General coevolution of topology and dynamics in networks

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(Dated: February 18, 2011)

We present a general framework for the study of coevolution in dynamical systems. This phenomenon consists of the coexistence of two dynamical processes on networks of interacting elements: node state change and rewiring of links between nodes. The process of rewiring is described in terms of two basic actions: disconnection and reconnection between nodes, both based on a mechanism of comparison of their states. Different rewiring rules can be expressed in this scheme. We assume that each process, rewiring and node state change, occurs with its own probability, independently from the other. The collective behavior of a coevolutionary system is characterized in the space of parameters given by these two probabilities. As an application, for a voterlike node dynamics we find that reconnections between nodes with similar states lead to network fragmentation. The critical boundaries for the onset of fragmentation in networks with different properties are calculated on this space. We show that coevolution models correspond to curves on this space, describing coupling relations between the probabilities for the two processes. The occurrence of network fragmentation transitions are predicted for diverse models, and agreement is found with some earlier results.

PACS numbers: 89.75.Fb, 87.23.Ge, 05.50.q, 64.60.Ak, 89.75.Hc

Many complex systems observed in nature can be described as dynamical networks of interacting elements or nodes where the connections and the states of the elements evolve simultaneously [1–5]. The links representing the interactions between nodes can change their strengths or appear and disappear as the system evolves on various timescales. In many cases, these modifications in the topology of the network occur as a feedback effect of the dynamics of the states of the nodes: the network changes in response to the evolution of those states which in turn determines the modification of the network. Systems that exhibit this coupling between the topology and states have been denominated as coevolutionary dynamical systems or adaptive networks [1, 3, 4].

Coevolution dynamics has been studied in the context of spatiotemporal dynamical systems, such as neural networks [6, 7], coupled map lattices [8, 9], motile elements [10], as well as in game theory [1, 3], spin dynamics [11], epidemic propagation [12–15], and models of social dynamics and opinion formation [16–21].

In many systems where this type of coevolution dynamics is implemented, a transition is often observed from a phase where most nodes are in the same state forming a large connected network to a phase where the network is fragmented into small disconnected components, each composed by nodes in a common state [22]. This network fragmentation transition is related to the difference in time scales of the processes that govern the two dynamics: the state of the nodes and the network of interactions [18]. In these models, the time scales of the processes of interaction between nodes and modification of their links are coupled and controlled by a single parameter in the system.

The phenomenon of coevolution raises one of the fundamental questions in dynamical networks, namely whether the dynamics of the nodes controls the topology of the network, or this topology controls the dynamics of the nodes. In this paper we propose a general framework to approach this question. We consider that the process by which a node changes its neighbors, called rewiring, takes place with a probability P_r, and that the process by which a node changes its state occurs with a probability P_c. We assume that these two processes that govern the evolution of a dynamical network are independent. As a consequence of this assumption, the collective behavior of the system can be studied on the space of the parameters (P_r, P_c) representing the time scales for both processes. A coevolutionary dynamics can be described by formulating a specific coupling condition or functional relation between the probabilities P_r and P_c of the two competing processes in the network. We shall show that the collective behavior and the existence of a network fragmentation transition for given coevolution functions can be predicted from the general phase diagram of the system on the space of parameters (P_r, P_c).

Each process in a coevolutionary system, rewiring of the network and change of states, may have its own dynamics. Here we focus on the mechanisms for the rewiring process of the coevolution phenomenon. For simplicity, we consider that the number of connections in the network is conserved. Then, we assume that the rewiring process consists of two basic actions: disconnection and reconnection between nodes. Both connecting and disconnecting interactions are often found in social relations, biological systems, and economic dynamics [4, 5, 16, 21].

In general, either action, disconnection or reconnection, is driven by some mechanism of comparison of the states of the nodes. We define a parameter $d \in [0, 1]$ that measures the tendency to disconnect between nodes in identical states; i.e., d represents the probability that two nodes in identical states become disconnected and $1-d$ is the probability that two nodes in different states disconnect from each other. Similarly, we define another parameter $r \in [0,1]$ that describes the probability to connect between nodes in identical states; then, $1-r$ is the probability that two nodes in different states connect to each other. A rewiring process can be characterized by the label dr, where d indicates the probability for the disconnection action between nodes sharing the same state, and r assigns the probability for reconnection between nodes possesing the same state. Thus, we can construct a plane dr where any rewiring process subject to disconnection-reconnection actions between nodes can be represented as a point on this plane.

