Equilibrium States on Toeplitz Algebras

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Abstract

This thesis describes the equilibrium states (the KMS states) of dynamical systems arising from local homeomorphisms. It has two main components.

First, we consider a local homeomorphism on a compact space and the associated Hilbert bimodule. This Hilbert bimodule has both a Toeplitz algebra and a Cuntz-Pimsner algebra, which is a quotient of the Toeplitz algebra. Both algebras carry natural gauge actions of the circle, and hence one can obtain natural dynamics by lifting these actions to actions of the real numbers. We study KMS states of these dynamics at, above, and below a certain critical value. For inverse temperature larger than the critical value, we find a large simplex of KMS states on the Toeplitz algebra. For the Cuntz-Pimsner algebra the KMS states all have inverse temperatures below the critical value. Our results for the Cuntz-Pimsner algebra overlap with recent work of Thomsen, but our proofs are quite different. At the critical value, we build a KMS state of the Toeplitz algebra which factors through the Cuntz-Pimsner algebra.

To understand KMS states below the critical value, we study the backward shift on the infinite path space of an ordinary directed graph. Merging our results for the Cuntz-Pimsner algebra of shifts with the recent work about KMS states of the graph algebras, we show that Thomsen's bounds on of the possible inverse temperature of KMS states are sharp.

In the second component, we consider a family of $*$-commuting local homeomorphisms on a compact space, and build a compactly aligned product system of Hilbert bimodules (in the sense of Fowler). This product system also has two interesting algebras, the Nica-Toeplitz algebra and the Cuntz-Pimsner algebra. For these algebras the gauge action is an action of

a higher-dimensional torus, and there are many possible dynamics obtained by composing with different embeddings of the real line in the torus.

We use the techniques from the first component of the thesis to study the KMS states for these dynamics. For large inverse temperature, we describe the simplex of the KMS states on the Nica-Toeplitz algebra. To study KMS states for smaller inverse temperature, we consider a preferred dynamics for which there is a single critical inverse temperature, which we can normalise to be 1. We then find a KMS_1 state for the Nica-Toeplitz algebra which factors through the Cuntz-Pimsner algebra. We then illustrate our results by considering different backward shifts on the infinite path space of some higher-rank graphs.

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Contents

Introduction

Given an action α of the real line $\mathbb{R}$ by automorphisms of a C^*-algebra A, the C^*-dynamical system $(A, \mathbb{R}, \alpha)$ provides an algebraic model for studying a physical system in quantum statistical physics [5]. In this framework, the observables are the self-adjoint elements of the C^*-algebra A, the states are positive linear functionals on A with norm 1, and the time evolution is given by the action α. Work of Kubo, Martin and Schwinger shows that equilibrium states of the physical system are exactly those states on A which satisfy a certain commutation relation (the so called KMS condition). This relation involves a real number β, which is interpreted as the inverse temperature of the physical system.

The KMS condition makes sense for abstract dynamical systems and operator algebraists study KMS states of dynamical systems regardless of applications in physics. Many authors have studied KMS states in different contexts. For example: in systems constructed from number theory [4, 32, 33, 34], in systems associated to graph algebras [12, 15, 28, 29], in systems arising from groupoids [31, 39], and in topological systems built from local homeomorphisms [56, 57].

In most of the contexts mentioned above, there are two main C^*-algebras: a Cuntz-Pimsner type algebra and its Toeplitz extension. There has been profound progress in characterising KMS states of Cuntz-Pimsner algebras in the literature [12, 13, 43, 56], and interesting work of Exel, Laca and Neshveyev [15, 34] shows that Toeplitz algebras are expected to have a much greater supply of KMS states.

This thesis focuses on characterising KMS states on Toeplitz algebras associated to local homeomorphisms. It is organised in two main parts. The first part is allocated to dynamical systems arising from a single local homeomorphism and their KMS states. The result of this part is published in [1] and here we provided it as an Appendix chapter (see Appendix A). In the second part, we study KMS states of dynamical systems associated to a family of local homeomorphisms in the context of product systems of Hilbert bimodules. This part occupies the main body of this thesis.

The notion of a product system was initially introduced by Arveson as a continuous product system of Hilbert spaces [2]. Then several authors generalised this to discrete product systems in [11, 20, 22]. We follow Fowler's extension [20] which is about discrete product systems of Hilbert bimodules over semigroups [20]. Roughly speaking, for a semigroup P with identity e, a product system of Hilbert bimodules over P is a semigroup $X = \bigsqcup_{p\in P} X_p$ such that each X_p is a right Hilbert bimodule and $x \otimes y \mapsto xy$ implements an isomorphism from $X_p \otimes X_q$ onto X_{pq} for all $p, q \in P \setminus \{e\}$.

For such a product system X, Fowler defined Toeplitz representations of X as multiplicative maps whose restriction on each fibre X_p is a Toeplitz representation in the sense of [21]. Then he associated the Toeplitz algebra $\mathcal{T}(X)$ as the universal algebra for Toeplitz representations of X. He defined the Cuntz-Pimsner algebra $\mathcal{O}(X)$ as a quotient of $\mathcal{T}(X)$. When (G, P) is a quasi-lattice ordered group in the sense of Nica [40], he imposed a covariance condition (Nica-covariance) on Toeplitz representations, and defined the Nica-Toeplitz algebra $\mathcal{NT}(X)$[1] as the universal algebra for Nica-covariant Toeplitz representations. He noticed that $\mathcal{NT}(X)$ is only tractable for certain class of product systems called compactly aligned product systems. For such a product system, he showed that

$$\mathcal{NT}(X) = \overline{\operatorname{span}}\{\psi_p(x)\psi_q(y)^* : p, q \in P, x \in X_p, y \in X_q, \} \tag{1}$$

where ψ is the universal Nica-covariant representation.

Viewing $\mathbb{N}^k$ as an additive semigroup, there are many interesting examples for the product systems over $\mathbb{N}^k$ in the literature. For these examples, by universal properties of $\mathcal{NT}(X)$, and $\mathcal{O}(X)$, respectively we can get strongly continuous gauge actions of k-torus $\mathbb{T}^k$ on these algebras. Then we can lift these actions to the actions of the real line via the embedding $t \mapsto e^{itr} = (e^{itr_1}, e^{itr_2}, \ldots, e^{itr_k})$ for some $r \in (0, \infty)^k$.

Well known examples of product systems over $\mathbb{N}^k$ are the ones constructed from the higher-rank graph of Kumjian-Pask [30]. It is observed in [22, page 1492] that we can view a k-graph Λ as a product system over $\mathbb{N}^k$. Soon after Sims and Raeburn showed that by putting particular combinatorial condition on the underlying higher-ranks graph we can get a compactly aligned product system over the quasi-lattice ordered group $(\mathbb{Z}^k, \mathbb{N}^k)$ [45]. They imposed a Nica-covariance condition by adding an extra relation to the usual Cuntz-Krieger relations. They called the associated Nica-Toeplitz algebras the Cuntz-Krieger-Toeplitz algebra $\mathcal{T}C^*(\Lambda)$. The Cuntz-Krieger $C^*(\Lambda)$ can be viewed as a quotient of $\mathcal{T}C^*(\Lambda)$. Thus the C^*-algebras of higher-rank

[1] In Fowler's paper the Nica-Toeplitz algebra is denoted by $\mathcal{T}_{\mathrm{cov}}(X)$.

graphs and their KMS states can be a rich supply of test examples for analysing KMS states of product systems. In particular there has been recently great progress in analysing the KMS structure of these dynamics (for example [26, 28]).

There are also intriguing examples for Nica-Toeplitz algebras in number theory, for example, the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ studied by Laca and Raeburn in [35]. It is observed in [7] that $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ and the associated additive and multiplicative quotients are all Nica-Toeplitz algebras. Then the KMS structure of these algebras is analysed by applying the technique developed in [35]. Following the same approach, Hong, Larsen and Szymański characterized the KMS structure of a product system over a general semigroup [24]. But the authors of [24] used the strong condition "finite type product system" in their hypothesis. This condition requires the existence of a finite orthonormal basis for all fibres in the product system.

In [53, 54], Solel used different notation to study the product systems over $\mathbb{N}^k$. He used the term "c.c. (completely contractive covariant) representation" for Fowler's Toeplitz representation (see [54, Defnition 2.3, Definition 3.1]) and defined the "doubly commuting relation" ([54, Defnition 3.8])). He showed in [54, Lemma 3.11] that this relation is equivalent to Fowler's Nica-covariance relation and that the universal Nica-covariant representation ψ satisfies his doubly commuting relation.

Here we are interested in the dynamical systems arising from local homeomorphisms. We first show that a family of surjective and commuting local homeomorphisms $h_1, \ldots, h_k$ on a compact Hausdorff space Z induces a compactly aligned product system X over $\mathbb{N}^k$ (see Chapter 2). Letting $h^m := h_1^{m_1} \circ \cdots \circ h_k^{m_k}$, each fibre X_m in this product system is the graph correspondence associated to the topological graph (Z, Z, id, h^m). We know very well from our work in [1] what each fibre looks like. So we think about generalizing the results of [1] from one Hilbert bimodule to a product system of Hilbert bimodules.

Our approach is inspired by [28] which is again a refinement of original technique introduced in [34]. So we first look for a characterization of KMS states of $\mathcal{NT}(X)$ which makes it easier to recognise the KMS states. To do this, having looked at similar results in the literature (for example [28, Proposition 3.1] and [24, Theorem 4.6]), we noticed that it is crucial to express elements of the form $\psi_n(y)^*\psi_m(x)$ in terms of usual spanning elements $\psi_p(s)\psi_q(t)^*$ in the algebra $\mathcal{NT}(X)$. For a general product system over a semigroup, Fowler provided an approximation [20, Proposition 5.10], but this is not enough because we need an exact formula; in the dynamics associated to a higher-rank graph [28] this formula already exists as one of the Toeplitz-Cuntz-Krieger relations; in [24], since each fibre in the product system has an orthonormal basis, it is easier to find such a formula (see [24, Lemma 4.7]).

To solve this problem, we impose an extra hypothesis of $*$-commutativity on the local homeomorphisms. Two maps $f, g : Z \to Z$, $*$-commute if for every $z, z' \in Z$ such that $f(z) = g(z')$, there exists unique $z'' \in Z$ such that $z = g(z'')$ and $z' = f(z'')$ (see [3]). Recently, there have been great interest in studying C^*-algebras of $*$-commuting maps and associated dynamics [16, 37, 55].

The $*$-commutativity hypothesis allows us to find Parseval frames for each fibre. Given $m \in \mathbb{N}^k$, since the fibre X_m is the graph correspondence associated to the local homeomorphisms h^m, there is a well-known Parseval frame $\{\tau_i\}_{i=0}^d$ for X_m which comes from a partition of unity ([17, Proposition 8.2]). We observed that for $n \in \mathbb{N}^k$ with $m \wedge n = 0$, the composition of elements of this Parseval frame with h^n form another Parseval frame for X_m. Then we prove

$$\psi_n(y)^*\psi_m(x) = \sum_{0 \le i,j \le d} \psi_m\big(\langle y, \tau_j \circ h^m\rangle \cdot \tau_i\big)\psi_n\big(\langle x, \tau_i \circ h^n\rangle \cdot \tau_j\big)^*, \tag{2}$$

getting the formula we need. This formula is for fibres X_m and X_n with $m \wedge n = 0$. However by using proper isomorphisms between fibres we can apply (2) and rewrite $\psi_n(y)^*\psi_m(x)$ in terms of elements of the for $\psi_p(s)\psi_q(t)^*$ for general $m, n \in \mathbb{N}^k$. Then we use the formula (2) and provide a characterization of KMS states in Proposition 3.1.6.

In fact the equation (2) is a translation of Solel's doubly commuting relation from his notation to Fowler's notation. The difficulty of this translation is that the doubly commuting relation contains a flip map between fibres. Notice that the existence of such a flip map is a consequence of definition of the product system. Solel used the doubly commuting relation in his approach without any explicit formula for the flip map. We find a nice formula for this flip map in Lemma 3.1.1(c) and therefore we can translate the doubly commuting relation to get (2) (see Appendix A).

Let Λ be a k-graph and $A_i (1 \le i \le k)$ be the associated vertex matrices. The vectors that are subinvariant for all A_i in the sense of Perron-Frobenius theory [50], play a very important role in analysing KMS states of $\mathcal{T}C^*(\Lambda)$. For dynamics determined by $r \in (0, \infty)^k$, we follow the same idea and define a subinvariance relation using a family of Ruelle operators. When β is large enough, that is $\beta > \beta_c$ for

$$\beta_c := \max_i \{r_j^{-1}\beta_{c_i}\} \text{ and } \beta_{c_i} := \limsup_{j\to\infty} \big(j^{-1} \ln \big(\max_{z\in Z} |h_i^{-j}(z)|\big)\big),$$

we describe all solutions of our subinvariance relation in Proposition 3.2.7. If in addition r has rationally independent coordinates, we show that there is a bijection between the simplex of KMS_β states on $\mathcal{NT}(X)$ and the probability measures satisfying our subinvariance relation (Theorem 3.3.1). A rational independency condition on r is crucial when we prove the surjectivity of our isomorphism in Theorem 3.3.1. So whenever we need to get a probability measure (satisfying the subinvariance relation) from a KMS sates we have to impose this hypothesis.

To study KMS states for smaller β, in order to have satisfactory results, we pay careful attention in choosing $r \in (0, \infty)^k$. Following recent conventions in graph algebras [26, 28, 59, 60], we consider a preferred dynamics where $r := (\beta_{c_1}, \dots, \beta_{c_k})$. Notice that in this case $\beta_c = 1$. We call $\beta_c = 1$ the critical inverse temperature. At the critical inverse temperature, we show that by taking limits of KMS_{β_j} states as the β_j decrease to 1, there is a KMS_1 state on $\mathcal{NT}(X)$, and at least one such a state factors through $\mathcal{O}(X)$ (Theorem 3.4.1).

Finally, we provide an example of $*$-commuting maps. Let Λ be a 1-coaligned k-graph in the sense that for each pair of paths (μ, ν) with the same source there is a unique pair of paths (ξ, η) such that $\xi\mu = \eta\nu$. It is observed in [37, Theorem 2.3] that the shift maps on the infinite path space of Λ $*$-commute. Now writing $X(\Lambda^\infty)$ for the associated product system, we apply our result in the previous chapters to study the KMS structure of the associated Nica-Toeplitz algebra $\mathcal{NT}(X(\Lambda^\infty))$ and the Cuntz-Pimsner algebra $\mathcal{O}(X(\Lambda^\infty))$. We first prove that, as we expect from our results for a 1-graph, the Cuntz-Pimsner algebra $\mathcal{O}(X(\Lambda^\infty))$ is isomorphic to the Cuntz-Krieger algebra $C^*(\Lambda)$. We also prove that the Nica-Toeplitz algebra $\mathcal{NT}(X(\Lambda^\infty))$ contains an injective copy of $\mathcal{T}C^*(\Lambda)$ (Proposition 4.2.7). Furthermore, we prove that every KMS state of $\mathcal{T}C^*(\Lambda)$ is the restriction of a KMS state of $\mathcal{NT}(X(\Lambda^\infty))$ (Proposition 4.3.3).

Thesis outline

This thesis is broken up to 4 chapters and 2 appendices:

In Chapter 1, we provide an overview of product systems of Hilbert bimodules and the associated dynamical systems. We present the basic definitions and notation and discuss the properties of these dynamical systems in details. In Chapter 2, we show that a family of commuting and surjective local homeomorphisms gives a compactly aligned product system of Hilbert bimodules. Chapter 3 allocated to characterising KMS states and ground states of dynamical systems arising from a family of $*$-commuting and surjective local homeomorphisms. In Chapter 4, we discuss the shifts on the infinite path space of 1-coaligned higer-rank graphs. We show the relationships between the

KMS states of graph algebras and the KMS states of the C^*-algebras of the shifts.

In appendix A, we reconcile our results with those of Solel's. We show that for the dynamical system considered in Chapter 3, the universal Nica-covariant representation satisfies Solel's doubly commuting relation. Finally, we attach our published paper [1] as Appendix B. This appendix presents our results about the KMS states of dynamical systems associated to a single local homeomorphism.

Chapter 1

Preliminaries

1.1 Hilbert bimodules

The following definitions are taken from chapter 2 of [46].

Given a complex vector space X and a C^*-algebra A, by a *right action of A on X* we mean a pairing $(x, a) \mapsto x \cdot a : X \times A \to X$ satisfying the consistency conditions: $(x + x') \cdot a = x \cdot a + x' \cdot a$; $x \cdot (aa') = (x \cdot a) \cdot a'$ and $\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)$ for all $\lambda \in \mathbb{C}$, $x, x' \in X$ and $a, a' \in A$.

Definition 1.1.1. Let A be a C^*-algebra and X be a complex vector space with a right action of A on X. A *right A-valued inner product on X* is a function $\langle \cdot, \cdot \rangle_A : X \times X \to A$ which is linear in the second variable and satisfies:

(a) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$,

(b) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$,

(c) $\langle x, y \rangle_A$ is a positive element of A, and

(d) $\langle x, x \rangle_A = 0$ implies that $x = 0$.

We may write $\langle x, y \rangle$ for $\langle x, y \rangle_A$ if it is clear from the context which C^*-algebra A is meant.

Remark 1.1.2. Since $\langle \cdot, \cdot \rangle_A$ is linear in second variable, we deduce that $x = 0$ implies $\langle x, x \rangle_A = 0$. It also follows from condition (b) that $\langle \cdot, \cdot \rangle_A$ is conjugate linear in the first variable.

It follows from [46, Corollary 2.7] that the formula $\|x\|_A := \|\langle x, x\rangle_A\|^{\frac{1}{2}}$ defines a norm on X. If X is complete in this norm we call it a *right Hilbert A-module*.

Suppose X is a right Hilbert A-module. An operator $T : X \to X$ is *adjointable*, if there is an operator $T^* : X \to X$ such that $\big\langle T(x), y\big\rangle_A = \big\langle x, T^*(y)\big\rangle_A$ for all $x, y \in X$. We denote by $\mathcal{L}(X)$ the set of all adjointable operators on X.

It follows from [46, Lemma 2.18] that every adjointable operator T on a right Hilbert A-module X is a linear bounded operator. [46, Proposition 2.21] says that the adjoint T^* is unique and the set $\mathcal{L}(X)$ is a C^*-algebra with respect to the operator norm, and with the involution given by $T \mapsto T^*$.

Given $x, y \in X$, we define $\Theta_{x,y} : X \to X$ by

$$\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_A.$$

Then $\Theta_{x,y}$ is adjointable and $\Theta^*_{x,y} = \Theta_{y,x}$ (see [46, page 18]). The set

$$\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X\}.$$

is a C^*-algebra and we call it the *algebra of compact operators on X*.

Definition 1.1.3. Let A be a C^*-algebra. A *right Hilbert A–A bimodule X* (or a *correspondence over A*) is a right Hilbert A-module X together with a homomorphism $\varphi : A \to \mathcal{L}(X)$. We view φ as implementing a left action of A on X and we usually write $a \cdot x$ for $\varphi(a)(x)$. We say X is *essential* if $X = \overline{\operatorname{span}}\{\varphi(a)x : a \in A, x \in X\}$.

Remark 1.1.4. Since $\varphi(a) \in \mathcal{L}(X)$ for all $a \in A$, it follows that $\langle a\cdot x, y\rangle_A = \langle x, a^*\cdot y\rangle_A$. Now let $x, y \in X$ and $a, a' \in A$. The statements (b) and (c) of Definition 1.1.1 imply that

$$\begin{aligned}\langle a \cdot (x \cdot a'), y\rangle_A = \langle x \cdot a', a^* \cdot y\rangle_A = \langle a^* \cdot y, x \cdot a'\rangle_A^* = \big(\langle a^* \cdot y, x\rangle_A a'\big)^*\\ = \big(\langle y, a \cdot x\rangle_A a'\big)^* = \langle y, (a \cdot x) \cdot a'\rangle_A^* = \langle (a \cdot x) \cdot a', y\rangle_A.\end{aligned}$$

Thus $a \cdot (x \cdot a') = (a \cdot x) \cdot a'$ and the actions of A on X are compatible.

Example 1.1.5. Let A be a C^*-algebra. The multiplication in A gives a right action of A on itself. The formula $\langle a, a'\rangle_A = a^*a'$ defines a right A-valued inner product on A. To see this, first note that it is linear in the second variable. 2nd, conditions (a)$-$(c) of Definition 1.1.1 are immediate. Third, to check (d), let $\langle a, a'\rangle_A = a^*a = 0$. It follows that $\|aa^*\| = \|a\|^2 = 0$. This implies $a = 0$. Thus $\langle a, a'\rangle_A = a^*a'$ is a right A-valued inner product on A. Since $\|a\|_A = \|a\|$, A is complete in the norm $\|\cdot\|_A$ and therefore is a Hilbert A-module. Next define $\varphi : A \to \mathcal{L}(A)$ by $\varphi(a)(a') = aa'$. To see that φ is adjointable, observe that

$$\langle \varphi(a)(a), a'' \rangle_A = (aa')^* a'' = a'^*(a^* a'') = \langle a', a^* a'' \rangle_A = \langle a', \varphi(a^*)(a'') \rangle_A.$$

Thus $\varphi(a)$ is adjointable and $\varphi(a)^* = \varphi(a^*)$. Clearly φ is a homomorphism. Thus A is a right Hilbert A–A bimodule which we call the *standard bimodule* and denote by ${}_AA_A$.

Example 1.1.6. Let A be a unital C^*-algebra with identity I_A and suppose X is a right Hilbert A–A bimodule. If $\varphi(\mathrm{I}_A)x = x$ for all $x \in X$, then X is essential.

Definition 1.1.7. Let A be a C^*-algebra and X be a right Hilbert A–A bimodule. A *representation* (ψ, π) of X in a C^*-algebra B consists of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

$$\psi(a \cdot x \cdot b) = \pi(a)\psi(x)\pi(b) \text{ and } \pi(\langle x, y \rangle_A) = \psi(x)^*\psi(y).$$

for every $x, y \in X$ and $a, b \in A$.

Remark 1.1.8. A representation (ψ, π) induces a homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}(X) \to \mathcal{T}(X)$ such that $(\psi, \pi)^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$ (see page 202 of [42]).

Definition 1.1.9. Suppose X is a right Hilbert A–A bimodule. Following [17, 23], we refer to a sequence $\{x_i\}_{i=0}^d$ in X such that

$$\sum_{i=0}^{d} x_i \cdot \langle x_i, x \rangle_A = x \quad \text{for all } x \in X. \tag{1.1}$$

as a *finite Parseval frame for* X. The formula (1.1) is known as *the reconstruction formula.*

1.2 Internal tensor products of Hilbert bimodules

In this section, we show how we can define the internal tensor product $X \otimes_A Y$ for right Hilbert A–A bimodules X, Y. We also show that $X \otimes_A Y$ has a right Hilbert A–A bimodule structure.

We write $X \odot Y$ for the algebraic tensor product of X and Y. We use $X \odot_A Y$ for the quotient of $X \odot Y$ by the subspace

$$N := \overline{\operatorname{span}}\{(x \cdot a) \odot y - x \odot (a \cdot y) : x \in X, y \in Y, a \in A.\} \tag{1.2}$$

To avoid possible confusion, we temporary write $x \odot y$ for the elements of $X \odot Y$ and $x \odot_A y$ for the elements $X \odot_A Y$. Then by definition each $x \odot_A y$ has the form $x \odot y + N$.

Lemma 1.2.1. *Let A be a C^*-algebra and let X, Y be two right Hilbert A–A bimodules. Then there is a well defined right action $(x \odot_A y, a) \mapsto (x \odot_A y) \cdot a : (X \odot_A Y) \times A \to X \odot_A Y$ such that*

$$(x \odot_A y) \cdot a = x \odot_A y \cdot a \quad \textit{for all } x \odot_A y \in X \odot_A Y,\ a \in A.$$

Proof. Fix $a \in A$. The map $(x, y) \mapsto x \odot_A y \cdot a$ is a bilinear map from $X \times Y$ into $X \odot_A Y$. Then the universal property of $X \odot Y$ gives us a linear map $L_a : X \odot Y \to X \odot_A Y$ satisfying $L_a(x \odot y) = x \odot y \cdot a$. Since L_a vanishes on N, it induces a linear map $\tilde{L}_a : X \odot_A Y \to X \odot_A Y$ such that $\tilde{L}_a(x \odot_A y) = x \odot_A y \cdot a$. Now $\big(x \odot_A y, a\big) \mapsto \tilde{L}_a(x \odot_A y)$ is a well defined map from $(X \odot_A Y) \times A$ into $X \odot_A Y$. Write $(x \odot_A y) \cdot a := \tilde{L}_a(x \odot_A y)$. To see that this map is a right action, let $x \odot_A y, x' \odot_A y' \in X \odot_A Y$ and $a, a' \in A$. Since $\tilde{L}_a$ is linear, it follows that

$$\begin{aligned}(x \odot y + x' \odot y') \cdot a = L_a(x \odot y + x' \odot y') &= \tilde{L}_a(x \odot y) + \tilde{L}_a(x' \odot y') \\ &= (x \odot y) \cdot a + (x' \odot y') \cdot a.\end{aligned}$$

We also have

$$(\lambda(x \odot y)) \cdot a = \tilde{L}_a(\lambda(x \odot y)) = \lambda \tilde{L}_a(x \odot y) = \lambda((x \odot y) \cdot a).$$

A similar calculation shows $(x \odot y) \cdot (\lambda a) = (\lambda(x \odot y)) \cdot a$.

Finally, we have

$$\begin{aligned}(x \odot y) \cdot (aa') = \tilde{L}_{aa'}(x \odot y) &= x \odot y \cdot (aa') = x \odot (y \cdot a) \cdot a' \\ &= \tilde{L}_{a'}(\tilde{L}_a(x \odot y)) = ((x \odot y) \cdot a)) \cdot a',\end{aligned}$$

as required. □

The next lemma shows that we can equip the space $X \odot_A Y$ with a right A-valued inner product.

Proposition 1.2.2 ([36, Proposition 4.5])**.** *Let A be a C^*-algebra and let X, Y be two right Hilbert A–A bimodules. Suppose that $\varphi_Y : A \to \mathcal{L}(Y)$ is the homomorphism which defines the left action of A on Y. Then there is a unique right A-valued inner product on $X \odot_A Y$ such that*

$$\big\langle x \odot_A y, z \odot_A w \big\rangle = \big\langle y, \varphi_Y\big(\langle x, z\rangle\big) w \big\rangle \quad \textit{for } x \odot_A y, z \odot_A w \in X \odot_A Y. \tag{1.3}$$

Let $X \otimes_A Y$ be the completion of $X \odot_A Y$ with respect to the inner product (1.3). It then follows from [46, Lemma 2.16] that (1.3) is a right A-valued inner product on $X \otimes_A Y$ as well. Thus $X \otimes_A Y$ is a right Hilbert A-module.

The next lemma shows that we can define a left action of A on $X \otimes_A Y$.

Proposition 1.2.3 ([58, Proposition I.1]). *Let A be a C^*-algebra and let X, Y be two right Hilbert A–A bimodules. Suppose that $\varphi_Y : A \to \mathcal{L}(Y)$ is the homomorphism which defines the left action of A on Y. Then for every $S \in \mathcal{L}(X)$, there is a unique operator $S \otimes 1_Y \in \mathcal{L}(X \otimes_A Y)$ such that*

$$S \otimes 1_Y(x \otimes y) = Sx \otimes y \qquad \text{for } x \otimes y \in X \otimes_A Y. \tag{1.4}$$

The map $S \to S \otimes 1_Y$ is a homomorphism of $\mathcal{L}(X)$ into $\mathcal{L}(X \otimes_A Y)$. In particular the map $a \mapsto \varphi_X(a) \otimes 1_Y$ determines a homomorphism of A into $\mathcal{L}(X \otimes_A Y)$.

We can view the homomorphism $a \mapsto \varphi_X(a) \otimes 1_Y$ as a left action of A on $X \otimes_A Y$. Thus $X \otimes_A Y$ is a right Hilbert A–A bimodule. We call $X \otimes_A Y$ the *balanced tensor product* of right Hilbert A–A bimodules X, Y.

For convenience, in the rest of thesis we keep $x \odot y$ for the elements of $X \odot Y$ and we write $x \otimes y$ for the elements of both $X \odot_A Y$ and $X \otimes_A Y$.

1.2.1 Product systems of Hilbert bimodules

We use conventions of [20] for the basics of product systems of Hilbert bimodules. For convenience, we use the following equivalent formulation from ([52, page 6]).

Definition 1.2.4. Suppose P is a multiplicative semigroup with identity e, and let A be a C^*-algebra. For each $p \in P$ let X_p be a right Hilbert A–A bimodule and suppose that $\varphi_p : A \to \mathcal{L}(X_p)$ is the homomorphism which defines the left action of A on X_p. A *product system over P of right Hilbert A–A bimodules* (or a *product system over P with fibres X_p*) is the disjoint union $X := \bigsqcup_{p \in P} X_p$ such that:

(P1) The identity fibre X_e equals the standard bimodule ${}_A A_A$.

(P2) X is a semigroup and for each $p, q \in P \backslash \{e\}$ the map $(x, y) \mapsto xy : X_p \times X_q \to X_{pq}$, extends to an isomorphism $\sigma_{p,q} : X_p \otimes_A X_q \to X_{pq}$.

