathbf{x}_0\right)$ and $\phi\left(s; s_1, \mathbf{x}_1\right)$ it follows that $\gamma\left(\mathbf{x}_1\right) = \gamma\left(\mathbf{x}_0\right)$. Therefore, orbits are independent on the point of reference, thus we can avoid the reference to such point when no confusion can arise.

Suppose that $\mathbf{x}_2 \in \gamma\left(\mathbf{x}_1\right) \cap \gamma\left(\mathbf{x}_0\right) \neq \emptyset$; that is $\mathbf{x}_2 \in \gamma\left(\mathbf{x}_0\right)$ and $\mathbf{x}_2 \in \gamma\left(\mathbf{x}_1\right)$. Since orbits do not depend on the point of reference we obtain $\gamma\left(\mathbf{x}_0\right) = \gamma\left(\mathbf{x}_1\right) = \gamma\left(\mathbf{x}_2\right)$. Therefore, *if two orbits intersect at a point, then they are equal*, assuming uniqueness of solution.

2.3 The flow of a differential equation

Consider the differential equation
$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right), \tag{2.3.1}$$

where $\mathbf{f} : U \to \mathbb{R}^n$ is locally Lipschitz in U and U is an open subset of $\mathbb{R}^n$. Suppose that for every $\mathbf{x} \in U$, the solution $\phi\left(s; 0, \mathbf{x}\right)$ is defined in whole $\mathbb{R}$; i.e. $I_{\left(0, \mathbf{x}\right)} = \mathbb{R}$. The *flow of the differential Equation* (2.3.1) is defined to be the function
$$\Phi : \mathbb{R} \times U \to \mathbb{R}^n,$$

given by $\Phi\left(s, \mathbf{x}\right) = \phi\left(s; 0, \mathbf{x}\right)$. The notion of flow introduced here is referred by other authors as *complete flow* because the maximal interval of definition of the solutions is $\mathbb{R}$. But, since the system to which this work is devoted to is autonomous, there exists rescaling on time such that the new system has maximal solution defined in whole $\mathbb{R}$. In particular, if $\mathbf{f}$ is polynomial, then the flow of the differential equation $\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right)$ is complete.

Some other authors denote by flow of a differential equation the pair formed by the function Φ and the phase space U. It is also usual to denote by $\Phi_s\left(\mathbf{x}\right)$ the application $\Phi\left(s, \mathbf{x}\right)$ (see [3] and [13]). Some properties of the flows are the following.

Proposition 2.3.1 *Let* $\Phi\left(s, \mathbf{x}\right)$ *be the flow defined by the differential Equation (2.3.1).*

(a) The flow satisfies that $\Phi\left(0, \mathbf{x}\right) = \mathbf{x}$ *for every* $\mathbf{x} \in U$.

(b) The flow satisfies that $\Phi\left(s + t, \mathbf{x}\right) = \Phi\left(s, \Phi\left(t, \mathbf{x}\right)\right)$ *for every* $s, t \in \mathbb{R}$ *and* $\mathbf{x} \in U$.

(c) The flow Φ *is a continuous function with respect to the first variable.*

Proof. Statement (a) follows easily from the definition of Φ.

Taking $\mathbf{x}_1 = \mathbf{x}$, $\mathbf{x}_2 = \phi\left(t; 0, \mathbf{x}\right)$, $\tau_1 = t$, $\tau_2 = 0$ and $s_1 = s_2 = 0$ and applying Proposition 2.1.3.1(b), statement (b) follows straightforward.

Statement (c) is a consequence of the Theorem 2.1.2.1 of continuous dependence of solutions on initial conditions and parameters. ∎

Under the classical point of view, the objective of the differential equations theory is to find explicit expressions for the flow $\Phi\left(s, \mathbf{x}\right)$. On the other hand, in the qualitative theory it is more important to describe the topological properties of the flow and the asymptotical behavior of orbits; i.e. the behavior of orbits when s tends to $\pm\infty$. The *phase portrait* of a differential Equation (2.3.1) is defined as the union of all the orbits of Equation (2.3.1).

Let $\Phi\left(s, \mathbf{x}\right)$ be the flow of the differential Equation (2.3.1) and take $\mathbf{p} \in U$. By the Theorem of continuos dependence of solutions on the initial conditions and parameters, the function $\Phi_{\mathbf{p}} : \mathbb{R} \to U$ given by $\Phi_{\mathbf{p}}\left(s\right) :=$

$\Phi(s, \mathbf{p})$ is $r \geq 1$ times continuously differentiable with respect to s. Furthermore, since $\dot{\Phi}_{\mathbf{p}}(s) = \mathbf{f}(\Phi_{\mathbf{p}}(s))$, if there exists s_0 such that $\dot{\Phi}_{\mathbf{p}}(s_0) = \mathbf{0}$, then (by the uniqueness of the solutions) we have $\Phi_{\mathbf{p}}(s) = \mathbf{p}$ for every $s \in \mathbb{R}$, and the orbit $\gamma(\mathbf{p}) = \{\mathbf{p}\}$ is called a *singular point*. To simplify the notation, if $\gamma(\mathbf{p})$ is a singular point, we denote by $\mathbf{p}$ the orbit $\gamma(\mathbf{p})$. Therefore, $\mathbb{R}^n \setminus \gamma(\mathbf{p})$, $\mathbb{R}^n \setminus \{\mathbf{p}\}$ and $\mathbb{R}^n \setminus \mathbf{p}$ are identical expressions. If $\dot{\Phi}_{\mathbf{p}}(s_0) \neq \mathbf{0}$ for some $s_0 \in \mathbb{R}$, then $\Phi_{\mathbf{p}}(\mathbb{R}) = \gamma(\mathbf{p})$ is a one dimensional manifold and we call $\mathbf{p}$ a *regular point*. The flow in a neighborhood of a regular point is approximated to a *parallel flow*, by applying the Flow Box Theorem, [7] . From the classification of one dimensional manifolds, see [7], $\gamma(\mathbf{p})$ is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$. When $\gamma(\mathbf{p})$ is diffeomorphic to $\mathbb{S}^1$ the orbit $\gamma(\mathbf{p})$ is said to be a *periodic orbit*.

From this we conclude the following theorem; for a proof of it we refer the reader to [7].

Theorem 2.3.1 *Every orbit of a differential Equation (2.3.1) is diffeomorphic either to a point, or to a circle* $\mathbb{S}^1$*, or to a straight line* $\mathbb{R}$*.*

2.4   Basic ideas in the qualitative theory

After analyzing the topology of the orbits, we present some basic definitions to study their asymptotical behavior. Consider the differential Equation (2.3.1) and let E be a subset of U. The set E is *positively invariant (by the flow)* if for every $\mathbf{q} \in E$ we have $\Phi(s, \mathbf{q}) \in E$ for $s \geq 0$. The set E is *negatively invariant (by the flow)* if for every $\mathbf{q} \in E$ we have $\Phi(s, \mathbf{q}) \in E$ for $s \leq 0$. A set E *is invariant (by the flow)* when it is positively and negatively invariant (by the flow).

An invariant set E is *stable* if for any neighborhood W of E there exists a neighborhood V of E such that for every $\mathbf{p} \in V$ and $s > 0$ it follows that $\Phi(s, \mathbf{p}) \in W$. An invariant set E is u*unstable* when it is not stable.

Given $\mathbf{p}, \mathbf{q} \in U$, the point $\mathbf{q}$ is called an α–*limit point of* the orbit through $\mathbf{p}$, if there exists a sequence $\{s_n\}_{n=0}^{+\infty}$ satisfying $\lim_{n \nearrow +\infty} s_n = -\infty$ and such that $\lim_{n \nearrow +\infty} \Phi(s_n, \mathbf{p}) = \mathbf{q}$. The point $\mathbf{q}$ is called an ω–*limit point of* the orbit through $\mathbf{p}$, if there exists a sequence $\{s_n\}_{n=0}^{+\infty}$ satisfying $\lim_{n \nearrow +\infty} s_n = +\infty$ and such that $\lim_{n \nearrow +\infty} \Phi(s_n, \mathbf{p}) = \mathbf{q}$.

The α–*limit set* of the orbit through $\mathbf{p} \in U$, denoted by $\alpha(\mathbf{p})$, is defined as the union of all the α–limit points of $\mathbf{p}$. Analogously, the ω–*limit set* of a point $\mathbf{p} \in U$, denoted by $\omega(\mathbf{p})$, is defined as the union of all the ω–limit point of $\mathbf{p}$.

Let $\gamma(\mathbf{p})$, or simply γ, be the orbit passing through the point $\mathbf{p} \in U$, the α–*limit set of the orbit* γ is the α–limit set of the the orbit through point $\mathbf{p}$, the ω–*limit set of the orbit* γ is the ω–limit set of the orbit through $\mathbf{p}$. As it is easy to check these definitions do not depend on the chosen point $\mathbf{p}$ of the orbit. Therefore, we denote the α– and the ω–limit sets of an orbit by $\alpha(\gamma)$ and $\omega(\gamma)$, respectively.

Given an invariant set E, the *stable manifold* of E, denoted by $W^s(E)$, is the set of the points in the phase space U having their ω–limit set contained in E, and the *unstable manifold* of E, denoted by $W^u(E)$, is the set of the points in U having their α–limit set contained in E.

A set E is called *asymptotically stable* if its stable manifold, $W^s(E)$, is a neighborhood of E. A set E is called *asymptotically unstable* if its unstable manifold, $W^u(E)$, is a neighborhood of E. In particular, every asymptotically stable set (respectively, unstable) is stable (respectively, unstable).

By *limit cycle* we mean a periodic orbit isolated in the set of all the periodic orbits. A limit cycle is called a *stable limit cycle* (respectively, *unstable limit cycle*) if it is asymptotically stable (respectively, unstable). Another kind of limit cycle, called semistable limit cycle, can be also defined.

Let γ be an orbit of the flow $\Phi(s, \mathbf{x})$ and $\mathbf{p}$ be a point in γ. We define the positive and negative semiorbit of γ as the sets $\gamma^+(\mathbf{p}) := \{\Phi(s, \mathbf{p}) : s \geq 0\}$ and $\gamma^-(\mathbf{p}) := \{\Phi(s, \mathbf{p}) : s \leq 0\}$, respectively. The orbit γ is called *positively bounded* if there exists a point $\mathbf{p} \in \gamma$ and a compact subset K of U such that $\gamma^+(\mathbf{p}) \subset K$. The orbit γ is called *negatively bounded* if there exists a point $\mathbf{p} \in \gamma$ and a compact subset K of U such that $\gamma^-(\mathbf{p}) \subset K$. The orbit γ is said to be *bounded* if it is positively and negatively bounded.

For a proof of the following result we refer the reader to [14, p. 245].

Proposition 2.4.1 *Let γ be an orbit of the differential System (2.3.1). If γ is positively bounded (respectively, negatively bounded), then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is a non–empty set.*

2.5   Linear systems

Linear systems of differential equations, or more briefly, linear systems, are one of the families of differential equations for which there exists a quite complete theory. We review some of the standard facts on linear systems because, as we will see later, there exists a close relationship between linear systems and general non–linear differential ones. This occurs in such a way that linear systems can be considered the natural first step for the study of differential systems.

As usual, $L(\mathbb{R}^n)$ denotes the vector space of the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $GL(\mathbb{R}^n)$ the group of the invertible linear maps. Consider $T \in L(\mathbb{R}^n)$ and let A be the matricial representation of T. In the sequel, we will identify the linear map T with its matricial representation A, and we express this identification by $A \in L(\mathbb{R}^n)$. If T is invertible (equivalently, $\det(A) \neq 0$) then we will write $A \in GL(\mathbb{R}^n)$.

If $A \in L(\mathbb{R}^n)$, we denote by t or $\operatorname{trace}(A)$ the trace of A, and by d or $\det(A)$ the determinant of A. This is why we use the variable s to denote the time in the differential equation instead of the more usual variable t. Take $A \in L(\mathbb{R}^n)$, for every $s \in \mathbb{R}$ we define *the exponential matrix of sA* by the formal power series

$$e^{sA} := \sum_{k=0}^{\infty} \frac{s^k A^k}{k!},$$

where A^0 denotes the identity matrix Id and $A^k = A^{k-1}A$ for $k \geq 1$. We summarize some properties of the exponential matrix in the following proposition whose proof can be found in [1, Chapter 3] and [13, pp. 10–13].

Two matrices $A, B \in L(\mathbb{R}^n)$ are said to be *equivalent* if there exists $P \in GL(\mathbb{R}^n)$ such that $B = PAP^{-1}$.

Proposition 2.5.1 *Take $A \in L(\mathbb{R}^n)$.*

(a) For every $s \in \mathbb{R}$ the serie

$$\sum_{k=0}^{\infty} \frac{s^k A^k}{k!},$$

is absolutely convergent. Moreover, if $s_0 > 0$, the serie is uniformly convergent in $(-s_0, s_0)$.

(b) If A and $B \in L(\mathbb{R}^n)$ are equivalent i.e there exist $P \in GL(\mathbb{R}^n)$ such that $B = PAP^{-1}$, then $e^{sB} = Pe^{sA}P^{-1}$ for every $s \in \mathbb{R}$.

(c) If there exists $B \in L(\mathbb{R}^n)$ such that $AB = BA$, then $e^{s(A+B)} = e^{sA}e^{sB}$ for every $s \in \mathbb{R}$.

(d) For every $s \in \mathbb{R}$, it is satisfied that $\left(e^{sA}\right)^{-1} = e^{-sA}$.

(e) For every $s \in \mathbb{R}$, it is satisfied that $de^{sA}/ds = Ae^{sA}$.

(f) Let $\mathbf{v} \in \mathbb{R}^n$ be an eigenvector of A of eigenvalue $\lambda \in \mathbb{R}$, then $\mathbf{v}$ is a eigenvector of e^{sA} of eigenvalue $e^{s\lambda}$.

Along this section we consider the linear system (more precisely, *the homogeneous linear system*)

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2.5.1}$$

where $A \in L(\mathbb{R}^n)$, $d = \det(A)$ and $t = \operatorname{trace}(A)$.

The linear vector field, $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, is a globally Lipschitz function with Lipschitz constant $L = ||A||$. From the existence and uniqueness Theorem, it follows that for every $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a unique solution of System (2.5.1) passing through $\mathbf{x}_0$ at $s = 0$. Moreover, this solution is defined for every $s \in \mathbb{R}$, see Proposition 2.1.1.1. In the following result we show the expression of the linear flow. A proof of this theorem can be obtained as a corollary of Proposition 2.5.1(e).