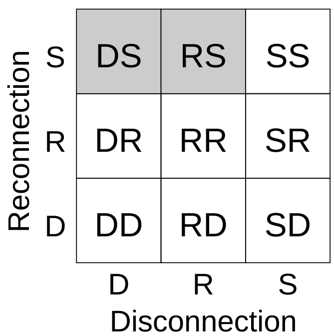

FIG. 1: Discrete rewiring processes on the disconnection-reconnection action space, dr. Either action can occur via three mechanisms: similarity (S), randomness (R), or dissimilarity (D). The two-letter labels describe the resulting rewiring processes. Rewirings that lead to a fragmentation transition in our model are colored in grey.

In a simpler approach, we may consider a discrete expression of the plane dr as follows. We assume that either action of the rewring, disconnection or reconnection, can be driven by three distinct mechanisms: similarity S (interaction between nodes sharing the same state), randomness R (interaction between nodes regardless of their states), and dissimilarity D (interaction between nodes having different states). Then both r and d can only take the values $0(D)$, $0.5(R)$, and $1(S)$. This gives rise to nine possible rewiring processes based on the combinations of these actions and their mechanisms, as shown in Fig. 1. For example, $dr = RS$ denotes a rewiring where node i is disconnected from node j chosen at random and then reconnected to a node m that possesses a state equal to that of i. The RS process corresponds to that assumed in Ref. [16], while the rewiring employed in Refs. [17–20] can be regarded as type DR. Note that only the process RR is completely independent of the states of the nodes.

For the node state dynamics, we choose a simple imitation rule such as a voterlike model that has been used in various contexts [16, 23–26]. The state of node i is denoted by g_i, where g_i can take any of G possible options. Then, consider a random network of N nodes having average degree of edges $\bar{k}$, i.e., $\bar{k}$ is the average number of neighbors of a node. Let ν_i be the set of neighbors of node i, possessing k_i elements. The states g_i are initially assigned at random with a uniform distribution.

Let us assume that the network topology is subject to a rewiring process dr. The coevolution dynamics in this system is defined by iterating the following steps:

1. Chose randomly a node i such that $k_i > 0$.

2. With probability P_r, apply rewiring process dr: break the edge between i and a neighbor $j \in \nu_i$ that satisfies mechanism d, and set a new connection between node i and a node $l \notin \nu_i$ that satisfies mechanism r.

3. Chose randomly a node $m \in \nu_i$ such that $g_i \neq g_m$. With probability P_c, set $g_i = g_m$.

Step 2 describes the rewiring process that allows the acquisition of new connections, while step 3 specifies the process of node state change; in this case the states of the nodes becoming similar as a result of connections. We have verified that the collective behavior of this system is statistically invariant if steps 2 and 3 are interchanged.

In this paper we concentrate on the discrete rewiring processes indicated in Fig. 1. The network size N, the average degree $\bar{k}$, and the number of options G remain constant during the evolution of the system. In our simulations we fixed $N/G = 10$. Thus, the parameters of our model are the probability of rewiring, P_r, and the probability of changing the state of a node, P_c.

The chosen imitation dynamics of the nodes tends to increase the number of connected pairs of nodes with equal states, while some rewiring processes may favor the fragmentation of the network. Therefore, the time evolution of the system should eventually lead to the formation of a set of separate components, or subgraphs, disconnected from each other, with all members of a subgraph sharing the same state. We call *domains* such subgraphs.

To characterize the collective behavior of the system, we employ, as an order parameter, the normalized average size of the largest domain in the system, $S_{\rm m}$. Figure 2 shows $S_{\rm m}$ as a function of the probability P_r for the nine rewiring processes in Fig. 1 on a network having $\bar{k} = 4$, with a fixed value of the probability P_c.

We observe that most rewiring processes in our model lead to collective states characterized by values $S_{\rm m} \to 1$ and corresponding to a large domain whose size is comparable to the system size. However, the rewiring processes DS and RS exhibit a transition at some critical value of P_r, from a regime having a large domain, to a state consisting of only small domains for which $S_{\rm m} \to 0$. Those rewirings dr with $r = S$ can sustain a stable regime consisting of many small domains (SS leaves the initial network structure statistically invariant). The critical point P_r^* for the domain fragmentation transition in each case is estimated by the value of P_r for which the largest fluctuation of the order parameter $S_{\rm m}$ occurs. For the rewiring process RS on a network with