(P3) The multiplications $X_e \times X_p \to X_p$ and $X_p \times X_e \to X_p$ satisfy

$$ax = \varphi_p(a)z, \quad xa = x \cdot a \ \text{ for } a \in X_e \text{ and } x \in X_p.$$

If each fibre X_p is essential, then we call X a *product system over P of essential right Hilbert A–A bimodules.*

Let $p, q \in P \setminus \{e\}$ and $S \in \mathcal{L}(X_p)$. Then the isomorphism $\sigma_{p,q} : X_p \otimes_A X_q \to X_{pq}$ together with Proposition 1.2.3 give us a homomorphism $\iota_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ defined by

$$\iota_p^{pq}(S) = \sigma_{p,q} \circ (S \otimes 1_{X_q}) \circ \sigma_{p,q}^{-1}.$$

Definition 1.2.5. Suppose P is a subsemigroup of a group G such that $P \cap P^{-1} = \{e\}$. Then $p \leq q \Leftrightarrow p^{-1}q \in P$ defines a partial order on G. Following [40], we say (G, P) is a *quasi-lattice ordered group* if for any two elements $p, q \in G$ which have a common upper bound in P there is a least upper bound $p \vee q \in P$. We write $p \vee q = \infty$ when $p, q \in G$ have no common upper bound.

Example 1.2.6. $(\mathbb{Z}^k, \mathbb{N}^k)$ is a quasi-lattice ordered group. Observe that for all $m, n \in \mathbb{N}^k$, there is a least upper bound $m \vee n$ with ith coordinate $(m \vee n)_i := \max\{m_i, n_i\}$.

Definition 1.2.7. Let (G, P) be a quasi-lattice ordered group. A product system over P of right Hilbert A–A bimodules is *compactly aligned*, if for all $p, q \in P$ with $p \vee q < \infty$, $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$.

Proposition 1.2.8 ([20, Proposition 5.8])**.** *Let (G, P) be a quasi-lattice ordered group and suppose that X is a compactly aligned product system over P of right Hilbert A–A bimodules. Suppose that the left action of A on each fibre X_p is by compact operators. Then X is compactly aligned.*

1.3 C^*-algebras associated to product systems of Hilbert bimodules

Definition 1.3.1. Let P be a multiplicative semigroup with identity e, and let X be a product system over P of right Hilbert A–A bimodules. Let B be a C^*-algebra, and let ψ be a function from X to B. Write ψ_p for the restriction of ψ to X_p. We call ψ a *Toeplitz representation* of X if:

(T1) For each $p \in P \setminus \{e\}$, $\psi_p : X_p \to B$ is linear, and $\psi_e : A \to B$ is a homomorphism,

(T2) $\psi_p(x)^*\psi_p(y) = \psi_e(\langle x, y\rangle)$ for $p \in P$, and $x, y \in X_p$,

(T3) $\psi_{pq}(xy) = \psi_p(x)\psi_q(y)$ for $p, q \in P$, $x \in X_p$, and $y \in X_q$.

Remark 1.3.2. Conditions (T1) and (T2) imply that (ψ_p, ψ_e) is a Toeplitz representation for the fibre X_p (which is a right Hilbert A–A bimodule). Then Remark 1.1.8 gives us a homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to B$ such that $\psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^*$.

Fowler showed in [20, Proposition 2.8] that there exists a C^*-algebra $\mathcal{T}(X)$ and a Toeplitz representation ω of X in $\mathcal{T}(X)$ such that:

(U1) For any other Toeplitz representation T of X in a C^*-algebra B, there exists a unique homomorphism $T_* : \mathcal{T}(X) \to B$ such that $T_* \circ \omega = T$, and

(U2) $\mathcal{T}(X)$ is generated by $\{\omega(x) : x \in X\}$.

It then follows that the pair $(\mathcal{T}(X), \omega)$ is unique up to canonical isomorphism. We say the pair $(\mathcal{T}(X), \omega)$ is universal for the Toeplitz representations. The C^*-algebra $\mathcal{T}(X)$, is called *the Toeplitz algebra of* X and the representation ω is known as *the universal Toeplitz representation of* X. We keep ω for the universal Toeplitz representation of X.

Definition 1.3.3. Let P be a semigroup with identity e, and let X be a product system over P of right Hilbert A–A bimodules. A Toeplitz representation ψ of X is *Cuntz-Pimsner-covariant* if

$$\psi_e(a) = \psi^{(p)}(\varphi_p(a)) \quad \text{for all } p \in P, a \in \varphi_p^{-1}(\mathcal{K}(X_p)). \tag{1.5}$$

The *Cuntz-Pimsner algebra* $\mathcal{O}(X)$ is the quotient of $\mathcal{T}(X)$ by the ideal

$$\Big\{\omega(a) - \omega^{(p)}(\varphi_p(a)) : p \in P, a \in \varphi_p^{-1}(\mathcal{K}(X_p))\Big\}. \tag{1.6}$$

Let $q_{\mathcal{O}} : \mathcal{T}(X) \to \mathcal{O}(X)$ be the quotient map. It is observed in [20, Proposition 2.9] that $q_{\mathcal{O}} \circ \omega$ is a Cuntz-Pimsner-covariant representation of X in $\mathcal{O}(X)$. Moreover the pair $(\mathcal{O}(X), q_{\mathcal{O}} \circ \omega)$ is universal for the Cuntz-Pimsner-covariant representations of X.

Let (G, P) be a quasi-lattice ordered group and suppose that X is a product system of essential right Hilbert A–A bimodules over P. Suppose that ψ is Toeplitz representation of X on a Hilbert space $\mathcal{H}$. It follows from [20, Proposition 4.1] that there is a unique action $\alpha^\psi : P \to \operatorname{End} \psi_e(A)'$ such that

$$\alpha_p^\psi(T)\psi_p(x) = \psi_p(x)T \text{ for all } T \in \psi_e(A)',\ x \in X_p, \text{ and} \tag{1.7}$$

$$\alpha_p^\psi(1_p)r = 0 \text{ for } r \in (\psi_p(X_p)\mathcal{H})^\perp, \tag{1.8}$$

where 1_p is the identity operator on X_p.

Lemma 1.3.4. *Let (G, P) be a quasi-lattice ordered group and X be a product system of essential right Hilbert A–A bimodules over P. Suppose that ψ is a Toeplitz representation on a Hilbert space $\mathcal{H}$. Let $p \in P$ and suppose that $\{x_i\}_{i=0}^d$ is a Parseval frame for the fibre X_p. Let $\alpha_p^\psi(1_p)$ be as in (1.8). Then*

$$\alpha_p^\psi(1_p) = \sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*.$$

Proof. By uniqueness in [20, Proposition 4.1], it suffices to prove that

$$\Big(\sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*\Big)\psi_p(x) = \psi_p(x) \text{ for all } x \in X_p, \text{ and} \tag{1.9}$$

$$\Big(\sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*\Big) r = 0 \text{ for all } r \in (\psi_p(X_p)\mathcal{H})^\perp = 0. \tag{1.10}$$

To see (1.9), let $x \in X_p$. We compute by applying the reconstruction formula for x:

$$\begin{aligned}
\Big(\sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*\Big)\psi(x) &= \Big(\sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*\Big)\psi\Big(\sum_{j=0}^{d} x_j \cdot \langle x_j, x\rangle\Big) \\
&= \sum_{0\le i,j\le d} \psi_p(x_i)\psi_p(x_i)^*\psi(x_j)\psi_0(\langle x_j, x\rangle) \qquad \text{using (T3)} \\
&= \sum_{0\le i,j\le d} \psi_p(x_i)\psi_0(\langle x_i, x_j\rangle)\psi_0(\langle x_j, x\rangle) \qquad \text{using (T2)} \\
&= \sum_{0\le i,j\le d} \psi_p(x_i \cdot \langle x_i, x_j\rangle)\psi_0(\langle x_j, x\rangle).
\end{aligned}$$

Rearranging this and two applications of the reconstruction formula give

$$\begin{aligned}
\Big(\sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*\Big)\psi(x) &= \sum_{j=0}^{d} \psi_p\Big(\sum_{i=0}^{d} x_i \cdot \langle x_i, x_j\rangle\Big)\psi_0(\langle x_j, x\rangle) = \sum_{j=0}^{d} \psi_p(x_j)\psi_0(\langle x_j, x\rangle) \\
&= \sum_{j=0}^{d} \psi_p(x_j \cdot \langle x_j, x\rangle) = \psi_p(x).
\end{aligned}$$

This is precisely (1.9).

To check (1.10), fix $r \in (\psi_p(X_p)\mathcal{H})^\perp$. Notice that for all $r' \in \mathcal{H}$ we have

$$\Big(\Big(\sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*\Big) r \Big| r'\Big) = \sum_{i=0}^{d} \Big(r \Big| \psi_p(x_i)\psi_p(x_i)^* r'\Big) = 0.$$

It then follows $\Big(\sum_{i=0}^{d} \psi_p(x_i)\psi_p(x_i)^*\Big) r = 0$ and we have proven (1.10). $\square$

Definition 1.3.5. Let (G, P) be a quasi-lattice ordered group and suppose that X is a product system of essential right Hilbert A–A bimodules over P. Suppose that ψ is Toeplitz representation of X on a Hilbert space $\mathcal{H}$. We say ψ is *Nica-covariant* if for every $p, q \in P$, we have

$$\alpha_p^\psi(1_p)\alpha_q^\psi(1_q) = \begin{cases} \alpha_p^\psi(1_{p\vee q}) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Fowler showed in [20, Proposition 5.6] that the Nica-covariance condition can be expressed in terms of compact operators. Then for the class of compactly-aligned product systems, he extended the Nica-covariance condition for the representations over C^*-algebras.

Definition 1.3.6. Let (G, P) be a quasi-lattice ordered group and suppose that X is a compactly aligned product system over P of right Hilbert A–A bimodules. A Toeplitz representation ψ of X is *Nica-covariant* if for every $p, q \in P$, $S \in \mathcal{K}(X_p)$, and $T \in \mathcal{K}(X_q)$, we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}\big(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\big) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [20, Theorem 6.3] that there exists a C^*-algebra $\mathcal{NT}(X)$ and a Nica-covariant representation ψ of X in $\mathcal{NT}(X)$ such that $(\mathcal{NT}(X), \psi)$ is universal for the Nica-covariant representations of X. Moreover, we have

$$\mathcal{NT}(X) = \overline{\operatorname{span}}\{\psi_p(x)\psi_q(y)^* : p, q \in P, x \in X_p, y \in X_q\}. \tag{1.11}$$

The C^*-algebra $\mathcal{NT}(X)$, is called *the Nica-Toeplitz algebra of* X. Throughout we will keep ψ for the universal Nica-covariant representation of X.

The next lemma shows that $\mathcal{NT}(X)$ is a quotient of $\mathcal{T}(X)$.

Lemma 1.3.7. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right Hilbert A–A bimodules. Suppose $\mathcal{J}$ is the ideal in $\mathcal{T}(X)$ such that*

$$\mathcal{J} := \bigcap\{\ker\theta_* : \theta \textit{ is Nica-covariant representation of } X\}, \tag{1.12}$$

and let $q_{\mathcal{NT}} : \mathcal{T}(X) \to \mathcal{T}(X)/\mathcal{J}$ be the quotient map. Then $\big(\mathcal{T}(X)/\mathcal{J}, q_{\mathcal{NT}} \circ \omega\big)$ is universal for Nica-covariant representation, and is canonically isomorphic to $(\mathcal{NT}(X), \psi)$.

Proof. Since $q_{\mathcal{NT}}$ is a homomorphism and ω satisfies (T1)$-$(T3), it follows that $q_{\mathcal{NT}} \circ \omega$ satisfies (T1)$-$(T3) as well. To see that $q_{\mathcal{NT}} \circ \omega$ is Nica-covariant, let $p, q \in P$, $S \in \mathcal{K}(X_p)$, and $T \in \mathcal{K}(X_q)$. Notice that $(q_{\mathcal{NT}} \circ \omega)^{(p)} = q_{\mathcal{NT}} \circ \omega^{(p)}$. Then

$$(q_{\mathcal{NT}} \circ \omega)^{(p)}(S)(q_{\mathcal{NT}} \circ \omega)^{(q)}(T) = q_{\mathcal{NT}}\big(\omega^{(p)}(S)\omega^{(q)}(T)\big). \tag{1.13}$$

If $p \vee q < \infty$, since $\omega^{(p)}(S)\omega^{(q)}(T) - \omega^{(p\vee q)}\big(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\big) \in \mathcal{J}$, it follows that

$$q_{\mathcal{NT}}\big(\omega^{(p)}(S)\omega^{(q)}(T)\big) = q_{\mathcal{NT}}\Big(\omega^{(p\vee q)}\big(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\big)\Big).$$

Now (1.13) implies that

$$(q_{\mathcal{NT}} \circ \omega)^{(p)}(S)(q_{\mathcal{NT}} \circ \omega)^{(q)}(T) = (q_{\mathcal{NT}} \circ \omega)^{(p\vee q)}\big(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\big).$$

Similarly, for $p \vee q = \infty$, we have $q_{\mathcal{NT}}\big(\omega^{(p)}(S)\omega^{(q)}(T)\big) = 0$. Putting this in (1.13) gives $(q_{\mathcal{NT}} \circ \omega)^{(p)}(S)(q_{\mathcal{NT}} \circ \omega)^{(q)}(T) = 0$. Thus $q_{\mathcal{NT}} \circ \omega$ is a Nica-covariant representation. Since $\{\omega(x) : x \in X\}$ generates $\mathcal{T}(X)$, we have that $\{q(\omega(x)) : x \in X\}$ generates $\mathcal{T}(X)/\mathcal{J}$.

To see (U1), suppose that T is another Nica-covariant representation of X in a C^*-algebra B. Notice that T is in particular a Toeplitz representation of X. Then the universal property of pair $(\mathcal{T}(X), \omega)$ gives a unique homomorphism $T_* : \mathcal{T}(X) \to B$ such that $T_* \circ \omega = T$. Notice that T_* vanishes on $\mathcal{J}$ because by definition $\mathcal{J} \subset \ker T_*$. Thus there is a homomorphism $T_* : \mathcal{T}(X)/\mathcal{J} \to B$ such that $T_*(q_{\mathcal{NT}} \circ \omega) = T$. $\square$

The Cuntz-Pimsner algebra $\mathcal{O}(X)$ is by definition a quotient of $\mathcal{T}(X)$. Since we are interested in studying the C^*-algebra $\mathcal{NT}(X)$, it would be very helpful to explain $\mathcal{O}(X)$ as a quotient of $\mathcal{NT}(X)$. The next lemma shows that, under some assumptions, we can express $\mathcal{O}(X)$ as a quotient of $\mathcal{NT}(X)$.

Lemma 1.3.8. *Let (G, P) be a quasi-lattice ordered group, and let X be a compactly aligned product system over P of right Hilbert A–A bimodules. Suppose that every Cuntz-Pimsner-covariant representation of X is a Nica-covariant representation. Then $\mathcal{O}(X)$ is the quotient of $\mathcal{NT}(X)$ by the ideal generated by*

$$\Big\{\psi_e(a) - \psi^{(p)}(\varphi_p(a)) : p \in P, a \in \varphi_p^{-1}(\mathcal{K}(X_p))\Big\}. \tag{1.14}$$

Proof. Let

$$\mathcal{I} := \bigcap \Big\{ \ker \pi_* : \pi \text{ is a Cuntz-Pimsner-covariant representation of } X\Big\}. \tag{1.15}$$

Following the same argument of Lemma 1.3.7, we can view $\mathcal{O}(X)$ as the quotient of $\mathcal{T}(X)$ by the ideal $\mathcal{I}$. Let $\mathcal{J}$ and $q_{\mathcal{NT}}$ be the ideal and the quotient map as in Lemma 1.3.7. It then follows $q_{\mathcal{NT}}(b) = b + \mathcal{J}$ for all $b \in \mathcal{T}(X)$ and

$$q_{\mathcal{NT}} \circ \omega = \psi.$$

Since every Cuntz-Pimsner-covariant representation of X is also a Nica-covariant representation, it follows that $\mathcal{J} \subseteq \mathcal{I}$. An application of the third isomorphism theorem in algebra gives us a quotient map $q : \mathcal{NT}(X) \to \mathcal{O}(X)$ such that $\ker q = \mathcal{I}/\mathcal{J}$. We now have

$$\mathcal{I}/\mathcal{J} = \big\{ i + \mathcal{J} : i \in \mathcal{I} \big\} = \big\{ q_{\mathcal{NT}}(i) : i \in \mathcal{I} \big\}. \tag{1.16}$$

An argument in set theory shows that $\mathcal{I}$ is the same as the ideal (1.6). Now using elements (1.6) in (1.16) and applying $q_{\mathcal{NT}} \circ \omega = \psi$, we have

$$\mathcal{I}/\mathcal{J} = \Big\langle \psi_e(a) - \psi^{(p)}(\varphi_p(a)) : p \in P, a \in \varphi_p^{-1}(\mathcal{K}(X_p)) \Big\rangle.$$

Thus we can consider $\mathcal{O}(X)$ as the quotient of $\mathcal{NT}(X)$, by the ideal

$$\Big\langle \psi_e(a) - \psi^{(p)}(\varphi_p(a)) : p \in P, a \in \varphi_p^{-1}(\mathcal{K}(X_p)) \Big\rangle.$$ □

Proposition 1.3.9 ([20, Proposition 5.4])**.** *Let (G, P) be a quasi-lattice ordered group such that every $p, q \in P$ have a common upper bound. Let X be a compactly aligned product system over P of right Hilbert A–A bimodules. Suppose that each fibre X_p is essential and the left action of A on X_p is by compact operators. Then every Toeplitz representation of X which is Cuntz-Pimsner-covariant is also Nica-covariant.*

Remark 1.3.10. Let (G, P) be a quasi-lattice ordered group and X be a compactly aligned product system over P of right Hilbert A–A bimodules. In [52, Proposition 3.12], Sims and Yeend defined their Cuntz-Pimsner algebra $\mathcal{NO}(X)$ as a quotient of $\mathcal{NT}(X)$. In general $\mathcal{NO}(X)$ and $\mathcal{O}(X)$ are different. But we can deduce from [52, Remark 3.14, Proposition 5.1] that if

(a) each pair (p, q) in P, has an upper bound (and automatically a least upper bound),

(b) for each $p \in P$ the homomorphism $\varphi_p : A \to \mathcal{L}(X_p)$ is injective, and

(c) the Cuntz-Pimsner-covariance (1.5) implies the Nica-covariance,

then the two C^*-algebras $\mathcal{NO}(X)$ and $\mathcal{O}(X)$ coincide. In our set-up these conditions are satisfied (see Remark 2.1.3). But we found it easier to work with $\mathcal{O}(X)$ and the quotient map mentioned in Lemma 1.3.8.

1.4 The Fock representation

We take the definition of Fock representation from [20, page 340].

Let P be a semigroup with identity e and suppose X is a product system over P of right Hilbert A–A bimodules. Define $r : X \to P$ by $r(x) := p$ for $x \in X_p$. Let $\bigoplus_{p\in P} X_p$ be the subset of $\prod_{p\in P} X_p$ consisting of all elements (x_p) such that $\sum_{p\in P}\langle x_p, x_p\rangle_A$ converges in norm. Write $\oplus x_p$ for elements of $\bigoplus_{p\in P} X_p$. It follows from [20, page 340] that $\bigoplus_{p\in P} X_p$ is a right Hilbert A–A bimodule with the right action given by $(\oplus x_p)\cdot a := (\oplus x_p \cdot a)$, the inner product by $\langle \oplus x_p, \oplus y_p\rangle := \sum_{p\in P}\langle x_p, y_p\rangle$, and the left action by the map $\oplus\varphi_p : A \to \mathcal{L}(F(X))$ defined by

$$\oplus\varphi_p(\oplus x_p) = \oplus\varphi_p(x_p) \text{ for } \oplus x_p \in F(X).$$

We write $F(X) := \bigoplus_{p\in P} X_p$ and call it the *Fock module.*

Fowler shows in [20, page 340] that for $x \in X$ there is an adjointable operator $T(x)$ such that

$$T(x)(\oplus x_p) = \oplus(x x_p) \text{ for } \oplus x_p \in F(X).$$

The adjoint $T(x)^*$ is zero on any summand X_p for which $p \notin r(x)P$. When $p \in r(x)P$, there is an isomorphism $\sigma_{r(x),p-r(x)} : X_{r(x)} \otimes_A X_{p-r(x)} \to X_p$, and the adjoint $T(x)^*$ is determined by the formula

$$T(x)^*\big(\sigma_{r(x),p-r(x)}(y\otimes z)\big) = \langle x, y\rangle \cdot z. \tag{1.17}$$

He also shows that T is a Toeplitz representation of X and calls it the *Fock representation.*

Remark 1.4.1. Let X be a compactly aligned product system over $\mathbb{N}^k$ of right Hilbert A–A bimodules and suppose the left action of A on each fibre is by compact operators. Then the homomorphism $T_* : \mathcal{NT}(X) \to \mathcal{L}(F(X))$ induced from the Fock representation is faithful (see [24, Remark 4.8]).

1.5 Topological graphs

A *topological graph* $E = (E^0, E^1, r, s)$ consists of two locally compact Hausdorff spaces, a continuous map $r : E^1 \to E^0$ and a local homeomorphism $s : E^1 \to E^0$. The map r is called the *range map* and s is called the *source map.* Given such a graph, let

$A := C_0(E^0)$. It is observed in [44, Chapter 9] that there is a right action of A on $C_c(E^1)$ and there is a well-defined right A-valued inner product on $C_c(E^1)$ such that

$$(x \cdot a)(z) = x(z)a(s(z)), \text{ and } \langle x, y\rangle_A(z) = \sum_{s(w)=z} \overline{x(w)}y(w).$$

It follows that the completion $X(E)$ is a right Hilbert A-module. The formula

$$(a \cdot x)(z) := a(r(z))x(z),$$

defines an action of A by adjointable operators on $X(E)$ (see [44, Chapter 9]). Then $X(E)$ becomes a right Hilbert A–A bimodule. We call $X(E)$ the *graph correspondence* associated to the topological graph E.

In topological graphs of interest to us, the spaces E^0 and E^1 are always compact. Then $C_c(E^1) = C(E^1)$. Since s is a local homeomorphism on the compact space E^1, $D := \max_{z\in E^0} |s^{-1}(z)| < \infty$. Now we have

$$\|x\|_A^2 = \sup_{z\in E^0} \Big| \sum_{s(w)=z} \overline{x(w)}x(w)\Big| \leq \|x\|_{\sup}^2 D.$$

On the other hand, since E^0 is compact, $\|x\|_{\sup} = |x(z_0)|$ for some $z_0 \in E^0$. Then

$$\|x\|_{\sup} = |x(z_0)|^2 \leq \sum_{s(w)=s(z_0)} \overline{x(w)}x(w) \leq \sup_{z\in E^0} \Big| \sum_{s(w)=z} \overline{x(w)}x(w)\Big| = \|x\|_A^2.$$

Thus the norm $\|\cdot\|_A$ on $X(E)$ is equivalent (as a vector-space norm) to the supremum norm on $C(E^1)$. Thus there is no completion required here and it makes sense to write $X(E) = C(E^1)$.

Example 1.5.1. Let Z be a locally compact Hausdorff space and $\mathrm{id} : Z \to Z$ be the identity map on Z. Let E be the topological graph $(Z, Z, \mathrm{id}, \mathrm{id})$. Then $X(E) = C(Z) = A$. The actions of A on $X(E)$ are by pointwise multiplication which are the same as the actions in ${}_AA_A$. Notice that

$$\langle x, y\rangle(z) = \sum_{\mathrm{id}(w)=z} \overline{x(w)})y(w) = \overline{x(z)}y(z).$$

This is precisely the inner product in the standard bimodule ${}_AA_A$. Thus $X(E) =_A A_A$.

1.6 Measures

All the measures we consider here are positive in the sense that they take values in $[0, \infty)$. We write $M(Z)_+$ for the set of finite regular Borel measures on Z. For us, a *probability measure* is a Borel measure with total mass 1.

Chapter 2

A product system associated to a family of local homeomorphisms

In this chapter we show that a family of surjective and commuting local homeomorphisms $h_1, \dots, h_k$ on a compact Hausdorff space Z induces a compactly aligned product system of Hilbert bimodules over $\mathbb{N}^k$. We also prove that the C^*-algebras of product systems of Hilbert bimodules over $\mathbb{N}^k$ carry gauge actions of $\mathbb{T}^k$.

2.0.1 Notations

We consider $\mathbb{N}^k$ as a monoid under addition with identity 0. We write $\mathbb{N}^k_+$ for the nonzero elements of $\mathbb{N}^k$. We use $e_1, \dots, e_k$ for the standard generators and write n_i for i-th coordinate of n. We denote $\leq$ for the partial order in $\mathbb{N}^k$ defined by $m \leq n$ if and only if $m_i \leq n_i$ for all $1 \leq i \leq k$. We write $m \vee n$ for the coordinate-wise maximum of m and n in the sense that $(m \vee n)_i := \max\{m_i, n_i\}$. Similarly we denote by $m \wedge n$ the coordinate-wise minimum of m and n.

Let $h_1, \dots, h_k$ be surjective and commuting local homeomorphisms on a compact Hausdorff space Z. Then for $m \in \mathbb{N}^k$ we write $h^m := h_1^{m_1} \circ \cdots \circ h_k^{m_k}$ and $h^{-m} := \left(h_1^{m_1} \circ \cdots \circ h_k^{m_k}\right)^{-1}$.

2.1 Building a product system from local homeomorphisms

In [1, Lemma 5.2] we proved that for a local homeomorphism f and the associated graph correspondence $X(E)$, there is an isomorphism from $X(E)^{\otimes 2}$ onto the graph correspondence associated to $f \circ f$. The next lemma generalizes this to graph correspondences of two different local homeomorphisms. There is also a similar result in the dynamics arising from graph algebras (see [6, Proposition 2.2]).

Lemma 2.1.1. *Let f, g be surjective local homeomorphisms on a compact Hausdorff space Z. Let $A := C(Z)$ and suppose $X(E_1)$, $X(E_2)$ and $X(F)$ are the graph correspondences related to topological graphs $E_1 = (Z, Z, \mathrm{id}, f)$, $E_2 = (Z, Z, \mathrm{id}, g)$, and $F = (Z, Z, \mathrm{id}, g \circ f)$. Then there is an isomorphism $\sigma_{f,g}$ from $X(E_1) \otimes_A X(E_2)$ onto $X(F)$ such that*

$$\sigma_{f,g}(x \otimes y)(z) = x(z)y(f(z)) \text{ for all } z \in Z. \tag{2.1}$$

Proof. Define the map $\sigma : C(Z) \times C(Z) \to C(Z)$ by

$$\sigma(x, y)(z) = x(z)y(f(z)) \text{ for all } x, y \in C(Z). \tag{2.2}$$

We first show that σ is bilinear and onto. Take $c, c' \in \mathbb{C}$ and $x, x', y, y' \in C(Z)$. Then

$$\begin{aligned}
\sigma\big(cx + c'x', y\big)(z) &= (cx + c'x')(z)y(f(z)) \\
&= cx(z)y(f(z)) + c'x'(z)y(f(z)) \\
&= c\sigma(x, y)(z) + c'\sigma(x, y)(z).
\end{aligned}$$

Similarly we have $\sigma\big(x, cy + c'y'\big) = c\sigma(x, y) + c'\sigma(x, y')$. So σ is bilinear. Taking $y = 1$ in (2.2) implies that σ is surjective. Now the universal property of the algebraic tensor product $\odot$ gives a unique surjective linear map $\tilde{\sigma} : C(Z) \odot C(Z) \to C(Z)$ satisfying $\tilde{\sigma}(x \odot y)(z) = x(z)y(f(z))$ for all $x \odot y \in C(Z) \odot C(Z)$. Since $\tilde{\sigma}$ vanishes on the element of the form (1.2), we can extend it to a surjective linear map $\sigma_{f,g} : C(Z) \odot_A C(Z) \to C(Z)$ such that $\sigma_{f,g}(x \otimes y)(z) = x(z)y(f(z))$ for all $x \otimes y \in C(Z) \odot_A C(Z)$.