Theorem 2.5.1 (Linear flow) *The linear differential equation* $\dot{\mathbf{x}} = A\mathbf{x}$ *where* $A \in L(\mathbb{R}^n)$, *defines a flow* $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *given by* $\Phi(s, \mathbf{x}) = e^{sA}\mathbf{x}$.

The vector subspace generated by the *kernel of the linear map* A, $\ker(A)$, is formed by the singular points of the linear System (2.5.1). In particular, the origin is always a singular point. Moreover, if $A \in GL(\mathbb{R}^n)$, then the origin is the unique singular point.

Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n_s}$ be the eigenvectors of A associated to the eigenvalues with a negative real part, let $\mathbf{u}_1, \mathbf{u}_2..., \mathbf{u}_{n_u}$ be the eigenvectors of A associated to the eigenvalues with a positive real part, and let $\mathbf{w}_1, ..., \mathbf{w}_{n_c}$ be the eigenvectors of A associated to the eigenvalues with a null real part. The *stable subspace* of the origin is the vector subspace

$$E^s := \langle \mathbf{v}_1, ..., \mathbf{v}_{n_s} \rangle,$$

the *unstable subspace* is the vector subspace

$$E^u := \langle \mathbf{u}_1, ..., \mathbf{u}_{n_u} \rangle,$$

and the *center subspace* is the vector subspace

$$E^c := \langle \mathbf{w}_1, ..., \mathbf{w}_{n_c} \rangle,$$

where $\langle \mathbf{e}_1, ..., \mathbf{e}_l \rangle$ denotes the vector subspace generated by the vectors $\mathbf{e}_1, ..., \mathbf{e}_l \in \mathbb{R}^n$.

For a proof of the following result see [1] and [14].

Theorem 2.5.2 (Dynamical behavior of linear systems) *Consider the linear differential system* $\dot{\mathbf{x}} = A\mathbf{x}$ *where* $A \in GL(\mathbb{R}^n)$.

(a) $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$.

(b) $W^s(\mathbf{0}) = (E^s \oplus E^c) \smallsetminus (E^c \smallsetminus \mathbf{0})$.

(c) $W^u(\mathbf{0}) = (E^u \oplus E^c) \smallsetminus (E^c \smallsetminus \mathbf{0})$.

2.5.1 Jordan canonical real form of System (1.3.2)

The dynamics of a linearized differential system and in particular the LVS (1.3.2), is reduced by Theorem 2.5.2, to the direct sum $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$. To represent the system in the orthonormal basis of the eigenvectors, we need to find the Jordan form of the Jacobian matrix of the system evaluated at the corresponding singular point. This is the reason for enunciating the Jordan canonical form of 3D differential systems. This theorem is proved in [6].

Theorem 2.5.1.1 (The Jordan Canonical Form) *Let* A *be a real matrix with real eigenvalues* λ_j, $j = 1, \ldots, k$ *and complex eigenvalues* $\lambda_j = a_j + ib_j$ *and* $\bar{\lambda}_j = a_j - ib_j$, $j = k+1, \ldots, n$. *Then there exists a basis* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{u}_{k+1}, \ldots, \mathbf{v}_n, \mathbf{u}_n\}$ *for* $\mathbb{R}^{2n-k}$, *where* $\mathbf{v}_j$, $j = 1, \ldots, k$ *and* $\mathbf{w}_j$, $j = k+1, \ldots, n$ *are generalized eigenvectors of* A, $\mathbf{u}_j = \mathrm{Re}(\mathbf{w}_j)$ *and* $\mathbf{v}_j = \mathrm{Im}(\mathbf{w}_j)$ *for* $j = k+1, \ldots, n$, *such that the matrix* $P = [\mathbf{v}_1, \ldots, \mathbf{v}_k \mathbf{v}_{k+1} \mathbf{u}_{k+1} \ldots \mathbf{v}_n \mathbf{u}_n]$ *is invertible and*

$$P^{-1}AP = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{pmatrix},$$

where the elementary Jordan blocks $B = B_j$, $j = 1, \ldots, r$ are either of the form

$$B = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \ldots & & & & \\ 0 & \ldots & & \lambda & 1 \\ 0 & \ldots & & 0 & \lambda \end{pmatrix},$$

for λ one of the real eigenvalues of A or of the form

$$B = \begin{pmatrix} D & I_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & D & I_2 & \ldots & \mathbf{0} \\ \ldots & & & & \\ \mathbf{0} & \ldots & & D & I_2 \\ \mathbf{0} & \ldots & & \mathbf{0} & D \end{pmatrix},$$

with

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad and \; \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

for $\lambda = a + ib$ one of the complex eigenvalues of A.

For the purpose of this Thesis, we will describe the case of the 3D System (1.3.2) whose matrix A obtained by linearization about each of the singular points of the straight line $\mathcal{R}$ formed by singular points (4.1.5) in Chapter 4, Section 4.1, has one eigenvalue $\lambda = 0$, and a pair of complex conjugate eigenvalues of the form $\alpha \pm i\beta$, with $\alpha \in \mathbb{R}$ and $\beta > 0$.
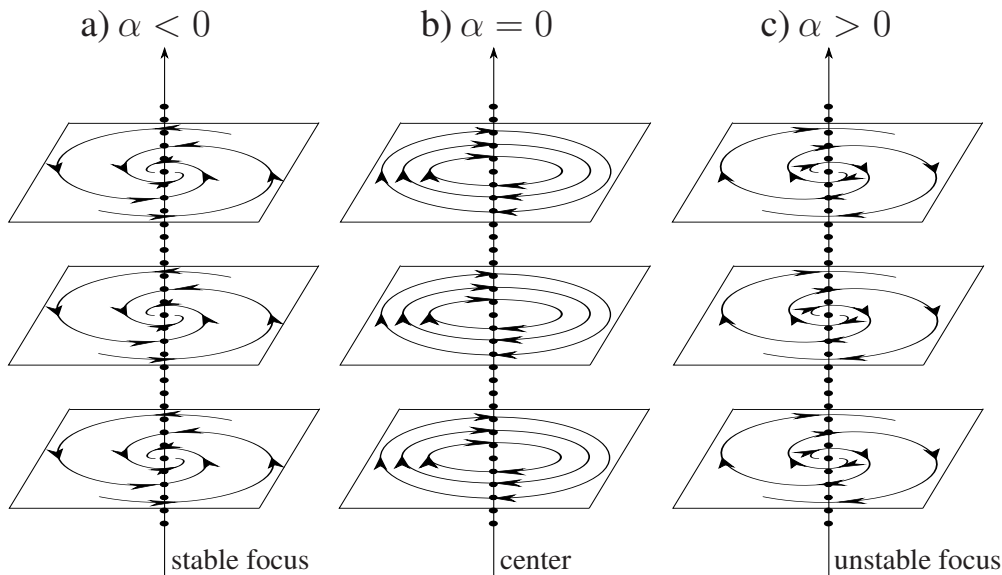
2.5.2 Linear 3D phase portraits

Figure 2.5.2.1. Linear 3D phase portraits of a system with one eigenvalue $\lambda = 0$ and a pair of complex conjugate eigenvalues. (a) Stable focus corresponding to $\alpha < 0$; (b) center corresponding to $\alpha = 0$ and (c) unstable focus corresponding to $\alpha > 0$. The dotted z axis corresponds to $\lambda = 0$. In each case, (a), (b) or (c), through every point of the z axis there is a perpendicular plane, and the flow of the vector field is invariant in every plane of the corresponding case. For the clarity of the illustration, we have sketched only three of these parallel planes, in every case.

By applying the Jordan Canonical Form to the 3D case of System (1.3.2), there exists a change of coordinates such that $P^{-1}AP = J$. The real Jordan normal form of A is

$$J = \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}_{\lambda=0}.$$

Therefore, the time-dependent solutions of the linearization of System (1.3.2) are of the form

$$\mathbf{x}(t) = e^{At}\mathbf{x_0} = Pe^{Jt}P^{-1}\mathbf{x_0} = \begin{pmatrix} e^{\alpha t}\cos\beta t & -e^{\alpha t}\sin\beta t & 0 \\ e^{\alpha t}\sin\beta t & -e^{\alpha t}\cos\beta t & 0 \\ 0 & 0 & e^{\lambda t} \end{pmatrix}_{\lambda=0} \mathbf{x_0}.$$

Figure 2.5.2.1 shows the 3D phase portraits of a system with a zero eigenvalue and a pair of complex conjugated eigenvalues. We know that the invariant flow in each one of the parallel planes, which are perpendicular to the $z-$axis, is in correspondence with the zero value that the corresponding eigenvalue takes in System (1.3.2).

2.6 Classification of flows

In general, every classification criteria implies the previous definition of the invariant items along one class. If the list of the selected items is large, then the number of elements in each class is small and the classification is not effective. On the contrary, if the list of such items is small, then we can put into the same class systems with a very different behavior. Thus, a first problem is to find an optimal criteria of classification. In the theory of flows, the criteria chosen is the preservation of the "orbits structure". With this end, in the following we introduce the topological conjugacy and equivalence of flows.

Consider the systems of differential equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$, with $\mathbf{f} : U \to \mathbb{R}^n$ a locally Lipschitz function defined on an open subset U of $\mathbb{R}^n$, and $\mathbf{g} : V \to \mathbb{R}^n$ a locally Lipschitz function defined on an open subset V of $\mathbb{R}^n$. Let $\Phi(s, \mathbf{x})$ and $\Phi^*(s, \mathbf{y})$ be the flows of the differential equations, respectively. Recall that in this chapter we consider the flows as completed flows; i.e. the interval of definition of the solutions is $\mathbb{R}$.

Definition 2.6.1 (Topological conjugation of flows) *Two flows are said to be topologically conjugated if there exists a homeomorphism* $\mathbf{h} : U \to V$ *(called conjugacy) such that* $\Phi^*(s, \mathbf{h}(\mathbf{x})) = \mathbf{h}(\Phi(s, \mathbf{x}))$ *for every* $s \in \mathbb{R}$ *and* $\mathbf{x} \in U$.

Definition 2.6.2 (Topological equivalence of flows) *The flows are said to be topologically equivalent if there exists a homeomorphism* $\mathbf{h} : U \to V$ *(called equivalence) such that* γ *is an orbit of the first system if and only if* $\mathbf{h}(\gamma)$ *is an orbit of the second one, and* $\mathbf{h}$ *preserves the sense of the orbit.*

It is easy to check that if two flows are conjugated, then they are equivalent. Moreover, an equivalence $\mathbf{h}$ transforms singular points into singular points and periodic orbits into periodic orbits (see Proposition 2.6.1.1). When $\mathbf{h}$ is a conjugacy the period of the periodic orbits are also preserved (see Proposition 2.6.1.1).

Two differential equations are said to be topologically equivalent (respectively, conjugated) if their flows are topologically equivalent (respectively, conjugated). Moreover, they are said to present the same *qualitative behavior* or the same *dynamical behavior* if they are topologically equivalents.

In the next result we present a characterization of the topological conjugacy of linear flows. For a proof of this result see Arnold [1, pp. 172–182].

Proposition 2.6.1 (Topological conjugacy of linear flows) *The flows of two linear systems whose eigenvalues have not zero real part are topologically conjugated if and only if the number of the eigenvalues with positive and negative real part are equal for both systems.*

2.6.1   Topological equivalence of non–linear flows

In Proposition 2.6.1, we have stated a characterization of topological conjugacy in the linear case. As far as we know, only for planar non–linear flows there exists a characterization of the topologically equivalence criteria, see Theorem 2.6.1.1. To present it we need to introduce some notation an previous results. Essentially, all of these results and definitions can be found in [9, pp. 127–148] and [10, pp. 73–81], but in these works they are applied to a more general context. Similar results are due to Peixoto [11].

Consider a differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{f}$ a Lipschitz function defined in $\mathbb{R}^2$ and let $\Phi(s, \mathbf{x})$ be the flow defined by the differential equation. Following Markus and Newmann, we denote by $(\mathbb{R}^2, \Phi)$ the flow of the differential equation.

An orbit $\gamma(\mathbf{p})$ of the flow $(\mathbb{R}^2, \Phi)$ is called a *separatrix* if either it is:

(a) a singular point, or

(b) a limit cycle, or

(c) the orbit is homeomorphic to $\mathbb{R}$ and there does not exist any tubular neighborhood N of $\gamma(\mathbf{p})$ satisfying the following statements.

   (c.1) Every point $\mathbf{q}$ in N has the same α–limit and ω–limit sets of $\mathbf{p}$; i.e. $\alpha(\mathbf{q}) = \alpha(\mathbf{p})$ and $\omega(\mathbf{q}) = \omega(\mathbf{p}), \forall q \in N$.

   (c.2) The boundary of N i.e. $Cl(N) \setminus N$, is formed by $\alpha(\mathbf{p})$, $\omega(\mathbf{p})$ and two orbits $\gamma(\mathbf{q}_1)$ and $\gamma(\mathbf{q}_2)$ such that $\alpha(\mathbf{p}) = \alpha(\mathbf{q}_1) = \alpha(\mathbf{q}_2)$ and $\omega(\mathbf{p}) = \omega(\mathbf{q}_1) = \omega(\mathbf{q}_2)$, see Figure 2.6.1.1. Where, $Cl(N)$ denotes the clousure of N; i.e. the smallest closed set containing N.
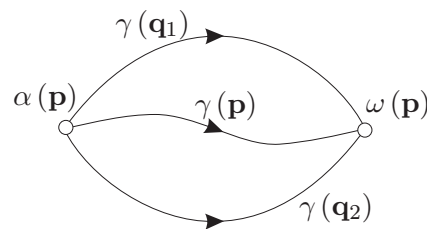

Figure 2.6.1.1. The boundary of N.

Let S be the union of the separatrices of the flow $(\mathbb{R}^2, \Phi)$. It is easy to check that S is a closed and an invariant set. If N is a connected component of $\mathbb{R}^2 \setminus S$, then N is also an invariant set and the flow $(N, \Phi|_N)$ is called a *canonical region* of the flow $(\mathbb{R}^2, \Phi)$.

Given a flow $(\mathbb{R}^2, \Phi)$ the *separatrix configuration* of the flow is, by definition, the union of every separatrix of the flow together with an orbit belonging to each of the canonical regions. Given two flows $(\mathbb{R}^2, \Phi)$ and $(\mathbb{R}^2, \Phi^*)$, let S and S^* be the union of their separatrices, respectively. The separatrix configuration C of the flow $(\mathbb{R}^2, \Phi)$ is said to be topologically equivalent to the separatrix configuration C^* of the flow $(\mathbb{R}^2, \Phi^*)$ if there exists a preserving sense homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ which transforms orbits of C into orbits of C^*, and orbits of S into orbits of S^*. For a proof of the following result we refer to the reader to [10].