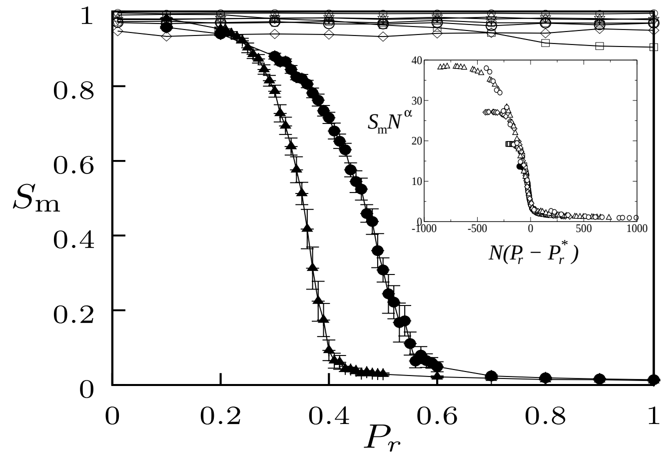

FIG. 2: $S_{\rm m}$ as a function of P_r for the 9 rewiring processes in Fig. 1, with fixed $P_c = 0.6$. Network parameters are $N = 3200$ and $\bar{k} = 4$. Only rewiring processes DS (triangles) and RS (solid circles) exhibit a fragmentation transition. Error bars indicate standard deviations obtained over 100 realizations of initial conditions for each value of P_r. Inset: Scaling collapse found with the exponent $\alpha = 0.5$, for the rewiring process RS with $P_c = 0.6$. Sizes N are 3200 (circles), 1800 (triangles), 800 (diamonds), 400 (squares), 200 (solid circles

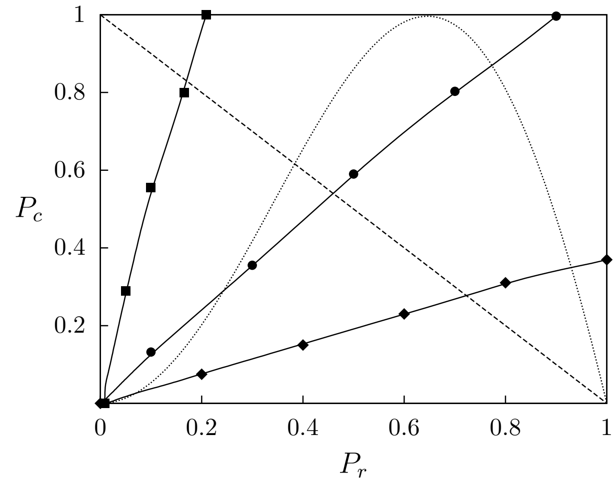

FIG. 3: Critical boundaries on the space of parameters (P_r, P_c) for the fragmentation transition associated to the rewiring process RS, on a network with $\bar{k} = 2$ (line with squares); $\bar{k} = 4$ (circles); $\bar{k} = 8$ (diamonds). Each curve indicates the corresponding boundary that separates the regions where a state having a large domain (above the curve) and a state consisting of many small domains (below the curve) occur. All the numerical data points are averaged over 100 realizations of initial conditions. The slashed line is the relation $P_c = 1 - P_r$, and the dotted line is $P_c = 1.72P_r \sin(\pi P_r)$.

$\bar{k} = 4$, a finite size scaling analysis is shown in the inset in Fig. 2, where $N^{\alpha}S_{\rm m}$ is plotted versus $N(P_r - P_r^*)$, with $P_r^* = 0.541 \pm 0.007$, and for various system sizes. We find that the data collapses in the critical region when $\alpha = 0.50 \pm 0.05$. A similar scaling analysis for the rewiring DS in Fig. 2 yields $P_r^* = 0.380 \pm 0.007$ and $\alpha = 0.20 \pm 0.05$. Thus, there exists a universal scaling function F such that $S_{\rm m} = N^{-\alpha}F(N(P_r - P_r^*))$ associated to each process RS and DS.

For a given rewiring process, the collective behavior of the coevolving system can be characterized in terms of the quantity $S_{\rm m}$ on the space of parameters (P_r, P_c). Figure 3 shows the phase diagrams arising on the plane (P_r, P_c) when the rewiring process RS is employed on networks having different values of $\bar{k}$. For each value of $\bar{k}$, two phases appear in the system as the parameters P_c and P_r are varied: one phase consists of the presence of only small domains and characterized by $S_{\rm m} \to 0$, and the other is distinguished by the formation of a large domain and characterized by larger values of $S_{\rm m}$. These two regimes are separated by a critical curve (P_c^*, P_r^*), as indicated in Fig. 3.

Figure 3 expresses the general phase diagram of a coevolving system subject to a given node state dynamics and a given rewiring process. Diverse coevolution models can be represented in this diagram by formulating specific coupling relations between the rewiring and the node state dynamics. In general, such a coupling can be expressed as a functional relation $P_c(P_r)$ that describes a curve on the space of parameters in Fig. 3. For example, consider the relation $P_c = 1 - P_r$ on the phase diagram in Fig. 3. This corresponds to the coevolution model proposed in Ref. [16]. In this case, the transition from a large domain regime to a fragmented phase on a network characterized by a value of $\bar{k}$ should occur when this straight line intersects the corresponding critical boundary curve in Fig. 3. These intersections yield the values $P_r^* = 0.171$ for $\bar{k} = 2$, $P_r^* = 0.458$ for $\bar{k} = 4$, and $P_r^* = 0.722$ for $\bar{k} = 8$, which agree with the critical values found in [16].