Next we show that $\sigma_{f,g}$ preserves the actions and the inner products. Let $x \otimes y \in C(Z) \odot_A C(Z), a \in C(Z)$ and $z \in Z$. To check that $\sigma_{f,g}$ preserves the right action, we have

$$\begin{aligned}
\sigma_{f,g}\big(x \otimes y \cdot a\big)(z) &= x(z)(y \cdot a)(f(z)) \\
&= x(z)y(f(z))a(g \circ f(z)) \\
&= \sigma_{f,g}(x \otimes y)(z)a(g \circ f(z)) \\
&= \big(\sigma_{f,g}(x \otimes y) \cdot a\big)(z).
\end{aligned}$$

Similarly for the left action, we have

$$\sigma_{f,g}\big(a \cdot (x \otimes y)\big)(z) = (a \cdot x)(z)y(f(z))$$

$$
\begin{aligned}
&= a(z)x(z)y(f(z)) \\
&= a(z)\sigma_{f,g}(x \otimes y)(z) \\
&= \big(a \cdot \sigma_{f,g}(x \otimes y)\big)(z).
\end{aligned}
$$

To see that $\sigma_{f,g}$ preserves the inner products, take $x \otimes y, x' \otimes y' \in C(Z) \odot_A C(Z)$. Then remembering that the range functions are identity and the source functions are f, g, we have

$$
\begin{aligned}
\Big\langle \sigma_{f,g}(x \otimes y), \sigma_{f,g}(x' \otimes y') \Big\rangle(z) &= \sum_{g \circ f(w) = z} \overline{\sigma_{f,g}(x \otimes y)(w)} \sigma_{f,g}(x' \otimes y')(w) \\
&= \sum_{g \circ f(w) = z} \overline{x(w) y(f(w))} x'(w) y'(f(w)) \\
&= \sum_{g(v) = z} \Big[\sum_{f(w) = v} \overline{x(w)} x'(w) \Big] \overline{y(f(w))} y'(f(w)) \\
&= \sum_{g(v) = z} \langle x, x' \rangle(v) \overline{y(v)} y'(v) \\
&= \sum_{g(v) = z} \overline{y(v)} \big(\langle x, x' \rangle \cdot y' \big)(v) \\
&= \big\langle y, \langle x, x' \rangle \cdot y' \big\rangle(z) \\
&= \big\langle x \otimes y, x' \otimes y' \big\rangle(z). \qquad (2.3)
\end{aligned}
$$

Next a quick calculation shows that $\sigma_{f,g}$ is an isometry. Take a typical element $v = \sum_{i=0}^{d} x_i \otimes y_i \in C(Z) \odot_A C(Z)$. We have

$$
\begin{aligned}
\Big\| \sigma_{f,g}(v) \Big\|^2 &= \Big\langle \sigma_{f,g}(v), \sigma_{f,g}(v) \Big\rangle \\
&= \Big\langle \sum_{i=0}^{d} \sigma_{f,g}(x_i \otimes y_i), \sum_{j=0}^{d} \sigma_{f,g}(x_j \otimes y_j) \Big\rangle \\
&= \sum_{0 \le i,j \le d} \Big\langle \sigma_{f,g}(x_i \otimes y_i), \sigma_{f,g}(x_j \otimes y_j) \Big\rangle \\
&= \sum_{0 \le i,j \le d} \Big\langle x_i \otimes y_i, x_j \otimes y_j \Big\rangle \qquad \text{by (2.3)} \\
&= \Big\langle \sum_{i=0}^{d} x_i \otimes y_i \, , \sum_{j=0}^{d} x_j \otimes y_j \Big\rangle \\
&= \|v\|^2.
\end{aligned}
$$

Thus $\sigma_{f,g}$ is an isometry on $C(Z) \odot_A C(Z)$, and then it extends to an isomorphism $\sigma_{f,g}$ of $X(E_1) \otimes_A X(E_2)$ onto $X(F)$ which satisfies (2.1). $\square$

Corollary 2.1.2. *Let $h_1, \dots, h_k$ be surjective and commuting local homeomorphisms on a compact Hausdorff space Z. For each $m \in \mathbb{N}^k$, let X_m be the graph correspondence associated to the topological graph (Z, Z, id, h^m). Suppose $X := \bigsqcup_{m\in\mathbb{N}^k} X_m$ and $A := C(Z)$. Let $\sigma_{m,n} : X_m \otimes_A X_n \to X_{m+n}$ be the isomorphism obtained by applying Lemma 2.1.1 with the local homeomorphisms h^m, h^n. Then X is a compactly aligned product system over $\mathbb{N}^k$ of essential right Hilbert A–A bimodules with the multiplication given by*

$$xy := \sigma_{m,n}(x \otimes y) \text{ for } x \in X_m, y \in Y_n, \tag{2.4}$$

that is $(xy)(z) = x(z)y(h^m(z))$ for all $z \in Z$. Furthermore, the left action of A on each fibre X_m is by compact operators.

Proof. To see that X is a semigroup, let $m, n, p \in \mathbb{N}^k$ and take $x \in X_m, x' \in X_n$ and $x'' \in X_p$. Then by applying the definition of multiplication, we have

$$\begin{aligned}\Big((xx')x''\Big)(z) &= \Big(\sigma_{m+n,p}\big(\sigma_{m,n}(x \otimes x') \otimes x''\big)\Big)(z) \\ &= \big(\sigma_{m,n}(x \otimes x')\big)(z)x''\big(h^{m+n}(z)\big) \\ &= x(z)x'(h^m(z))x''\big(h^{m+n}(z)\big).\end{aligned}$$

A similar computation shows

$$\begin{aligned}\Big(x(x'x'')\Big)(z) &= \Big(\sigma_{m,n+p}\big(x \otimes \sigma_{n,p}(x' \otimes x'')\big)\Big)(z) \\ &= x(z)\sigma_{n,p}(x' \otimes x'')(h^m(z)) \\ &= x(z)x'(h^m(z))x''\big(h^{m+n}(z)\big).\end{aligned}$$

Thus $(xx')x'' = x(x'x'')$ and X is a semigroup. Next we check conditions (P1)−(P3) of the Definition 1.2.4. (P1) follows from Example 1.5.1 which says that $X_e =_A A_A$. (P2) is immediate by definition of X. To check (P3), let $a \in A$ and $x \in X_m$. Then

$$ax(z) = \sigma_{0,m}(a \otimes x)(z) = a(z)x(z) = (a \cdot x)(z),$$

similarly

$$xa(z) = \sigma_{m,0}(x \otimes a)(z) = x(z)a(h^m(z)) = (x \cdot a)(z).$$

To see that the fibre X_m is essential, notice that $A = C(Z)$ is unital with the identity $\mathrm{I}_{C(Z)} : Z \to \mathbb{C}$ defined by $\mathrm{I}_{C(Z)}(z) = 1$ for all $z \in Z$. Since the left action is by pointwise multiplication, $\varphi_m(\mathrm{I}_{C(Z)})x = x$ for all $x \in X_m$. Thus X_m is essential.

To prove that the left action of A on the fibre X_m is by compact operators, let $\{U_j\}_{j=0}^d$ be an open cover of Z such that $h^m|_{U_j}$ is injective. Choose a partition of unity $\{\rho_j\}$ subordinate to $\{U_j\}$ and define $\xi_j := \sqrt{\rho_j}$. We claim that for each $a \in A$,

$$\varphi_m(a) = \sum_{j=0}^{d} \Theta_{a\cdot\xi_j,\xi_j}. \tag{2.5}$$

Take $x \in X_m$ and $z \in Z$, we compute the right-hand side of (2.5)

$$\begin{aligned}
\Big(\sum_{j=0}^{d} \Theta_{a\cdot\xi_j,\xi_j}(x)\Big)(z) &= \sum_{j=0}^{d} \Big((a\cdot\xi_j)\cdot\langle\xi_j, x\rangle\Big)(z) \\
&= \sum_{j=0}^{d} (a\cdot\xi_j)(z)\langle\xi_j, x\rangle(h^m(z)) \\
&= \sum_{j=0}^{d} a(z)\xi_j(z) \sum_{h^m(w)=h^m(z)} \overline{\xi_j(w)}x(w).
\end{aligned}$$

Since h^m is injective on each $\operatorname{supp}\xi_j$,

$$\begin{aligned}
\Big(\sum_{j=0}^{d} \Theta_{a\cdot\xi_j,\xi_j}(x)\Big)(z) &= \sum_{j=0}^{d} a(z)\xi_j(z)\overline{\xi_j(z)}x(z) \\
&= a(z)x(z)\sum_{j=0}^{d} |\xi_j(z)|^2 \\
&= a(z)x(z),
\end{aligned}$$

which is equal to the left-hand side of (2.5), as we required.

Finally, it follows from [20, Proposition 5.8]that X is a compactly aligned product system. □

Remark 2.1.3. Let $h_1, \dots, h_k$ be surjective and commuting local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. We aim to show that the two Cuntz-Pimsner algebra $\mathcal{NO}(X)$ and $\mathcal{O}(X)$ coincide. We check the conditions (a)$-$(b) of Remark 1.3.10.

Condition (a) is clear because each pair in $\mathbb{N}^k$ has an upper bound. To prove (b), notice that for each $m \in \mathbb{N}^k$ the homomorphism $\varphi_m : A \to \mathcal{L}(X_m)$ is injective. To see this, let $\varphi_m(a) = \varphi_m(a')$ for $a, a' \in A$. Let $I_{C(Z)}$ be the identity in $C(Z)$. Then $\varphi_m(a)(I_{C(Z)}) = \varphi_m(a')(I_{C(Z)})$. It follows that $a(z) = a'(z)$ for all $z \in Z$ and therefore $a = a'$. To check (c), notice that X is a compactly aligned product system of essential Hilbert A–A bimodule and the left action is by compact operators. Then

Proposition 1.3.9 implies that every Cuntz-Pimsner covariant representation is a Nica-covariant representation. Thus we have checked all conditions Remark 1.3.10, and hence the two Cuntz-Pimsner algebra $\mathcal{NO}(X)$ and $\mathcal{O}(X)$ coincide.

2.2 The gauge action

By a *strongly continuous action* of a locally compact group G on a C^*-algebra A, we mean a homomorphism $g \mapsto \alpha_g : G \to \mathrm{Aut}(A)$ such that $g \mapsto \alpha_g(a)$ is continuous for each fixed $a \in A$.

It is well known that the Nica-Toeplitz algebra of a product system over $\mathbb{N}^k$ of right Hilbert A–A bimodules carries an action of the k-torus $\mathbb{T}^k$. But we could not find an explicit reference for this. The next lemma shows this fact.

Lemma 2.2.1. *Let A be a C^*-algebra and X be a compactly aligned product system over $\mathbb{N}^k$ of right Hilbert A–A bimodules. Then there is a strongly continuous action $\gamma : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{NT}(X))$, called the gauge action, such that*

$$\gamma_z(\psi_n(x)) = z^n \psi_n(x) \textit{ for all } n \in \mathbb{N}^k, z \in \mathbb{T}^k, x \in X_n.$$

Proof. Fix $z \in \mathbb{T}^k$ and define $\theta : X \to \mathcal{NT}(X)$ by

$$\theta_n(x) = z^n \psi_n(x) \text{ for } n \in \mathbb{N}^k, x \in X_n.$$

We claim that θ is a Toeplitz representation of X. To see this, we check the conditions (T1)$-$(T3) of Definition 1.3.1. That θ is a Toeplitz representation follows because ψ is. Each θ_n is linear and θ_e is a homomorphism. We have

$$\theta_n(x)^* \theta_m(y) = \overline{z^n} \psi_n(x)^* z^n \psi_n(y) = \psi_0(\langle x, y \rangle) = \theta_0(\langle x, y \rangle).$$

and

$$\theta_n(x)\theta_m(y) = z^{n+m} \psi_n(x)\psi_m(y) = z^{n+m} \psi_{n+m}(xy) = \theta_{n+m}(xy).$$

Thus conditions (T1)$-$(T3) of Definition 1.3.1 are satisfied.

To see that it is Nica-covariant, we consider $\theta^{(n)} : \mathcal{K}(X_n) \to \mathcal{NT}(X)$. For $x, x' \in X_n$, we have

$$\theta^{(n)}(\Theta_{x,x'}) = \theta_n(x)\theta_n(x')^* = z^n \psi_n(x) \overline{z^n} \psi_n(x')^* = \psi^{(n)}(\Theta_{x,x'}).$$

Thus

$$\theta^{(n)}(S) = \psi^{(n)}(S) \text{ for all } S \in \mathcal{K}(X).$$

Now let $S \in \mathcal{K}(X_n), T \in \mathcal{K}(X_m)$. Since ψ is Nica-covariant, we have

$$\begin{aligned}\theta^{(n)}(S)\theta^{(m)}(T) &= \psi^{(n)}(S)\psi^{(m)}(T)\\ &= \psi^{(m\vee n)}\big(\iota_n^{m\vee n}(S)\iota_m^{m\vee n}(T)\big)\\ &= \theta^{(n\vee m)}\big(\iota_n^{m\vee n}(S)\iota_m^{m\vee n}(T)\big).\end{aligned}$$

Now it follows from the universal property of $\mathcal{NT}(X)$ that there is a homomorphism $\gamma_z : \mathcal{NT}(X) \to \mathcal{NT}(X)$ such that $\theta = \gamma_z \circ \psi$. This gives an explicit formula for γ_z on the generators of $\mathcal{NT}(X)$:

$$\gamma_z \circ \psi_n = \theta_n = z^n \psi_n. \tag{2.6}$$

Notice that $\gamma_{\overline{z}} \circ \gamma_z(\psi_n(x)) = \gamma_z \circ \gamma_{\overline{z}}(\psi_n(x)) = \psi_n(x)$. But the universal property of $\mathcal{NT}(X)$ implies that the identity map on $\mathcal{NT}(X)$ is the only homomorphism with this property. It then follows $(\gamma_z)^{-1} = \gamma_{\overline{z}}$ and hence $\gamma_z \in \mathrm{Aut}(\mathcal{NT}(X))$.

Next let $\mathrm{I}_{\mathbb{T}^k}$ be the identity element in $\mathbb{T}^k$. Then

$$\gamma_{\mathrm{I}_{\mathbb{T}^k}}(\psi_n(x)) = (\mathrm{I}_{\mathbb{T}^k})^n \psi_n(x) = \psi_n(x).$$

Then $\gamma_{\mathrm{I}_{\mathbb{T}^k}}$ is the identity map on $\mathcal{NT}(X)$. Finally, for $z, w \in \mathbb{T}^k$, we have

$$\gamma_z \circ \gamma_w(\psi_n) = (zw)^n \psi_n = \gamma_{zw}(\psi_n).$$

Thus γ is a homomorphism of $\mathbb{T}^k$ into the $\mathrm{Aut}(\mathcal{NT}(X))$.

To see that γ is strongly continuous, we must prove that $z \mapsto \gamma_z(b)$ is continuous for all $b \in \mathcal{NT}(X)$. Fix $\epsilon > 0$ and b. There is a linear combination c of generators in $\mathcal{NT}(X)$ such that $\|b - c\| < \frac{\epsilon}{3}$. Equation (2.6) implies that, $z \mapsto \gamma_z(c)$ is continuous. Then there exists some $\delta > 0$ such that $|z - w| < \delta \Rightarrow \|\gamma_w(c) - \gamma_z(c)\| < \frac{\epsilon}{3}$. Now for $|z - w| < \delta$ we have

$$\|\gamma_w(b) - \gamma_z(b)\| \leq \|\gamma_w(b - c)\| + \|\gamma_w(c) - \gamma_z(c)\| + \|\gamma_z(b - c)\| < \epsilon,$$

as we require. $\square$

Remark 2.2.2. Let $q : \mathcal{NT}(X) \to \mathcal{O}(X)$ be the quotient map as in Lemma 1.3.8. Since the gauge action on $\mathcal{NT}(X)$ fixes the kernel of q, it then induces a natural gauge action $\tilde{\gamma}$ of $\mathbb{T}^k$ on the quotient $\mathcal{O}(X)$.

Chapter 3

KMS states on the C^*-algebras of product systems associated to $*$-commuting local homeomorphisms

In this chapter we consider a family of $*$-commuting local homeomorphisms and the associated product system as in Corollary 2.1.2. We study KMS states and ground states on the C^*-algebras of this product system. Our object here is to generalize the results in [1] to our product system. When we have only one local homeomorphism, the results here (except those associated to ground states) reduce to those in [1].

3.0.1 KMS states

A *C^*-algebraic dynamical system* is a triple $(A, \mathbb{R}, \alpha)$ consisting of a C^*-algebra A, the real line $\mathbb{R}$ and an action $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$. Given such a C^*-algebraic dynamical system, we say an element a of A is *analytic* if $t \mapsto \alpha_t(a)$ is the restriction of an entire function $z \mapsto \alpha_z(a)$ on $\mathbb{C}$. It follows from [41, Sec. 8.12] that the analytic elements form a dense subalgebra of A.

Definition 3.0.3. Let $(A, \mathbb{R}, \alpha)$ be a C^*-algebraic dynamical system and ϕ be a state of A. We say ϕ is a *KMS state with inverse temperature* $\beta \in (0, \infty)$ (or a *KMS$_\beta$ state*) of (A, α) if it satisfies the following *KMS condition*:

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) \text{ for all analytic elements } a, b. \tag{3.1}$$

It suffices to check the KMS condition on a set of analytic elements which span a dense subspace of A (see [41, Proposition 8.12.4]).

We now look at the product system X associated to the local homeomorphisms $h_1, \ldots, h_k$ as in Corollary 2.1.2. We have shown in Lemma 2.2.1 that the Nica-Toeplitz algebra $\mathcal{NT}(X)$ carries a gauge action of $\mathbb{T}^k$. We can lift this action to an action of $\mathbb{R}$ on $\mathcal{NT}(X)$ as follows: Fix $r \in (0, \infty)^k$ and embed $\mathbb{R}$ in $\mathbb{T}^k$ via the map

$$t \mapsto e^{itr} = (e^{itr_1}, e^{itr_2}, \ldots, e^{itr_k}).$$

Then define $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X))$ by $\alpha_t = \gamma_{e^{itr}}$.

Considering the system $(\mathcal{NT}(X), \alpha)$, notice that for each $\psi_m(x)\psi_n(y) \in \mathcal{NT}(X)$, the function $t \mapsto \alpha_t\big(\psi_m(x)\psi_n(y)\big) = e^{itr\cdot(m-n)}\psi_m(x)\psi_n(y)$ on $\mathbb{R}$ extends to an entire function on all of $\mathbb{C}$. Thus each $\psi_m(x)\psi_n(y)$ is an analytic element of $\mathcal{NT}(X)$. The elements $\psi_m(x)\psi_n(y)$ span a dense subalgebra of $\mathcal{NT}(X)$ as in (1.11). Thus it suffices for us to check the KMS condition on these spanning elements.

Remark 3.0.4. We could get the action α directly (without passing through $\mathbb{T}^k$) by applying [24, Proposition 3.1] with the homomorphism $N : \mathbb{Z}^K \to (0, \infty)$ defined by $N(n) = n \cdot r = \sum_i^k n_i r_i$.

3.0.2 $*$-commuting local homeomorphisms

The notion of $*$-commuting maps was first introduced in [3] and then expanded by Exel and Renault in [16, §10]. The next definition is taken from [16, §10].

Definition 3.0.5. Let f, g be commuting maps on a set Z. We say f, g *$*$-commute*, if for every $x, y \in Z$ satisfying $f(x) = g(y)$, there exists a unique $z \in Z$ such that $x = g(z)$ and $y = f(z)$. The following digram illustrates this property beautifully.

$$\begin{tikzcd}
 & \overset{z}{\square} \arrow[dl, "f"'] \arrow[dr, "g"] & \\
y \arrow[dr, "g"'] & & x \arrow[dl, "f"] \\
 & f(x) = g(y) &
\end{tikzcd}$$

We also say that a family of maps $*$-commute if any two of them $*$-commute.

Lemma 3.0.6. *Let f, g and h be $*$-commuting maps on a space Z. Then*

(a) For $i, j \in \mathbb{N}$, f^i and g^j $*$-commute,

(b) f and $g \circ h$ $*$-commute.

Proof. For part (a), see the proof of [16, Proposition 10.2].

To prove (b), we apply the method used in [16, Proposition 10.2]. Suppose $u, v \in Z$ satisfying $f(u) = g \circ h(v)$. We have to show that there exists a unique $z \in Z$ such that

$$u = g \circ h(z) \text{ and } v = f(z). \tag{3.2}$$

Since f, g $*$-commute, it follows from $f(u) = g \circ h(v)$ that there exists a unique $w \in Z$ such that

$$u = g(w) \text{ and } h(v) = f(w). \tag{3.3}$$

Similarly, since h, f $*$-commute, the equation $h(v) = f(w)$ gives a unique $z \in Z$ satisfying

$$v = f(z) \text{ and } w = h(z). \tag{3.4}$$

Now combining (3.3) and (3.4), we deduce that z satisfies (3.2).

To see the uniqueness, suppose $z' \in Z$ satisfies (3.2). Let $w' := h(z')$. It follows from (3.2) that

$$u = g(w') \text{ and } h(v) = h(f(z')) = f(w').$$

The uniqueness property in (3.3) implies that $w = w'$. Considering this with the fact that z' satisfies (3.2), we have

$$w = w' = h(z') \text{ and } v = f(z').$$

Now the uniqueness in (3.4) implies that $z = z'$. □

Remark 3.0.7. There is another proof for part (b) of Lemma 3.0.6 in [55, Lemma 1.3].

Corollary 3.0.8. *Let $h_1, \ldots, h_k$ be $*$-commuting local homeomorphisms on a space Z. Fix $m, n \in \mathbb{N}^k$ such that $m \wedge n = 0$. Then h^m and h^n $*$-commute.*

Proof. Remember that $h^m = h_1^{m_1} \circ \cdots \circ h_k^{m_k}$ and $h^n = h_1^{n_1} \circ \cdots \circ h_k^{n_k}$. Since $m \wedge n = 0$, the local homeomorphisms appearing in $h^m = h_1^{m_1} \circ \cdots \circ h_k^{m_k}$ do not appear in $h^n = h_1^{n_1} \circ \cdots \circ h_k^{n_k}$. Now applying Lemma 3.0.6 finitely many times gives the proof. □

Remark 3.0.9. The condition $m \wedge n = 0$ in Corollary 3.0.8 is crucial. When the local homeomorphisms $h_1, \ldots, h_k$ $*$-commute, it does not imply that they $*$-commute with themselves. Thus we can not deduce from Lemma 3.0.6 that h^m and h^n $*$-commute for all $m, n \in \mathbb{N}^k$.

3.1 A characterization of KMS states

In this section we provide a characterization of KMS$_\beta$ states on $(\mathcal{NT}(X), \alpha)$ in Proposition 3.1.6. The characterization formula (3.19) says that KMS states vanish on most of the spanning elements of $\mathcal{NT}(X)$. Thus Proposition 3.1.6 enables us to recognise KMS states easier. To prove this proposition, we first show that the $*$-commutativity condition on $h_1, \dots, h_k$ allows us to find interesting Parseval frames for each fibre in X. Then we use these Parseval frames to find a formula which expresses elements of the form $\psi_n(y)^*\psi_m(x)$ as linear combinations of the elements $\psi_m(s)\psi_n(t)^*$ for suitable $s \in X_m, t \in X_n$ (Proposition 3.1.2(b)). This formula plays an important role in proving that the KMS condition holds. We also provide two simple lemmas which are again helpful when we discuss KMS condition.

Lemma 3.1.1. *Let f, g be $*$-commuting local homeomorphisms on a compact Hausdorff space Z. Suppose $X(E_1)$, $X(E_2)$ are the graph correspondences related to topological graphs $E_1 = (Z, Z, \mathrm{id}, f)$ and $E_2 = (Z, Z, \mathrm{id}, g)$. Let $\{\rho_i\}_{i=0}^d$ be a partition of unity such that $f|_{\operatorname{supp}\rho_i}, g|_{\operatorname{supp}\rho_i}$ are injective and suppose that $\tau_i := \sqrt{\rho_i}$. Then*

(a) $\{\tau_i\}_{i=0}^d$,$\{\tau_i \circ g\}_{i=0}^d$ are Parseval frames for $X(E_1)$,

(b) $\{\tau_i\}_{i=0}^d$,$\{\tau_i \circ f\}_{i=0}^d$ are Parseval frames for $X(E_2)$, and

(c) there exists an isomorphism $t_{f,g} : X(E_1) \otimes_A X(E_2) \to X(E_2) \otimes_A X(E_1)$ such that

$$t_{f,g}(\tau_i \circ g \otimes \tau_j) = \tau_j \circ f \otimes \tau_i \text{ for } 0 \le i, j \le d. \tag{3.5}$$

We will call this isomorphism the flip map.

Proof. Parts (a) and (b) are quite similar. We only prove (a). It follows from [17, Proposition 8.2] that $\{\tau_i\}_{i=0}^d$ is a Parseval frame for $X(E_1)$. To see that $\{\tau_i \circ g\}_{i=0}^d$ is a Parseval frame for $X(E_1)$, we take $x \in X(E_1)$ and check the reconstruction formula:

$$\sum_{i=0}^d (\tau_i \circ g) \cdot \big\langle (\tau_i \circ g), x \big\rangle = x.$$

Take $z \in Z$. Using the definition of the left action and the inner product, we have

$$\begin{aligned}\sum_{i=0}^d (\tau_i \circ g) \cdot \big\langle (\tau_i \circ g), x \big\rangle(z) &= \sum_{i=0}^d (\tau_i \circ g)(z) \big\langle (\tau_i \circ g), x \big\rangle (f(z)) \\ &= \sum_{i=0}^d \tau_i\big(g(z)\big) \big[\sum_{f(w)=f(z)} \overline{\tau_i(g(w))} x(w). \big]\end{aligned} \tag{3.6}$$

Suppose $f(w) = f(z)$. Notice that the i-summand vanishes unless $g(z), g(w) \in \operatorname{supp} \tau_i$. So suppose that $g(z), g(w) \in \operatorname{supp} \tau_i$. Then

$$\begin{aligned} f(w) = f(z) &\Rightarrow g \circ f(z) = g \circ f(w) \\ &\Rightarrow f \circ g(z) = f \circ g(w) \\ &\Rightarrow g(w) = g(z) && (f \text{ is one-to-one on } \operatorname{supp} \tau_i). \end{aligned}$$

Now we consider the digram

$$\begin{tikzcd} & \square \arrow[dl, "f"'] \arrow[dr, "g"] & \\ f(z) \arrow[dr, "g"] & & g(z) \arrow[dl, "f"'] \\ & g(f(z)) = f(g(z)) & \end{tikzcd}$$

Notice that both w, z fit in the box. Then the $*$-commutativity of f, g implies that $w = z$. Thus the interior sum in the last line of (3.6) will collapse to $\overline{(\tau_i \circ g)(z)} x(z)$ and hence the reconstruction formula follows from

$$\begin{aligned} \sum_{i=0}^{d} (\tau_i \circ g) \cdot \langle (\tau_i \circ g), x \rangle(z) &= \sum_{i=0}^{d} \overline{\tau_i(g(z))} \tau_i(g(z)) x(z) \\ &= x(z) \sum_{i=0}^{d} \tau_i(g(z))^2 \\ &= x(z). \end{aligned}$$

Next we look at part (c). Applying Lemma 2.1.1 implies that there are isomorphisms $\sigma_{f,g} : X(E_1) \otimes_A X(E_2) \to X(F)$ and $\sigma_{g,f} : X(E_2) \otimes_A X(E_1) \to X(F)$. Now set $t_{f,g} := \sigma_{g,f}^{-1} \circ \sigma_{f,g}$. It is clear that $t_{f,g}$ is an isomorphism from $X(E_1) \otimes_A X(E_2)$ onto $X(E_2) \otimes_A X(E_1)$. To check (3.5), note that

$$\sigma_{f,g}(\tau_i \circ g \otimes \tau_j) = \sigma_{g,f}(\tau_j \circ f \otimes \tau_i). \tag{3.7}$$

Thus $t_{f,g}(\tau_i \circ g \otimes \tau_j) = \sigma_{g,f}^{-1} \circ \sigma_{f,g}(\tau_i \circ g \otimes \tau_j) = \tau_j \circ f \otimes \tau_i$, as required. $\square$

The next Proposition is an analogue of [20, Proposition 5.10] and [24, Lemma 4.7]. In fact Proposition 3.1.2 is more general because the formula of [20, Proposition 5.10] is an approximation and [24, Lemma 4.7] holds only for product systems where each fibre is required to have an orthonormal basis.

Proposition 3.1.2. *Let $h_1, \dots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. Take $m, n \in \mathbb{N}^k$ such that $m \wedge n = 0$. Let $\{\rho_i\}_{i=0}^d$ be a partition of unity such that $h^m|_{\operatorname{supp}\rho_i}, h^n|_{\operatorname{supp}\rho_i}$ are injective and suppose that $\tau_i := \sqrt{\rho_i}$.*

(a) Let $\sigma_{m,n} : X_m \otimes_A X_n \to X_{m+n}$ and $\sigma_{n,m} : X_n \otimes_A X_m \to X_{m+n}$ be the isomorphisms induced by the multiplication in X. Then for all $x \otimes y \in X_m \otimes_A X_n$, we have

$$\sigma_{m,n}(x \otimes y) = \sum_{0 \le i,j \le d} \sigma_{n,m}\big(\tau_j \circ h^m \otimes \tau_i\big) \cdot \Big\langle \big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j \, , \, y \Big\rangle. \tag{3.8}$$

(b) Then for all $x \in X_m, y \in X_n$, we have

$$\psi_n(y)^* \psi_m(x) = \sum_{0 \le i,j \le d} \psi_m\big(\big\langle y, \tau_j \circ h^m \big\rangle \cdot \tau_i\big) \psi_n\big(\big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j\big)^*. \tag{3.9}$$

Proof. For part (a), it suffices to prove (3.8) for $x \otimes y \in X_m \odot_A X_n$. Notice that X_m, X_n are graph correspondences associated to the topological graphs (Z, Z, id, h^m) and (Z, Z, id, h^n). Since $m \wedge n = 0$, h^m and h^n are $*$-commuting. It then follows from Lemma 3.1.1 that $\{\tau_i \circ h^n\}_{i=0}^d$ and $\{\tau_j\}_{j=0}^d$ form Parseval frames for X_m, X_n respectively. Also notice that the formula for multiplication in X implies that

$$\sigma_{m,n}(\tau_i \circ h^n \otimes \tau_j) = \sigma_{n,m}(\tau_j \circ h^m \otimes \tau_i). \tag{3.10}$$

We use this to prove (3.8). So we must write $x \otimes y$ in terms of the elements $\{\tau_i \circ h^n \otimes \tau_j\}_{i,j}$.

To do this we start by writing the reconstruction formulas for the Parseval frames $\{\tau_i \circ h^n\}_{i=0}^d$ and $\{\tau_j\}_{j=0}^d$.

$$x \otimes y = \Big(\sum_{i=0}^d \tau_i \circ h^n \cdot \big\langle \tau_i \circ h^n, x \big\rangle\Big) \otimes \Big(\sum_{j=0}^d \tau_j \cdot \langle \tau_j, y \rangle\Big).$$

Since the tensors are balanced, we have

$$x \otimes y = \sum_{0 \le i,j \le d} \Big(\tau_i \circ h^n \otimes \big\langle \tau_i \circ h^n, x \big\rangle \cdot \tau_j \cdot \langle \tau_j, y \rangle\Big). \tag{3.11}$$

We then claim that

$$\big\langle \tau_i \circ h^n, x \big\rangle \cdot \tau_j \cdot \langle \tau_j, y \rangle = \tau_j \cdot \Big\langle \big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j \, , \, y \Big\rangle. \tag{3.12}$$

To see the claim, we evaluate both sides of (3.12) on $z \in Z$. For the left-hand side we have

$$\Big(\big\langle \tau_i \circ h^n, x \big\rangle \cdot \tau_j \cdot \langle \tau_j, y \rangle\Big)(z) = \big\langle \tau_i \circ h^n, x \big\rangle(z) \tau_j(z) \langle \tau_j, y \rangle(h^n(z))$$

$$= \big\langle \tau_i \circ h^n, x \big\rangle(z)\tau_j(z) \sum_{h^n(w)=h^n(z)} \overline{\tau_j(w)}y(w)$$

$$= \big\langle \tau_i \circ h^n, x \big\rangle(z)\tau_j(z)\overline{\tau_j(z)}y(z) \qquad (h^n \text{ is injective on } \operatorname{supp}\tau_j).$$

Similarly, we compute the right-hand side of (3.12):

$$\begin{aligned}\Big(\tau_j \cdot \Big\langle \big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j\,,\, y \Big\rangle\Big)(z) &= \tau_j(z)\Big\langle \big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j\,,\, y \Big\rangle(h^n(z)) \\ &= \tau_j(z) \sum_{h^n(w)=h^n(z)} \overline{\big\langle x, \tau_i \circ h^n \big\rangle(w)\tau_j(w)}y(w) \\ &= \tau_j(z)\big\langle \tau_i \circ h^n, x \big\rangle(z)\overline{\tau_j(z)}y(z).\end{aligned}$$

So we have proven the claim.