Theorem 2.6.1.1 (Markus–Newmann–Peixoto) *Consider two continuous flows $(\mathbb{R}^2, \Phi)$ and $(\mathbb{R}^2, \Phi^*)$ with isolating singular points. The flows are topologically equivalents if and only if their separatrix configurations are topologically equivalent.*

In the case of differential systems of dimension higher than 2, there are no results that allow characterizing the topological equivalence. In the following we present an elementary result which affords the necessary conditions for the topological equivalence for such systems.

Proposition 2.6.1.1 *Consider the systems of differential equations in $\mathbb{R}^n$,*

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right),$$
$$\dot{\mathbf{y}} = \mathbf{g}\left(\mathbf{y}\right).$$

(a) *If they are topologically equivalent, then they have the same number of singular points and periodic orbits.*

(b) *Let us consider that they are topologically conjugate through the homeomorphism ϕ. Let γ be a periodic orbit of the first system, of period T. Then, $\delta = \phi(\gamma)$ is a periodic orbit of the second system, of period T.*

Proof.

(a) From Definition 2.6.2, there exists a homeomorphism $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$, that transforms a singular point into a singular point a periodic orbit into a periodic orbit; therefore, the two dynamical systems have the same number of singular points and periodic orbits.

(b) Since both dynamical systems are topologically conjugate, from Definition 2.6.1, there exists a homeomorphism $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ that transforms the trajectories of one system onto the trajectories of the other system and preserves the time parameter. Thus, we have

$$\mathbf{y}\left((s + T)\right) = \mathbf{h}\left(\mathbf{x}\left(s + T\right)\right) = \mathbf{h}\left(\mathbf{x}\left(s\right)\right) = \mathbf{y}(s),$$

which proves point (b).

∎

2.7   Non–linear systems

In this Section we return to non–linear flows. Let $U \subseteq \mathbb{R}^n$ be an open subset, $\mathbf{f} : U \to \mathbb{R}^n$ be a locally Lipschitz function in U and $\Phi\left(s, \mathbf{x}\right)$ be the flow defined by the differential equation $\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right)$. Recall that the flow is always complete; i.e. the solutions are defined for any time $s \in \mathbb{R}$.

2.7.1   Local phase portrait of singular points

We begin by studying the local behavior of the flow in a neighborhood of a singular point; i.e. a point $\mathbf{x} \in U$ such that $\mathbf{f}\left(\mathbf{x}\right) = \mathbf{0}$.

Now we classify singular points depending on the linear part of the vector field. Let $\mathbf{x}_0$ be a singular point of the differential system $\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right)$, where $\mathbf{f}$ is a C^1 function in a neighborhood of $\mathbf{x}_0$. Let $D\mathbf{f}\left(\mathbf{x}_0\right)$ be the Jacobian matrix of $\mathbf{f}$ evaluated at $\mathbf{x}_0$. The point $\mathbf{x}_0$ is said to be a *hyperbolic singular point* if all the eigenvalues of $D\mathbf{f}\left(\mathbf{x}_0\right)$ have non–zero real part.

For a planar differential system, we say that a singular point $\mathbf{x}_0$ is an *elementary non–degenerate* singular point if the determinant of $D\mathbf{f}\left(\mathbf{x}_0\right)$ is not zero. In particular, every hyperbolic singular point is an elementary non–degenerate one, the inverse implication is not true. Since elementary non–degenerate singular points with determinant of $D\mathbf{f}\left(\mathbf{x}_0\right)$ less than zero is a saddle point, we call *antisaddle* the non–degenerate singular points with positive determinant of the Jacobian matrix. The singular point $\mathbf{x}_0$ is said to be an *elementary degenerate* singular point if the determinant of $D\mathbf{f}\left(\mathbf{x}_0\right)$ is zero and the trace of $D\mathbf{f}\left(\mathbf{x}_0\right)$ is non–zero. The singular point $\mathbf{x}_0$ is said to be *nilpotent* if the determinant and the trace of the matrix $D\mathbf{f}\left(\mathbf{x}_0\right)$ are zero and $D\mathbf{f}\left(\mathbf{x}_0\right)$ is not indentically zero.

Since the definition of flow introduced in this work, see Subsection 2.3, corresponds with the idea of a complete flow used by other authors, in the following version of the Hartman–Grobman Theorem we impose on the maximal interval of definition of the solutions to be $\mathbb{R}$.

Theorem 2.7.1.1 (Hartman–Grobman) *Let U be an open subset of $\mathbb{R}^n$, $\mathbf{f} : U \to \mathbb{R}^n$ be a $C^1(U)$ function, $\Phi(s, \mathbf{x})$ be the flow of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and $\mathbf{x}_0$ be a hyperbolic singular point of the differential equation. There exist a neighborhood W of $\mathbf{x}_0$, a neighborhood V of the origin, a homeomorphism $\mathbf{h} : W \to V$ with $\mathbf{h}(\mathbf{x}_0) = \mathbf{0}$, and an interval $I \subseteq \mathbb{R}$ containing the origin such that*

$$\mathbf{h} \circ \Phi(s, \mathbf{x}) = e^{sD\mathbf{f}(\mathbf{x}_0)}\mathbf{h}(\mathbf{x}),$$

for every $s \in I$ and $\mathbf{x} \in U$.

For a proof of the previous theorem see [14, p. 294].

The Hartman–Grobman Theorem asserts that differential systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ are topologically equivalent in a neighborhood W of the hyperbolic singular point $\mathbf{x}_0$ and V of the origin. This is because we use the same names for non–linear hyperbolic singular points and for linear hyperbolic ones. Even for non–hyperbolic singular points when the non–linear differential system is topologically equivalent to a linear system we will use the same name for both singular points. Therefore, the singular point $\mathbf{x}_0$ of a non–linear differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is said to be a *stable (respectively, unstable) normally hyperbolic singular point* if $\mathbf{f}$ is topologically equivalent to the differential system $\dot{x} = 0; \dot{y} = -y$ (respectively, $\dot{x} = 0; \dot{y} = y$) in a neighborhood of $\mathbf{x}_0$ and $\mathbf{0}$. The singular point $\mathbf{x}_0$ is said to be a *non–isolated nilpotent singular point* if $\mathbf{f}$ is topologically equivalent to the differential system $\dot{x} = y; \dot{y} = 0$.

2.7.2 α and ω limit sets in the plane

After studying singular points and periodic orbits, in this section we deal with the asymptotic behavior of remainder orbits, which are diffeomorphic to straight lines, see Theorem 2.3.1. We restrict ourselves to planar flows because the following version of the Jordan's curve Theorem is needed in lots of arguments used here. This entire subsection concerns only with 2D flows; we have included it, because it is important in the analysis of the $\alpha-$ and $\omega-$limit sets on certain level surfaces of the 3D System (1.3.2), as it will be shown in Section 4, Subsection 4.3.

A curve in the plane is said to be a *Jordan's curve* if it is homeomorphic to $\mathbb{S}^1$; i.e. if it is a closed curve without autointersections.

Theorem 2.7.2.1 (Jordan's curve) *The complementary set of a Jordan's curve γ in the plane is the union of two open, disjoint and connected sets. Furthermore, one of these sets is bounded and its boundary is the curve γ.*

Since no orbit of a flow can intersect itself -because of the uniqueness of the solution, see Theorem 2.1.1-, from the Jordan's curve Theorem follows that a periodic orbit γ splits the phase plane in two regions invariant by the flow, one of them bounded. This bounded region will be called *inner region* of γ and denoted by Σ_γ.

Periodic orbits are not the unique Jordan's curves formed by solutions. We define a *separatrix loop* to be the finite union of n singular points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ (some of these points can be identified) and n orbits $\gamma_1, \gamma_2, \ldots, \gamma_n$, in such away that $\alpha(\gamma_k) = \{\mathbf{p}_k\}$ for $k = 1, 2, ..., n$, $\omega(\gamma_k) = \{\mathbf{p}_{k+1}\}$ if $k = 1, 2, ..., n-1$, and $\omega(\gamma_n) = \{\mathbf{p}_1\}$, see Figure 2.7.2.1. The singular points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ will be called the *vertex of the loop*.

We define a *homoclinic loop* to be a separatrix loop formed by one singular point (*homoclinic point*) and one orbit (*homoclinic orbit*), see Figure 2.7.2.1(a). We define a *double homoclinic loop* to be a separatrix loop formed by one singular point (in this case $\mathbf{p}_1$ and $\mathbf{p}_2$ are identified) and two orbits, see Figure 2.7.2.1(b). We define a *heteroclinic loop* to be a separatrix loop formed by two singular points and two orbits, see Figure 2.7.2.1(c).

The Poincaré–Bendixson Theorem asserts that the $\alpha-$ and $\omega-$limit sets of any orbit of a planar differential system are quite simple sets: either a singular point, or a periodic orbit or a separatrix loop.

Theorem 2.7.2.2 (Poincaré–Bendixson) *Let $\mathbf{f} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a locally Lipschitz function in the open subset U, and let γ be an orbit of the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Suppose that γ is positively bounded (respectively, negatively bounded) and the number of singular points in $\omega(\gamma)$ (respectively, in $\alpha(\gamma)$) is finite.*
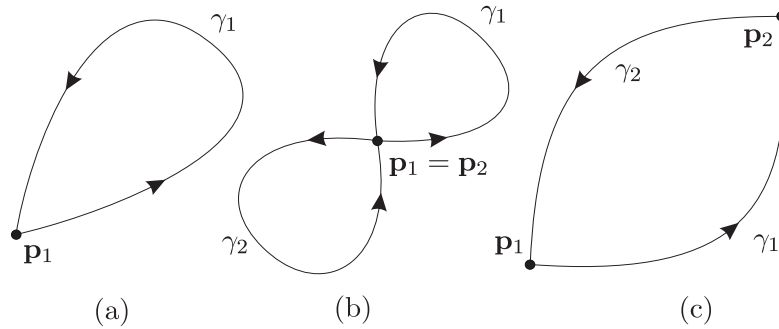
Figure 2.7.2.1. Separatrix loops: (a) homoclinic loop; (b) double homoclinic loop; (c) heteroclinic loop.

(a) *If $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) has not singular points, then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is a periodic orbit.*

(b) *If $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) has singular points and regular points, then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is a separatrix loop.*

(c) *If $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) has not regular points, then $\omega(\gamma)$ (respectively, $\alpha(\gamma)$) is singular point.*

A proof of this result can be found in the book of Hartman [4, Chapter 7]. Next result is a corollary of the Poincaré–Bendixson Theorem.

Corollary 2.7.2.1 *Let $\mathbf{f} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz function in an open and simply connected set U and let $\gamma \subset U$ be a periodic orbit of the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Therefore, there exists a singular point in Σ_γ.*

2.8   Local vs. global bifurcations

The qualitative behavior of a parametric family of differential equations, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda)$, can change by moving the value of the parameter vector λ; that is, the differential equation changes from one topological equivalence class to another. From Theorem 2.6.1.1, in the case of a two-dimensional differential system, a change of the topological equivalence class is equal to a change of the separatrices configuration. These changes in the separatrices configuration are called *bifurcations* and the value of the parameter in which it happens is called *bifurcation value*. In a more general context the word bifurcation applies also for other changes in the behavior of the flow, not only for changes in the topological equivalence class. See the books of J. Guckenheimer and P. Holmes [20], J. Hale and H. Kocak [22], and S. Chow and J. Hale [12] for details about the bifurcation Theory.

It is not our purpose to study details of the bifurcation theory. In this Section we introduce very few definitions related to this subject, that will serve us to characterize the global bifurcation showed by System (1.3.2), in Chapter 4, Section 4.4.

Bifurcations that take place in a neighborhood of a singular point, are referred as *local bifurcations*. By the contrary, bifurcations taking place when 'larger' invariant sets, such as periodic orbits, collide with equilibria, are called *global bifurcations*. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations. Examples of global bifurcations are homoclinic bifurcation, in which a limit cycle collides with a saddle point, or heteroclinic bifurcation, in which a limit cycle collides with two or more saddle points.

The set of all the bifurcation values in the parameter space is called the *bifurcation set* of the parametric family. When the bifurcation values define a manifold in the parameters space we refer to it as *bifurcation manifold*.

The collection of the phase portraits corresponding to each choice of the parameter vector, given not topologically equivalent behaviors, is called a bifurcation diagram.

Chapter 3

The Darboux theory of integrability

> *"But just as much as it is easy to find the differential of a given quantity, so it is difficult to find the integral of a given differential. Moreover, sometimes we cannot say with certainty whether the integral of a given quantity can be found or not."*
>
> – Johann Bernoulli –

The algebraic theory of integrability is a classical part of the qualitative theory of differential equations, and it is related with the first part of the Hilbert's 16th problem. This type of integrability, originally represented a link between the integrability of polynomial differential systems and the number of their invariant algebraic curves [18]. By the end of the nineteenth century, Darboux developed a method for constructing first integrals of planar polynomial differential systems which possess a sufficient number of invariant algebraic curves. He proved that if a planar polynomial differential system of degree m has at least $[m(m+1)m/2]$ invariant algebraic curves, then it has a first integral, and he explicitly calculated it as a function of the invariant algebraic curves.

Darboux's theory of integrability initially concerned planar polynomial systems with complex coefficients. His theory is fundamental for analyzing the dynamics of planar differential polynomial systems which contain invariant algebraic curves. In these systems, the existence of an adequate number of invariant algebraic curves, completely determines its phase portraits.

If the system dimension is higher than 2, the dynamics is far richer; chaotic motion and strange attractors appear, and in general it is not possible to characterize the phase portrait exclusively with its first integrals. However, first integrals, reduce the dimension of the system by one, which in many cases makes easier the analysis of the system.

Jouanoulou [16] extended the planar Darboux theory of integrability to polynomial differential systems in $\mathbb{R}^n$ or $\mathbb{C}^n$. More recently, the original Darboux's Theorem has been extended by the works by Christopher and Llibre [20], [19], who took into account exponential factors, independent singular points and rational first integrals. More important for the purpose of this Thesis are the extension of Darboux theory of integrability to certain families of polynomial systems in $\mathbb{R}^3$; see Llibre and Rodriguez [26] and Cairo and Llibre [17].

3.1 Basic notions of the Darboux theory of integrability for 3D polynomial systems

In order to maintain the connection with the purpose of this Thesis, closely related to polynomial systems in 3D of real coefficients, the results of the Darboux theory of integrability presented in this Section, are restricted to differential systems in $\mathbb{R}^3$ of Lotka Volterra type, and with real coefficients.