The phase diagram of Fig. 3 predicts the critical values (P_r^*, P_c^*) for the network fragmentation transition in more complicated coevolution models. For example, consider the nonlinear relation $P_c = aP_r \sin(\pi P_r)$ on the space of parameters of Fig. 3. For $a = 1.72$, this function crosses the critical boundary associated to $\bar{k} = 4$ in Fig. 3 twice, at the values $P_r^* = 0.25$, corresponding to a recombination of the network, and $P_r^* = 0.77$, signaling a fragmentation transition. In the range of parameters $P_r \in (0.25, 0.77)$, the function lies within the one-large domain region of the phase diagram. Thus, in a coevolution model described by this function on a network characterized by $\bar{k} = 4$, a regime of one large domain should exist for this range of parameters. For $\bar{k} = 2$, only a fragmented phase occurs for this coevolution function.

Figure 4 shows $S_{\rm m}$ as a function of P_r for the two coevolution models presented in Fig. 3 for a network with $\bar{k} = 4$. For the model in Ref. [16], the fragmentation transition takes place at the value P_r^* predicted from Fig. 3. Similarly, for the nonlinear model we confirm the existence of a one-large domain phase confined in the region

$P_r \in (0.25, 0.77)$.

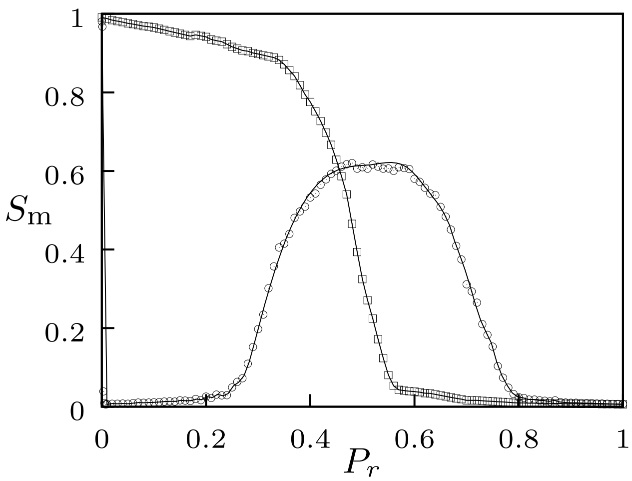

FIG. 4: S_m as a function of P_r for different coevolution curves subject to rewiring process RS, on a network with $\bar{k} = 4$. $P_c = 1 - P_r$ (squares); $P_c = 1.72 P_r \sin(\pi P_r)$ (circles). For each value of P_r, 100 realizations of initial conditions were performed.

In conclusion, we have presented a general framework for the study of the phenomenon of coevolution in dynamical networks. Coevolution consists of the coexistence of two processes, node state change and rewiring of links between nodes, that can occur with independent probabilities P_r and P_c, respectively. We have focused on the process of rewiring, which we have described in terms of the actions of disconnection and reconnection between nodes, both based on a mechanism of comparison of their states. For a voterlike node dynamics, we found that only reconnections between nodes with similar states can lead to network fragmentation.

The collective behavior of a coevolving system can be represented in the space of parameters (P_r, P_c). We have calculated the critical boundaries on this space for the fragmentation transition in networks having different values of $\bar{k}$. The size of the region for the fragmented phase in the space (P_r, P_c) decreases with increasing $\bar{k}$. This suggests that fragmentation is more likely to be observed in networks where $\bar{k} \ll N$. We have shown that coevolution models correspond to curves $P_c(P_r)$ on the plane (P_r, P_c). The occurrence of network fragmentation as well as recombination transitions for diverse models can be predicted in this framework.

We have limited our study to the case when then number of connections in the coevolving network is conserved. This condition is expressed in step 2 of the algorithm, where both actions of disconnection and reconnection occur with probability one. This condition can be generalized by considering different probabilities for each of these actions. Thus, our framework provides an scenario for studying coevolving dynamical networks with no conservation of the total number of links.

Other extensions to be investigated in the future include the characterization of the emergent topological properties of the network on the continuous plane dr, the consequences of preferential attachment rules for the reconnection action, the consideration of variable connection strengths, and the influence of the node dynamics on the collective behavior of coevolving systems.

This work was supported by project C-1692-10-05-B from CDCHTA, Universidad de Los Andes, Venezuela. J.C.G-A was supported by project FISICOS FIS2007-60327 from M.E.C., Spain.

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