Now putting (3.12) in (3.11) gives

$$x \otimes y = \sum_{0 \le i,j \le d} \Big(\tau_i \circ h^n \otimes \tau_j \cdot \Big\langle \big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j\,,\, y \Big\rangle\Big),$$

which express $x \otimes y$ in terms of the elements $\{\tau_i \circ h^n \otimes \tau_j\}_{i,j}$.

Next we compute $\sigma_{m,n}(x \otimes y)$ using (3.10). Notice that $\sigma_{m,n}$ is an isomorphism of correspondences. Then

$$\sigma_{m,n}(x \otimes y) = \sum_{0 \le i,j \le d} \sigma_{m,n}\big(\tau_i \circ h^n \otimes \tau_j\big) \cdot \Big\langle \big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j\,,\, y \Big\rangle.$$

Now applying (3.10) gives

$$\sigma_{m,n}(x \otimes y) = \sum_{0 \le i,j \le d} \sigma_{n,m}\big(\tau_j \circ h^m \otimes \tau_i\big) \cdot \Big\langle \big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j\,,\, y \Big\rangle,$$

as required.

For part (b), we use the Fock representation T of X. Remark 1.4.1 implies that the induced homomorphism

$$T_* : \mathcal{NT}(X) \to \mathcal{L}(F(X))$$

is an injection. Then by the universal property of ψ, it suffices for us to prove that

$$T_n(y)^* T_m(x) = \sum_{0 \le i,j \le d} T_m\big(\big\langle y, \tau_j \circ h^m \big\rangle \cdot \tau_i\big) T_n\big(\big\langle x, \tau_i \circ h^n \big\rangle \cdot \tau_j\big)^*. \tag{3.13}$$

To do this, we evaluate both sides of (3.13) on an arbitrary $s \in X_p$ where $p \in \mathbb{N}^k$. An application of the formula (1.17) for the adjoint shows that the right-hand side of (3.13) vanishes unless $p \ge n$. For the left hand-side, the definition of the Fock representation says that $\big(T_n(y)^* T_m(x)\big)(s) = T_n(y)^*\big(\sigma_{m,p}(x \otimes s)\big)$. Now equation (1.17) implies that the left hand side of (3.13) is zero unless $m + p \geq n$. Since $m \wedge n = 0$, $m + p \geq n$ is equivalent to $p \geq n$. Thus both sides of (3.13) are zero unless $p \geq n$. So we assume $p \geq n$ from now.

It suffices to check (3.13) for $s = \sigma_{n,p-n}(s' \otimes s'')$ where $s' \otimes s'' \in X_n \odot_A X_{p-n}$. To do this we first compute the right-hand side of (3.13) by using the adjoint formula (1.17) and the definition of the Fock representation:

$$\sum_{0 \leq i,j \leq d} T_m\big(\langle y, \tau_j \circ h^m \rangle \cdot \tau_i\big) T_n\big(\langle x, \tau_i \circ h^n \rangle \cdot \tau_j\big)^* (\sigma_{n,p-n}(s' \otimes s''))$$

$$= \sum_{0 \leq i,j \leq d} T_m\big(\langle y, \tau_j \circ h^m \rangle \cdot \tau_i\big) \Big(\Big\langle \langle x, \tau_i \circ h^n \rangle \cdot \tau_j \, , \, s' \Big\rangle \cdot s'' \Big)$$

$$= \sum_{0 \leq i,j \leq d} \sigma_{m,p-n} \Big(\langle y, \tau_j \circ h^m \rangle \cdot \tau_i \otimes \Big\langle \langle x, \tau_i \circ h^n \rangle \cdot \tau_j \, , \, s' \Big\rangle \cdot s'' \Big). \tag{3.14}$$

Next we evaluate the left-hand side of (3.13) at $\sigma_{n,p-n}(s' \otimes s'')$. For convenience, let $\dagger := T_n(y)^* T_m(x)\big(\sigma_{n,p-n}(s' \otimes s'')\big)$. We start by applying the definition of the Fock representation. Then

$$\dagger = T_n(y)^* \Big(\sigma_{m,p} \Big(x \otimes \sigma_{n,p-n}(s' \otimes s'') \Big) \Big).$$

The associativity of multiplication in X implies that

$$\dagger = T_n(y)^* \Big(\sigma_{m+n,p-n} \Big(\sigma_{m,n}\big(x \otimes s'\big) \otimes s'' \Big) \Big).$$

In order to apply the adjoint formula (1.17), we must write $\sigma_{m,n}(x \otimes s')$ in terms of the elements of $X_n \otimes X_m$. To do this, we apply part (a) for $x \otimes s' \in X_m \otimes_A X_n$. Then

$$\dagger = \sum_{0 \leq i,j \leq d} T_n(y)^* \Big(\sigma_{m+n,p-n} \Big(\sigma_{n,m}(\tau_j \circ h^m \otimes \tau_i) \cdot \Big\langle \langle x, \tau_i \circ h^n \rangle \cdot \tau_j \, , \, s' \Big\rangle \otimes s'' \Big) \Big).$$

Since the tensors are balanced, we have

$$\dagger = \sum_{0 \leq i,j \leq d} T_n(y)^* \Big(\sigma_{m+n,p-n} \Big(\sigma_{n,m}(\tau_j \circ h^m \otimes \tau_i) \otimes \Big\langle \langle x, \tau_i \circ h^n \rangle \cdot \tau_j \, , \, s' \Big\rangle \cdot s'' \Big) \Big).$$

Another application of associativity of the multiplication in X gives

$$\dagger = \sum_{0 \leq i,j \leq d} T_n(y)^* \Big(\sigma_{n,m+p-n} \Big(\tau_j \circ h^m \otimes \sigma_{m,p-n} \Big(\tau_i \otimes \Big\langle \langle x, \tau_i \circ h^n \rangle \cdot \tau_j \, , \, s' \Big\rangle \cdot s'' \Big) \Big) \Big).$$

Now, we can apply the adjoint formula (1.17)

$$\dagger = \sum_{0 \leq i,j \leq d} \langle y, \tau_j \circ h^m \rangle \cdot \sigma_{m,p-n} \Big(\tau_i \otimes \Big\langle \langle x, \tau_i \circ h^n \rangle \cdot \tau_j \, , \, s' \Big\rangle \cdot s'' \Big) \Big).$$

Since $\sigma_{m,p-n}$ is an isomorphism of correspondences,

$$\dagger = \sum_{0\le i,j\le d} \sigma_{m,p-n}\Big(\big\langle y, \tau_j \circ h^m\big\rangle \cdot \tau_i \otimes \Big\langle \big\langle x, \tau_i \circ h^n\big\rangle \cdot \tau_j\,,\, s'\Big\rangle \cdot s''\Big)\Big).$$

This equals (3.14). Thus (3.13) holds for all $n \in \mathbb{N}^k$ and $s \in X_n$. Then it holds for all elements of $F(X)$. Now the injectivity of T_* gives (3.9).

□

Remark 3.1.3. In [54], Solel studied the product systems over $\mathbb{N}^k$ via different notations (see Appendix A). He used the notion *doubly commuting representation* [54, (3.12)] as an alternative for Nica-covariance representation. Then he proved in [54, Theorem 3.15] that the universal Nica-covariant representation ψ satisfies his doubly commuting relation. The doubly commuting relation involves a flip map between fibres of the product system. Since we have an explicit formula for the flip as in (3.5), we can translate his results to our notation. In Appendix A, we reconcile our result with [54, Theorem 3.15]. We show that ψ satisfies [54, Lemma 3.9(i)] by using our formula (3.9) and the flip map (3.5).

Lemma 3.1.4. *Let $h_1, \dots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. Suppose $m, n, p, q \in \mathbb{N}^k$ $x \in X_m$, $y \in X_n$, $s \in X_p$, and $t \in X_q$. Then there exist $\{\xi_{i,j}\}_{0\le i,j\le d} \subset X_{m+p-n\wedge p}$ and $\{\eta_{i,j}\}_{0\le i,j\le d} \subset X_{n+q-n\wedge p}$ such that*

$$\psi_m(x)\psi_n(y)^*\psi_p(s)\psi_q(t)^* = \sum_{0\le i,j\le d} \psi_{m+p-n\wedge p}(\xi_{i,j})\psi_{n+q-n\wedge p}(\eta_{i,j})^*. \tag{3.15}$$

Proof. Let $N := n - n\wedge p$ and $P := p - n \wedge p$. It suffices for us to prove (3.15) for $y = \sigma_{n\wedge p,N}(y'' \otimes y')$ and $s = \sigma_{n\wedge p,P}(s'' \otimes s')$, where $y'' \otimes y' \in X_{n\wedge p} \odot_A X_N, s'' \otimes s' \in X_{n\wedge p} \odot_A X_P$. Routine calculation shows that

$$\begin{aligned}
\psi_n(y)^*\psi_p(s) &= \psi_N(y')^*\psi_{n\wedge p}(y'')^*\psi_{n\wedge p}(s'')\psi_P(s') \\
&= \psi_N(y')^*\psi_0\big(\langle y'', s''\rangle\big)\psi_P(s') \\
&= \psi_N(y')^*\psi_P\big(\langle y'', s''\rangle \cdot s'\big). \qquad (3.16)
\end{aligned}$$

Let $\{U_i\}_{i=0}^d$ be an open cover of Z such that $h^N|_{U_i}$ and $h^P|_{Ui}$ are injective. Choose a partition of unity $\{\rho_i\}_{i=0}^d$ subordinate to $\{U_i\}_{i=0}^d$ and define $\tau_i := \sqrt{\rho_i}$. Since $N \wedge P = 0$, applying Proposition 3.1.2 to $\{\tau_i\}_{i=0}^d$ implies that

(3.17)

$$\psi_N(y')^*\psi_P\big(\langle y'', s''\rangle \cdot s'\big) = \sum_{0\le i,j\le d} \psi_P\big(\langle y', \tau_j \circ h^P\rangle \cdot \tau_i\big)\psi_N\Big(\Big\langle \langle y'', s''\rangle \cdot s', \tau_i \circ h^N\Big\rangle \cdot \tau_j\Big)^*.$$

Combining equations (3.16) and (3.17), we have

$$\begin{aligned}
&\psi_m(x)\psi_n(y)^*\psi_p(s)\psi_q(t)^* \\
&\quad = \psi_m(x)\Big[\sum_{0\le i,j\le d} \psi_P\big(\langle y',\tau_j\circ h^P\rangle\cdot\tau_i\big)\psi_N\Big(\Big\langle\langle y'',s''\rangle\cdot s',\tau_i\circ h^N\Big\rangle\cdot\tau_j\Big)^*\Big]\psi_q(t)^* \\
&\quad = \sum_{0\le i,j\le d}\psi_{m+P}\Big(\sigma_{m,P}\Big(x\otimes\langle y',\tau_j\circ h^P\rangle\cdot\tau_i\Big)\Big)\psi_{q+N}\Big(\sigma_{N,q}\Big(t\otimes\Big\langle\langle y'',s''\rangle\cdot s',\tau_i\circ h^n\Big\rangle\cdot\tau_j\Big)\Big)^*.
\end{aligned}$$

Now labelling $\xi_{i,j} := \sigma_{m,P}\big(x\otimes\langle y',\tau_j\circ h^P\rangle\cdot\tau_i\big)$ and $\eta_{i,j} := \sigma_{N,q}\Big(t\otimes\Big\langle\langle y'',s''\rangle\cdot s',\tau_i\circ h^n\Big\rangle\cdot\tau_j\Big)$ completes the proof of (3.15). $\square$

Lemma 3.1.5. *Suppose $m,n,p,q\in\mathbb{N}^k$ satisfying $m+p=n+q$ and $n\wedge p=0$. Then*

$$m-m\wedge q=n \ \textit{ and } \ q-m\wedge q=p.$$

Proof. We first prove $m-m\wedge q=n$. Fix $1\le i\le k$. Since $n\wedge p=0$, either $n_i=0$ or $p_i=0$.

If $n_i=0$, then $m+p=n+q$ implies that $m_i\le q_i$ and hence $m_i-(m\wedge q)_i = m_i-m_i=0=n_i$. If $p_i=0$, then $m_i\ge q_i$ and $m_i-(m\wedge q)_i=m_i-q_i=n_i-p_i=n_i$. Thus $m_i-(m\wedge q)_i=n_i$ for all i, as required.

To prove $q-m\wedge q=p$, it suffices to apply the construction of the previous paragraph to the equality $q+n=p+m$. $\square$

Now we are ready to prove a generalization of [1, Proposition 3.1] to our product system. There is also a similar Proposition for the higher-rank graph algebras (see [27, Proposition 3.1]).

Proposition 3.1.6. *Let $h_1,\dots,h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. Suppose $r\in(0,\infty)^k$ and $\alpha:\mathbb{R}\to\mathrm{Aut}(\mathcal{NT}(X))$ is given in terms of the gauge action by $\alpha_t=\gamma_{e^{itr}}$. Let $\beta>0$ and ϕ be a state on $\mathcal{NT}(X)$.*

(a) If ϕ satisfies

$$\phi\big(\psi_m(x)\psi_n(y)^*\big)=\delta_{m,n}e^{-\beta r\cdot m}\phi\circ\psi_0(\langle y,x\rangle) \textit{ for } x\in X_m, y\in X_n, \tag{3.18}$$

then ϕ is a KMS$_\beta$ state of $(\mathcal{NT}(X),\alpha)$.

(b) If ϕ is a KMS$_\beta$ state of $(\mathcal{NT}(X),\alpha)$ and $r\in(0,\infty)^k$ has rationally independent coordinates, then ϕ satisfies (3.18).

Proof of (a). Suppose state ϕ satisfies (3.18). To show that ϕ is a KMS_β state, it suffices to check the KMS condition

$$\phi(bc) = e^{-\beta r\cdot(m-n)}\phi(cb) \tag{3.19}$$

for elements $b = \psi_m(x)\psi_n(y)^*$ and $c = \psi_p(s)\psi_q(t)^*$ from $\mathcal{NT}(X)$. Let $M := m - m\wedge q$, $N := n - n\wedge p$, $P := p - n\wedge p$ and $Q := q - m\wedge q$. It is also enough to prove (3.19) for elements of the form $x = \sigma_{m\wedge q,M}(x''\otimes x')$, $y = \sigma_{n\wedge p,N}(y''\otimes y')$, $s = \sigma_{n\wedge p,P}(s''\otimes s')$ and $t = \sigma_{m\wedge q,Q}(t''\otimes t')$ where $x''\otimes x' \in X_{m\wedge q}\odot_A X_M$, $y''\otimes y' \in X_{n\wedge p}\odot_A X_N$, $s''\otimes s' \in X_{n\wedge p}\odot_A X_P$, and $t''\otimes t' \in X_{m\wedge q}\odot_A X_Q$. During the proof, we will need the following equations occasionally

$$\psi_n(y)^*\psi_p(s) = \psi_N(y')^*\psi_P\big(\langle y'', s''\rangle\cdot s'\big), \text{ and} \tag{3.20}$$

$$\psi_q(t)^*\psi_m(x) = \psi_Q(t')^*\psi_M\big(\langle t'', x''\rangle\cdot x'\big); \tag{3.21}$$

they are obtained by a calculation similar to the one done to establish (3.16).

To prove (3.19), first note that Lemma 3.1.4 together with the equation (3.18) imply that both of $\phi(bc)$ and $\phi(cb)$ vanish unless $m+p = n+q$. So we assume this from now. Next we claim that it suffices for us to check (3.19) for $n\wedge p = 0$. To see this, suppose we have proven the case $n\wedge p = 0$ and consider m, n, p, q such that $m+p = n+q$. Then (3.20) implies that $\phi(bc) = \phi\big(\psi_m(x)\psi_N(y')^*\psi_P\big(\langle y'', s''\rangle\cdot s'\big)\psi_q(t)^*\big)$. Since $N\wedge P = 0$, we are back into the other case. Thus

$$\phi(bc) = e^{-\beta r\cdot(m-N)}\phi\Big(\psi_P\big(\langle y'', s''\rangle\cdot s'\big)\psi_q(t)^*\psi_m(x)\psi_N(y')^*\Big).$$

Applying a similar calculation twice (by using (3.21) and (3.20)) gives:

$$\begin{aligned}
\phi(cb) &= \phi\Big(\psi_p(s)\psi_Q(t')^*\psi_M\big(\langle t'', x''\rangle\cdot x'\big)\psi_n(y)^*\Big)\\
&= e^{-\beta r\cdot(p-Q)}\phi\Big(\psi_M\big(\langle t'', x''\rangle\cdot x'\big)\psi_n(y)^*\psi_p(s)\psi_Q(t')^*\Big) \quad \text{(since } Q\wedge M = 0\text{)}\\
&= e^{-\beta r\cdot(p-Q)}\phi\Big(\psi_M\big(\langle t'', x''\rangle\cdot x'\big)\psi_N(y')^*\psi_P\big(\langle y'', s''\rangle\cdot s'\big)\psi_Q(t')^*\Big)\\
&= e^{-\beta r\cdot(p-Q+M-N)}\phi\Big(\psi_P\big(\langle y'', s''\rangle\cdot s'\big)\psi_Q(t')^*\psi_M\big(\langle t'', x''\rangle\cdot x'\big)\psi_N(y')^*\Big).
\end{aligned}$$

Since $m+p = n+q$, we have $e^{-\beta r\cdot(m-N)} = e^{-\beta r\cdot(m-n)}e^{-\beta r\cdot(p-Q+M-N)}$. Now (3.21) and our calculations imply that $\phi(bc) = e^{-\beta r\cdot(m-n)}\phi(cb)$. So it is enough to prove (3.19) when $n\wedge p = 0$.

Now we assume that $m+p = n+q$ and $n\wedge p = 0$. Let $\{U_i\}_{i=0}^d$ be an open cover of Z such that $h^n|_{U_i}$ and $h^p|_{Ui}$ are injective. Choose a partition of unity $\{\rho_i\}_{i=0}^d$ subordinate to $\{U_i\}_{i=0}^d$ and define $\tau_i := \sqrt{\rho_i}$. To compute $\phi(bc)$, we start by using (3.9) to rewrite $\psi_n(y)^*\psi_p(s)$ to get

$$\begin{aligned}
\phi(bc) &= \phi\Big(\psi_m(x)\psi_n(y)^*\psi_p(s)\psi_q(t)^*\Big) \\
&= \phi\Big(\psi_m(x)\Big[\sum_{0\le i,j\le d} \psi_p\big(\langle y, \tau_j \circ h^p\rangle \cdot \tau_i\big)\psi_n\big(\langle s, \tau_i \circ h^n\rangle \cdot \tau_j\big)^*\Big]\psi_q(t)^*\Big) \\
&= \sum_{0\le i,j\le d} \phi\bigg(\psi_{m+p}\Big(\sigma_{m,p}\Big(x \otimes \langle y, \tau_j \circ h^p\rangle \cdot \tau_i\Big)\Big)\psi_{q+n}\Big(\sigma_{q,n}\Big(t \otimes \langle s, \tau_i \circ h^n\rangle \cdot \tau_j\Big)\Big)^*\bigg).
\end{aligned}$$

By our assumption (3.18), we get

$$\phi(bc) = e^{-\beta r\cdot(m+p)}\phi \circ \psi_0\bigg(\sum_{0\le i,j\le d} \Big\langle \sigma_{q,n}\Big(t \otimes \langle s, \tau_i \circ h^n\rangle \cdot \tau_j\Big), \sigma_{m,p}\Big(x \otimes \langle y, \tau_j \circ h^p\rangle \cdot \tau_i\Big)\Big\rangle\bigg).$$

To calculate $\phi(cb)$, notice that $\phi(cb) = \phi\Big(\psi_p(s)\psi_Q(t')^*\psi_M\big(\langle t'', x''\rangle \cdot x'\big)\psi_n(y)^*\Big)$ by (3.21). Since $m+p = n+q$ and $n \wedge p = 0$, Lemma 3.1.5 implies that $Q = p$ and $M = n$. Then $\phi(cb) = \phi\big(\psi_p(s)\psi_p(t')^*\psi_n\big(\langle t'', x''\rangle \cdot x'\big)\psi_n(y)^*\big)$. Now we use the formula (3.9) and the identity $\big(\psi(\xi)\psi(\eta)\big)^* = \psi(\eta)^*\psi(\xi)^*$ to rewrite $\psi_p(t')^*\psi_n\big(\langle t'', x''\rangle \cdot x'\big)$.

$$\begin{aligned}
\phi(cb) &= \phi\Big(\psi_p(s)\Big[\sum_{0\le i,j\le d} \psi_n\big(\langle t', \tau_i \circ h^n\rangle \cdot \tau_j\big)\psi_p\big(\big\langle\langle t'', x''\rangle \cdot x', \tau_j \circ h^p\big\rangle \cdot \tau_i\big)^*\Big]\psi_n(y)^*\Big) \\
&= \phi\bigg(\sum_{0\le i,j\le d} \psi_{p+n}\Big(\sigma_{p,n}\Big(s \otimes \langle t', \tau_i \circ h^n\rangle\cdot\tau_j\Big)\Big) \\
&\qquad\qquad \psi_{n+p}\Big(\sigma_{n,p}\Big(y \otimes \big\langle\langle t'', x''\rangle\cdot x', \tau_j \circ h^p\big\rangle\cdot\tau_i\Big)\Big)^*\bigg)
\end{aligned}$$

Our assumption (3.18) implies that

$$\begin{aligned}
\phi(cb) = e^{-\beta r\cdot(n+p)}\phi \circ \psi_0\bigg(&\sum_{0\le i,j\le d}\Big\langle\sigma_{n,p}\Big(y \otimes \big\langle\langle t'', x''\rangle \cdot x', \tau_j \circ h^p\big\rangle \cdot \tau_i\Big), \\
&\sigma_{p,n}\Big(s \otimes \langle t', \tau_i \circ h^n\rangle \cdot \tau_j\Big)\Big\rangle\bigg).
\end{aligned}$$

Since $m+p = m-n+n+p$, it follows that $e^{-\beta r\cdot(m+p)} = e^{-\beta r\cdot(m-n+n+p)}$. Now to check KMS condition (3.19), it suffices to prove that

$$\dagger := \sum_{0\le i,j\le d} \Big\langle\sigma_{q,n}\Big(t \otimes \langle s, \tau_i \circ h^n\rangle \cdot \tau_j\Big), \sigma_{m,p}\Big(x \otimes \langle y, \tau_j \circ h^p\rangle\cdot\tau_i\Big)\Big\rangle$$

and

$$\ddagger := \sum_{0\le i,j\le d}\Big\langle\sigma_{n,p}\Big(y \otimes \big\langle\langle t'', x''\rangle \cdot x', \tau_j \circ h^p\big\rangle \cdot \tau_i\Big), \sigma_{p,n}\Big(s \otimes \langle t', \tau_i \circ h^n\rangle \cdot \tau_j\Big)\Big\rangle$$

are equal. To do this we compute $\dagger(z)$ and $\ddagger(z)$ for $z \in Z$. Since the calculation for $\ddagger(z)$ is easier, we compute it first. We start by applying the multiplication formula (2.4) in X:

$$\begin{aligned}
\ddagger(z) &= \sum_{0\le i,j\le d} \sum_{h^{n+p}(w)=z} \overline{\sigma_{n,p}\Big(y \otimes \big\langle \langle t'', x''\rangle \cdot x', \tau_j \circ h^p\big\rangle \cdot \tau_i\Big)(w)}\sigma_{p,n}\Big(s \otimes \langle t', \tau_i \circ h^n\rangle \cdot \tau_j\Big)(w) \\
&= \sum_{0\le i,j\le d} \sum_{h^{n+p}(w)=z} \overline{y(w)\big\langle\langle t'', x''\rangle \cdot x', \tau_j \circ h^p\big\rangle\big(h^n(w)\big)\tau_i\big(h^n(w)\big)} \\
&\qquad\qquad s(w)\langle t', \tau_i \circ h^n\rangle\big(h^p(w)\big)\tau_j\big(h^p(w)\big) \\
&= \sum_{h^{n+p}(w)=z} \overline{y(w)}s(w) \sum_{i=0}^{d} \langle t', \tau_i \circ h^n\rangle\big(h^p(w)\big)\tau_i\big(h^n(w)\big)
\end{aligned}$$

(3.22)

$$\sum_{j=0}^{d} \tau_j\big(h^p(w)\big)\big\langle \tau_j \circ h^p\,, \langle t'', x''\rangle \cdot x'\big\rangle\big(h^n(w)\big).$$

Since $n \wedge p = 0$, h^n and h^p are $*$-commuting. Now remembering that X_n, X_p are graph correspondences associated to the topological graphs (Z, Z, id, h^n) and (Z, Z, id, h^p) Lemma 3.1.1 implies that $\{\tau_j \circ h^p\}_{j=0}^d$ and $\{\tau_i \circ h^n\}_{i=0}^d$ are Parseval frames for X_n, X_p (respectively). We rearrange (3.22) by using the definition of the actions to apply the reconstruction formulas for these Parseval frames:

$$\begin{aligned}
\ddagger(z) &= \sum_{h^{n+p}(w)=z} \overline{y(w)}s(w) \sum_{i=0}^{d} \overline{\big(\tau_i \circ h^n \,\cdot\, \langle \tau_i \circ h^n, t'\rangle\big)(w)} \\
&\qquad \sum_{j=0}^{d} \big(\tau_j \circ h^p \cdot \big\langle \tau_j \circ h^p\,,\, \langle t'', x''\rangle \cdot x'\big\rangle\big)(w) \\
&= \sum_{h^{n+p}(w)=z} \overline{y(w)}s(w)\overline{t'(w)}(\langle t'', x''\rangle \cdot x')(w).
\end{aligned}$$

Next we compute $\dagger(z)$. Using the formula (2.4) for multiplication in X, we have

$$\begin{aligned}
\dagger(z) &= \sum_{0\le i,j\le d} \sum_{h^{m+p}(w)=z} \overline{\sigma_{q,n}\Big(t \otimes \langle s, \tau_i \circ h^n\rangle \cdot \tau_j\Big)(w)}\sigma_{m,p}\Big(x \otimes \langle y, \tau_j \circ h^p\rangle \cdot \tau_i\Big)(w) \\
&= \sum_{0\le i,j\le d} \sum_{h^{m+p}(w)=z} \overline{t(w)\langle s, \tau_i \circ h^n\rangle(h^q(w))\tau_j(h^q(w))}x(w)\langle y, \tau_j \circ h^p\rangle(h^m(w))\tau_i(h^m(w)) \\
&= \sum_{h^{m+p}(w)=z} \overline{t(w)}x(w) \sum_{i=0}^{d} \overline{\langle s, \tau_i \circ h^n\rangle(h^q(w))}\tau_i(h^m(w)) \sum_{j=0}^{d} \tau_j(h^q(w))\langle y, \tau_j \circ h^p\rangle(h^m(w)).
\end{aligned}$$

An application of Lemma 3.1.5 implies that $q = m \wedge q + p$ and $m = m \wedge q + n$. Then

$$\dagger(z) = \sum_{h^{n+p+m\wedge q}(w)=z} \overline{t(w)}x(w) \sum_{i=0}^{d} \big\langle \tau_i \circ h^n, s\big\rangle\big(h^{m\wedge q+p}(w)\big)\tau_i\big(h^{m\wedge q+n}(w)\big)$$

$$\sum_{j=0}^{d} \tau_j\big(h^{m\wedge q+p}(w)\big)\langle y, \tau_j \circ h^p\rangle\big(h^{m\wedge q+n}(w)\big).$$

We again rearrange this equation to apply the reconstruction formulas for the Parseval frames $\{\tau_i \circ h^n\}_{i=0}^d$ and $\{\tau_j \circ h^p\}_{j=0}^d$.

$$\begin{aligned}
\dagger(z) &= \sum_{h^{n+p+m\wedge q}(w)=z} \overline{t(w)}x(w) \sum_{0\le i\le d} \big[\tau_i \circ h^n \,\cdot\, \big\langle \tau_i \circ h^n\,,\, s\big\rangle\big]\big(h^{m\wedge q}(w)\big) \\
&\qquad\qquad \sum_{0\le j\le d} \overline{\big[\tau_j \circ h^p \cdot \langle \tau_j \circ h^p\,,\, y\rangle\big]\big(h^{m\wedge q}(w)\big)} \\
&= \sum_{h^{n+p+m\wedge q}(w)=z} \overline{t(w)}x(w)s\big(h^{m\wedge q}(w)\big)\overline{y\big(h^{m\wedge q}(w)\big)}.
\end{aligned}$$

Now writing $t = \sigma_{m\wedge q,Q}(t'' \otimes t')$, $x = \sigma_{m\wedge q,M}(x'' \otimes x')$ and splitting $\sum$, we have

$$\begin{aligned}
\dagger(z) &= \sum_{h^{n+p+m\wedge q}(w)=z} \overline{t''(w)t'\big(h^{m\wedge q}(w)\big)}x''(w)x'\big(h^{m\wedge q}(w)\big)s\big(h^{m\wedge q}(w)\big)\overline{y\big(h^{m\wedge q}(w)\big)} \\
&= \sum_{h^{n+p}(u)=z} s(u)x'(u)\overline{y(u)t'(u)} \sum_{h^{m\wedge q}(w)=u} \overline{t''(w)}x''(w) \\
&= \sum_{h^{n+p}(u)=z} s(u)x'(u)\overline{y(u)t'(u)}\langle t'', x''\rangle(u)
\end{aligned}$$

Thus $\dagger(z) = \ddagger(z)$ and hence ϕ satisfies (3.19). $\square$

Proof of (b). Suppose ϕ is a KMS$_\beta$ state on $\mathcal{NT}(X)$ and r has rationally independent coordinates. To show that ϕ satisfies (3.18), let $x \in X_m$ and $y \in X_n$. By two application of the KMS condition, we have

$$\begin{aligned}
\phi\big(\psi_m(x)\psi_n(y)^*\big) &= \phi\big(\psi_n(y)^*\alpha_{i\beta}(\psi_m(x))\big) \\
&= e^{-\beta r\cdot m}\phi\big(\psi_n(y)^*\psi_m(x)\big) \\
&= e^{-\beta r\cdot (m-n)}\phi\big(\psi_m(x)\psi_n(y)^*\big).
\end{aligned}$$

Now since r has rationally independent coordinates and $\beta > 0$, both sides will vanish for $m \neq n$. For $m = n$ the KMS condition and (T2) of Definition1.3.1 imply that

$$\phi\big(\psi_m(x)\psi_m(y)^*\big) = e^{-\beta r\cdot m}\phi\big(\psi_m(y)^*\psi_m(x)\big) = e^{-\beta r\cdot m}\phi\big(\psi_0(\langle y, x\rangle)\big),$$

and ϕ satisfies (3.18). $\square$

3.2 KMS states and subinvariance relation

In this section we introduce a subinvariance relation involving a family of "Ruelle operators". We characterize the solutions of this subinvariance relation in Proposition 3.2.7. We also show that every KMS$_\beta$ state for $\beta \in (0, \infty)$ gives a measure which satisfies our subinvariance relation (Proposition 3.2.8).