3.1.1 Polynomial vector fields and invariant algebraic surfaces of a 3D polynomial system

Let us first define a *3D polynomial system*

$$\frac{dx}{ds} = \dot{x} = P(x,y,z), \qquad \frac{dy}{ds} = \dot{y} = Q(x,y,z), \qquad \frac{dz}{ds} = \dot{z} = R(x,y,z), \qquad (3.1.1.1)$$

where P, Q and R are polynomials with coefficients in $\mathbb{R}^3$. We denote by δA the *degree of the polynomial A*. Moreover, we define the *degree of the polynomial differential system* as $m = \max\{\delta P, \delta Q\ \delta R\}$, and we write $\delta X = m$.

Associated to the polynomial differential System (3.1.1.1), there is the *polynomial vector field*

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}. \qquad (3.1.1.2)$$

The starting point in Darboux's theory of integrability in 3D, is the calculation of *invariant algebraic surfaces*, that is, algebraic surfaces which are invariant by the flow of the vector field associated to the polynomial differential System (3.1.1.1). Let $f \in \mathbb{R}[x,y,z]$, where as usual $\mathbb{R}[x,y,z]$ denotes the ring of the polynomials in the variables x, y and z with real coefficients. An algebraic surface $f(x) = 0$ in $\mathbb{R}^3$ with $f \in \mathbb{R}[x,y,z]$ is an *invariant algebraic surface* of a polynomial System (3.1.1.1) if

$$\dot{f} = Xf = P(x,y,z)\frac{\partial f}{\partial x} + Q(x,y,z)\frac{\partial f}{\partial y} + R(x,y,z)\frac{\partial f}{\partial z} = Kf. \qquad (3.1.1.3)$$

for some polynomial $K \in \mathbb{R}[x,y,z]$, which is called a cofactor of the invariant algebraic surface $f = 0$. The polynomial K is called the cofactor of the invariant algebraic surface $f = 0$. Since the polynomial vector field has degree m, then any cofactor has at most degree $m-1$.

At the first glance, it it is not easy to see that an algebraic variety $f = 0$ is invariant algebraic surface of a polynomial system.
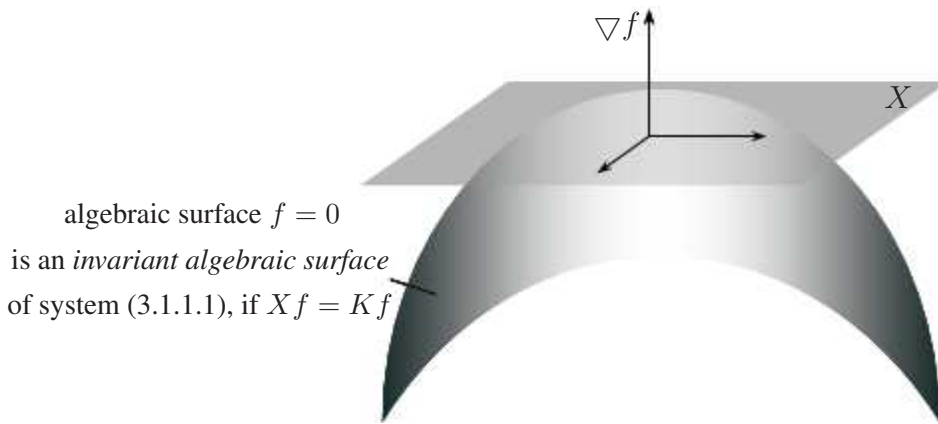

Figure 3.1.1.1. The algebraic surface $f(x,y,z) = 0$ is an invariant algebraic surface of the polynomial System (3.1.1.1), if for some polynomial K(x,y,z) we have $Xf = Kf$. The polynomial K is called the cofactor of the algebraic surface $f = 0$. The gradient $\triangledown f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z,)$ is orthogonal to the vector field (P,Q,R) at the points of the surface where $f = 0$.

In Figure 3.1.1.1 it is shown that on the points of an invariant algebraic surface $f = 0$, the gradient of f, $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ is orthogonal to the polynomial vector field $X = (P,Q,R)$(see (Equation 3.1.1.2)). It follows that at every point (x,y,z) of the surface $f = 0$ the vector field X is contained into the tangent plane to the surface $f = 0$ at that point. Hence, the surface $f = 0$ is formed by trajectories of the vector field X. This justifies the name 'invariant algebraic surface' given to the algebraic surface $f = 0$ satisfying (3.1.1.3) for

some polynomial K, because it is *invariant* under the flow defined by X.

We would like to know if is it possible to reduce the study of an invariant algebraic surface, to the study of irreducible invariant algebraic surfaces in $\mathbb{R}[x, y, z]$. The answer is yes, and it is given by Proposition 3.1.1.1. To prove it, we will need the following lemma.

Lemma 3.1.1.1 *Let $f, g \in \mathbb{R}[x, y, z]$. We assume that f and g are relatively prime in the ring $\mathbb{R}[x, y, z]$. Then, for a polynomial System* (3.1.1.1), *$fg = 0$ is an invariant algebraic surface with cofactor K_{fg} if and only if $f = 0$ and $g = 0$ are invariant algebraic surfaces with cofactors K_f and K_g, respectively. Moreover, $K_{fg} = K_f + K_g$.*

Proof. It is clear that

$$X(fg) = (Xf)g + f(Xg). \tag{3.1.1.4}$$

We assume that $fg = 0$ is an invariant algebraic surface with cofactor K_{fg} of System (3.1.1.1). Then, $X(fg) = K_{fg}fg$ and from the equality (3.1.1.4) we get $K_{fg}fg = (Xf)g + fXg$. Therefore, since f and g are relatively prime, we obtain that f divides Xf, and g divides Xg. If we denote by $K_f = Xf/f$ and by $K_g = Xg/g$, then $f = 0$ and $g = 0$ are invariant algebraic surfaces of System (3.1.1.1) with cofactors K_f and K_g respectively, and $K_{fg} = K_f + K_g$. ∎

Proposition 3.1.1.1 *We suppose that $f \in \mathbb{R}[x, y, z]$ and let $f = f_1^{n_1} \cdots f_i^{n_i} \cdots f_q^{n_r}$ be its factorization in irreducible factors over $\mathbb{R}[x, y, z]$. Then for a polynomial System* (3.1.1.1), *$f = 0$ is an invariant algebraic surface with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic surface for each $i = 1, \ldots, r$ with cofactor K_{f_i}. Moreover, $K_f = n_1 K_{f_1} + \ldots + n_r K_{f_r}$.*

Proof. From Lemma 3.1.1.1, we have that $f = 0$ is an invariant algebraic surface with cofactor K_f if and only if $f_i^{n_i} = 0$ is an invariant algebraic surface for each $i = 1, \ldots, r$ with cofactor $K_{f_i^{n_i}}$, furthermore $K_f = K_{f_1^{n_1}} + \cdots + K_{f_r^{n_r}}$.

Now for proving the proposition it is sufficient to show, for each $i = 1, \ldots, r$ that $f_i^{n_i} = 0$ is an invariant algebraic surface with cofactor $K_{f_i^{n_i}}$ if and only if $f_i = 0$ is an invariant algebraic surface with cofactor K_{f_i}, and that $K_{f_i^{n_i}} = \lambda_i K_{f_i}$. We assume that $f_i^{n_i} = 0$ is an invariant algebraic surface with cofactor $K_{f_i^{n_i}}$. Then

$$K_{f_i^{n_i}} f_i^{n_i} = X\left(f_i^{n_i}\right) = n_i f_i^{n_i - 1} X\left(f_i\right), \tag{3.1.1.5}$$

or equivalently

$$X\left(f_i\right) = \tfrac{1}{n_i} K_{f_i^{n_i}} f_i. \tag{3.1.1.6}$$

So defining $K_{f_i} = K_{f_i^{n_i}}/n_i$, we obtain that $f_i = 0$ is an invariant algebraic surface with cofactor K_{f_i}, such that $K_{f_i^{n_i}} = \lambda_i K f_i$. The proof in the converse direction follows in a similar way. ∎

In the following Subsection, we try to give a more detailed characterization of algebraic surfaces of the 3D polynomial systems of the form (3.1.1.1).

3.1.2   First integrals and Darboux functions

Darboux's original version of the theory of integrability relates the integrability of a polynomial vector field with its number of invariant algebraic curves which are independent. Once a sufficient number of such curves are known, he showed how to use them to construct first integrals of planar vectorial fields. We will use his method here, to find functions which are invariant algebraic surfaces for the System (1.3.2). We recall that our interest is not only on the Darboux first integrals but also on the invariant algebraic surfaces of System (1.3.2), which we call *Darboux functions*. We will use both first integrals and Darboux functions to analyze the dynamics of System (1.3.2).

We say that $H : U \to \mathbb{R}^3$ is a *first integral* of the vector field X of System (3.1.1.1), if the Lebesgue measure of $\mathbb{R}^3 \setminus U$ is zero and H is a non-constant analytic function which is constant on all solution surfaces $(x(s), y(s), z(s))$ of the vector field X on U; i.e. $H(x(s), y(s), z(s)) = $ constant for all values of s for which the solution $(x(s), y(s), z(s))$ is defined on U. This means that $H(x(s), y(s), z(s)) = c$ with $c \in \mathbb{R}$ for every time s for which the solution $(x(s), y(s), z(s))$ is defined on U, is a Darboux function. Hence, function $H(x(s), y(s), z(s))$ is a *Darboux first integral* in U if and only if $XH \equiv 0$ on all the points $(x(s), y(s), z(s))$ of U, where X has been defined in Equation (3.1.1.2).

For a polynomial differential System (3.1.1.1), the existence of two independent first integrals $H(x, y, z)$ implies that, by drawing surfaces $H(x, y, z) = constant$, we obtain important information about the phase portrait of the system. In the meantime, the existence of invariants will provide information about the $\alpha-$ or the $\omega-$limit sets of the orbits of the system.

To define Darboux functions which are not first integrals, we say that function $H(x(s), y(s), z(s))$ is a *Darboux function* in U if $XH \neq 0$ on U.

If H is a first integral of X, then we can reduce the study of the trajectories of X on the invariant sets $H(x(s), y(s), z(s)) = h$, when h varies in $\mathbb{R}$. We note that if $h \in \mathbb{R}$ is a regular value of the function H, then $H(x(s), y(s), z(s)) = h$ is a surface of $\mathbb{R}^3$, and that by Sard's Theorem the regular values are dense in $\mathbb{R}$.

We say that the vector field X is integrable if X has two independent first integrals; i.e. if X has two first integrals $H_i : U_i \to \mathbb{R}^3$ for $i = 1, 2$ such that the two vectors

$$\left(\frac{\partial H_1}{\partial x}, \frac{\partial H_1}{\partial y}, \frac{\partial H_1}{\partial z} \right) \qquad \left(\frac{\partial H_2}{\partial x}, \frac{\partial H_2}{\partial y}, \frac{\partial H_2}{\partial z} \right)$$

are independent at all the points $(x, y, z) \in U_1 \cap U_2$ except perhaps into a subset of zero Lebesgue measure.

If X is integrable with the two independent first integrals H_1 and H_2, then its trajectories are determined by intersecting the invariant sets $H_1(x, y, z) = h_1$ and $H_2(x, y, z) = h_2$ when h_1 and h_2 vary in $\mathbb{R}$. Hence, the dynamics (i.e. the trajectories) of an integrable system is determined by the functions H_1 and H_2.

3.2 The method of Darboux

In this Section we present the part of the Darboux theory on integrability which tells us how to construct a first integral for a d-dimensional polynomial system using his invariant algebraic varieties. Since its proof is not difficult, we do it. It is a global formulation of the Darboux theory of integrability, for polynomial differential system of arbitrary dimension, therefore it encloses the case of our 3D LVS (1.3.2). Let us first define the differential system

$$\frac{dx_i}{ds} = P_i(x_1, \ldots, x_d), \qquad i = 1, \ldots, d, \tag{3.2.1}$$

for any d natural number and real polynomial $P_i \in \mathbb{R}_{m_i}[x_1, \ldots, x_d]$, of degree m_i. We will now recall the following known result.

Lemma 3.2.1 *The real vector space* $\mathbb{R}_m[x_1, \ldots, x_d]$ *of degree at most* m, *has dimension*

$$\Delta(d, m) = \binom{d + m}{m}. \tag{3.2.2}$$

Without loss of generality, we may assume that the degree of the polynomial system is $\mathbf{m} = \max\{m_1, \ldots, m_d\}$ we write $\Delta(d; \mathbf{m}) = \Delta(d; m_1)$.

Proof. We will use induction over $\mathbf{m}$ to prove Lemma 3.2.1. Let us first verify Equation (3.2.2) for a polynomial ring in d variables, of degree 1. In this case the polynomial P can be written as

$$P_i(x_1, \ldots, x_d) = a_0 + \textstyle\sum_{i=1}^{d} a_i x_i. \tag{3.2.3}$$

From Equation (3.2.3), it is easy to see that the vector field defined over the polynomial ring $\mathbb{R}_{\mathbf{m}}[x_1, \ldots, x_d]$, has dimension $d+1$. One the other hand, we have that $\binom{d+1}{1} = d+1$, so the first statement is proved.

Now, based on the assumption that Lemma 3.2.1 is true for the vector field defined over the polynomial ring up to degree m, $\mathbb{R}_m[x_1, \ldots, x_d]$, we would like to prove that it is also true for the $\mathbb{R}_{m+1}[x_1, \ldots, x_d]$. Hence, we assume that if the maximum degree of P is $\mathbf{m}$, then P has $\binom{d+m}{m}$ terms. When increasing the degree of P to $m+1$, we add $\binom{d+m}{m+1}$ new terms to P, so that the dimension of the vector field defined over the polynomial ring $\mathbb{R}_{m+1}[x_1, \ldots, x_d]$ will be

$$\Delta(d, m+1) = \binom{d+m}{m} + \binom{d+m}{m+1} = \binom{d+m+1}{m+1},$$

which concludes the proof. ∎

The Darboux theory of integrability restricted to the use of invariant algebraic varieties is summarized into the following theorem. For more general results using additionally independent singular points and exponential factors see Christopher and Llibre [20], Christopher [19], Chavarriga et al. [22] or Pantazi et al. [21].

Theorem 3.2.1 (Darboux) *Suppose that a polynomial System (3.2.1) of degree m in $\mathbb{R}^d$ admits q invariant algebraic varieties $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$.*

(a) *If there exist $\lambda_i \in \mathbb{R}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i = 0$, then $|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p}$ is a first integral of the system.*

(b) *If $p = \Delta(d, m_1 - 1) + 1$, then there exist $\lambda_i \in \mathbb{R}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i = 0$.*

Proof. From

$$X\left(|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p}\right) = \pm|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p} \left(\sum_{i=1}^p \lambda_i \frac{X(f_i)}{f_i}\right) = \pm|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p} \left(\sum_{i=1}^p \lambda_i K_i\right) \equiv 0,$$

statement (a) follows.