Lemma 3.2.1. *Let $h_1, \dots, h_k$ be commuting and surjective local homeomorphisms on a compact Hausdorff space Z. For $i \in \{1, \dots, k\}$, define $Q_i : C(Z) \to C(Z)$ by*

$$Q_i(a)(z) = \sum_{h_i(w)=z} a(w) \text{ for } a \in C(Z).$$

(a) The functions $Q_i : C(Z) \to C(Z)$ are commuting linear bounded operators.

(b) For $n \in \mathbb{N}^k$, set $Q^n := Q_k^{n_k} \circ \cdots \circ Q_1^{n_1}$. Then

$$Q^n(a)(z) = \sum_{h^n(w)=z} a(w) \text{ for } a \in C(Z). \tag{3.23}$$

(c) For each $1 \le i \le k$, there is a unique adjoint operator $Q_i^* : C(Z)^* \to C(Z)^*$ such that

$$\|Q_i^*\| = \|Q_i\| \text{ and } Q_i^*(f) = f \circ Q_i \text{ for } f \in C(Z)^*.$$

Proof. To prove (a), take $1 \le i \le k$ and $a \in C(Z)$. It is clear that Q_i is linear. Notice that

$$\begin{aligned}\|Q_i(a)\| = \sup_{z\in Z} |Q_i(a)(z)| = \sup_{z\in Z} \Big| \sum_{h_i(w)=z} a(w) \Big| &\le \max_{z\in Z} |h_i^{-1}(z)| \sup_{z\in Z} |a(z)| \\ &= \max_{z\in Z} |h_i^{-1}(z)| \|a\|.\end{aligned}$$

Since h_i is a local homeomorphism on the compact space Z, $\max_{z\in Z} |h_i^{-1}(z)| < \infty$. It then follows that Q_i is bounded and $\|Q_i\| \le \max_{z\in Z} |h_i^{-1}(z)|$.

For the commutativity, take $1 \le i, j \le k$. We have

$$\begin{aligned}\big(Q_iQ_j(a)\big)(z) = \Big(Q_i\big(Q_j(a)\big)\Big)(z) &= \sum_{h_i(w)=z} \big(Q_j(a)\big)(w) \\ &= \sum_{h_i(w)=z} \sum_{h_j(u)=w} a(u) = \sum_{h_i\circ h_j(u)=z} a(u).\end{aligned} \tag{3.24}$$

Since h_i, h_j are commuting, (3.24) implies $Q_iQ_j = Q_jQ_i$.

For part (b), notice that $\{Q_i\}$ are commuting and surjective local homeomorphisms. Then (3.23) follows from (3.24).

Finally part (c) follows from [19, page 160 (Exercise 22)] or from [47, Theorem 4.10]. □

Definition 3.2.2. Let $h_1, \dots, h_k$ be commuting and surjective local homeomorphisms on a compact Hausdorff space Z. Let $Q_1, \dots, Q_k$ be as in Lemma 3.2.1. For $1 \le i \le k$, we define $R^{e_i} : C(Z)^* \to C(Z)^*$ by $R^{e_i} := Q_i^*$. Then $R^{e_1}, \dots, R^{e_k}$ are commuting, linear bounded operators. We write $R^0 := \mathrm{id}_{C(Z)^*}$ and for $n \in \mathbb{N}_+^k$, we use

$$R^n := R^{n_k e_k} \circ \cdots \circ R^{n_1 e_1}.$$

The operators $R^{e_1}, \dots, R^{e_k}$ are sometimes called *Ruelle operators* (for example see [48, (2.3)],[49, (3.1)],[14, (2.1)]).

Remark 3.2.3. A finite regular Borel measure ν on Z can be viewed as an element of $C(Z)^*$ by

$$\nu(a) := \int a(z)\, d\nu(z) \text{ for } a \in C(Z).$$

We can then calculate a formula for $R^n(\nu)$. Lemma 3.2.1(c) implies that $R^n(\nu) = (Q^n)^*(\nu) = \nu(Q^n)$. It then follows

$$\int a\, d\big(R^n(\nu)\big) = \int \sum_{h^n(w)=z} a(w)\, d\nu(z) \quad \text{for } a \in C(Z). \tag{3.25}$$

Remark 3.2.4. The operation R in (3.25) is an analogue for the operation R studied in [1]. But here we define it as an operator on the whole of $C(Z)^*$, while in [1] it is only defined on measures (which are positive elements of $C(Z)^*$).

Definition 3.2.5. Let $h_1, \dots, h_k$ be commuting and surjective local homeomorphisms on a compact Hausdorff space Z and suppose ν is a finite regular Borel measure on Z. We say ν satisfies *the subinvariance relation* if for every subset K of $\{1, \dots, k\}$, we have

$$\int a\, d\Big(\prod_{i\in K}(1 - e^{-\beta r_i} R^{e_i})\nu\Big) \ge 0 \text{ for all positive } a \in C(Z). \tag{3.26}$$

Given $J \subseteq K$, we write $e_J := \sum_{j\in J} e_j$ and we interpret $R^{e_\varnothing}\nu = \nu$. The following identity is helpful when we work with the subinvariance relation.

$$\Big(\prod_{i\in K}(1 - e^{-\beta r_i} R^{e_i})\Big)\nu = \sum_{\varnothing\subseteq J\subseteq K} (-1)^{|J|} e^{-\beta r\cdot e_J} R^{e_J}\nu. \tag{3.27}$$

Remark 3.2.6. The subinvariance relation (3.26) is a generalization of the subinvariance relation [1, (4.2)] where we have only one local homeomorphism. It also is a variant of the subinvariance relation appearing in the analysis of KMS states of the Toeplitz-Cuntz-Krieger algebras of higer-rank graphs [28, (Proposition 4.1 (a)].

The next Proposition characterizes the solutions of the subinvariance relation (3.26). It is a generalization of [1, Proposition 4.2] and [28, Theoerem 6.1(a)].

Proposition 3.2.7. *Let $h_1, \dots, h_k$ be surjective and commuting local homeomorphisms on a compact Hausdorff space Z. For each $1 \leq i \leq k$, let*

$$\beta_{c_i} := \limsup_{j\to\infty} \Big(j^{-1} \ln \Big(\max_{z\in Z} |h_i^{-j}(z)| \Big) \Big). \tag{3.28}$$

Let $r \in (0,\infty)^k$, and suppose $\beta \in (0,\infty)$ satisfies $\beta r_i > \beta_{c_i}$.

(a) The series $\sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n} |h^{-n}(z)|$ converges uniformly for $z \in Z$ to a continuous function $f_\beta(z) \geq 1$.

(b) Suppose ε is a finite regular Borel measure on Z. Then the series $\sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n} R^n \varepsilon$ converges in norm in the dual space $C(Z)^*$ with sum μ, say. Then μ satisfies the subinvariance relation (3.26) *and we have $\varepsilon = \big(\prod_{i=1}^k \big(1 - e^{-\beta r_i} R^{e_i}\big)\big)\mu$. Then μ is a probability measure if and only if $\int f_\beta \, d\varepsilon = 1$.*

(c) Suppose μ is a probability measure which satisfies the subinvariance relation (3.26)*. Then $\varepsilon = \big(\prod_{i=1}^k \big(1 - e^{-\beta r_i} R^{e_i}\big)\big)\mu$ is a finite regular Borel measure satisfying $\sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n} R^n \varepsilon = \mu$, and we have $\int f_\beta \, d\varepsilon = 1$.*

Before starting the proof, notice that we regard a sum indexed by $\mathbb{N}^k$ as an integral over $\mathbb{N}^k$ with respect to the counting measure. All series here have positive summands. Then by Tonelli's theorem, we can consider a sum over $\mathbb{N}^k$ as iterated sums over $\mathbb{N}$. Moreover, if the iterated sums over $\mathbb{N}$ are convergent in one order, then the sum over $\mathbb{N}^k$ converges as well (see for example [19, Theorem 7.27]).

We will need the following algebraic identities occasionally:

$$\sum_{m\in\mathbb{N}^k} \prod_{i=1}^k f_i(m_i) = \prod_{i=1}^k \sum_{m_i\in\mathbb{N}} f_i(m_i). \tag{3.29}$$

Also notice that if $f_i g_j = g_j f_i$ for all $1 \leq i, j \leq k$, then

$$\prod_{i=1}^k f_i \prod_{j=1}^k g_j = \prod_{l=1}^k f_l g_l. \tag{3.30}$$

Proof of Proposition 3.2.7. For part (a), we first claim that for each $1 \le i \le k$, there exist $0 < \delta_i \in \mathbb{R}$ and $M_i \in \mathbb{N}$ such that

$$l \in \mathbb{N}, l \ge M_i \Rightarrow e^{-l\beta r_i} \max_z |h_i^{-l}(z)| < e^{-l\delta_i} \quad \text{for all } z \in Z. \tag{3.31}$$

To see the claim, since $\beta r_i > \beta_{c_i}$, applying the calculation of the first paragraph in the proof of [1, Proposition 4.2] with the local homeomorphism h_i gives δ_i and M_i satisfying (3.31). Now we take $M := (M_1, \dots, M_k)$ and calculate the N-th partial sum for $N \ge M$.

$$\begin{aligned}
\sum_{M \le n \le N} e^{-\beta r \cdot n} |h^{-n}(z)| &= \sum_{M \le n \le N} e^{-\beta r \cdot n} \big| \big(h_1^{n_1} \circ \cdots \circ h_k^{n_k}\big)^{-1}(z) \big| \\
&\le \sum_{M \le n \le N} e^{-\beta r \cdot n} \Big(\prod_{i=1}^{k} \max_z \big|h_i^{-n_i}(z)\big| \Big) \\
&= \sum_{M \le n \le N} \prod_{i=1}^{k} \Big(e^{-\beta r_i \cdot n_i} \max_z \big|h_i^{-n_i}(z)\big| \Big).
\end{aligned}$$

Using the identity (3.29) and the equation (3.31), we have

$$\begin{aligned}
\sum_{M \le n \le N} e^{-\beta r \cdot n} |h^{-n}(z)| &\le \prod_{i=1}^{k} \sum_{M_i \le n_i \le N_i} \Big(e^{-\beta r_i \cdot n_i} \max_z \big|h_i^{-n_i}(z)\big| \Big) \\
&\le \prod_{i=1}^{k} \sum_{M_i \le n_i \le N_i} e^{-\delta_i n_i}.
\end{aligned} \tag{3.32}$$

Now let $N \to \infty$ in $\mathbb{N}^k$. This means each $N_i \to \infty$ for $1 \le i \le k$. Since each sum $\sum_{n_i = M_i}^{\infty} e^{\delta_i n_i}$ is convergent, it follows that $\sum_{n=M}^{\infty} e^{-\beta r \cdot n} |h^{-n}(z)|$ converges uniformly for $z \in Z$.

Notice that $h^n = h_k^{n_k} \circ \cdots \circ h_1^{n_1}$ is a local homeomorphism on Z for all $n \in \mathbb{N}^k$ (because each h_i $(1 \le i \le k)$ is). Then [8, Lemma 2.2] implies that $z \mapsto |h^{-n}(z)|$ is locally constant and hence is continuous. Thus $f_\beta(z) := \sum_{n \in \mathbb{N}^k} e^{-\beta n} |h^{-n}(z)|$ is the uniform limit of a sequence of continuous functions, and is therefore continuous. The term corresponding to $n = 0$ is 1, so $f_\beta \ge 1$.

For part (b), take M and δ_i $(1 \le i \le k)$ as in part (a). We want to show that $\sum_{n \ge M} e^{-\beta r \cdot n} R^n \varepsilon$ converges in norm in the dual space $C(Z)^*$. To do this, we calculate the N-th partial sum using formula (3.25) for the definition of R^n. Let $g \in C(Z)$, we have

$$\Big| \sum_{M \le n \le N} e^{-\beta r \cdot n} \int g \, d(R^n \varepsilon) \Big| = \Big| \sum_{M \le n \le N} e^{-\beta r \cdot n} \int \sum_{h^n(w) = z} g(w) \, d\varepsilon(z) \Big|$$

$$\leq \sum_{M\leq n\leq N} e^{-\beta r\cdot n}|h^{-n}(z)|\,\|\varepsilon\|_{C(Z)^*}\|g\|_\infty$$

$$\leq \|\varepsilon\|_{C(Z)^*}\|g\|_\infty \prod_{i=1}^{k}\sum_{M_i\leq n_i\leq N_i} e^{-\delta_i n_i} \qquad \text{by (3.32).} \tag{3.33}$$

Now when $N\to\infty$, all the series $\sum_{n_i=M_i}^{\infty} e^{-\delta_i n_i}$ are convergent and hence the series $\sum_{n=0}^{\infty} e^{-\beta r\cdot n}R^n\varepsilon$ converges in the norm of $C(Z)^*$ to a measure μ, say.

Since ε is a measure on Z, it is a positive functional on $C(Z)$. The formula (3.25) for definition of R^n, says that μ is positive functional on $C(Z)$ and therefore is a Borel measure on Z by the Riesz-representation theorem.

To prove that μ satisfies the subinvariance relation (3.26), let $K\subset\{1,\dots,k\}$. We first simplify the N-th partial sum $\big(\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\big)\sum_{0\leq n\leq N} e^{-\beta r\cdot n}R^n$ for $N\in\mathbb{N}^k$. We have,

$$\Big(\prod\nolimits_{j\in K}(1-e^{-\beta r_j}R^{e_j})\Big)\sum_{0\leq n\leq N} e^{-\beta r\cdot n}R^n = \Big(\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\Big)\Big(\sum_{0\leq n\leq N}\prod_{i=1}^{k} e^{-\beta r_i n_i}R^{n_i e_i}\Big)$$

$$= \Big(\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\Big)\Big(\prod_{i=1}^{k}\sum_{n_i=0}^{N_i} e^{-\beta r_i n_i}R^{n_i e_i}\Big) \quad \text{by identity (3.29)}$$

$$= \Big(\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\Big)\Big(\prod_{i\in K}\sum_{n_i=0}^{N_i} e^{-\beta r_i n_i}R^{n_i e_i}\prod_{i\in\{1,\dots,k\}\setminus K}\sum_{n_i=0}^{N_i} e^{-\beta r_i n_i}R^{n_i e_i}\Big) \text{ by (3.30).}$$

Relabelling the indices in products, we have

$$\Big(\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\Big)\sum_{0\leq n\leq N} e^{-\beta r\cdot n}R^n$$

$$= \prod_{i\in\{1,\dots,k\}\setminus K}\Big(\sum_{n_i=0}^{N_i} e^{-\beta r_i n_i}R^{n_i e_i}\Big)\prod_{j\in K}\Big(\sum_{n_j=0}^{N_j} e^{-\beta r_j n_j}R^{n_j e_j} - \sum_{n_j=0}^{N_j} e^{-\beta r_j (n_j+1)}R^{(n_j+1)e_j}\Big). \tag{3.34}$$

Now we can compute $\big(\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\big)\sum_{n\geq 0} e^{-\beta r\cdot n}R^n\varepsilon$ by applying (3.34) to ε and letting $N\to\infty$. Notice that for each $j\in K$, we have

$$\sum_{n_j=0}^{\infty} e^{-\beta r_j n_j}R^{n_j e_j}\varepsilon - \sum_{n_j=0}^{\infty} e^{-\beta r_j(n_j+1)}R^{(n_j+1)e_j}\varepsilon = \varepsilon.$$

It then follows that

$$\Big(\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\Big)\sum_{n\geq 0} e^{-\beta r\cdot n}R^n\varepsilon = \prod_{i\in\{1,\dots,k\}\setminus K}\Big(\sum_{n_i=0}^{\infty} e^{-\beta r_i n_i}R^{n_i e_i}\Big)\varepsilon.$$

The argument in the last paragraph of the proof of [1, Proposition 4.2(b)], shows that applying each $\sum_{n_i=0}^{\infty} e^{-\beta r_i n_i}R^{n_i e_i}$ to a finite regular Borel measure gives a finite regular

Borel measure. It then follows that $\prod_{j\in K}(1-e^{-\beta r_j}R^{e_j})\Big)\sum_{n\geq 0}e^{-\beta r\cdot n}R^n\varepsilon$ is a finite regular Borel measure. Thus

$$\int a\,d\Big(\prod_{i\in K}(1-e^{-\beta r_i}R^{e_i})\sum_{0\leq n\leq N}e^{-\beta r\cdot n}R^n\varepsilon\Big)\geq 0 \text{ for all positive } a\in C(Z).$$

Thus μ satisfies the subinvariance relation (3.26).

To prove that $\varepsilon=\big(\prod_{i=1}^k\big(1-e^{-\beta r_i}R^{e_i}\big)\big)\mu$, it suffices to apply the argument of the previous two paragraphs with $K=\{1,\ldots,k\}$.

To see the relation between μ and f_β, we compute using (3.25):

$$\begin{aligned}\mu(Z)&=\sum_{n\in\mathbb{N}^k}e^{-\beta r\cdot n}(R^n\varepsilon)(Z)\\&=\sum_{n\in\mathbb{N}^k}e^{-\beta r\cdot n}\int 1\,d(R^n\varepsilon)\\&=\sum_{n\in\mathbb{N}^k}e^{-\beta r\cdot n}\int|h^{-n}(z)|\,d\varepsilon(z).\end{aligned}$$

An application of Tonelli's theorem implies that

$$\mu(Z)=\int\sum_{n\in\mathbb{N}^k}(e^{-\beta r\cdot n}|h^{-n}(z)|)\,d\varepsilon(z)=\int f_\beta\,d\varepsilon.$$

Since Z is compact and f_β is continuous on Z, $\mu(Z)=\int f_\beta\,d\varepsilon<\infty$. Also μ is a probability measure if and only if $\int f_\beta\,d\varepsilon=1$.

We now look at (c). First note that the measure ε is obtained by finitely many times applications of the bounded operators $R^{e_i}(1\leq i\leq k)$ on the measure μ. Since μ is a finite measure, ε is a finite measure as well. The subinvariance relation (3.26) says that ε is a positive measure. An application of the Riesz-representation theorem implies that ε is a Borel measure on Z. Since ε is finite, it is regular as well (see [19, Theorem 7.8]).

To check

$$\sum_{n\in\mathbb{N}^k}e^{-\beta r\cdot n}R^n\varepsilon=\mu, \tag{3.35}$$

we calculate the N-th partial sum using the identity (3.29):

$$\begin{aligned}\sum_{0\leq n\leq N}e^{-\beta r\cdot n}R^n\varepsilon&=\Big(\sum_{0\leq n\leq N}e^{-\beta r\cdot n}R^n\prod_{i=1}^k\big(1-e^{-\beta r_i}R^{e_i}\big)\Big)\mu\\&=\Big(\sum_{0\leq n\leq N}\prod_{i=1}^k e^{-\beta r_i n_i}R^{n_i e_i}\big(1-e^{-\beta r_i}R^{e_i}\big)\Big)\mu\end{aligned}$$

$$(3.36) \qquad = \Big(\prod_{i=1}^{k}\sum_{n_i=0}^{N_i}\big(e^{-\beta r_i n_i}R^{n_i e_i} - e^{-\beta r_i(n_i+1)}R^{(n_i+1)e_i}\big)\Big)\mu.$$

Let $i \in \{1, \dots, k\}$ and $N_i \to \infty$. Applying each sum in the last line of (3.36) to μ, we have

$$\sum_{n_i=0}^{\infty} e^{-\beta r_i n_i}R^{e_i n_i}\mu - \sum_{n_i=0}^{\infty} e^{-\beta r_i(n_i+1)}R^{e_i(n_i+1)}\mu = \mu.$$

Taking the product over $1 \le i \le k$, completes the proof of (3.35). □

The next Proposition shows that every KMS states on $(\mathcal{NT}(X), \alpha)$ gives a probability measure on Z satisfying the subinvariance relation (3.37). This proposition is an extension of our result in [1, Proposition 4.1] for a single local homeomorphism. There is also a similar result for the Toeplitz-Cuntz-Krieger algebra of a higher-rank graph in [28, Proposition 4.1(a)].

Proposition 3.2.8. *Let $h_1, \dots, h_k$ be commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system over $\mathbb{N}^k$, as in Corollary 2.1.2. Let $r \in (0, \infty)^k$ and suppose that $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathcal{NT}(X))$ is given in terms of the gauge action by $\alpha_t = \gamma_{e^{itr}}$. Suppose ϕ is a KMS$_\beta$ state of $(\mathcal{NT}(X), \alpha)$, and μ is the probability measure on Z such that $\phi(\psi_0(a)) = \int a\,d\mu$ for all $a \in C(Z)$. Let K be a subset of $\{1, \dots, k\}$ and write $e_J := \sum_{j\in J} e_j$ for all $J \subseteq K$. Then*

$$(3.37) \qquad \int a\,d\mu + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} e^{-\beta r\cdot e_J} \int a\,d\big(R^{e_J}\mu\big) \ge 0 \text{ for all positive } a \in C(Z).$$

To prove the Proposition 3.2.8, we need the following simple lemma.

Lemma 3.2.9. *Let $h_1, \dots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. Let T be the Fock representation of X. Take $n \in \mathbb{N}^k$, and let $\{\rho_l\}_{l=0}^d$ be a partition of unity such that $h^n|_{\operatorname{supp}\rho_l}$ is injective for each l. Set $\tau_l := \sqrt{\rho_l}$. Then the restriction of $\sum_{l=0}^d T_n(\tau_l)T_n(\tau_l)^*$ to each m-summand X_m of the Fock module is the identity map if $m \ge n$, and is otherwise 0.*

Proof. Let $m \in N^k$. If $m \not\ge n$, then the adjoint formula for the Fock representation (1.17) implies that $\sum_{l=0}^d T_n(\tau_l)T_n(\tau_l)^*$ vanishes on X_m. Now let $m \ge n$. It suffices to prove

$$\sum_{l=0}^{d} T_n(\tau_l)T_n(\tau_l)^*(x) = x,$$

for $x = \sigma_{n,,m-n}(x' \otimes x'')$ where $x' \otimes x'' \in X_n \odot_A X_{m-n}$.

To see this, we compute by using the definition of the Fock representation and the adjoint formula (1.17):

$$\Big(\sum_{l=0}^{d} T_n(\tau_l)T_n(\tau_l)^*\Big)\Big(\sigma_{n,m-n}(x' \otimes x'')\Big) = \sum_{l=0}^{d} T_n(\tau_l)\big(\langle \tau_l, x'\rangle \cdot x''\big)$$
$$= \sum_{l=0}^{d} \sigma_{n,n-m}\big(\tau_l \otimes \langle \tau_l, x'\rangle \cdot x''\big).$$

Lemma 3.1.1(a) implies that $\{\tau_l\}_{l=0}^{d}$ is a Parseval frame for the fibre X_m. Applying the reconstruction formula for $\{\tau_l\}_{l=0}^{d}$, we have

$$\Big(\sum_{l=0}^{d} T_n(\tau_l)T_n(\tau_l)^*\Big)\Big(\sigma_{n,m-n}(x' \otimes x'')\Big) = \sigma_{m,m-n}\Big(\sum_{l=0}^{d} \tau_l \cdot \langle \tau_l, x'\rangle \otimes x''\Big)$$
$$= \sigma_{n,m-n}\big(x' \otimes x''\big)$$

which is precisely x as required. □

Proof of Proposition 3.2.8. Let a be a positive element of $C(Z)$. If $K = \varnothing$, since a is positive, $\int a\, d\mu \geq 0$. So we assume $K \neq \varnothing$. We apply the method of the proof of [1, Proposition 4.1]. So we first write each integral in (3.37) in terms of elements of $\mathcal{NT}(X)$ and then use the Fock representation to show that the sum of these integrals is positive.

The first integral in (3.37) by assumption is

$$\int a\, d\mu = \phi\big(\psi_0(a)\big). \tag{3.38}$$

Now consider J-summand. To write the integral $\int a\, d\big(R^{e_J}\mu\big)$ in terms of elements of $\mathcal{NT}(X)$, let $\{U_l^J\}_{l=0}^{d}$ be an open cover of Z such that $h^{e_J}|_{U_l^J}$ is injective and choose a partition of unity $\{\rho_l^J\}_{l=0}^{d}$ subordinate to $\{U_l^J\}_{l=0}^{d}$. Define $\tau_l^J := \sqrt{\rho_l}$. Remember that the fibre X_{e_J} in X is the graph correspondence $(Z, Z, \mathrm{id}, h^{e_J})$. Then applying the calculation in the first two paragraphs of [1, Proposition 4.1], to X_{e_J} shows that

$$\int a\, d\big(R^{e_J}\mu\big) = e^{\beta r \cdot e_J}\phi\Big(\sum_{l=0}^{d} \psi_{e_J}(a \cdot \tau_l^J)\psi_{e_J}(\tau_l^J)^*\Big)$$
$$= e^{\beta r \cdot e_J}\phi\Big(\sum_{l=0}^{d} \psi_0(a)\psi_{e_J}(\tau_l^J)\psi_{e_J}(\tau_l^J)^*\Big). \tag{3.39}$$

Putting (3.38) and (3.39) in the left-hand side of (3.37), we have

$$\int a\,d\mu + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} e^{-\beta r\cdot e_J} \int a\, d\big(R^{e_J}\mu\big)$$
$$= \phi(\psi_0(a)) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \phi\Big(\sum_{l=0}^{d} \psi_0(a)\psi_{e_J}(\tau_l^J)\psi_{e_J}(\tau_l^J)^*\Big)$$
$$= \phi\Big(\psi_0(a) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \sum_{l=0}^{d} \psi_0(a)\psi_{e_J}(\tau_l^J)\psi_{e_J}(\tau_l^J)^*\Big), \tag{3.40}$$

which express the integrals in (3.37) in terms of elements of $\mathcal{NT}(X)$.

Next we show that the right-hand side of (3.40) is positive. Since ϕ is a state, it suffices to show that

$$\psi_0(a) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \psi_0(a) \sum_{l=0}^{d} \psi_{e_J}(\tau_l^J)\psi_{e_J}(\tau_l^J)^* \geq 0. \tag{3.41}$$

To do this, we use the Fock representation T of X. We aim to prove

$$\Big(T_0(a) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} T_0(a) \sum_{l=0}^{d} T_{e_J}(\tau_l^J)T_{e_J}(\tau_l^J)^*\Big)(x_n) \geq 0 \tag{3.42}$$

for all $x_n \in X_n, n \in \mathbb{N}^k$.

Fix $n \in \mathbb{N}^k$ and $x_n \in X_n$. Let $I := \{i \mid i \in K, n_i \neq 0\}$. Applying Lemma 3.2.9 with $\{\tau_l^J\}_{l=0}^d$ implies that the J-summands with $n \ngeq e_J$ vanishes. Since $n \geq e_J$ is equivalent to $J \subseteq I$, the outer sum in (3.42) reduces to

$$\Big(T_0(a) + \sum_{\varnothing \subsetneq J \subseteq I} (-1)^{|J|} T_0(a) \sum_{l=0}^{d} T_{e_J}(\tau_l^J)T_{e_J}(\tau_l^J)^*\Big)(x_n).$$

Now we compute using Lemma 3.2.9

$$\Big(T_0(a) + \sum_{\varnothing \subsetneq J \subseteq I} (-1)^{|J|} T_0(a) \sum_{l=0}^{d} T_{e_J}(\tau_l^J)T_{e_J}(\tau_l^J)^*\Big)(x_n)$$
$$= T_0(a)(x_n) + \sum_{\varnothing \subsetneq J \subseteq I} (-1)^{|J|} T_0(a)(x_n)$$
$$= \sum_{\varnothing \subset J \subseteq I} (-1)^{|J|} T_0(a)(x_n).$$

This vanishes because the number of subsets with odd cardinality equals with the number of subsets with even cardinality. Thus

$$\Big(T_0(a) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \sum_{l=0}^{d} T_{e_J}(a\cdot\tau_l^J)T_{e_J}(\tau_l^J)^*\Big)(x_n) \geq 0.$$

We now deduce that $T_0(a) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \sum_{l=0}^{d} T_{e_J}(a \cdot \tau_l^J) T_{e_J}(\tau_l^J)^*$ is a positive operator on $F(X)$. Since the induced homomorphism $T_* : \mathcal{NT}(X) \to \mathcal{L}(F(X))$ is an injection (see Remark 1.4.1), it follows that

$$\psi_0(a) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \psi_0(a) \sum_{l=0}^{d} \psi_{e_J}(\tau_l^J) \psi_{e_J}(\tau_l^J)^* \geq 0,$$

as required. $\square$

3.3 KMS states at large inverse temperatures

In this section we prove our main theorem which characterizes the KMS$_\beta$ states of $(\mathcal{NT}(X), \alpha)$ for large β. We found a one-to-one correspondence between the KMS states on $(\mathcal{NT}(X), \alpha)$ and the probability measures on Z satisfying the subinvariance relation (3.26). This theorem is a generalization of our result in [1, Theorem 6.1] for dynamical system associated to a single local homeomorphism. There is also a similar characterization in [27, Theorem 6.1] for the dynamics arising from higher-rank graphs. As a corollary, we also obtain some results for the dynamical system $(\mathcal{O}(X), \tilde{\alpha})$.