Suppose that $p = \Delta(d, m_1 - 1) + 1 = \dim \mathbb{R}_{m_1-1}[x_1, \ldots, x_d] + 1$, see Lemma 3.2.1. Since $K_i \in \mathbb{R}_{m_1-1}[x_1, \ldots, x_d]$, for $i = 1, \ldots, p$, the polynomials K_i must be linearly dependent on $\mathbb{R}_{m_1-1}[x_1, \ldots, x_d]+1$. Hence, they are $\lambda_i \in \mathbb{R}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i = 0$. Consequently, from (a) statement (b) holds. ∎

In the following Section we apply the Darboux theory of integrability, to the 3D LVS (1.3.2).

3.3   Invariant algebraic surfaces of System (1.3.2)

In this section we study the invariant algebraic surfaces of the three-dimensional LVS (1.3.2), of degree two. In the following Proposition we present the invariant planes (invariant algebraic surfaces of degree one) together with their cofactors, and the conditions for their existence. In System (1.3.2) we have $\mathbf{k} = (k_1, k_2, k_3, k_4) \in \mathbb{R}^4$, and we will prove that the four faces that define the simplex $\mathcal{T}$, are invariant algebraic surfaces by the flow of the polynomial System (1.3.2).

Proposition 3.3.1 *The algebraic surfaces $f_1(x, y, z) = x$, $f_2(x, y, z) = y$, $f_3(x, y, z) = z$ and $f_4(x, y, z) = x + y + z - 1$, are invariant algebraic surfaces of System (1.3.2).*

Proof. It is easy to check that $Xf_i = f_iK_i$, in System (1.3.2), with $i = 1, 2, 3, 4$, where $K_1(x, y, z) = k_4x + (k_1 + k_4)y + k_4z - k_4$, $K_2(x, y, z) = -k_1x + k_2z$, $K_3(x, y, z) = -k_3x - (k_2 + k_3)y - k_3z + k_3$ and $K_4(x, y, z) = k_4x - k_3z$. Therefore, by Proposition 3.3.1, $f_i = 0$ is an invariant surface with cofactor K_i, with $i = 1, 2, 3, 4$. ∎

Now we can apply the Darboux Theorem 3.2.1 and for simplicity, we reformulate only the point (i) of it, which helps us in the search for the first integrals.

Proposition 3.3.2 *Consider the parameter sets* $\mathcal{S} = \{\mathbf{k} \in \mathbb{R}^4 \setminus \{\mathbf{0}\} : k_1k_3 - k_2k_4 = 0\}$ *and* $\mathcal{NZ} = \{\mathbf{k} \in \mathbb{R}^4 : k_1k_2k_3k_4 \neq 0\}$. *Moreover, consider the functions* $H(x, y, z) = x^{k_2}z^{k_1}$, $\widetilde{H}(x, y, z) = x^{k_3}z^{k_4}$, $V(x, y, z) = y^{k_3}(1 - x - y - z)^{k_2}$ *and* $\widetilde{V}(x, y, z) = y^{k_4}(1 - x - y - z)^{k_1}$. *Then the following statements follow.*

- *(a) If* $\mathbf{k} \in \mathcal{S} \cap \mathcal{NZ}$, *then* $H, V, \widetilde{H}$ *and* $\widetilde{V}$ *are first integrals which satisfy that* $\widetilde{H}^{k_1} = H^{k_4}$ *and* $\widetilde{V}^{k_3} = V^{k_4}$. *Moreover* H *and* V *are independent.*
- *(b) If* $\mathbf{k} \in \mathcal{S} \setminus \mathcal{NZ}$, *then two of the previous functions are first integrals and they are independent.*
- *(c) If* $\mathbf{k} \notin \mathcal{S}$, *then none of the previous functions is a first integral in* $\mathcal{T}$.

Proof. From Proposition 3.3.1, we have that $f_i = 0$ is an invariant surface with cofactor K_i, with $i = 1, 2, 3, 4$.

From Theorem 3.2.1, if there exist λ_i not all zero and such that $\sum_{i=1}^{4} \lambda_i K_i = 0$, then $F = f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3} f_4^{\lambda_4}$ is a first integral of System (1.3.2). Since

$$\sum_{i=1}^{4} \lambda_i K_i = (\lambda_4k_4 - \lambda_2k_1)x + (\lambda_1k_1 - \lambda_3k_2)y + (\lambda_2k_2 - \lambda_4k_3)z + (\lambda_3k_3 - \lambda_1k_4)(1 - x - y - z)$$

the existence of such λ_i is equivalent to the existence of non–trivial solutions of the homogeneous linear systems

$$\begin{pmatrix} k_1 & -k_2 \\ -k_4 & k_3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} k_2 & -k_3 \\ -k_1 & k_4 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (3.3.1)$$

Note that the determinant of both previous systems is equal to $k_1k_3 - k_2k_4$. Therefore when $\mathbf{k}$ belongs to the set $\mathcal{S}$ there exist Darboux type first integrals of System (1.3.2).

Under the assumption $\mathbf{k} \in \mathcal{S}$ linear System (3.3.1) has the following non–trivial solutions $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (k_2, 0, k_1, 0), (0, k_3, 0, k_2), (k_3, 0, k_4, 0)$ and $(0, k_4, 0, k_1)$. As a consequence, the functions $H(x, y, z) = x^{k_2}z^{k_1}$, $V(x, y, z) = y^{k_3}(1 - x - y - z)^{k_2}$, $\widetilde{H}(x, y, z) = x^{k_3}z^{k_4}$ and $\widetilde{V} = y^{k_4}(1 - x - y - z)^{k_1}$ are first integrals. In fact

$$\begin{aligned} XH &= x^{k_2}z^{k_1}(1 - x - y - z)(k_1k_3 - k_2k_4) & XV &= y^{k_3}(1 - x - y - z)^{k_2}x(k_2k_4 - k_1k_3), \\ X\widetilde{H} &= x^{k_3}z^{k_4}(k_1k_3 - k_2k_4)y & X\widetilde{V} &= y^{k_4}(1 - x - y - z)^{k_1}(k_2k_4 - k_1k_3)z \end{aligned} \qquad (3.3.2)$$

which vanish in the whole region $\mathcal{T}$ only when $\mathbf{k} \in \mathcal{S}$.

Consider that $\mathbf{k} \in \mathcal{S} \cap \mathcal{NZ}$. Since every coordinate of $\mathbf{k}$ is different from zero it follows that $H, V, \widetilde{H}$ and $\widetilde{V}$ are not constant in $\mathcal{T}$. Therefore all of these functions are first integrals. It is easy to check that $\widetilde{H}^{k_1} = H^{k_4}$ and $\widetilde{V}^{k_3} = V^{k_4}$. Moreover since $\nabla H(x, y, z) = x^{(k_2-1)}z^{(k_1-1)}(k_2z, 0, k_1x)$ and $\nabla V(x, y, z) = y^{(k_3-1)}(1 - x - y - z)^{(k_2-1)}(-k_2y, k_3(1 - x - y - z) - k_2y, -k_2y)$, both integrals are dependent only at the points satisfying that $k_3(1 - x - y - z) = k_2y$ and $k_2z = k_1x$. Taking into account that $k_2 \neq 0$ it follows that this set has zero Lebesgue measure. Then H and V are two independent first integrals.

Consider now that $\mathbf{k} \in \mathcal{S} \setminus \mathcal{NZ}$. Hence $\mathbf{k}$ has one coordinate which is different from zero. Without lost of generality we assume that $k_1 \neq 0$, the remaining cases follow in a similar way. It is easy to check that H and $\widetilde{V}$ are not constant in $\mathcal{T}$, and therefore they are first integrals. Since $\nabla H(x, y, z) = x^{(k_2-1)} z^{(k_1-1)} \left(k_2 z, 0, k_1 x\right)$ and $\nabla \widetilde{V}(x, y, z) = y^{(k_4-1)}(1 - x - y - z)^{(k_1-1)} \left(-k_1 y, k_4(1 - x - y - z) - k_1 y, -k_1 y\right)$, both integrals are dependent only at the points satisfying that $k_4(1 - x - y - z) = k_1 y$ and $k_2 z = k_1 x$. Therefore H and $\widetilde{V}$ are independent. ∎

Chapter 4

Flow analysis and first integrals of a family of 3D Lotka-Volterra Systems in Chemical Interactions

> *"We come finally, however, to the relation of the ideal theory to real world, or "real" probability. If he is consistent a man of the mathematical school washes his hands of applications. To someone who wants them he would say that the ideal system runs parallel to the usual theory: "If this is what you want, try it: it is not my business to justify application of the system; that can only be done by philosophizing; I am a mathematician". In practice he is apt to say: "try this; if it works that will justify it". But now he is not merely philosophizing; he is committing the characteristic fallacy. Inductive experience that the system works is not evidence."*
>
> – John Edensor Littlewood, *A Mathematician's Miscellany*–

In this Chapter we will apply elements of the qualitative theory of the dynamical systems to the analysis of the polynomial differential System (1.3.2). As mentioned in Chapter 1, this systems models a closed path of four chemical equilibria interconnecting four coexisting macromolecular species denoted by X, Y, Z and V. The system represents autocatalytic chemical reactions, in which reaction rates are defined as functions of the time dependent product concentrations, multiplied by their reaction rate constants.

Numerical simulations of the system as well as reported results in dynamical systems of similar structure [23] suggested us the possibility of self-sustained oscillations. Linearization of System (1.3.2), as it will be shown further, leads to a zero eigenvalue plus a pair of complex conjugate eigenvalues with zero real part. All these facts indicate the possibility of equilibria in the form of centers in the interior of $\mathcal{T}$. Let us recall here that, as it is given in Expression 1.3.3, $\mathcal{T} = \{x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1\}$. However, the fact that linearization yields a unique zero eigenvalue indicates that System (1.3.2) is not hyperbolic, so more sophisticated methods for proving the existence of periodic orbits are necessary. These methods consist in proving the existence and calculating the explicit expression of two independent first integrals for the System (1.3.2).

More importantly, in this case, the finding of first integrals does not only serves for proving the existence of periodic orbits in System (1.3.2). It represents a major tool for analyzing the flow inside the invariant and bounded region $\mathcal{T}$. In addition, we use this approach to completely describe the direction of the flow, as well as for characterizing the global bifurcation taking place at the boundary.

As it will be shown further, our invariant algebraic varieties divided the interior of the invariant region into two disjoint subsets. The case of the flow tangent to the invariant surfaces corresponds to a particular manifold in the parameter space; in the remaining regions of the parameter space, the flow is transversal to

these invariant surfaces; once the parameters of the system have been defined, the flow has only one possible direction, either approaching the invariant surfaces or getting away of it. Thus in this Thesis, we completely describe the direction of the flow in the parameter space, and this description has been possible by working out in first place, the invariant algebraic surfaces of the system.

The Subsections are organized as follows. In Subsection 4.1, we analyze the existence of singular points in both the interior and the boundary of $\mathcal{T}$. In Subsections 4.2 and 4.3, we analyze the flow at the boundary and in the interior of $\mathcal{T}$, respectively. In Subsection 4.4, we describe the bifurcation taking place at the boundary.

4.1   Preliminary definitions and singular points

The flow of System (1.3.2) is restricted to the the invariant region $\mathcal{T}$. Note that the boundary $\partial\mathcal{T}$ of the region is a three dimensional simplex which is invariant by the flow. This boundary is formed by the union of the following invariant subsets: the invariant faces $\mathcal{X} = \{(x,0,z) : x > 0, z > 0, x + z < 1\}$, $\mathcal{Y} = \{(0,y,z) : y > 0, z > 0, y + z < 1\}$, $\mathcal{Z} = \{(x,y,0) : x > 0, y > 0, x + y < 1\}$ and $\Sigma = \{(x,y,z) : x > 0, y > 0, z > 0, x + y + z < 1\}$; and the invariant edges $\mathcal{R}_{xz} = \{(x,0,0) : 0 < x < 1\}$, $\mathcal{R}_{xy} = \{(0,0,z) : 0 < z < 1\}$, $\mathcal{R}_{yz} = \{(0,y,0) : 0 \le y \le 1\}$, $\mathcal{R}_{px} = \{(x,0,1-x) : 0 \le x \le 1\}$, $\mathcal{R}_{py} = \{(0,y,1-y) : 0 < y < 1\}$ and $\mathcal{R}_{pz} = \{(x,1-x,0) : 0 < x < 1\}$. We remark that the edges $\mathcal{R}_{yz}$ and $\mathcal{R}_{px}$ are closed segments formed by singular points.

In order to make easier the analysis we consider the following subsets in the parameter space: $\mathcal{S}^- = \{\mathbf{k} \in \mathbb{R}^4 : k_1 k_3 - k_2 k_4 < 0\}$, $\mathcal{S}^+ = \{\mathbf{k} \in \mathbb{R}^4 : k_1 k_3 - k_2 k_4 > 0\}$, $\mathcal{NZ} = \{\mathbf{k} \in \mathbb{R}^4 : k_1 k_2 k_3 k_4 \ne 0\}$ and $\mathcal{PS} = \{\mathbf{k} \in \mathbb{R}^4 : k_1 k_2 > 0, k_1 k_3 > 0, k_1 k_4 > 0\}$. We note that $\mathcal{S}^-$ and $\mathcal{S}^+$ together with $\mathcal{S}$ (defined above) form a partition of the parameter space $\mathbb{R}^4$. We also note that the parameter set $\mathcal{PS}$ is a subset of $\mathcal{NZ}$.

In the following Proposition we summarize the results about the existence, location and stability of the singular points of System (1.3.2).

Proposition 4.1.1 *The half straight lines $\mathcal{R}_{px}$ and $\mathcal{R}_{yz}$ are formed by singular points.*

(a) *If $\mathbf{k} \in \mathcal{NZ}$ there are no other singular points in the boundary of the simplex.*

(a-1) *Suppose that $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$. The open segment*

$$R = \left\{ \left(\frac{k_3}{k_4} z, \frac{k_4 - (k_4 + k_3)z}{k_4 + k_1}, z \right) : 0 < z < \frac{k_4}{k_3 + k_4} \right\}$$

is formed by all the singular points in the interior of the region $\mathcal{T}$. Moreover the jacobian matrix of the vector field evaluated at each of these points has one real eigenvalue equal to zero and two purely imaginary eigenvalues.

(a-2) *Suppose that $\mathbf{k} \in \mathcal{NZ} \setminus \{\mathcal{PS} \cap \mathcal{S}\}$. There are no singular points in the interior of region $\mathcal{T}$.*

(b) *Suppose that $\mathbf{k} \notin \mathcal{NZ}$ and $\mathbf{k} \ne \mathbf{0}$. There are no singular points in the interior of region $\mathcal{T}$.*

Proof. Straight forward computations show that the half straight lines $\mathcal{R}_{px}$ and $\mathcal{R}_{yz}$ are formed by singular points.

Suppose now that $\mathbf{k} \in \mathcal{NZ}$. Hence none of the components of the parameter $\mathbf{k}$ is zero. In this case the singular points are given by the solutions to the following systems

$$\left.\begin{array}{r} x = 0 \\ yz = 0 \\ z(-k_2 y + k_3(1 - y - z)) = 0 \end{array}\right\} \quad (4.1.1) \qquad\qquad \left.\begin{array}{r} -x(1 - x - z) = 0 \\ y = 0 \\ z(1 - x - z) = 0 \end{array}\right\} \quad (4.1.3)$$

$$\left.\begin{array}{r} x(k_1 y - k_4(1 - x - y)) = 0 \\ -yx = 0 \\ z = 0 \end{array}\right\} \quad (4.1.2) \qquad\qquad \left.\begin{array}{r} k_4 x + (k_1 + k_4)y + k_4 z = k_4 \\ (k_1 + k_4)y + (k_3 + k_4)z = k_4 \\ (k_2 k_4 - k_1 k_3)(y + z) = k_2 k_4 - k_1 k_3 \end{array}\right\} \quad (4.1.4)$$

where $x, y, z \in \mathcal{T}$ and in System (4.1.4) we impose $xyz \neq 0$ to avoid repetitions. From Systems (4.1.1), (4.1.3) and (4.1.2), it is easy to conclude that there are no other singular points that those in the half straight lines $\mathcal{R}_{px}$ and $\mathcal{R}_{yz}$. With respect to System (4.1.4), we distinguish two situations.

First let us suppose that $\mathbf{k} \notin S$, that is $k_2 k_4 - k_1 k_3 \neq 0$. From the third equation it follows that $y + z = 1$, and therefore $x = 0$. Since $k_1 k_4 \neq 0$ from the first equation we conclude that $y = 0$ and $z = 1$. Hence the singular point is one of the endpoints of the edge $\mathcal{R}_{yz}$; i.e. it does not belong to the interior of $\mathcal{T}$.

Suppose now that $\mathbf{k} \in S$, that is $k_2 k_4 - k_1 k_3 = 0$. Thus the linear system is equivalent to the following one

$$\left.\begin{array}{l} k_4 x + (k_1 + k_4)y + k_4 z = k_4, \\ (k_1 + k_4)y + (k_3 + k_4)z = k_4. \end{array}\right\}$$

where in the last one we impose $xyz \neq 0$ to avoid repetitions. From the three first systems it is easy to conclude that there are no other singular points that those in the half straight lines $\mathcal{R}_{px}$ and $\mathcal{R}_{yz}$. With respect to the last one we distinguish two situations.

First let us suppose that $\mathbf{k} \notin S$, that is $k_2 k_4 - k_1 k_3 \neq 0$. From the third equation it follows that $y + z = 1$, and therefore $x = 0$. Since $k_1 k_4 \neq 0$ from the first equation we conclude that $y = 0$ and $z = 1$. Hence the singular point is one of the endpoints of the edge $\mathcal{R}_{yz}$; i.e. it does not belong to the interior of $\mathcal{T}$.

Suppose now that $\mathbf{k} \in S$, that is $k_2 k_4 - k_1 k_3 = 0$. Thus the linear system is equivalent to the following one

$$\left.\begin{array}{l} k_4 x + (k_1 + k_4)y + k_4 z = k_4, \\ (k_1 + k_4)y + (k_3 + k_4)z = k_4. \end{array}\right\}$$

If $k_1 + k_4 = 0$, then from the first equation we obtain $x + z = 1$. Therefore $y = 0$ and the singular point belongs to $\mathcal{R}_{px}$. On the contrary, if $k_1 + k_4 \neq 0$, then there exists a straight line of singular points parametrically defined by $x = z k_3 / k_4$ and $y = (k_4 - (k_3 + k_4)z)/(k_1 + k_4)$. Since the singular points in the interior of $\mathcal{T}$ must satisfy that $x > 0, y > 0, z > 0$ and $x + y + z < 1$, then there exist singular points in the interior of $\mathcal{T}$ if and only if

$$\frac{k_3}{k_4} > 0, \quad \frac{k_3 + k_4}{k_1 + k_4}z < \frac{k_4}{k_1 + k_4}, \quad \frac{k_1}{k_4}\left(\frac{k_3 + k_4}{k_1 + k_4}z\right) < \frac{k_1}{k_1 + k_4}, \quad z > 0.$$

It is easy to check that the previous inequalities are equivalent to

$$\frac{k_3}{k_4} > 0, \quad \frac{k_3 + k_4}{k_1 + k_4}z < \frac{k_4}{k_1 + k_4}, \quad \frac{k_1}{k_4} > 0, \quad z > 0.$$

Since $\mathbf{k} \in S$ we have $k_1 / k_4 = k_2 / k_3$. Therefore we conclude that there exist singular points in the interior of $\mathcal{T}$ if and only if all the components of $\mathbf{k}$ have the same sign; that is $\mathbf{k} \in \mathcal{PS}$. In such case these singular points are given by

$$x = \frac{k_3}{k_4}z, \quad y = \frac{k_4 - (k_3 + k_4)z}{k_1 + k_4}, \quad 0 < z < \frac{k_4}{k_1 + k_4}, \tag{4.1.5}$$

which proves statement (a-1).

The Jacobian matrix of the vector field defined by the differential Equation (1.3.2) evaluated at the singular points (4.1.5) is given by

$$\begin{pmatrix} k_3 z & (k_2 + k_3)z & k_3 z \\ -k_1 y & 0 & k_2 y \\ -k_3 z & -(k_2 + k_3)z & -k_3 z \end{pmatrix}$$

The characteristic polynomial is equal to $\lambda(\lambda^2 + b) = 0$, where $b = zy(k_1 + k_2)(k_2 + k_3)$. Since $\mathbf{k} \in \mathcal{PS}$ coefficient b is positive. Then we get one zero eigenvalue and a pair of complex conjugated eigenvalues with zero real part. From this we conclude the statement (a-1).

If $\mathbf{k} \notin \mathcal{NZ}$ and $\mathbf{k} \neq \mathbf{0}$, then at least one of the coordinates of $\mathbf{k}$ is equal to zero and at least one is different from zero. Without lost of generality we suppose that $k_1 = 0$ and $k_2 \neq 0$. From the second equation in (1.3.2) we conclude that the singular points are contained in the boundary of $\mathcal{T}$. This proves the statement (b). ∎

Proposition 4.1.2 *If* $\mathbf{k} \in \mathcal{PS} \setminus \mathcal{S}$, *then no singular point neither in* $\mathcal{R}_{px} \setminus s_{px}$ *nor in* $\mathcal{R}_{yz} \setminus s_{yz}$ *is the limit set of an orbit in the interior of the region* $\mathcal{T}$.

Proof. Let $\mathbf{p}$ be a point in the set $\mathcal{R}_{px} \setminus s_{px}$, that is $\mathbf{p} = (x_0, 0, 1 - x_0)$ where either

$$x_0 > \max \left\{ \frac{k_2}{k_1 + k_2}, \frac{k_3}{k_3 + k_4} \right\} \quad \text{or} \quad x_0 < \min \left\{ \frac{k_2}{k_1 + k_2}, \frac{k_3}{k_3 + k_4} \right\}, \tag{4.1.6}$$

see Expression (4.3.1). If we consider a point $\mathbf{p}$ in the set $\mathcal{R}_{yz} \setminus s_{yz}$, the following arguments can be applied in a similar way.

Through the change of variables $\bar{x} = x - x_0$, $\bar{y} = y$ and $\bar{z} = z - 1 + x_0$, System (1.3.2) can be written as system $\dot{\bar{\mathbf{x}}} = A\bar{\mathbf{x}} + \mathbf{Q}(\bar{\mathbf{x}})$ where $\bar{\mathbf{x}} = (\bar{x}, \bar{y}, \bar{z})^T$,

$$A = \begin{pmatrix} k_4 x_0 & (k_1 + k_4) x_0 & k_4 x_0 \\ 0 & k_2 - (k_1 + k_2) x_0 & 0 \\ k_3(x_0 - 1) & (k_2 + k_3)(x_0 - 1) & k_3(x_0 - 1) \end{pmatrix} \text{ and } \mathbf{Q}(\bar{\mathbf{x}}) = \begin{pmatrix} \bar{x}(k_4\bar{x} + (k_1 + k_4)\bar{y} + k_4\bar{z}) \\ \bar{y}(k_2\bar{z} - k_1\bar{x}) \\ \bar{z}(-k_3\bar{x} - (k_2 + k_3)\bar{y} - k_3\bar{z}) \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda_1 = 0$, $\lambda_2 = (k_3 + k_4) x_0 - k_3$ and $\lambda_3 = k_2 - x_0(k_1 + k_2)$. From (4.1.6) it is easy to conclude that $\lambda_2 \lambda_3 < 0$. Therefore there exists a regular matrix P such that $PAP^{-1} = diag\{0, \lambda_2, \lambda_3\}$.

Going through the change of coordinates $\mathbf{x}_p = P\bar{\mathbf{x}}$ the system can be rewritten as

$$\begin{cases} \dot{x}_p = \frac{k_2 z_p - k_3 y_p}{k_1 k_4 x_0} \left(k_4 k_1 x_p + k_1(x_0 - 1)(k_3 + k_4) y_p + k_4(x_0 - 1)(k_2 + k_4) z_p \right) \\[2mm] \dot{y}_p = \frac{y_p}{k_1 k_4 x_0} \left((\lambda_2 - (k_3 + k_4) x_p) k_1 k_4 x_0 + k_1(k_3^2(1 - x_0) + k_4^2 x_0) y_p + k_4(k_1 k_4 x_0 + k_2 k_3(1 - x_0)) z_p \right) \\[2mm] \dot{z}_p = \frac{z_p}{k_1 k_4 x_0} \left((\lambda_3 + (k_2 + k_1) x_p) k_1 k_4 x_0 - k_1(k_1 k_4 x_0 + k_2 k_3(1 - x_0)) y_p - k_2(k_2^2(1 - x_0) + k_1^2 x_0) z_p \right) \end{cases} \tag{4.1.7}$$

System (4.1.7) has two invariant planes $\{y_p = 0\}$ and $\{z_p = 0\}$ intersecting at a straight line formed by singular points, which corresponds with the segment $\mathcal{R}_{px}$. The direction of the vector field, in a sufficiently small neighborhood of the origin, satisfies that

$$\text{sign}(\dot{y}_p) = \text{sign}(y_p)\text{sign}(\lambda_2),$$

$$\text{sign}(\dot{z}_p) = \text{sign}(z_p)\text{sign}(\lambda_3).$$

We conclude that the origin is neither the α–limit set nor the ω–limit set of any orbit in the interior of the regions $\{y_p > 0, z_p > 0\}, \{y_p > 0, z_p < 0\}, \{y_p < 0, z_p > 0\}$ and $\{y_p < 0, z_p < 0\}$. From this we conclude the Proposition. ∎

In the following two Sections we will analyze the flow of System (1.3.2), at the the boundary and in the interior of the region $\mathcal{T}$, respectively. Useful information about the phase portrait of the system is given not only by the first integrals presented in Proposition 3.3.2, but also by the Darboux functions.

4.2   Behavior at the boundary of $\mathcal{T}$

As we have proved in Proposition 3.3.2 some of the functions $H, \widetilde{H}, V$ and $\widetilde{V}$ are first integrals over the whole region $\mathcal{T}$ only when $\mathbf{k} \in \mathcal{S}$. Nevertheless the restriction of these functions to a particular face of $\mathcal{T}$ results in a first integral even when $\mathbf{k} \notin \mathcal{S}$. In fact, denoting by $\widetilde{H}|_{\mathcal{X}}$ the restriction of the function $\widetilde{H}$ to the face $\mathcal{X}$, from Expression (3.3.2) it follows that $X\widetilde{H}|_{\mathcal{X}} = 0$. Therefore the level curves $\widetilde{H}|_{\mathcal{X}} = C^{k_4}$ are invariant by the flow. Under the assumption $k_3 k_4 > 0$, these level curves define a foliation of $\mathcal{X}$ whose leaves are given by the arcs

of hyperbola $\left\{ z = Cx^{-\frac{k_3}{k_4}} \right\}_{0<C<C^*}$ where

$$C^* = \frac{k_4}{k_4 + k_3} \left(\frac{k_3}{k_4 + k_3} \right)^{\frac{k_3}{k_4}}. \tag{4.2.1}$$

Furthermore, every leaf with $0 < C < C^*$ intersects the segment $\mathcal{R}_{px}$ at exactly two points, see Figure 4.2.1(a). The value $C = C^*$ leads to a unique intersection point with coordinates $x = k_3/(k_3 + k_4)$ and $z = k_4/(k_3 + k_4)$. Since in the face $\mathcal{X}$ we have $y = 0$, it follows that the point corresponding to C^* is the point $\mathbf{q}_{px} \frac{k_3}{k_3+k_4}, 0, \frac{k_4}{k_3+k_4}$.