Theorem 3.3.1. *Let $h_1, \ldots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z. Let X be the associated product system over $\mathbb{N}^k$, as in Corollary 2.1.2. For $1 \leq i \leq k$ let β_{c_i} be as in (3.28), and suppose that $r \in (0, \infty)^k$ satisfies $\beta r_i > \beta_{c_i}$ for all i. Let f_β be the function in Proposition 3.2.7(a) and define $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathcal{NT}(X))$ by $\alpha_t = \gamma_{e^{itr}}$.*

(a) Suppose that ε is a finite regular Borel measure on Z such that $\int f_\beta \, d\varepsilon = 1$, and take $\mu := \sum_{n \in \mathbb{N}^k} e^{-\beta r \cdot n} R^n \varepsilon$. Then there is a KMS$_\beta$ state ϕ_ε on $(\mathcal{NT}(X), \alpha)$ such that

$$\phi_\varepsilon\big(\psi_m(x)\psi_p(y)^*\big) = \begin{cases} 0 & \text{if } m \neq p \\ e^{-\beta r \cdot m} \int \langle y, x \rangle \, d\mu & \text{if } m = p. \end{cases} \tag{3.43}$$

(b) If in addition r has rationally independent coordinates, then the map $\varepsilon \mapsto \phi_\varepsilon$ is an affine isomorphism of

$$\Sigma_\beta := \left\{ \varepsilon \in M(Z)_+ : \int f_\beta \, d\varepsilon = 1 \right\}$$

onto the simplex of KMS$_\beta$ states of $(\mathcal{NT}(X), \alpha)$. Given a state ϕ, let μ be the probability measure such that $\phi(\psi_0(a)) = \int a \, d\mu$ for $a \in C(Z)$. Then the inverse of $\varepsilon \mapsto \phi_\varepsilon$ takes ϕ to $\varepsilon := \prod_{i=1}^{k} (1 - e^{-\beta r_i} R^{e_i}) \mu$.

Proof of (a). Let ε be a finite regular Borel measure on Z. We follow the structure of the proof of [1, Theorem 5.1]. Thus we aim to construct the KMS state ϕ_ε by using a representation θ of X on $H_\theta := \bigoplus_{n\in\mathbb{N}^k} L^2(Z, R^n\varepsilon)$. Notice that here each R^n is a bounded operator on $C(Z)^*$ while in [1, Theorem 5.1] the operation R was defined only on measures (positive functionals). We write $\xi = (\oplus\xi_n)$ for the elements of the direct sum. For $m \in \mathbb{N}^k$ and $x \in X_m$, we claim that there is a well-defined operator $\theta_m(x)$ on H_θ such that

$$(\theta_m(x)\xi)_n(z) = \begin{cases} 0 & \text{if } n \not\geq m \\ x(z)\xi_{n-m}(h^m(z)) & \text{if } n \geq m. \end{cases} \tag{3.44}$$

Let $\xi = (\oplus\xi_n) \in \bigoplus_{n\in\mathbb{N}^k} L^2(Z, R^n\varepsilon)$. Then

$$\begin{aligned}
\|\theta_m(x)\xi\|^2 &= \sum_{n\in\mathbb{N}^k} \|(\theta_m(x)\xi)_n\|^2 \\
&= \sum_{n\geq m} \int |x(z)|^2 |\xi_{n-m}(h^m(z))|^2 \, d(R^n\varepsilon)(z) \\
&= \sum_{n\in\mathbb{N}^k} \int |x(z)|^2 |\xi_n(h^m(z))|^2 \, d(R^{n+m}\varepsilon)(z) \\
&\leq \sum_{n\in\mathbb{N}^k} \|x\|_\infty^2 \int \sum_{h^m(w)=z} |\xi_n(h^m(w))|^2 \, d(R^n\varepsilon)(z) \\
&= \sum_{n\in\mathbb{N}^k} \|x\|_\infty^2 \int \sum_{h^m(w)=z} |\xi_n(z)|^2 \, d(R^n\varepsilon)(z) \\
&\leq \sum_{n\in\mathbb{N}^k} \|x\|_\infty^2 c_m \int |\xi_n(z)|^2 \, d(R^n\varepsilon)(z) \quad (\text{where } c_m = \max_z |h^{-m}(z)|) \\
&= c_m \|x\|_\infty^2 \|\xi\|^2.
\end{aligned} \tag{3.45}$$

Thus $\theta_m(x) \in B(H_\theta)$.

Next we apply a similar calculation to compute the adjoint $\theta(x)^*$. Take $\eta \in H_\theta$, then

$$\begin{aligned}
\big(\theta_m(x)\xi \,|\, \eta\big) &= \sum_{n\in\mathbb{N}^k} \Big(\big(\theta_m(x)\xi\big)_n \,\Big|\, \eta_n \Big) \\
&= \sum_{n\in\mathbb{N}^k} \int (\theta_m(x)\xi)_n(z)\overline{\eta_n(z)} \, d(R^n\varepsilon)(z) \\
&= \sum_{n\geq m} \int x(z)\xi_{n-m}(h^m(z))\overline{\eta_n(z)} \, d(R^n\varepsilon)(z) \\
&= \sum_{n\in\mathbb{N}^k} \int x(z)\xi_n(h^m(z))\overline{\eta_{n+m}(z)} \, d(R^{n+m}\varepsilon)(z)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{N}^k} \int \sum_{h^m(w)=z} x(w)\xi_n(h^m(w))\overline{\eta_{n+m}(w)}\, d(R^n\varepsilon)(z) \\
&= \sum_{n\in\mathbb{N}^k} \int \xi_n(z) \sum_{h^m(w)=z} x(w)\overline{\eta_{n+m}(w)}\, d(R^n\varepsilon)(z).
\end{aligned}
$$

Thus $\theta_m(x)^*$ satisfies

$$
\big(\theta_m(x)^*\eta\big)_n(z) = \sum_{h^m(w)=z} \overline{x(w)}\eta_{n+m}(w) \quad \text{for } \eta \in H_\theta. \tag{3.46}
$$

Next we claim that θ is a Toeplitz representation of X. We check conditions (T1)$-$(T3) of Definition 1.3.1. For (T1), since each $\theta_m : X_m \to B(H_\theta)$ is clearly linear, we need only check that $\theta_0 : A \to B(H_\theta)$ is a homomorphism on $A = C(Z)$ here. Since the multiplication in A is pointwise multiplication, for $a, a' \in A$, we have

$$
\big(\theta_0(aa')\xi\big)_n(z) = a(z)a'(z)\xi_n(z) = a(z)\big(\theta_0(a')\xi\big)_n(z) = \big(\theta_0(a)\theta_0(a')\xi\big)_n(z).
$$

Thus $\theta_0 : A \to B(H_\theta)$ is a homomorphism.

To check (T2), fix m and $x_1, x_2 \in X_m$. Then

$$
\begin{aligned}
\big(\theta_0(\langle x_1, x_2\rangle)\xi\big)_n(z) &= \langle x_1, x_2\rangle(z)\xi_n(z) \\
&= \sum_{h^m(w)=z} \overline{x_1(w)}x_2(w)\xi_n(z) \\
&= \sum_{h^m(w)=z} \overline{x_1(w)}x_2(w)\xi_n(h^m(w)).
\end{aligned}
$$

Since $(\theta_m(x_2)\xi)_{n+m}(w) = x_2(w)\xi_n(h^m(w))$,

$$
\big(\theta_0(\langle x_1, x_2\rangle)\xi\big)_n(z) = \sum_{h^m(w)=z} \overline{x_1(w)}(\theta_m(x_2)\xi)_{n+m}(w).
$$

Now formula (3.46) implies that

$$
\big(\theta_0(\langle x_1, x_2\rangle)\xi\big)_n(z) = \big(\theta_m(x_1)^*\theta_m(x_2)\xi\big)_n(z).
$$

Thus $\theta_0(\langle x_1, x_2\rangle) = \theta_m(x_1)^*\theta_m(x_2)$, giving (T2).

For (T3), let $x \in X_m$ and $y \in X_p$. If $n \not\geq m + p$, then $\big(\theta_{m+p}(xy)\xi\big)_n(z) = 0$. Also we have

$$
\big(\theta_m(x)\theta_p(y)\xi\big)_n(z) = x(z)\big(\theta_p(y)\xi\big)_{n-m}\big(h^m(z)\big),
$$

which vanishes for $n - m \not\geq p$. So we assume $n \geq m + p$. Using the definition of multiplication in X, we have

$$
\big(\theta_{m+p}(xy)\xi\big)_n(z) = \big(\theta_{m+p}\big(\sigma_{m,p}(x \otimes y)\big)\xi\big)_n(z)
$$

$$
\begin{aligned}
&= \big(\sigma_{m,p}(x \otimes y)\big)(z)\xi_{n-(m+p)}\big(h^{m+p}(z)\big)\\
&= x(z)y\big(h^m(z)\big)\xi_{n-(m+p)}\big(h^{m+p}(z)\big)\\
&= x(z)\big(\theta_p(y)\xi\big)_{n-m}(z)\big(h^m(z)\big)\\
&= \big(\theta_m(x)\theta_p(y)\xi\big)_n(z).
\end{aligned}
$$

This complete our proof of (T3).

Next we show that θ is Nica-covariant. Let 1_m be the identity operator on the fibre X_m and $\alpha^\theta : \mathbb{N}^k \to \operatorname{End}\theta_0(A)'$ be the action as in [20, Proposition 4.1]. Since each fibre X_m is essential and θ is a representation on the Hilbert space H_θ, we must show that

$$
\alpha^\theta_m(1_m)\alpha^\theta_p(1_p) = \begin{cases} \alpha^\theta_{m\vee p}(1_{m\vee p}) & \text{if } m \vee p < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{3.47}
$$

for all $m, p \in \mathbb{N}^k$. To do this, fix $m, p \in \mathbb{N}^k$. Clearly $m \vee p < \infty$. So we check $\alpha^\theta_m(1_m)\alpha^\theta_p(1_p) = \alpha^\theta_{m\vee p}(1_{m\vee p})$. Choose a partition of unity $\{\rho_j : 1 \leq j \leq d\}$ for Z such that $h^{m\vee p}$ is injective on each $\operatorname{supp}\rho_j$ and take $\tau_j := \sqrt{\rho_j} \in X_m$. Notice that $\{\tau_j\}_{j=0}^d$ can be viewed as a Parseval frame for the fibres X_m, X_p and $X_{m\vee p}$. Now to check (3.47), Lemma 1.3.4 implies that it suffices to prove that

$$
\Big(\sum_{i=1}^d \theta_m(\tau_i)\theta_m(\tau_i)^*\Big)\Big(\sum_{j=1}^d \theta_p(\tau_j)\theta_p(\tau_j)^*\Big) = \Big(\sum_{l=1}^d \theta_{m\vee p}(\tau_l)\theta_{m\vee p}(\tau_l)^*\Big). \tag{3.48}
$$

To see this, let $\xi \in H_\theta$ and $z \in Z$. We evaluate both sides of (3.48) at ξ:

For the right-hand side of (3.48), notice that the definition of $\theta_{m\vee p}$ implies that $\big(\big(\sum_{l=1}^d \theta_{m\vee p}(\tau_l)\theta_{m\vee p}(\tau_l)^*\big)\xi\big)_n$ vanishes unless $n \geq m \vee p$. So we assume $n \geq m \vee p$ and compute using the definition of $\theta_{m\vee p}$ and the adjoint formula (3.46):

$$
\begin{aligned}
\Big(\Big(\sum_{l=1}^d \theta_{m\vee p}(\tau_l)\theta_{m\vee p}(\tau_l)^*\Big)\xi\Big)_n(z) &= \sum_{l=1}^d \Big(\theta_{m\vee p}(\tau_l)\big(\theta_{m\vee p}(\tau_l)^*\xi\big)\Big)_n(z)\\
&= \sum_{l=1}^d \Big(\tau_l(z)\big(\theta_{m\vee p}(\tau_l)^*\xi\big)_{n-m\vee p}\big(h^{m\vee p}(z)\big)\Big)\\
&= \sum_{l=1}^d \Big(\tau_l(z)\sum_{h^{m\vee p}(w)=h^{m\vee p}(z)} \overline{\tau_l(w)}\xi_n(w)\Big).
\end{aligned}
$$

Since $h^{m\vee p}$ is injective on each $\operatorname{supp}\tau_l$, we have

$$
\Big(\Big(\sum_{l=1}^d \theta_{m\vee p}(\tau_l)\theta_{m\vee p}(\tau_l)^*\Big)\xi\Big)_n(z) = \sum_{l=1}^d \Big(\tau_l(z)\overline{\tau_l(z)}\xi_n(z)\Big) = \xi_n(z)\sum_{l=1}^d \big|\tau_l(z)\big|^2 = \xi_n(z).
$$

Thus

$$(3.49)\qquad \Big(\Big(\sum_{l=1}^{d}\theta_{m\vee p}(\tau_l)\theta_{m\vee p}(\tau_l)^*\Big)\xi\Big)_n = \begin{cases}\xi_n & \text{if } n\geq m\vee p\\ 0 & \text{otherwise.}\end{cases}$$

For the left-hand side of (3.48), notice that $\{\tau_i\}_{i=0}^d$ is a Parseval frame for X_m and h^m is injective on each $\operatorname{supp}\tau_i$. Then applying the same calculation of the previous paragraph (using formula for $\theta_m(\tau_i)$ and $\theta_m(\tau_i)^*$), we have

$$\Big(\Big(\sum_{i=1}^{d}\theta_m(\tau_i)\theta_m(\tau_i)^*\sum_{j=1}^{d}\theta_p(\tau_j)\theta_p(\tau_j)^*\Big)\xi\Big)_n = \begin{cases}\big(\big(\sum_{j=1}^{d}\theta_p(\tau_j)\theta_p(\tau_j)^*\big)\xi\big)_n & \text{if } n\geq m\\ 0 & \text{otherwise.}\end{cases}$$

Now suppose $n\geq m$. Again since $\{\tau_j\}_{j=0}^d$ is a Parseval frame for X_p and h^p is injective on each $\operatorname{supp}\tau_j$. A similar computation for $\big(\big(\sum_{j=1}^{d}\theta_p(\tau_j)\theta_p(\tau_j)^*\big)\xi\big)_n$, implies that

$$(3.50)\qquad \Big(\Big(\sum_{i=1}^{d}\theta_m(\tau_i)\theta_m(\tau_i)^*\sum_{j=1}^{d}\theta_p(\tau_j)\theta_p(\tau_j)^*\Big)\xi\Big)_n = \begin{cases}\xi_n & \text{if } n\geq m\vee p\\ 0 & \text{otherwise.}\end{cases}$$

Comparing (3.49) and (3.50) gives (3.48), and hence θ is a Nica-covariant representation.

Now the universal property of $\mathcal{NT}(X)$ ([20, Theorem 6.3]), gives us a homomorphism $\theta_*:\mathcal{NT}(X)\to B(H_\theta)$ such that $\theta_*\circ\psi=\theta$.

For each $q\in\mathbb{N}^k$, we choose a finite partition $\{Z_{q,i}: 1\leq i\leq I_q\}$ of Z by Borel sets such that h^q is one-to-one on each $Z_{q,i}$.[1] We take $Z_{0,1}=Z$ and write $I_0=1$. Let $\chi_{q,i}=\chi_{Z_{q,i}}$, and define $\xi^{q,i}\in\bigoplus_{n\in\mathbb{N}^k}L^2(Z,R^n\varepsilon)$ by

$$\xi^{q,i}_n=\begin{cases}0 & \text{if } n\neq q\\ \chi_{q,i} & \text{if } n=q.\end{cases}$$

We now define $\phi_\varepsilon:\mathcal{NT}(X)\to\mathbb{C}$ by

$$(3.51)\qquad \phi_\varepsilon(b)=\sum_{q\in\mathbb{N}^k}\sum_{i=1}^{I_q}e^{-\beta r\cdot q}\big(\theta_*(b)\xi^{q,i}\,\big|\,\xi^{q,i}\big)\quad\text{for } b\in\mathcal{NT}(X),$$

To see ϕ_ε is well-defined, we need to show that the series converges. Notice that elements of C^*-algebras can be written as a linear combination of positive elements,

[1]To see that there is such a partition, notice that since h^q is a local homeomorphism on Z, there is an open cover $\{U_l\}_{l=0}^d$ of Z such that each $h^q|_{U_l}$ is injective. Now set $V_0:=U_0$ and for each l let $V_l:=U_l\setminus\cup_{j=0}^{l-1}V_j$. Clearly $\{V_l\}_{l=0}^d$ is a Borel partition of Z. Since this partition is dependent on q, we relabel it as $\{Z_{q,i}\}_{i=1}^{I_q}$.

and a positive element b satisfies $b \leq \|b\|1$. Thus it suffices for us to show that the series defining $\phi_\varepsilon(1)$ is convergent. By definition $\phi_\epsilon(1)$ is

$$\sum_{q\in\mathbb{N}^k}\sum_{i=1}^{I_q} e^{-\beta r\cdot q}\big(\chi_{Z_{q,i}}\,\big|\,\chi_{Z_{q,i}}\big) = \sum_{q\in\mathbb{N}^k}\sum_{i=1}^{I_q} e^{-\beta r\cdot q}\int \overline{\chi_{Z_{q,i}}(z)}\chi_{Z_{q,i}}(z)\,d(R^q\varepsilon)(z)$$
$$= \sum_{q\in\mathbb{N}^k}\sum_{i=1}^{I_q} e^{-\beta r\cdot q}R^q\varepsilon(Z_{q,i}).$$

Since $\{Z_{q,i}\}_i$ is a partition of Z, we have

$$\sum_{q\in\mathbb{N}^k}\sum_{i=1}^{I_q} e^{-\beta r\cdot q}\big(\chi_{Z_{q,i}}\,\big|\,\chi_{Z_{q,i}}\big) = \sum_{q\in\mathbb{N}^k} e^{-\beta r\cdot q}R^q\varepsilon(Z).$$

By Proposition 3.2.7(b), the sum $\sum_{q\in\mathbb{N}^k} e^{-\beta r\cdot q}R^q\varepsilon$ converges to a measure μ. Since $\int f_\beta\,d\varepsilon = 1$, μ is a probability measure. Then

$$\sum_{q\in\mathbb{N}^k}\sum_{i=1}^{I_q} e^{-\beta r\cdot q}\big(\chi_{Z_{q,i}}\,\big|\,\chi_{Z_{q,i}}\big) = \mu(Z) = 1.$$

Thus $\phi_\epsilon(1) = 1$, and the formula (3.51) gives us a well-defined state on $\mathcal{T}(X(E))$.

To see that ϕ_ε satisfies (3.43), take $x \in X_m$, $y \in X_p$ and $b = \psi_m(x)\psi_p(y)^*$. Since $\xi^{q,i}$ is zero in all except the qth summand of $\bigoplus_{n\in\mathbb{N}^k} L^2(Z, R^n\varepsilon)$,

$$\theta_*(b)\xi^{q,i} = \theta_*(\psi_m(x)\psi_p(y)^*)\xi^{q,i} = \theta_m(x)\theta_p(y)^*\xi^{q,i}$$

is zero in all but the $(q-p+m)$th summand. Thus

$$\big(\theta_*(b)\xi^{q,i}\,\big|\,\xi^{q,i}\big) = 0 \quad \text{for all } q, i \text{ whenever } p \neq m,$$

and ϕ_ε satisfies (3.43) when $p \neq m$. Then we assume $p = m$. If $q \not\geq m$, then $\theta_m(x)\theta_m(y)^*\xi^{q,i} = 0$. Now suppose $q \geq m$. Since h^q is injective on $Z_{q,i}$, it follows that h^m is injective on each $Z_{q,i}$. Then

$$\big(\theta_m(x)\theta_m(y)^*\xi^{q,i}\,\big|\,\xi^{q,i}\big) = \int \Big(x(z)\sum_{h^m(w)=h^m(z)}\overline{y(w)}\chi_{q,i}(w)\Big)\overline{\chi_{q,i}(z)}\,d(R^q\varepsilon)(z)$$
$$= \int x(z)\overline{y(z)\chi_{q,i}(z)}\,d(R^q\varepsilon)(z).$$

Since the $Z_{q,i}$ partition Z, summing over i, we have

$$\sum_{i=1}^{I_q}\big(\theta_*(\psi_m(x)\psi_m(y)^*)\xi^{q,i}\,\big|\,\xi^{q,i}\big) = \int x(z)\overline{y(z)}\,d(R^q\varepsilon)(z).$$

Now using the formula (3.25) for R^m, we have

$$\begin{aligned}
(3.52)\qquad \phi_\varepsilon\big(\psi_m(x)\psi_m(y)^*\big) &= \sum_{q\geq m} e^{-\beta r\cdot q}\int x(z)\overline{y(z)}\,d(R^q\varepsilon)(z)\\
&= \sum_{q\geq m} e^{-\beta r\cdot q}\int \sum_{h^m(w)=z} x(w)\overline{y(w)}\,d(R^{q-m}\varepsilon)(z)\\
&= \sum_{q\in\mathbb{N}^k} e^{-\beta r\cdot(m+q)}\int \langle y,x\rangle(z)\,d(R^q\varepsilon)(z)\\
&= e^{-\beta r\cdot m}\int \langle y,x\rangle\,d\Big(\sum_{q\in\mathbb{N}^k} e^{-\beta q}R^q\varepsilon\Big).
\end{aligned}$$

Recall that $\sum_{q\in\mathbb{N}^k} e^{-\beta r\cdot q}R^q\varepsilon = \mu$. Then

$$\phi_\varepsilon\big(\psi_m(x)\psi_m(y)^*\big) = e^{-\beta r\cdot m}\int \langle y,x\rangle\,d\mu.$$

Thus ϕ_ε satisfies (3.43).

To see that ϕ_ε is a KMS$_\beta$ state, we apply Proposition 3.1.6 with $m=p=0$ and $x=y=a'\in A$ to get

$$\phi_\varepsilon\big(\psi_0(a'a'^*)\big) = \phi_\varepsilon\big(\psi_0(a')\psi_0(a')^*\big) = \int \langle a',a'^*\rangle_A\,d\mu = \int a'a'^*\,d\mu.$$

This implies that $\phi_\varepsilon(\psi_0(a)) = \int a\,d\mu$ for all positive $a\in A$ and so for all elements of A. It then follows that $\phi_\varepsilon\big(\psi_0\big(\langle y,x\rangle\big)\big) = \int \langle y,x\rangle\,d\mu$. Now

$$\phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big) = \delta_{m,n}e^{-\beta r\cdot m}\phi_\varepsilon\big(\psi_0\big(\langle y,x\rangle\big)\big),$$

and the Proposition 3.1.6(a) says that ϕ_ε is a KMS$_\beta$ state. □

Proof of Theorem 3.3.1 (b). Now assume that r has rationally independent coordinates. We first claim that Σ_β is a compact subset of $C(Z)^*$ in the weak* norm. Then to prove that $\varepsilon\mapsto\phi_\varepsilon$ is an isomorphism, it suffices to show that it is injective, surjective, and continuous.

For the claim, we show that Σ_β is a closed subset of the compact unit ball of $C(Z)^*$. Let $\varepsilon\in\Sigma_\beta$. Recall that $f_\beta = |f_\beta|\geq 1$. Thus

$$\|\varepsilon\|_{C(Z)^*} = \sup_{\substack{|f|\leq 1,\\ f\in C(Z)}} \Big|\int f\,d\varepsilon\Big| \leq \sup_{\substack{|f|\leq 1,\\ f\in C(Z)}} \int |f|\,d\varepsilon \leq \int f_\beta\,d\varepsilon = 1.$$

Then Σ_β is a subset of the unit ball in $C(Z)^*$. To check that it is closed, take a sequence $\{\varepsilon_j\}_{j=1}^\infty\subseteq\Sigma_\beta$ and $\varepsilon\in C(Z)^*$ such that $\varepsilon_j\to\varepsilon$ in weak* topology. Since

$$\varepsilon(f) = \lim_{j\to\infty}\varepsilon_j(f)\geq 0 \text{ for all positive } f\in C(Z),$$

the Riesz-representation theorem implies that $\varepsilon \in M(Z)_+$. Also note that

$$\int f_\beta \, d\varepsilon = \lim_{j\to\infty} \int f_\beta \, d\varepsilon_j = 1.$$

Then $\varepsilon \in \Sigma_\beta$ and that Σ_β is closed, as required.

For the surjectivity of $\varepsilon \mapsto \phi_\varepsilon$, let ϕ be a KMS$_\beta$ state, and let μ be the probability measure such that $\phi\big(\psi_0(a)\big) = \int a \, d\mu$ for $a \in C(Z)$. Since r has rationally independent coordinates, Proposition 3.1.6 implies that ϕ satisfies

$$\phi\big(\psi_m(x)\psi_n(y)^*\big) = \delta_{m,n} e^{-\beta r\cdot m} \phi\big(\psi_0\big(\langle y, x\rangle\big)\big) = e^{-\beta r\cdot m} \int \langle y, x\rangle \, d\mu. \tag{3.53}$$

On the other hand, since μ satisfies subinvariance relation (3.26) (by Proposition 3.2.8), Proposition 3.2.7(c) implies that $\varepsilon := \big(\prod_{i=1}^k \big(1 - e^{-\beta r_i} R^{e_i}\big)\big)\mu$ belongs to Σ_β and satisfies $\sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n} R^n \varepsilon = \mu$. Now applying part (a) to ε gives a KMS$_\beta$ state ϕ_ε such that

$$\phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big) = \begin{cases} 0 & \text{if } m \neq n \\ e^{-\beta r\cdot m} \int \langle y, x\rangle \, d\mu & \text{if } m = n. \end{cases} \tag{3.54}$$

Comparing equations (3.54) and (3.53), we have $\phi = \phi_\varepsilon$. This shows that $\varepsilon \mapsto \phi_\varepsilon$ is surjective.

To show the injectivity of $\varepsilon \mapsto \phi_\varepsilon$, let $\phi_{\varepsilon_1} = \phi_{\varepsilon_2}$ be two KMS$_\beta$ states. Suppose μ_1, μ_2 are probability measures such that $\phi_{\varepsilon_1} \circ \psi_0(a) = \int a \, d\mu_1$ and $\phi_{\varepsilon_2} \circ \psi_0(a) = \int a \, d\mu_2$ for all $a \in A$. Then $\mu_1 = \mu_2$. Now the construction of the previous paragraph shows that

$$\varepsilon_1 = \big(\prod_{i=1}^k \big(1 - e^{-\beta r_i} R^{e_i}\big)\big)\mu_1 = \big(\prod_{i=1}^k \big(1 - e^{-\beta r_i} R^{e_i}\big)\big)\mu_2 = \varepsilon_2.$$

Thus $\varepsilon \mapsto \phi_\varepsilon$ is one-to-one.

Finally, to check the continuity of $\varepsilon \mapsto \phi_\varepsilon$, suppose $\varepsilon_j \to \varepsilon$ in Σ_β. Let $\mu := \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n} R^n \varepsilon$ and $\mu_j := \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n} R^n \varepsilon_j$. Remember from the calculation (3.33) that

$$\Big\| \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n} R^n \varepsilon \Big\|_{C(Z)^*} \leq \|\varepsilon\|_{C(Z)^*}.$$

It then follows $\mu_j \to \mu$ in weak* topology. Now the formula (3.43) for ϕ_ε shows that $\phi_{\varepsilon_j} \to \phi_\varepsilon$ in weak* topology. □

The next Corollary is a generalization of [1, Corollary 5.3] to the dynamical system $(\mathcal{O}(X), \alpha)$.

Corollary 3.3.2. *Let $h_1, \dots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z. Let X be the associated product system over $\mathbb{N}^k$, as in Corollary 2.1.2. For each $1 \le i \le k$, take β_{c_i} as in (3.28) and suppose $r \in (0, \infty)^k$ has rationally independent coordinates. Define $\tilde{\alpha} : \mathbb{R} \to \operatorname{Aut}\big(\mathcal{O}(X)\big)$ in terms of the gauge action $\tilde{\gamma}$ by $\tilde{\alpha}_t = \tilde{\gamma}_{e^{itr}}$. If there is a KMS$_\beta$ state of $(\mathcal{O}(X), \tilde{\alpha})$, then there exists an $1 \le i \le k$ such that $\beta r_i \le \beta_{c_i}$.*

Proof. Suppose ϕ is a KMS$_\beta$ state of $(\mathcal{O}(X), \tilde{\alpha})$. Aiming for a contradiction suppose that $\beta r_i > \beta_{c_i}$ for all $1 \le i \le k$. Let $q : \mathcal{NT}(X) \to \mathcal{O}(X)$ be the quotient map as in Lemma 1.3.8. Then $\phi \circ q$ is a KMS$_\beta$ state for the system $\big(\mathcal{NT}(X), \alpha\big)$ considered in Theorem 3.3.1. Since r has rationally independent coordinates part (b) of Theorem 3.3.1 gives a measure ε on Z such that $\int f_\beta \, d\varepsilon = 1$ and $\phi \circ q = \phi_\varepsilon$. Since $f_\beta \ge 1$, $\int f_\beta \, d\varepsilon = 1$ implies that $\varepsilon(Z) > 0$.