Similarly, the restriction of $V, \widetilde{V}$ and H to the faces Y, Z and Σ respectively, are first integrals even when $\mathbf{k} \notin \mathcal{S}$, see Expression (3.3.2). Consider the changes of variables $(u, v, \alpha, \beta) \to (y, z, k_2, k_3), (y, x, -k_1, -k_4)$ or (x, y, k_1, k_2), depending on the face $\mathcal{Y}, \mathcal{Z}$ or Σ we are looking at. Under the assumption $\alpha\beta > 0$, the level curves $V|_{\mathcal{Y}} = C^{k_2}, \widetilde{V}|_{\mathcal{Z}} = C^{k_1}$ and $H|_{\Sigma} = C^{k_1}$ define a foliation on the corresponding face, whose leaves are given by the unimodal curves $\left\{ v = 1 - u - Cu^{-\frac{\beta}{\alpha}} \right\}_{0<C<C^*}$ where

$$C^* = \frac{\alpha}{\alpha + \beta} \left(\frac{\beta}{\alpha + \beta} \right)^{\frac{\beta}{\alpha}}.$$

Every leaf with $0 < C < C^*$ intersects the segment $\{v = 0, 0 < u < 1\}$ at exactly two points, see Figure 4.2.1(b). The value $C = C^*$ leads to a unique intersection point $(\beta/(\alpha + \beta), 0)$. Going back through the change of variables and adding the variable which does not appear in such change, that intersection point coincides with $\mathbf{q}_{yz}, \mathbf{p}_{yz}$ or $\mathbf{p}_{px}$ defined in (4.3.1), depending on the change of variables.
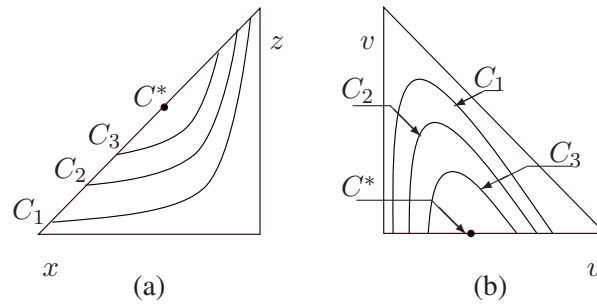

Figure 4.2.1. (a) Foliation over the face $\mathcal{X}$ defined by the level curves $\widetilde{H}|_{\mathcal{X}} = C^{k_4}$ where $0 < C_1 < C_2 < C_3 < C^*$ and $\mathbf{k}$ in $\mathcal{PS} \setminus \mathcal{S}$. (b) Foliation over the corresponding face of the level curves $H_\Sigma = C^{k_1}$ or $V_{\mathcal{Y}} = C^{k_2}$ or $\widetilde{V}_{\mathcal{Z}} = C^{k_1}$ where $0 < C_1 < C_2 < C_3 < C^*$ and $\mathbf{k}$ in $\mathcal{PS} \setminus \mathcal{S}$. Note that Figure (b) is represented in (u, v)–coordinates.

Using the geometric information of the aforementioned foliations, in the next result we summarize the behavior of the flow of System (1.3.2) at the boundary $\partial\mathcal{T}$ for $\mathbf{k} \in \mathcal{PS}$.

Lemma 4.2.1 (a) If $\mathbf{k} \in \mathcal{PS}$, then each of the two limit sets of every orbit contained in $\mathcal{X} \cup \Sigma$ (respectively, $\mathcal{Y} \cup \mathcal{Z}$) is formed by a singular point contained in the edge $\mathcal{R}_{px}$ (respectively, $\mathcal{R}_{yz}$).

(b) If $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$, then for every pair of orbits $\gamma_1 \subset \mathcal{X}$ and $\gamma_2 \subset \Sigma$ (respectively, $\gamma_1 \subset \mathcal{Y}$ and $\gamma_2 \subset \mathcal{Z}$) satisfying that $\omega(\gamma_1) = \alpha(\gamma_2)$, it follows that $\alpha(\gamma_1) = \omega(\gamma_2)$.

(c) If $\mathbf{k} \in \mathcal{PS} \setminus \mathcal{S}$, then for every pair of orbits $\gamma_1 \subset \mathcal{X}$ and $\gamma_2 \subset \Sigma$ (respectively, $\gamma_1 \subset \mathcal{Y}$ and $\gamma_2 \subset \mathcal{Z}$) satisfying that $\omega(\gamma_1) = \alpha(\gamma_2)$, it follows that $\alpha(\gamma_1) \neq \omega(\gamma_2)$.

Proof. We restrict ourselves to consider orbits in the faces $\mathcal{X} \cup \Sigma$. The study of the orbits in the faces $\mathcal{Y} \cup \mathcal{Z}$ follows in a similar way.

Suppose that $\mathbf{k} \in \mathcal{PS}$. Hence $k_3 k_4 > 0$. Therefore every orbit γ_1 in $\mathcal{X}$ is contained in a leaf of the foliation $z = Cx^{-\frac{k_3}{k_4}}$ with $0 < C < C^*$, which is an arc of hyperbola intersecting the edge $\mathcal{R}_{px}$ at exactly two points. Since there are not other singular points in $\mathcal{X}$, see Proposition 4.1.1(a), we conclude that each of the two limit sets of γ_1 is one of these intersection points.

On the other hand we have $k_2 k_1 > 0$. Therefore every orbit γ_2 in Σ is contained in a leaf of the foliation $y = 1 - x - Cx^{-\frac{k_2}{k_1}}$, which is an unimodal curve intersecting the edge $\mathcal{R}_{px}$ at exactly two points. We conclude again that each of the limit sets of γ_2 is one of these intersection points. From this we conclude the statement (a).

Taking into account that Σ is given by the relation $z = 1 - x - y$, we express the leaves in Σ as a function $z(x)$ in the following way $z = Cx^{-\frac{k_2}{k_1}}$.

Let $\mathbf{p} = (x_0, 0, 1 - x_0)$ be a point in the edge $\mathcal{R}_{px}$. There exist two positive values C_1 and C_2 such that both the leaf $z = C_1 x^{-\frac{k_3}{k_4}}$ in the face $\mathcal{X}$ and the leaf $z = C_2 x^{-\frac{k_2}{k_1}}$ in the face Σ contain the point $\mathbf{p}$. On the other hand the leaf in the face $\mathcal{X}$ intersects $\mathcal{R}_{px}$ at a new point $(x_1, 0, 1 - x_1)$ and the leaf in the face Σ intersects $\mathcal{R}_{px}$ at a new point $(x_2, 0, 1 - x_2)$. Since two arcs of hyperbola either intersect at most at one point or they coincide, we conclude that $k_3 k_1 = k_4 k_2$ if and only if $x_1 = x_2$. This proves the statements (b) and (c). ∎

4.3   Behavior in the interior of $\mathcal{T}$

We are now prepared to formulate and prove the main results of this Thesis.

Theorem 4.3.1   *(a) Suppose that $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$.*

(a-1) The open segment

$$R = \left\{ \left(\frac{k_3}{k_4} z, \frac{k_4 - (k_4 + k_3)z}{k_4 + k_1}, z \right) : 0 < z < \frac{k_4}{k_3 + k_4} \right\}$$

is contained in the interior of $\mathcal{T}$ and every point in R is a singular point.

(a-2) Let $\mathbf{p}$ be a point contained in the interior of $\mathcal{T}$ but not in R. Then the orbit $\gamma_{\mathbf{p}}$ through the point $\mathbf{p}$ is a periodic orbit.

(a-3) Each of the two limit sets of every orbit in $\mathcal{X} \cup \Sigma$ is a singular point contained in the edge $\mathcal{R}_{px}$. Moreover, given two orbits $\gamma_1 \subset \mathcal{X}$ and $\gamma_2 \subset \Sigma$ such that $\omega(\gamma_1) = \alpha(\gamma_2)$, then $\omega(\gamma_2) = \alpha(\gamma_1)$.

(a-4) Each of the two limit sets of every orbit in $\mathcal{Y} \cup \mathcal{Z}$ is a singular point contained in the edge $\mathcal{R}_{yz}$. Moreover, given two orbits $\gamma_1 \subset \mathcal{Y}$ and $\gamma_2 \subset \mathcal{Z}$ such that $\omega(\gamma_1) = \alpha(\gamma_2)$, then $\omega(\gamma_2) = \alpha(\gamma_1)$.

(a-5) The behavior of the flow in $\mathcal{T}$ is topologically equivalent to the one draw in Figure 4.3.1.

(b) Suppose that $\mathbf{k} \notin \mathcal{PS} \cap \mathcal{S}$ and $\mathbf{k} \neq \mathbf{0}$. The limit sets of every orbit in $\mathcal{T}$ are contained in the boundary of $\mathcal{T}$. Thus the flow goes from one face to another.

We remark that Theorem 4.3.1 completely characterizes the region in the parameter space where the corresponding System (1.3.2) exhibits self sustained oscillations. Thus the necessary conditions $\mathbf{k} \in \mathcal{S}$ for the existence of such behavior, given by Di Cera et al. [23], are here completed with the necessary and sufficient condition $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$. Futhermore this oscillating behavior in the interior of $\mathcal{T}$ extends to a heteroclinic behavior at the boundary. Therefore the period function defined in the interior of $\mathcal{T}$ is a non–constant function; it grows by approaching the boundary.

The dynamic behavior of family (1.3.2) is very simple. In fact, as showed in Theorem 4.3.1(b), in absence of periodic orbits every orbit goes from one side of the boundary of $\mathcal{T}$ to another. Nevertheless, we can remark certain singular situations related to the form and location of the limit sets. One of these limit set configurations
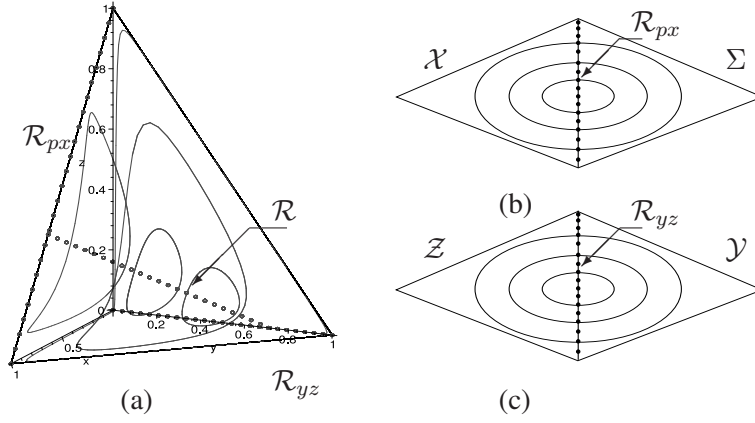
Figure 4.3.1. Behavior of the flow of System (1.3.2) for $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$: (a) in the interior of $\mathcal{T}$; (b) at the edges $\mathcal{X} \cup \Sigma$ and (c) at the edges $\mathcal{Y} \cup \mathcal{Z}$.

is described in the next result. Before stating it we consider the following singular points in the edges $\mathcal{R}_{px}$ and $\mathcal{R}_{yz}$, respectively

$$\mathbf{p}_{px} = \left(\frac{k_2}{k_1+k_2}, 0, \frac{k_1}{k_1+k_2} \right), \quad \mathbf{q}_{px} = \left(\frac{k_3}{k_3+k_4}, 0, \frac{k_4}{k_3+k_4} \right),$$

$$\mathbf{p}_{yz} = \left(0, \frac{k_4}{k_1+k_4}, 0 \right), \qquad \mathbf{q}_{yz} = \left(0, \frac{k_3}{k_3+k_2}, 0 \right).$$

(4.3.1)

When $\mathbf{k}$ is in the manifold $\mathcal{PS} \cap \mathcal{S}$, the points $\mathbf{p}_{px}$ and $\mathbf{q}_{px}$ are equal and they coincide with one of the endpoints of the segment R defined in Theorem 4.3.1(a-1). Similarly, the points $\mathbf{p}_{yz}$ and $\mathbf{q}_{yz}$ are also equal and they coincide with the other endpoint of R. On the other hand, when $\mathbf{k} \in \mathcal{PS} \setminus \mathcal{S}$, we define the following segments contained in the edges $\mathcal{R}_{px}$ and $\mathcal{R}_{yz}$, respectively

$$s_{px} = \{\mathbf{p}_{px} + r(\mathbf{q}_{px} - \mathbf{p}_{px}) : r \in [0,1]\}, s_{yz} = \{\mathbf{p}_{yz} + r(\mathbf{q}_{yz} - \mathbf{p}_{yz}) : r \in [0,1]\}. \quad (4.3.2)$$

To clarify the exposition of the next result we introduce the subsets $\mathcal{PS}_+ = \{\mathbf{k} \in \mathbb{R}^4 : k_i > 0\}$ and $\mathcal{PS}_- = \{\mathbf{k} \in \mathbb{R}^4 : k_i < 0\}$ which form a partition of $\mathcal{PS}$.

Proof. We start by setting the parameter $\mathbf{k}$ in the condition of the Theorem 4.3.1(a); that is, $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$. Under this assumption System (1.3.2) is integrable and the functions H and V are two independent first integrals, see Proposition 3.3.2(a). Since any level surface $\mathcal{H}_C$ is invariant by the flow, we can consider the restriction of the flow to each of these surfaces. Of course this restricted flow is also integrable because the restriction of the function V to $\mathcal{H}_C$ is a first integral.

On the other hand there exists exactly one singular point in the interior of $\mathcal{H}_C$, which comes from the intersection of the manifold $\mathcal{H}_C$ and the segment R defined in the Proposition 4.1.1(a-1). These singular points have a zero eigenvalue and a pair of purely imaginary eigenvalues. We conclude that every orbit in the interior of $\mathcal{H}_C$, but the singular points, is a periodic orbit. Since this result is independent on the level surface we are working at, it follows that every orbit in the interior of the region $\mathcal{T}$, but the singular points, is a periodic orbit. The behavior of the flow at the boundary of $\mathcal{T}$ when $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$ can be obtained from Lemma 4.2.1(b). This completes the proof of Theorem 4.3.1(a).

To prove Theorem 4.3.1(b) we consider that $\mathbf{k} \notin \mathcal{PS} \cap \mathcal{S}$ and $\mathbf{k} \neq \mathbf{0}$. We distinguish two situations: first we suppose that $\mathbf{k} \in \mathcal{S} \setminus \mathcal{PS}$. In such case $\mathbf{k}$ belongs to the manifold $\mathcal{S}$. From Proposition 3.3.2 it follows that at least one of the functions $H, V, \widetilde{H}$ or $\widetilde{V}$ is a first integral. Without loss of generality we can assume that H is a first integral. Hence any level surface $\mathcal{H}_C$ is invariant by the flow and we can consider the restriction of the flow to $\mathcal{H}_C$. From Proposition 4.1.1 there are not singular points in the interior of $\mathcal{H}_C$. Applying the

Poincaré–Bendixson Theorem to the flow in the level surface $\mathcal{H}_C$, we conclude that the flow goes from the boundary of $\mathcal{H}_C$ to the boundary of $\mathcal{H}_C$. Since these arguments are independent on the level surface, it follows that the flow goes from the boundary of $\mathcal{T}$ to the boundary of $\mathcal{T}$.

Suppose now that $\mathbf{k} \notin \mathcal{S}$ and $\mathbf{k} \neq \mathbf{0}$. Since one of the coordinates of $\mathbf{k}$ is different from zero, the level surfaces of at least one of the functions $H, V, \widetilde{H}$ and $\widetilde{V}$ can be expressed as the graph of an explicit differentiable function. For instance if $k_4 \neq 0$, then $\widetilde{\mathcal{H}}_{C^{k_4}}$ is the graph of the function $z = Cx^{-\frac{k_3}{k_4}}$ defined over the face $\mathcal{Z}$. Each of these level surfaces split the interior of $\mathcal{T}$ into two disjoint connected components. On the other hand since $\mathbf{k} \notin \mathcal{S}$ these level surfaces are not invariant by the flow, see Proposition 3.3.2(c). In fact the flow is transversal to them and the direction of the flow through them depends on $\mathbf{k} \in \mathcal{S}^+$ or $\mathbf{k} \in \mathcal{S}^-$, see Expression (3.3.2).

Since as C tends to 0 or to C^* the level surfaces $\widetilde{\mathcal{H}}_{C^{k_4}}$ tend to the boundary of $\mathcal{T}$, we conclude that the flow in the interior of $\mathcal{T}$ goes from one part of the boundary to another part of the boundary. That is the limit sets of every orbit in the interior of $\mathcal{T}$ are contained in $\partial \mathcal{T}$. From this we conclude the Theorem 4.3.1(b). ■

Theorem 4.3.2 *Suppose that $\mathbf{k} \in \mathcal{PS} \setminus \mathcal{S}$.*

(a) *Each of the two limit sets of every orbit in the interior of $\mathcal{T}$ is formed by a singular point contained in the segments s_{px} and s_{yz}. In particular, given a point $\mathbf{p}$ in the interior of $\mathcal{T}$, if $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$ or $\mathbf{k} \in \mathcal{PS}_- \cap \mathcal{S}^-$, then $\alpha(\gamma_{\mathbf{p}}) \in s_{yz}$ and $\omega(\gamma_{\mathbf{p}}) \in s_{px}$; and if $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^-$ or $\mathbf{k} \in \mathcal{PS}_- \cap \mathcal{S}^+$ then $\alpha(\gamma_{\mathbf{p}}) \in s_{px}$ and $\omega(\gamma_{\mathbf{p}}) \in s_{yz}$.*

(b) *Each of the two limit sets of every orbit in $\mathcal{X} \cup \Sigma$ is a singular point contained in the edge $\mathcal{R}_{px}$. Moreover, given two orbits $\gamma_1 \subset \mathcal{X}$ and $\gamma_2 \subset \Sigma$ such that $\omega(\gamma_1) = \alpha(\gamma_2)$, then $\omega(\gamma_2) \neq \alpha(\gamma_1)$.*

(c) *Each of the two limit sets of every orbit in $\mathcal{Y} \cup \mathcal{Z}$ is a singular point contained in the edge $\mathcal{R}_{yz}$. Moreover, given two orbits $\gamma_1 \subset \mathcal{Y}$ and $\gamma_2 \subset \mathcal{Z}$ such that $\omega(\gamma_1) = \alpha(\gamma_2)$, then $\omega(\gamma_2) \neq \alpha(\gamma_1)$.*

(d) *The behavior of the flow in $\mathcal{T}$ is topologically equivalent to the one drawn in Figure 4.3.2.*
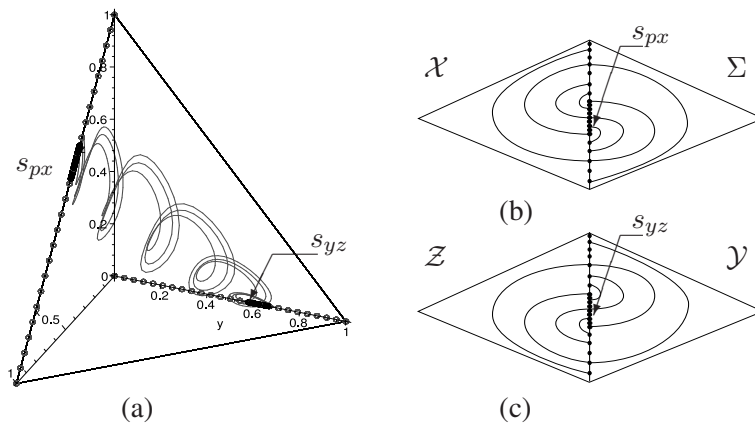

Figure 4.3.2. Behavior of the flow of System (1.3.2) for $\mathbf{k} \in \mathcal{PS} \setminus \mathcal{S}$: (a) in the interior of $\mathcal{T}$; (b) at the edges $\mathcal{X} \cup \Sigma$ and (c) at the edges $\mathcal{Y} \cup \mathcal{Z}$.

Note that in the previous proof we have only used that the flow crosses through the level surfaces of some of the functions $H, V, \widetilde{H}$ or $\widetilde{V}$, always in the same direction. This argumentation results enough to conclude that the limit sets of the orbits in $\mathcal{T}$ are contained in the boundary. To prove Theorem 4.3.2 we need to be more precise in the location of these limit sets. To reach this goal we will control the geometry of the level surfaces.

Proof. Suppose that $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$. Since $\mathbf{k} \notin \mathcal{S}$ the functions H and $\widetilde{H}$ are not first integrals and each level surface $\mathcal{H}_C$ and $\widetilde{\mathcal{H}}_C$ splits the region $\mathcal{T}$ into two disjoint regions in such a way that the flow goes from one to the other. In fact since $k_3 k_4 > 0$ the intersection of $\widetilde{\mathcal{H}}_C$ with any plane $\{y = y0 : 0 < y_0 < 1\}$ is an
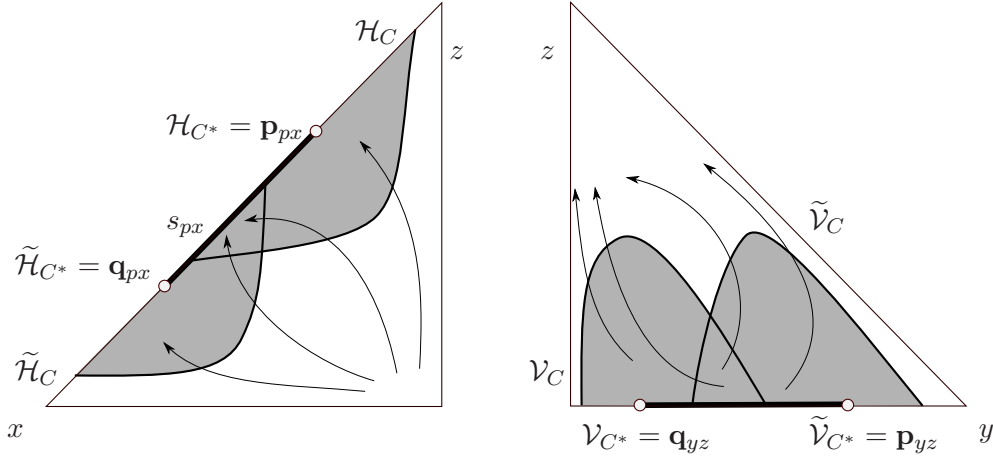
Figure 4.3.3. Representation when $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$ of the positive invariant regions limited by the level surfaces $\mathcal{H}_C$ and $\widetilde{\mathcal{H}}_C$; the negative invariant regions limited by the level surfaces $\mathcal{V}_C$ and $\widetilde{\mathcal{V}}_C$; the attractor set s_{px} and the repelor set s_{yz} of any given orbit in the interior of the region $\mathcal{T}$.

arc of hyperbola in the (x, z)–plane, see Figure 4.2.1(a). Similarly, since $k_1 k_2 > 0$ the intersection of $\mathcal{H}_C$ with any plane $\{y = y0 : 0 < y_0 < 1\}$ is an arc of hyperbola in the (x, z)–plane. The flow through $\mathcal{H}_C$ and through $\widetilde{\mathcal{H}}_C$ has the same direction as the vectors ∇H and $\nabla \widetilde{H}$ respectively, see Expression (3.3.2). Since $k_1 > 0$ the gradient ∇H points to the region containing the point $\mathbf{p}_{px}$, see the shadowed region in Figure 4.3.3. Moreover, since $k_4 > 0$ the gradient $\nabla \widetilde{H}$ points to the region containing the point $\mathbf{q}_{px}$, see Figure 4.3.3. Therefore, in Figure 4.3.3 the flow moves from the region containing the origin to the shadowed region. On the other hand points in $\mathcal{R}_{px} \setminus s_{px}$ are not limit set of orbits in the interior of $\mathcal{T}$, see Proposition 4.1.2. We conclude that the ω–limit set of any given orbit in the interior of $\mathcal{T}$ is a singular point contained in the segment s_{px}.

On the other hand, since $\mathbf{k} \notin \mathcal{S}$ the functions V and $\widetilde{V}$ are not first integrals. Moreover any of the level surfaces $\mathcal{V}_C$ and $\widetilde{\mathcal{V}}_C$ splits the region $\mathcal{T}$ into two disjoint regions in such a way that the flow goes from one to the other. The flow through these surfaces has opposite direction to that of the gradients ∇V and $\nabla \widetilde{V}$, see Expression (3.3.2). Since $k_2 > 0$ the gradient ∇V points to the region containing the point $\mathbf{q}_{yz}$. Similarly, since $k_1 > 0$ the gradient $\nabla \widetilde{V}$ points to the region containing the point $\mathbf{p}_{yz}$. Therefore the flow in the interior of $\mathcal{T}$ cames from the shadowed region in the Figure 4.3.3 to the region containing the point $(0, 0, 1)$. Since points in $\mathcal{R}_{yz} \setminus s_{yz}$ are not limit set of orbits in the interior of $\mathcal{T}$, see Proposition 4.1.2, we conclude that the α–limit set of any given orbit in the interior of $\mathcal{T}$ is a singular point contained in the segment s_{yz}, see Figure 4.3.3.

As we have just proved when $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$ the ω–limit set and the α–limit set of any given orbit in the interior of $\mathcal{T}$ is contained in the segments s_{px} and s_{yz}, respectively. Similar arguments apply when $\mathbf{k} \in \mathcal{PS}_- \cap \mathcal{S}^-$. In both cases the behavior of the flow at the boundary can be obtained from Lemma 4.2.1(c). This completes the proof of Theorem 4.3.2(a).

Theorem 4.3.2(b) follows by noting that a change of the sign of the parameter $\mathbf{k}$ is equivalent to a change in the sign of time in the differential System (1.3.2). ∎

4.4 Bifurcation analysis

From Theorem 4.3.2 we conclude that the bifurcation taking place at the manifold $\mathcal{S}$ is not only characterized by the behavior of the flow in the interior of $\mathcal{T}$. In addition it must be described by taking into account the changes of the limit sets s_{yz} and s_{px} at the boundary of $\mathcal{T}$.

Hence, when $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$ the orbits in the faces $\mathcal{Y} \cup \mathcal{Z}$ are organized in spirals around the segment s_{yz} moving away from it; and the orbits in the faces $\mathcal{X} \cup \Sigma$ are organized in spirals around the segment s_{px} approaching it. When $\mathbf{k} \in \mathcal{PS} \cap \mathcal{S}$, the segment s_{yz} reduces to the singular point $\mathbf{p}_{yz}$ and the segment s_{px}

reduce to the singular point $\mathbf{p}_{px}$; furthermore the flow in the faces $\mathcal{Y} \cup \mathcal{Z}$ and $\mathcal{X} \cup \Sigma$ describes heteroclinic orbits around them. Finally, when $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^-$ the orbits in $\mathcal{Y} \cup \mathcal{Z}$ are organized in spiral around the segment s_{yz} approaching it; and the orbits in the faces $\mathcal{X} \cup \Sigma$ are organized in spirals around the segment s_{px} moving away from it. The bifurcation set of System (1.3.2) is drawn in Figure 4.4.1. From this we conclude that the bifurcation at the boundary is similar to a focus–center–focus bifurcation.
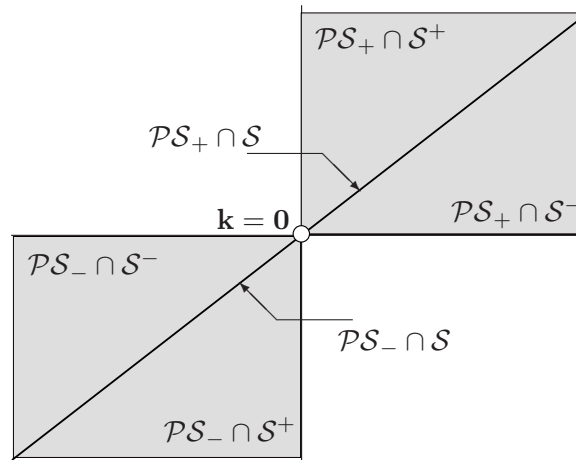

Figure 4.4.1. Representation of the bifurcation set in a two dimensional parameter space. The bifurcation parameter is $\mathbf{k}$.

The bifurcation set represented in Figure 4.4.1 can be used to describe the direction of the flow of the vector field of System (1.3.2). When $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$, the direction of the flow is from segment s_{yz} towards segment s_{px}. Since this case corresponds to $\{k_i > 0\} \cap \{k_1 k_3 - k_2 k_4 > 0\}$, the contraction of the spirals described by the orbits decreases as $k_1 k_3 - k_2 k_4 \to 0$, i.e. by approaching $\mathcal{PS}_+ \cap \mathcal{S}^+$, and the length of segments s_{yz} and s_{px}, decreases. A further change in $\mathbf{k}$ leads to $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$, where all orbits are periodic. This case corresponds to the reduction of segments s_{yz} and s_{px} to a single point, respectively, the intersection point between the straight line $\mathcal{R}$ with edges $\mathcal{R}_{yz}$ and $\mathcal{R}_{px}$, respectively. Crossing the 3D manifold $\mathcal{PS}_+ \cap \mathcal{S}^+$ in the bifurcation set, from $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^+$, to $\mathbf{k} \in \mathcal{PS}_+ \cap \mathcal{S}^-$, means a switch in the flow direction, from segment s_{px} towards segment s_{yz}. Similar changes in the flow behavior of System (1.3.2) occur if smooth changes of $\mathbf{k}$ lead to a change from the set $\mathbf{k} \in \mathcal{PS}_- \cap \mathcal{S}^-$ to the set $\mathbf{k} \in \mathcal{PS}_- \cap \mathcal{S}^+$. From this analysis, we can see that crossing the $\mathcal{PS} \cap \mathcal{S}$ 3D manifold in one sense or another, by a continuous change in the parameter $\mathbf{k}$, gives rise to a focus-center-focus like-bifurcation. It is represented not only by local changes, such as the singular points that form the straight line $\mathcal{R}$, and the periodic orbits, but also by global qualitative changes such as the direction of the flow in the interior of $\mathcal{T}$, the contraction of the spirals or the length of the segments s_{yz} and s_{px}.

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