We temporarily set $K := \{1, \dots, k\}$ and take an open cover $\{U_l : 1 \le l \le d\}$ of Z such that $h^{e_J}|_{U_l}$ is injective for all $J \subset K$ and $1 \le l \le d$. By applying [46, Lemma 4.32], we can find open cover $\{V_l : 1 \le l \le d\}$ for Z such that $\overline{V_l} \subset U_l$. Since $\varepsilon(Z) > 0$, there exists at least one l satisfying $\varepsilon(V_l) > 0$. Then we can find a function $f \in C(Z)$ such that $f(z) \ne 0$ for some $z \in V_l$ (see [47, Lemma 2.12], for example).

Next for each $J \subset K$, take $f_J := f \in X_{e_J}$ and view $|f|^2$ as an element of $A = C(Z)$. We aim to set up a contradiction by showing that

$$b := \psi_0(|f|^2) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \psi_J(f_J) \psi_J(f_J)^*$$

belongs to $\ker q$ while $\phi_\varepsilon = \phi \circ q$ does not vanish on it. Since the left action of $|f|^2$ on each fibre X_{e_J} is implemented by the finite-rank operator Θ_{f_J, f_J}, a routine calculation for b shows that

$$\begin{aligned} b &= \psi_0(|f|^2) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \psi^{(e_J)}\big(\Theta_{f_J, f_J}\big) \\ &= \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{(|J|+1)} \psi_0(|f|^2) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \psi^{(e_J)}\big(\varphi_{e_J}(|f_J|^2)\big) \\ &= \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} \Big(\psi_0(|f|^2) - \psi^{(J)}\big(\varphi_{e_J}(|f|^2)\big) \Big). \end{aligned}$$

Thus b belong to $\ker q$ (because each summand does).

Next we compute $\phi_\varepsilon(b)$ using the measure μ in part (b) of Theorem 3.3.1:

$$\phi_\varepsilon(b) = \int |f|^2(z) \, d\mu(z) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} e^{-\beta r \cdot e_J} \int \big\langle f_J, \overline{f_J} \big\rangle(z) \, d\mu(z)$$

$$= \int |f|^2(z)\, d\mu(z) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} e^{-\beta r \cdot e_J} \int \sum_{h^{e_J}(w)=z} |f|^2(w)\, d\mu(z).$$

Using definition of R at equation (3.25) and the notation $R^{e_\varnothing}\mu = \mu$, we have

$$\begin{aligned}
\phi_\varepsilon(b) &= \int |f|^2(z)\, d\mu(z) + \sum_{\varnothing \subsetneq J \subseteq K} (-1)^{|J|} e^{-\beta r \cdot e_J} \int |f|^2(z)\, d(R^{e_J}\mu)(z) \\
&= \sum_{\varnothing \subseteq J \subseteq K} (-1)^{|J|} e^{-\beta r \cdot e_J} \int |f|^2(z)\, d(R^{e_J}\mu)(z).
\end{aligned}$$

This is precisely $\int |f|^2(z)\, d\varepsilon(z)$. It follows that $\phi_\varepsilon(b) > 0$, and we have a contradiction. Thus there should be at least one $1 \leq i \leq k$ satisfying $\beta r_i \leq \beta_{c_i}$. $\square$

3.4 KMS states at the critical inverse temperature

In Theorem 3.3.1, we first chose an $r \in \mathbb{N}^k$ and then characterised KMS states of the dynamical system $\big(\mathcal{NT}(X), \alpha\big)$ for β satisfying

$$\beta > r_i^{-1}\beta_{c_i} \text{ for all } 1 \le i \le k. \tag{3.55}$$

Thus the range of possible inverse temperature is dependent on the choice of $r \in \mathbb{N}^k$. When r is a multiple of $(\beta_{c_1}, \dots, \beta_{c_k})$, following the recent conventions for the higher rank graph algebras (see [26, 28, 59, 60]), we call the common value $\beta_c := r_i^{-1}\beta_{c_i}$ the *critical inverse temperature*. In particular, we are interested in $r := (\beta_{c_1}, \dots, \beta_{c_k})$ which gives the critical inverse temperature $\beta_c = 1$. In this case, we refer to the associated dynamics $\alpha : t \mapsto \gamma_{e^{itr}}$ as *the preferred dynamics.*

In the next theorem, we consider the preferred dynamics α and discuss the KMS states at the critical inverse temperature. Theorem 3.4.1 is a generalization of [1, Theorem 6.1] and the proof follows a similar method.

Theorem 3.4.1. *Let $h_1, \dots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z. Let X be the associated product system over $\mathbb{N}^k$ as in Corollary 2.1.2. For each $1 \le i \le k$, let β_{c_i} be as in* (3.28) *and set $r := (\beta_{c_1}, \dots, \beta_{c_k})$. Define $\alpha : \mathbb{R} \to \operatorname{Aut}\big(\mathcal{NT}(X)\big)$ and $\tilde{\alpha} : \mathbb{R} \to \operatorname{Aut}\big(\mathcal{O}(X)\big)$ in terms of the gauge actions by $\alpha_t = \gamma_{e^{itr}}$ and $\tilde{\alpha}_t = \tilde{\gamma}_{e^{itr}}$. Then there is a KMS$_1$ state on $\big(\mathcal{NT}(X), \alpha\big)$, and at least one such state factors through a KMS$_1$ state of $\big(\mathcal{O}(X), \tilde{\alpha}\big)$.*

To prove this, we need the next lemma from [1].

Lemma 3.4.2 ([1], Lemma 6.2)**.** *Suppose $(A, \mathbb{R}, \alpha)$ is a dynamical system, and J is an ideal in A generated by a set P of positive elements which are fixed by α. If ϕ is a KMS$_\beta$ state of (A, α) and $\phi(p) = 0$ for all $p \in P$, then ϕ factors through a state of A/J.*

Proof of Theorem 3.4.1. Choose a decreasing sequence $\{\beta_j\}_{j\in\mathbb{N}}$ such that $\beta_j \to 1$ and a probability measure ν on Z. Then $K_j := \int f_{\beta_j}\, d\nu$ belongs to $[1, \infty)$, and $\varepsilon_j := K_j^{-1}\nu$ satisfies $\int f_{\beta_j}\, d\varepsilon_j = 1$. Thus for each j, part (a) of Theorem 3.3.1 gives a KMS$_{\beta_j}$ state ϕ_{ε_j} on $\big(\mathcal{NT}(X), \alpha\big)$. Since $\{\phi_{\varepsilon_j}\}_j$ is a sequence in the compact unit ball of $C(Z)^*$, by passing to a subsequence and relabelling, we may assume that $\phi_{\varepsilon_j} \to \phi$ for some state ϕ. Now [5, Proposition 5.3.23] implies that ϕ is a KMS$_1$ state.

To show that at least one such state factors through $\big(\mathcal{O}(X), \tilde{\alpha}\big)$, we apply the construction of the previous paragraph to a particular sequence of measures ε_j. Take one of the local homeomorphisms, for example h_1. Since for all $d \in \mathbb{N}$, $z \mapsto |h_1^{-d}(z)|$ is continuous (see [8, Lemma 2.2]), applying Proposition 2.3 of [18] gives $u \in Z$ such that[2]

$$|h_1^{-d}(u)| \geq e^{d\beta_{c_1}} \text{ for all } d \in \mathbb{N}. \tag{3.56}$$

Now let δ_u be the unit point mass at u, and take $\varepsilon_j = f_{\beta_j}(u)^{-1}\delta_u$. The argument of the first paragraph gives a sequence of KMS_{β_j} states on $\big(\mathcal{NT}(X), \alpha\big)$ which converges to a KMS_1 state ϕ of $\big(\mathcal{NT}(X), \alpha\big)$ in the weak* topology.

We aim to show that ϕ factors through $\big(\mathcal{O}(X), \tilde{\alpha}\big)$. By Lemma 3.4.2, it suffices for us to prove that the generators of the kernel of the quotient map $q : \mathcal{NT}(X) \to \mathcal{O}(X)$ are all positive, are fixed by α, and belong to $\ker \phi$. Remember from Lemma 1.3.8 that

$$\ker q = \Big\langle \psi_0(a) - \psi^{(n)}(\varphi_n(a)) : n \in \mathbb{N}^k, a \in \varphi_n^{-1}(\mathcal{K}(X_n)) \Big\rangle.$$

Fix $n \in \mathbb{N}^k$ and $a \in \varphi_n^{-1}(\mathcal{K}(X_n))$. Let $\{U_l^n\}_{l=0}^{L_n}$ be an open cover of Z such that $h^n|_{U_l^n}$ is injective. Choose a partition of unity $\{\rho_l^n\}$ subordinate to $\{U_l^n\}$ and define $\tau_l^n := \sqrt{\rho_l^n}$. The argument of the last paragraph in the proof of Corollary 2.1.2 shows that $\varphi_n(a) = \sum_{l=0}^{L_n} \Theta_{a\cdot\tau_l^n, \tau_l^n}$. Then

$$\begin{aligned}
\psi_0(a) - \psi^{(n)}(\varphi_n(a)) &= \psi_0(a) - \psi^{(n)}\Big(\sum_{l=0}^{L_n} \Theta_{a\cdot\tau_l^n, \tau_l^n}\Big) \\
&= \psi_0(a) - \sum_{l=0}^{L_n} \psi_n(a \cdot \tau_l^n)\psi_n(\tau_l^n)^* \\
&= \psi_0(a)\Big(1 - \sum_{l=0}^{L_n} \psi_n(\tau_l^n)\psi_n(\tau_l^n)^*\Big).
\end{aligned}$$

Thus the generators of $\ker q$ are of the form of $\big(1 - \sum_{l=0}^{L_n} \psi_n(\tau_l^n)\psi_n(\tau_l^n)^*\big)$. Clearly they are fixed by α.

Next we show that theses generators are positive. Writing T for the Fock representation, Lemma 3.2.9 says that $\big(\sum_{l=0}^{L_n} T_n(\tau_l^n)T_n(\tau_l^n)^*\big)(x)$ is either zero or x for all $x \in X$. Therefore,

$$1 - \sum_{l=0}^{L_n} T_n(\tau_l^n)T_n(\tau_l^n)^*$$

is positive in $\mathcal{L}(F(X))$. Since the induced homomorphism $T_* : \mathcal{NT}(X) \to \mathcal{L}(F(X))$ is injective, it follows that each generator $\big(1 - \sum_{l=0}^{L_n} \psi_n(\tau_l^n)\psi_n(\tau_l^n)^*\big)$ is positive in $\mathcal{NT}(X)$.

[2]As we mentioned in the proof of [1, Theorem 6.1], the results of [18] are mainly about metric spaces. But it seems that the argument for Proposition 2.3 in [18] does not need this hypothesis.

Now it remains to prove that

$$\phi\Big(\sum_{l=0}^{L_n} \psi_n(\tau_l^n)\psi_n(\tau_l^n)^*\Big) = 1. \tag{3.57}$$

Let μ_j be the measure $\sum_{m\in\mathbb{N}^k} e^{\beta_j r\cdot m} R^m \varepsilon_j$ of Theorem 3.3.1(a). We compute using the formula (3.43) for ϕ_{ε_j}:

$$\begin{aligned}\phi\Big(\sum_{l=0}^{L_n}\psi_n(\tau_l^n)\psi_n(\tau_l^n)^*\Big) &= \lim_{j\to\infty} \phi_{\varepsilon_j}\Big(\sum_{l=0}^{L_n} \psi_n(\tau_l^n)\psi_n(\tau_l^n)^*\Big)\\ &= \lim_{j\to\infty} e^{-\beta_j r\cdot n}\int \sum_{l=0}^{L_n} \big\langle \tau_l^n, \tau_l^n\big\rangle d\mu_j(z).\end{aligned}$$

Since h^n is injective on $\operatorname{supp}\tau_l^n$, we have

$$\sum_{l=0}^{L_n}\big\langle \tau_l^n,\tau_l^n\big\rangle(z) = \sum_{l=0}^{L_n}\sum_{h^n(w)=z} \overline{\tau_l^n(w)}\tau_l^n(w) = \sum_{h^n(w)=z}\sum_{l=0}^{L_n}\big|\tau_l^n(w)\big|^2 = \sum_{h^n(w)=z} 1 = \big|h^{-n}(z)\big|.$$

Thus

$$\begin{aligned}e^{-\beta_j r\cdot n}\int \sum_{l=0}^{L_n}\big\langle \tau_l^n,\tau_l^n\big\rangle\, d\mu_j(z) &= e^{-\beta_j r\cdot n}\int \big|h^{-n}(z)\big|\, d\mu_j(z)\\ &= \sum_{m\in\mathbb{N}^k} e^{-\beta_j r\cdot(n+m)}\int \big|h^{-n}(z)\big|\, d(R^m\varepsilon_j)(z).\end{aligned}$$

Using formula (3.25) for R^m, we have

$$e^{-\beta_j r\cdot n}\int \sum_{l=0}^{L_n}\big\langle \tau_l^n,\tau_l^n\big\rangle\, d\mu_j(z) = \sum_{m\in\mathbb{N}^k} e^{-\beta_j r\cdot(n+m)}\int \sum_{h^m(w)=z} \big|h^{-n}(w)\big|\, d\varepsilon_j(z).$$

Remember $\varepsilon_j = f_{\beta_j}(u)^{-1}\delta_u$. Then

$$\begin{aligned}e^{-\beta_j r\cdot n}\int \sum_{l=0}^{L_n}\big\langle \tau_l^n,\tau_l^n\big\rangle\, d\mu_j(z) &= \sum_{m\in\mathbb{N}^k} e^{-\beta_j r\cdot(m+n)}\big|h^{-(m+n)}(u)\big| f_{\beta_j}(u)^{-1}\\ &= \sum_{m\geq n} e^{-\beta_j r\cdot m}\big|h^{-m}(u)\big| f_{\beta_j}(u)^{-1}.\end{aligned}$$

Since $f_{\beta_j}(u) = \sum_{m\in\mathbb{N}^k} e^{-\beta_j r\cdot m}\big|h^{-m}(u)\big|$,

$$e^{\beta_j r\cdot n}\int \sum_{l=0}^{L_n}\big\langle \tau_l^n,\tau_l^n\big\rangle\, d\mu_j(z) = \Big(\frac{f_{\beta_j}(u) - \sum_{m<n} e^{-\beta_j r\cdot m}\big|h^{-m}(u)\big|}{f_{\beta_j}(u)}\Big).$$

Now to prove (3.57), it suffices to show that $f_{\beta_j}(u) \to \infty$ as $j \to \infty$. Fix j. Since the inverse image $\left|h^{-m}(u)\right| \geq \left|h^{-m_1}(u)\right|$, we have

$$f_{\beta_j}(u) \geq \sum_{m \in \mathbb{N}^k} e^{-\beta_j r \cdot m} \left|h_1^{-m_1}(u)\right|.$$

Recall that $r = (\beta_{c_1}, \ldots, \beta_{c_k})$. Since $\sum_{m \in \mathbb{N}^k} e^{-\beta_j r \cdot m} = \prod_{i=1}^k \sum_{m_i \in \mathbb{N}} e^{-\beta_j r_i m_i}$, we have

$$f_{\beta_j}(u) \geq \Big[\sum_{m_1 \in \mathbb{N}} e^{-\beta_j \beta_{c_1} m_1} \left|h_1^{-m_1}(u)\right| \Big] \prod_{i=2}^k \sum_{m_i \in \mathbb{N}} (e^{-\beta_j \beta_{c_i} m_i}).$$

It follows from the equation (3.56) that

$$f_{\beta_j}(u) \geq \Big[\sum_{m_1 \in \mathbb{N}} (e^{-\beta_j \beta_{c_1} + \beta_{c_1}})^{m_1} \Big] \prod_{i=2}^k \sum_{m_i \in \mathbb{N}} (e^{-\beta_j \beta_{c_i}})^{m_i}.$$

Since $\beta_j > 1$, all series in the right-hand side are convergent geometric series. Computing these series, we have

$$f_{\beta_j}(u) \geq \big(\frac{1}{1 - e^{-\beta_j \beta_{c_1} + \beta_{c_1}}} \big) \prod_{i=2}^k \big(\frac{1}{1 - e^{-\beta_j \beta_{c_i}}} \big) \quad \text{for all } j.$$

Now if $j \to \infty$, the right hand side goes to infinity. Thus $f_{\beta_j}(u) \to \infty$, as required. $\square$

3.5 Ground states and KMS$_\infty$ states

In this section we describe the ground states and KMS$_\infty$ states of $(\mathcal{NT}(X), \alpha)$. We first provide a characterization for the ground states in Lemma 3.5.4. Then in Proposition 3.5.5 we prove that there is a bijection between the simplex of the probability measures on Z and the ground states of $(\mathcal{NT}(X), \alpha)$. We also show that every ground state on $(\mathcal{NT}(X), \alpha)$ is a KMS$_\infty$ state.

The following definition and remarks have been taken from [35, page 19].

Definition 3.5.1. Let $(A, \mathbb{R}, \alpha)$ be a dynamical system. Following [10], we say a state ϕ is a *KMS$_\infty$ state* if it is the weak* limit of a sequence of KMS$_{\beta_i}$ states as $\beta_i \to \infty$. A state ϕ is said to be a *ground state*, if the entire functions $z \mapsto \phi(a\alpha_z(b))$ are bounded on the upper half-plan for all analytic elements a, b.

Remark 3.5.2. Here we distinguish between ground states and the KMS$_\infty$ states. But in older literature (for example in [5, 41]), there was not such a distinction. Considering our set-up, it follows from [10, Theorem 5.3.23] that every KMS$_\infty$ state is a ground state. But a ground state need not be a KMS$_\infty$ state (see for example [10, page 447] or [35, Theoerem 7.1]).

Remark 3.5.3. Given a dynamical system $(A, \mathbb{R}, \alpha)$, [41, Proposition 8.12.4]) implies that it suffices to check the ground state condition on a set of analytic elements which span a dense subspace of A. Note that the definition of ground states in [41] is slightly different: A state ϕ is said to be ground state if all the functions $z \mapsto \phi(a\alpha_z(b))$ are bounded by $\|a\|\|b\|$. But it is shown in the proof of (2) $\Rightarrow$ (5) in [5, Proposition 5.3.19] that an entire function which is bounded on the upper half-plane is bounded by the sup norm of its restriction to the real axis.

Fortunately, in the dynamical system $(\mathcal{NT}(X), \alpha)$, the sup norm of the restriction of the functions $z \mapsto \phi(a\alpha_z(b))$ to the real line is bounded by $\|a\|\|b\|$. To see this, let $a := \psi_m(x)\psi_n(y), b := \psi_p(s)\psi_q(t)$ and ϕ be a state of $\mathcal{NT}(X)$. Notice that for each $t \in \mathbb{R}$

$$\big|\phi(a\alpha_t(b))\big| \leq \|\phi\|\big\|b\alpha_t(a)\big\| \leq \|a\|\|b\|.$$

Since ϕ is bounded linear functional, we can extend this to all of $\mathcal{NT}(X)$. Thus any ground state in our set-up is a ground state of [41]. Now [41, Proposition 8.12.4] implies that it is enough to check the ground state condition on a set of analytic elements which span a dense subspace of $\mathcal{NT}(X)$.

The following lemma is a generalisation of [28, Proposition 3.1(c)] and [27, Proposition 2.1(b)] in the dynamical systems of graph algebras.

Lemma 3.5.4. *Let $h_1, \ldots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. Suppose $r \in (0, \infty)^k$ and $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathcal{NT}(X))$ is given in terms of the gauge action by $\alpha_t = \gamma_{e^{itr}}$. Suppose $\beta > 0$ and let ϕ be a state on $\mathcal{NT}(X)$. Then ϕ is a ground state of $(\mathcal{NT}(X), \alpha)$ if and only if*

$$\phi\big(\psi_m(x)\psi_n(y)^*\big) = 0 \text{ whenever } r \cdot m > 0 \text{ or } r \cdot n > 0. \tag{3.58}$$

Proof. First notice that for every state ϕ, $a + ib \in \mathbb{C}$ and $m, n, p, q \in \mathbb{N}^k$, the definition of α implies that

$$\begin{aligned} \phi\Big(\psi_m(x)\psi_n(y)^*\alpha_{a+ib}\big(\psi_p(s)\psi_q(t)^*\big)\Big) &= \Big|e^{i(a+ib)r\cdot(p-q)}\phi\Big(\psi_m(x)\psi_n(y)^*\psi_p(s)\psi_q(t)^*\Big)\Big| \\ &= e^{-br\cdot(p-q)}\Big|\phi\Big(\psi_m(x)\psi_n(y)^*\psi_p(s)\psi_q(t)^*\Big)\Big|. \end{aligned} \tag{3.59}$$

Now suppose ϕ is a ground state. Then

$$\Big|\phi\big(\psi_m(x)\alpha_{a+ib}\big(\psi_n(y)^*\big)\big)\Big| = e^{br\cdot n}\Big|\phi\big(\psi_m(x)\psi_n(y)^*\big)\Big|,$$

is bounded on the upper half plane $b > 0$. Thus $\phi\big(\psi_m(x)\psi_n(y)^*\big) = 0$ whenever $r \cdot n > 0$. Since $\phi\big(\psi_n(y)\psi_m(x)^*\big) = \overline{\phi\big(\psi_m(x)\psi_n(y)^*\big)}$, a symmetric calculation shows that $\phi\big(\psi_n(y)\psi_m(x)^*\big) = 0$ whenever $r \cdot m > 0$.

Next suppose that ϕ satisfies (3.58). It follows from Lemma 3.1.4 that there exist $\{\xi_{i,j}\}_{0\le i,j\le d} \subset X_{m+p-n\wedge p}$ and $\{\eta_{i,j}\}_{0\le i,j\le d} \subset X_{q+n-n\wedge p}$ such that

$$\psi_m(x)\psi_n(y)^*\psi_p(s)\psi_q(t)^* = \sum_{0\le i,j\le d} \psi_{m+p-n\wedge p}(\xi_{i,j})\psi_{q+n-n\wedge p}(\eta_{i,j})^*.$$

Putting this in (3.59), we have

$$\phi\Big(\psi_m(x)\psi_n(y)^*\alpha_{a+ib}\big(\psi_p(s)\psi_q(t)^*\big)\Big) = e^{-br\cdot(p-q)}\Big|\phi\Big(\sum_{0\le i,j\le d} \psi_{m+p-n\wedge p}(\xi_{i,j})\psi_{q+n-n\wedge p}(\eta_{i,j})^*\Big)\Big|.$$

The assumption (3.58) implies that the right-hand side is zero (consequently is bounded) unless

$$r \cdot (m + p - n \wedge p) = 0 = r \cdot (q + n - n \wedge p).$$

So suppose $r \cdot (m + p - n \wedge p) = 0 = r \cdot (q + n - n \wedge p)$. Since $r \in (0,\infty)^k$, it follows that

$$m + p - n \wedge p = 0 = q + n - n \wedge p.$$

Then

$$\phi\Big(\psi_m(x)\psi_n(y)^*\alpha_{a+ib}\big(\psi_p(s)\psi_q(t)^*\big)\Big) = e^{-br\cdot(p-q)}\Big|\sum_{0\le i,j\le d} \phi\big(\psi_0\big(\langle\eta_{i,j},\xi_{i,j}\rangle\big)\big)\Big|.$$

Notice that q and $n - n \wedge p$ are both positive. Then $q + n - n \wedge p = 0$ implies that $q = 0$. Now we have

$$\phi\Big(\psi_m(x)\psi_n(y)^*\alpha_{a+ib}\big(\psi_p(s)\psi_q(t)^*\big)\Big) = e^{-br\cdot p}\Big|\sum_{0\le i,j\le d} \phi\big(\psi_0\big(\langle\eta_{i,j},\xi_{i,j}\rangle\big)\big)\Big|.$$

Thus ϕ is bounded on the upper half plane $b > 0$, and hence it is a ground state. $\square$

The next Proposition is an extension of [28, Proposition 8.1] and [27, Proposition 5.1] from dynamical systems of graph algebras to the dynamical system $(\mathcal{NT}(X), \alpha)$.

Proposition 3.5.5. *Let $h_1, \dots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. Suppose $r \in (0,\infty)^k$ and $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X))$ is given in terms of the gauge action by $\alpha_t = \gamma_{e^{itr}}$. For each probability measure ε on Z there is a unique KMS$_\infty$ state ϕ_ε such that*

$$\text{(3.60)} \qquad \phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big) = \begin{cases} \int \langle y, x\rangle \, d\varepsilon & \textit{if } m = n = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

The map $\varepsilon \mapsto \phi_\varepsilon$ is an affine isomorphism of the simplex of probability measures on Z onto the ground states of $(\mathcal{NT}(X), \alpha)$, and that every ground state of $(\mathcal{NT}(X), \alpha)$ is a KMS$_\infty$ state.

Proof. Suppose ε is a probability measure on Z. For each $1 \le i \le k$, let β_{c_i} be as in (3.28). Choose a sequence $\{\beta_j\}_{j\in\mathbb{N}}$ such that $\beta_j \to \infty$ and each $\beta_j > \max_i r_i^{-1}\beta_{c_i}$. For each β_j, let $f_{\beta_j}(z)$ be the function in Proposition 3.28 (a) and set $K_j := \int f_{\beta_j}\, d\varepsilon$. Then K_j belongs to $[1, \infty)$, and $\varepsilon_j := K_j^{-1}\varepsilon$ satisfies $\int f_{\beta_j}\, d\varepsilon_j = 1$. Now part (a) of Theorem 3.3.1 gives a KMS$_{\beta_j}$ state ϕ_{ε_j} on $\big(\mathcal{NT}(X), \alpha\big)$. Since $\{\phi_{\varepsilon_j}\}_j$ is a sequence in the compact unit ball of $C(Z)^*$, by passing to a subsequence and relabelling, we may assume that $\phi_{\varepsilon_j} \to \phi_\varepsilon$. Then ϕ_ε is by definition a KMS$_\infty$ state.

We now show that ϕ_ε satisfies (3.60). For each ϕ_{ε_j}, we have

$$\phi_{\varepsilon_j}\big(\psi_m(x)\psi_n(y)^*\big) = \delta_{m,n} e^{-\beta_j r\cdot n} \int \langle y, x\rangle \, d\varepsilon_j \text{ for all } m, n \in \mathbb{N}^k.$$

Thus $\phi_{\varepsilon_j}\big(\psi_m(x)\psi_n(y)^*\big) = 0$ for $m \neq n$ and hence $\phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big) = 0$ if $m \neq n$. So we suppose that $m = n$. If $n \neq 0$, then $r \in (0, \infty)^k$ implies that $e^{-\beta_j r\cdot n} \to 0$, and again $\phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big) = \lim_{j\to\infty} \phi_{\varepsilon_j}\big(\psi_m(x)\psi_n(y)^*\big) = 0$. So we assume that $m = n = 0$.

Fix $z \in Z$ and let $f_{\beta_j}(z) = \sum_{p\in\mathbb{N}^k} e^{-\beta_j r\cdot p}|h^{-p}(z)|$ as in Proposition 3.2.7(a). We first show that $f_{\beta_j}(z) \to 1$ as $j \to \infty$. For each $p \in \mathbb{N}^k$ let

$$g(p) = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \,. \end{cases}$$

Clearly $e^{-\beta_j r\cdot p}|h^{-p}(z)| \to g(p)$ as $j \to \infty$. Since for each j, $e^{-\beta_j r\cdot p}|h^{-p}(z)|$ is dominated by $e^{-\beta_0 r\cdot p}|h^{-p}(z)|$, the dominated convergence theorem implies that

$$f_{\beta_j}(z) = \sum_{p\in\mathbb{N}^k} e^{-\beta_j r\cdot p}|h^{-p}(z)| \to \sum_{p\in\mathbb{N}^k} g(p) = 1,$$

as $j \to \infty$. Also notice that each f_{β_j} is dominated by f_{β_0} and ε is a probability measure. Then another application of dominated convergence theorem implies that

$$K_j = \int f_{\beta_j}\, d\varepsilon \to \int 1 \, d\varepsilon = 1$$

as $j \to \infty$.

We now compute using the formula (3.43) for ϕ_{ε_j}:

$$\begin{aligned}\phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big) &= \lim_{j\to\infty} \phi_{\varepsilon_j}\big(\psi_m(x)\psi_n(y)^*\big)\\ &= \lim_{j\to\infty} \int \langle y, x\rangle \, d\varepsilon_j.\end{aligned}$$

Since $\varepsilon_j = K_j^{-1}\varepsilon$,

$$\phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big) = \lim_{j\to\infty} K_j{}^{-1} \int \langle y, x\rangle(z)\, d\varepsilon(z) = \int \langle y, x\rangle(z)\, d\varepsilon(z).$$

Thus ϕ satisfies (3.60). Since $\phi_\varepsilon\big(\psi_m(x)\psi_n(y)^*\big)$ vanishes for all $m \neq 0$ or $n \neq 0$, it also does for $r \cdot m \neq 0$ or $r \cdot n \neq 0$. Then Lemma 3.5.4 says that ϕ_ε is a ground state.

Next let ϕ be a ground state and suppose that ε is the probability measure satisfying $\phi(\psi_0(a)) = \int a\, d\varepsilon$ for all $a \in A$. Then the formulas (3.58) and (3.60) for ϕ and ϕ_ε imply that $\phi = \phi_\varepsilon$. Thus $\varepsilon \mapsto \phi_\varepsilon$ maps the simplex of the probability measures of Z onto the ground states, and it is clearly affine and injective. Since each ϕ_ε is by construction a KMS_∞ state, it follows that every ground state is a KMS_∞ state. $\square$

Chapter 4

The shifts on the infinite path spaces of 1-coaligned higher rank graphs

In this chapter we consider a special type of k-graph called 1-coaligned k-graph. The shift maps on the infinite path space of this kind of graph $*$-commute. So by Corollary 2.1.2 we have a product system over $\mathbb{N}^k$. We study the relationships between the C^*-algebras associated to this product system and the C^*-algebras of the k-graph. Then we use our results in Chapter 3 and some others from the literature to compare the KMS states of these C^*-algebras.

4.1 Basics of Higher-rank graphs

Most of the following definitions have been taken from [44, Chapter 10] and [30].

A *countable category* $\mathcal{C}$ consists of two countable sets $\mathcal{C}^0$ and $\mathcal{C}^*$, two functions $r_{\mathcal{C}}, s_{\mathcal{C}} : \mathcal{C}^* \to \mathcal{C}^0$, a partially defined product $(f, g) \mapsto fg$ from

$$\{(f, g) \in \mathcal{C}^* \times \mathcal{C}^* : s_{\mathcal{C}}(f) = r_{\mathcal{C}}(g)\}$$

to $\mathcal{C}^*$, and an injective map $\mathrm{id} : \mathcal{C}^0 \to \mathcal{C}^*$, which satisfies

(*a*) $r_{\mathcal{C}}(fg) = r_{\mathcal{C}}(f)$ and $s_{\mathcal{C}}(fg) = s_{\mathcal{C}}(g)$,

(*b*) $(fg)h = f(gh)$ when $s_{\mathcal{C}}(f) = r_{\mathcal{C}}(g)$ and $s_{\mathcal{C}}(g) = r_{\mathcal{C}}(h)$,

(*c*) $r_{\mathcal{C}}(\mathrm{id}(v)) = v = s_{\mathcal{C}}(\mathrm{id}(v))$ for all $v \in \mathcal{C}^0$, and

(d) $\mathrm{id}(v)f = f$ and $g\,\mathrm{id}(v) = g$ when $r_{\mathcal{C}}(f) = v$ and $s_{\mathcal{C}}(g) = v$.

The elements of $\mathcal{C}^0$ are called the *objects* of the category, the elements of $\mathcal{C}^*$ are called the *morphisms* of the category, $r_{\mathcal{C}}$ is the *range map*, $s_{\mathcal{C}}$ is the *source map*, the operation $(f, g) \mapsto fg$ is called *composition*, and $\mathrm{id}(v)$ is called the *identity morphism on the object* v. When it is clear from the context we may write r, s for $r_{\mathcal{C}}, s_{\mathcal{C}}$.

Example 4.1.1. Let $k \in \mathbb{N}$. We can view $\mathbb{N}^k$ as morphisms of a countable category with a single object $\{v\}$. For each $m, n \in \mathbb{N}^k$, we can define $r(n) := v$, $s(n) := v$, $mn := m + n$, and $\mathrm{id}(v) := 0$.

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two countable categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of maps $F^0 : \mathcal{C}^0 \to \mathcal{D}^0$ and $F^* : \mathcal{C}^* \to \mathcal{D}^*$ such that

(a) $F^0(r_{\mathcal{C}}(f) = r_{\mathcal{C}}(F^*(f))$ and $F^0(s_{\mathcal{C}}(f) = s_{\mathcal{C}}(F^*(f))$ for all $f \in \mathcal{C}^*$,

(b) $F^*(fg) = F^*(g)F^*(f)$ for all $(f, g) \in \mathcal{C}^* \times \mathcal{C}^*$, and

(c) $\mathrm{id}(F^0(v) = \mathrm{id}(F^0(v))$ for all $v \in \mathcal{C}^0$.

Definition 4.1.2. Let $k \in \mathbb{N} \setminus \{0\}$. A *$k$-graph* (Λ, d) consists of a countable category Λ and a functor d from Λ to $\mathbb{N}^k$ (view $\mathbb{N}^k$ as the category of Example 4.1.1) satisfying the *factorization property*:

For all $\lambda \in \Lambda^*$ and $m, n \in \mathbb{N}^k$ such that $d^*(\lambda) = m + n$, there exist unique elements $\mu \in \Lambda^*$ and $\nu \in \Lambda^*$ such that $\lambda = \mu\nu$.

Since the category $\mathbb{N}^k$ has only one object, the map d^0 of the functor d is trivial. So we write d for both d^* and d^0 and call it the *degree map*. We usually use Λ for (Λ, d).

Let Λ be a k-graph. For any $n \in \mathbb{N}^k$, we define $\Lambda^n := \{\lambda \in \Lambda^* : d(\lambda) = n\}$ and we say Λ is a *finite* k-graph if Λ^n is finite for all $n \in \mathbb{N}^k$. We say Λ has *no sinks* if for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, there is a $\lambda \in \Lambda^*$ such that $s(\lambda) = v$ and $d(\lambda) = n$. Similarly, Λ has *no sources* if for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, there is a $\lambda \in \Lambda^*$ such that $r(\lambda) = v$ and $d(\lambda) = n$.

For $\mu, \nu \in \Lambda$, we write $\Lambda^{\min}(\mu, \nu)$ for the set of $(\xi, \eta) \in \Lambda \times \Lambda$ such that $\mu\xi = \nu\eta$ and $d(\mu\xi) = d(\mu) \vee d(\nu)$.

Given $v, w \in \Lambda^0$, $v\Lambda^n w$ denotes the $\{\lambda \in \Lambda^n : r(\lambda) = v \text{ and } s(\lambda) = w\}$. For $1 \leq i \leq k$, let A_i be the matrix in $M_{\Lambda^0}(\mathbb{N})$ with entries $A_i(v, w) = |v\Lambda^{e_i} w|$. We call the $A_i(1 \leq i \leq k)$ the *vertex matrices*. Notice that $(A_iA_j)(v, w) = |v\Lambda^{e_i+e_j} w|$. Then the factorisation property in Λ implies that $A_iA_j = A_jA_i$, and therefore we can define $A^n := \prod_{i=1}^k A_i^{n_i}$ for all $n \in \mathbb{N}^k$.

Example 4.1.3. Let Ω_k be the category with objects $\Omega_k^0 = \mathbb{N}^k$, morphisms $\Omega_k^* : \{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$, range and source maps $r(m,n) = m, s(m,n) = n$, identity morphisms $\mathrm{id}(m) = (m,m)$, and the composition $(m,n)(n,p) = (m,p)$. If we equip the category Ω_k with the degree map $d(m,n) = n - m$, then (Ω_k, d) becomes a k-graph.

Let (Λ_1, d_1) and (Λ_2, d_2) be two k-graphs. A *k-graph morphism* is a functor F form the category Λ_1 to the category Λ_2 preserving the degree maps, in the sense that $d_2 \circ F = d_1$.

For a k-graph Λ, we refer to the *infinite path space of* Λ as

$$\Lambda^\infty := \{z : \Omega_k \to \Lambda : \ z \text{ is a } k\text{-graph morphism}\}.$$

For $p \in \mathbb{N}^k$, we define the shift map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(z)(m,n) = z(m+p, n+p)$ for all $z \in \Lambda^\infty$ and $(m,n) \in \Omega_k$. Clearly $\sigma^p \circ \sigma^q = \sigma^{p+q} = \sigma^q \circ \sigma^p$. Notice that for every $z \in \Lambda^\infty$ and $p \in \mathbb{N}^k$ we have

$$z = z(0,p)\sigma^p(z). \tag{4.1}$$

For each $\lambda \in \Lambda$, let

$$Z(\lambda) := \{z \in \Lambda^\infty : z(0, d(\lambda)) = \lambda\}.$$

Endow Λ^∞ with the topology generated by the collection $\{Z(\lambda) : \lambda \in \Lambda\}$. For finite Λ, [30, Lemma 2.6] shows that Λ^∞ is compact in this topology. For each $p \in \mathbb{N}^k$, [30, Remark 2.5] implies that the shift map σ^p is a local homeomorphism on Λ^∞.

4.1.1 C^*-algebras associated to higher rank graphs

Definition 4.1.4. Let Λ be a finite k-graph. Following [27, 45], we say a collection of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B forms a *Toeplitz-Cuntz-Krieger Λ-family* if

(TCK1) $\{S_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections,

(TCK2) $S_\lambda S_\mu = S_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$,

(TCK3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for all λ,

(TCK4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have

$$S_v \geq \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*,$$

(TCK5) for all $\mu, \nu \in \Lambda$, we have

$$S_\mu^* S_\nu = \sum_{(\xi,\eta)\in\Lambda^{\min}(\mu,\nu)} S_\xi S_\eta^*.$$

They form a *Cuntz-Krieger* Λ*-family* if they also satisfy

(CK) $S_v = \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

We interpret any empty sums as 0.

Remark 4.1.5. Conditions (TCK1)$-$(TCK3) and (CK) implies (TCK5) (see [30, Lemma 3.1]). Then to see that a family of partial isometries is a Toeplitz-Cuntz-Krieger Λ-family, we can either check (TCK1)$-$(TCK5) or check (TCK1)$-$(TCK4) together with (CK).

The next lemma shows that it suffices to check (TCK5) for a subset of $\{S_\lambda : \lambda \in \Lambda\}$.

Lemma 4.1.6. *Let Λ be a finite k-graph. Suppose that $\{S_\lambda : \lambda \in \Lambda\}$ is a collection of partial isometries in a C^*-algebra B which satisfies (TCK1)$-$(TCK3). Suppose that for all $\mu, \nu \in \Lambda$ with $d(\mu) \wedge d(\nu) = 0$ we have $S_\mu^* S_\nu = \sum_{(\xi,\eta)\in\Lambda^{\min}(\mu,\nu)} S_\xi S_\eta^*$. Then $\{S_\lambda : \lambda \in \Lambda\}$ satisfies (TCK5).*

Proof. Fix $\mu, \nu \in \Lambda$. By the factorisation property we can write $\mu = \mu'\mu''$ and $\nu = \nu'\nu''$ such that

$$\begin{aligned} d(\mu') = d(\nu') = d(\mu)\wedge d(\nu),\ \ d(\mu'') = d(\mu) - d(\mu)\wedge d(\nu),\ \text{and} \\ d(\nu'') = d(\nu) - d(\mu)\wedge d(\nu). \end{aligned} \tag{4.2}$$

Now using (TCK1)$-$(TCK3) and the identity $S_{r(\lambda)}S_\lambda = S_\lambda$, we have

$$\begin{aligned} S_\mu^* S_\nu &= S_{\mu''}^* S_{\mu'}^* S_{\nu'} S_{\nu''} \\ &= S_{\mu''}^* \delta_{\mu',\nu'} S_{s(\mu')} S_{\nu''} \\ &= \delta_{\mu',\nu'} S_{\mu''}^* S_{r(\mu'')} S_{\nu''} \qquad \text{since } s(\mu') = r(\mu'') \\ &= \delta_{\mu',\nu'} S_{\mu''}^* S_{\nu''}. \end{aligned}$$

Since $d(\mu'') \wedge d(\nu'') = 0$, applying (TCK5) for μ'', ν'' gives

$$S_\mu^* S_\nu = \delta_{\mu',\nu'} \sum_{(\xi,\eta)\in\Lambda^{\min}(\mu'',\nu'')} S_\xi S_\eta^*. \tag{4.3}$$

Next we aim to show that

$$\Big(\mu' = \nu' \text{ and } (\xi,\eta) \in \Lambda^{\min}(\mu'',\nu'')\Big) \Longleftrightarrow (\xi,\eta) \in \Lambda^{\min}(\mu,\nu). \tag{4.4}$$

Suppose $\mu' = \nu'$ and $(\xi,\eta) \in \Lambda^{\min}(\mu'',\nu'')$. Then $\mu''\xi = \nu''\eta$ implies that $\mu\xi = \nu\eta$. Since $d(\mu'') \wedge d(\nu'') = 0$, it follows from $d(\mu''\xi) = d(\mu'') \vee d(\nu'')$ that $d(\mu''\xi) = d(\mu'') + d(\nu'')$ and hence $d(\xi) = d(\nu'')$. Now (4.2) implies that

$$d(\mu\xi) = d(\mu) + d(\nu'') = d(\mu) + d(\nu) - d(\mu) \wedge d(\nu) = d(\mu) \vee d(\nu).$$

Thus $(\xi,\eta) \in \Lambda^{\min}(\mu,\nu)$.

Next let $(\xi,\eta) \in \Lambda^{\min}(\mu,\nu)$. Since $\mu\xi = \nu\eta$, the factorization property implies that $\mu' = \nu'$. Notice that

$$d(\mu''\xi) = d(\mu\xi) - d(\mu') = d(\mu) \vee d(\nu) - d(\mu') = d(\mu) + d(\nu) - d(\mu) \wedge d(\nu) - d(\mu').$$

Now (4.2) implies that

$$d(\mu''\xi) = [d(\mu) - d(\mu) \wedge d(\nu)] + [d(\nu) - d(\mu')] = d(\mu'') + d(\nu'').$$

Thus $(\xi,\eta) \in \Lambda^{\min}(\mu'',\nu'')$.

Next we finish off by putting (4.4) in (4.3). We have

$$S_\mu^* S_\nu = \sum_{(\xi,\eta)\in\Lambda^{\min}(\mu,\nu)} S_\xi S_\eta^*,$$

which is precisely (TCK5) for μ, ν. □

Kumjian and Pask showed in [30] that for a finite k-graph Λ, there is a C^*-algebra $C^*(\Lambda)$ and a Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ on $C^*(\Lambda)$ such that

(U1) For any other Cuntz-Krieger Λ-family $\{T_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B, there exists a unique homomorphism $\pi_T : C^*(\Lambda) \to B$ such that $\pi_T(t_\lambda) = T_\lambda$.

(U2) $C^*(\Lambda)$ is generated by $\{t_\lambda : \lambda \in \Lambda\}$.

We say the pair $(C^*(\Lambda), t_\lambda)$ is universal for Cuntz-Krieger Λ-families. The C^*-algebra $C^*(\Lambda)$ is called the *C^*-algebra of* Λ and the family $\{t_\lambda : \lambda \in \Lambda\}$ is called a *universal Cuntz-Krieger Λ-family*.

The universal property shows that there exists a strongly continuous gauge action $\tilde{\gamma} : \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ such that $\tilde{\gamma}_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ (in multi-indexed notation, so that

$z^n = \prod_{i=1}^k z_i^{n_i}$ for $z = (z_1 \dots, z_k) \in \mathbb{T}^k$ and $n \in \mathbb{Z}^k$). It also follows from [30, Lemma 3.1] that

$$C^*(\Lambda) = \overline{\operatorname{span}}\{t_\lambda t_\mu^* : s(\lambda) = s(\mu)\}.$$

Raeburn and Sims showed in [45, Corollary 7.5] that there exists a C^*-algebra $\mathcal{T}C^*(\Lambda)$ and a Toeplitz-Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ on $\mathcal{T}C^*(\Lambda)$ such that $(\mathcal{T}C^*(\Lambda), s_\lambda)$ is universal for Toeplitz-Cuntz-Krieger Λ-families. We call $\mathcal{T}C^*(\Lambda)$ the *Toeplitz-Cuntz-Krieger algebra* and call $\{s_\lambda : \lambda \in \Lambda\}$ a *universal Toeplitz-Cuntz-Krieger Λ-families.*

The universal property shows that there is a strongly continuous gauge action $\gamma : \mathbb{T}^k \to \operatorname{Aut}(\mathcal{T}C^*(\Lambda))$ such that $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ (using multi-indexed notation). Furthermore, by a standard argument and using $(TCK5)$, we can show that

$$\mathcal{T}C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, s(\lambda) = s(\mu)\}.$$

(see [51, Lemma 3.1.2, Proposition 3.2.1]).

Remark 4.1.7. We can lift the gauge actions of $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$ to actions of $\mathbb{R}$ via the maps $t \mapsto \gamma_{e^{itr}}$ (and $t \mapsto \tilde{\gamma}_{e^{itr}}$) for some $r \in (0, \infty)^k$. Notice that for each $s_\lambda s_\mu^* \in \mathcal{T}C^*(\Lambda)$, the function $t \mapsto \gamma_{e^{itr}}\big(s_\lambda s_\mu^*\big) = e^{itr\cdot(d(\mu)-d(\nu)} s_\lambda s_\mu^*$ on $\mathbb{R}$ extends to an entire function on all of $\mathbb{C}$. Thus $s_\lambda s_\mu^*$ is an analytic element of $\mathcal{T}C^*(\Lambda)$. The elements $s_\lambda s_\mu^*$ span a dense subalgebra of $\mathcal{T}C^*(\Lambda)$. So when we study the KMS states of the system $(\mathcal{T}C^*(\Lambda), \gamma_{e^{itr}})$, it suffices to check KMS condition on these elements. Similarly, we can show that $\{t_\lambda t_\mu^* : s(\lambda) = s(\mu)\}$ spans a dense subspace of analytic elements of the system $(C^*(\Lambda), \tilde{\gamma}_{e^{itr}})$.

The next lemma shows that we can view $C^*(\Lambda)$ as a quotient of $\mathcal{T}C^*(\Lambda)$.

Lemma 4.1.8. *Let Λ be a finite k-graph. Suppose $\mathcal{I}$ is the ideal in $\mathcal{T}C^*(\Lambda)$ generated by*

$$\Big\{s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*, v \in \Lambda^0, n \in \mathbb{N}^k\Big\},$$

and let $q : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)/\mathcal{I}$ be the quotient map. Then $\big(\mathcal{T}C^*(\Lambda)/\mathcal{I}, q(s_\lambda)\big)$ is universal for Cuntz-Krieger Λ-families, and is canonically isomorphic to $(C^*(\Lambda), t_\lambda)$.

Proof. Since q is a homomorphism and $\{s_\lambda : \lambda \in \Lambda\}$ satisfy (TCK1)$-$(TCK3), the family $\{q(s_\lambda) : \lambda \in \Lambda\}$ satisfies (TCK1)$-$(TCK3) as well. Clearly $\{q(s_\lambda) : \lambda \in \Lambda\}$ satisfies (CK). Since $\{s_\lambda : \lambda \in \Lambda\}$ generates $\mathcal{T}C^*(\Lambda)$, we have that $\{q(s_\lambda) : \lambda \in \Lambda\}$ generates $\mathcal{T}C^*(\Lambda)/\mathcal{I}$.

To see $(U2)$, suppose that $\{T_\lambda : \lambda \in \Lambda\}$ is another Cuntz-Krieger Λ-family, in a C^*-algebra B. Notice that $\{T_\lambda : \lambda \in \Lambda\}$ is in particular a Toeplitz-Cuntz-Krieger Λ-family. Then the universal property of the pair $(\mathcal{T}C^*(\Lambda), s_\lambda)$ gives a unique homomorphism $\pi_T : \mathcal{T}C^*(\Lambda) \to B$ such that $\pi_T(s_\lambda) = T_\lambda$. Notice that $\{T_\lambda : \lambda \in \Lambda\}$ satisfies (CK). Then we can descends π_T to a homomorphism of $\mathcal{T}C^*(\Lambda)/\mathcal{I}$ such that $\pi_T(q(s_\lambda)) = T_\lambda$ for all $\lambda \in \Lambda$. □

4.2 1-coaligned higher rank graphs and the associated C^*-algebras

Definition 4.2.1 ([37, Definition 2.2]). A k-graph Λ is 1-*coaligned* if for all $1 \le i \ne j \le k$ and $(\lambda, \mu) \in \Lambda^{e_i} \times \Lambda^{e_j}$ with $s(\lambda) = s(\mu)$ there exists a unique pair $(\eta, \zeta) \in \Lambda^{e_j} \times \Lambda^{e_i}$ such that $\eta\lambda = \zeta\mu$.

It is observed in [37, Theorem 2.3] that a k-graph Λ is 1-coaligned if and only the shift maps $\sigma^{e_1}, \dots, \sigma^{e_k}$ on the infinite path space Λ^∞ $*$-commute. Let Λ be a 1-coaligned k-graph and let $X(\Lambda^\infty)$ be the product system associated to $\sigma^{e_1}, \dots, \sigma^{e_k}$ as in Corollary 2.1.2. We write $\mathcal{NT}(X(\Lambda^\infty))$ and $\mathcal{O}(X(\Lambda^\infty))$ for the Nica-Toeplitz algebra and the Cuntz-Pimsner algebra of $X(\Lambda^\infty)$. In this section, we show that the Cuntz-Pimsner algebra $\mathcal{O}(X(\Lambda^\infty))$ is isomorphic to the Cuntz-Krieger algebra $C^*(\Lambda)$ and the Nica-Toeplitz algebra $\mathcal{NT}(X(\Lambda^\infty))$ contains an isomorphic copy of the Toeplitz Cuntz-Krieger algebra $\mathcal{T}C^*(\Lambda)$.

The next lemma is contained in [37, Theorem 2.3]; since [37, Theorem 2.3] has not been published, we provide a brief proof here.

Lemma 4.2.2. *Let Λ be a finite 1-coaligned k-graph. Suppose $0 \le i \ne j \le k$. Then the shift maps σ^{e_i} and σ^{e_j} $*$-commute.*

Proof. Let $w, z \in \Lambda^\infty$ such that

$$\sigma^{e_i}(z) = \sigma^{e_j}(w). \tag{4.5}$$

It follows from (4.1) that $z = z(0, e_i)\sigma^{e_i}(z)$ and $w = w(0, e_j)\sigma^{e_j}(w)$. Now equation (4.5) implies that $z(0, e_i)$ and $w(0, e_j)$ have the same sources. Since Λ is 1-coaligned there exists a unique pair $(\eta, \zeta) \in \Lambda^{e_j} \times \Lambda^{e_i}$ such that $\eta z(0, e_i) = \zeta w(0, e_j)$. Let λ be the element of $\Lambda^{e_i+e_j}$ identified by $\eta z(0, e_i)$ (or $\zeta w(0, e_j)$), then $u := \lambda\sigma^{e_i}(z) \in \Lambda^\infty$ satisfies

$$\sigma^{e_j}(u) = z \text{ and } \sigma^{e_i}(u) = w.$$

Since λ is determined uniquely, so is u. Thus σ^{e_i} and σ^{e_j} $*$-commute. □

Notation 4.2.3. Let Λ be a finite 1-coaligned k-graph with no sinks. Then the shift maps $\sigma^{e_1}, \dots, \sigma^{e_k}$ are surjective $*$-commuting maps. As we mentioned before, we write $X(\Lambda^\infty)$ for the product system associated to $\sigma^{e_1}, \dots, \sigma^{e_k}$. We use ψ for the universal Nica-covariant representation. We write $X_m(\Lambda^\infty)$ for the fibre associated to $m \in \mathbb{N}^k$. We write φ_m for the left action of A on the fibre $X_m(\Lambda^\infty)$. Recall that the multiplication formula in $X(\Lambda^\infty)$ is[1]

$$xy(z) = x(z)y(\sigma^m(z)) \text{ for } x \in X_m, y \in X_n, z \in \Lambda^\infty.$$

In this section we work with four C^*-algebras: $\mathcal{NT}(X(\Lambda^\infty)), \mathcal{O}(X(\Lambda^\infty)), \mathcal{T}C^*(\Lambda)$, and $C^*(\Lambda)$. All of these C^*-algebras carry a gauge action of $\mathbb{T}^k$. To avoid possible clash of notation, we continue to write γ and $\tilde{\gamma}$ for the gauge actions on $\mathcal{NT}(X(\Lambda^\infty))$ and $\mathcal{O}(X(\Lambda^\infty))$, respectively. We write $\gamma_|$ and $\tilde{\gamma}_|$ for the actions on $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$, respectively.

Lemma 4.2.4. *Let Λ be a finite 1-coaligned k-graph with no sources. Suppose $\lambda \in \Lambda^m$ and $\mu \in \Lambda^n$ such that $m \wedge n = 0$ and $s(\lambda) = s(\mu)$. Then there exists a unique pair $(\eta, \xi) \in \Lambda \times \Lambda$ such that $\eta\lambda = \xi\mu$.*

Proof. We first show that there is such a pair $(\eta, \xi) \in \Lambda \times \Lambda$. Since Λ has no source, there exists $z \in \Lambda^\infty$ such that $z(0,0) = s(\lambda)$. Let $w' := \mu z$ and $w'' := \lambda z$. Notice that $\sigma^n(w') = z = \sigma^m(w'')$. Since $m \wedge n = 0$, Corollary 3.0.8 implies that σ^m and σ^n are $*$-commuting. Then there exists unique $w \in \Lambda^\infty$ such that $w' = \sigma^m(w)$ and $w'' = \sigma^n(w)$. Now let $\eta := w(0,n)$ and $\xi := w(0,m)$. Clearly $\eta\lambda = \xi\mu$. The uniqueness of pair (η, ξ) follows from the uniqueness of w. □

Remark 4.2.5. We could have proved the Lemma 4.2.4 for a finite 1-coaligned k-graph with sources by the way of induction. But all the k-graphs that we work with have no sources and with this hypothesis the proof of Lemma 4.2.4 is easier.

Lemma 4.2.6. *Let Λ be a finite k-graph and suppose $m, n \in \mathbb{N}^k$. Then the collection $\{\chi_{Z(\mu)}\}_{\mu \in \Lambda^{m+n}}$ is a partition of unity such that $\sigma^m|_{\operatorname{supp}\chi_{Z(\mu)}}$ and $\sigma^n|_{\operatorname{supp}\chi_{Z(\mu)}}$ are injective for all $\mu \in \Lambda^{m+n}$.*

[1] In previous chapters we wrote the multiplication in terms of isomorphisms between fibres. For example $xy(z) = \sigma(x \otimes y)(z)$. Unfortunately in this chapter we use letter σ for shifts. Thus here we do not use σ when writing products.

Proof. Fix $m, n \in \mathbb{N}^k$. Remark 2.5 in [30] says that, the sets $\{Z(\mu) : d(\mu) = m + n\}$ form a partition of Λ^∞. Then $\{\chi_{Z(\mu)}\}_{\mu \in \Lambda^{m+n}}$ is a partition of unity. Fix $\mu \in \Lambda^{m+n}$. To see that $\sigma^m|_{\operatorname{supp} \chi_{Z(\mu)}}$ is injective, let $\sigma^m(w) = \sigma^m(z)$ for $w, z \in \operatorname{supp} \chi_{Z(\mu)}$. Notice that $w(0, m) = \mu(0, m) = z(0, m)$. On the other hand, (4.1) implies that $w = w(0, m)\sigma^m(w)$ and $z = z(0, m)\sigma^m(z)$. Comparing these equations, we deduce that $w = z$. Thus $\sigma^m|_{\operatorname{supp} \chi_{Z(\mu)}}$ is injective. A similar argument shows that $\sigma^n|_{\operatorname{supp} \chi_{Z(\mu)}}$ is injective as well. $\square$

Proposition 4.2.7. *Let Λ be a finite 1-coaligned k-graph with no sinks or sources. For each $\lambda \in \Lambda$, let $S_\lambda := \psi_{d(\lambda)}(\chi_{Z(\lambda)})$. Then*

(a) The elements $\{S_\lambda\}_{\lambda \in \Lambda}$ form a Toeplitz-Cuntz-Krieger Λ-family in $\mathcal{NT}(X(\Lambda^\infty))$. Then the corresponding homomorphism $\pi_S : \mathcal{T}C^*(\Lambda) \to \mathcal{NT}(X(\Lambda^\infty))$ is injective and intertwines the respective gauge actions of $\mathbb{T}^k$ (in the sense that $\pi_S \circ \gamma_| = \gamma \circ \pi_S$).

(b) Let $q : \mathcal{NT}(X(\Lambda^\infty)) \to \mathcal{O}(X(\Lambda^\infty))$ be the quotient map as in Lemma 1.3.8. Then $\{q \circ S_\lambda\}_{\lambda \in \Lambda}$ is a Cuntz-Krieger Λ-family in $\mathcal{O}(X(\Lambda^\infty))$. The corresponding homomorphism $\pi_{q \circ S} : C^*(\Lambda) \to \mathcal{O}(X(\Lambda^\infty))$ is an isomorphism and intertwines the respective gauge actions of $\mathbb{T}^k$.

Proof of (a). Let $\lambda \in \Lambda$. Notice that $\chi_{Z(\lambda)} \in X_{d(\lambda)}(\Lambda^\infty)$. We will need the next formula is our proof:

$$\begin{aligned}
\big\langle \chi_{Z(\lambda)}, \chi_{Z(\lambda)} \big\rangle(z) &= \sum_{\sigma^{d(\lambda)}(w) = z} \overline{\chi_{Z(\lambda)}(w)} \chi_{Z(\lambda)}(w) \\
&= \Big| \Big\{ w : \sigma^{d(\lambda)}(w) = z \text{ and } w \in Z(\lambda) \Big\} \Big| \\
&= \begin{cases} 0 & \text{if } z \notin Z\big(s(\lambda)\big) \\ 1 & \text{if } z \in Z\big(s(\lambda)\big) \end{cases} \\
&= \chi_{Z(s(\lambda))}(z). \qquad (4.6)
\end{aligned}$$

Next we show that S_λ is a partial isometry:

$$\begin{aligned}
S_\lambda S_\lambda^* S_\lambda &= \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big) \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)^* \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big) \\
&= \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big) \psi_0\big(\langle \chi_{Z(\lambda)}, \chi_{Z(\lambda)} \rangle\big) \\
&= \psi_{d(\lambda)}\big(\chi_{Z(\lambda)} \cdot \chi_{Z(s(\lambda))}\big) && \text{by (4.6).}
\end{aligned}$$

Now the calculation

$$\chi_{Z(\lambda)} \cdot \chi_{Z(s(\lambda))}(z) = \chi_{Z(\lambda)}(z) \chi_{Z(s(\lambda))}\big(\sigma^{d(\lambda)}(z)\big) = \chi_{Z(\lambda)}(z) \text{ for } z \in \Lambda^\infty,$$

implies $S_\lambda S_\lambda^* S_\lambda = \psi_{d(\lambda)}(\chi_{Z(\lambda)}) = S_\lambda$. Thus S_λ is a partial isometry.

Next we aim to check properties (TCK1)−(TCK5). To see (TCK1), let $v \in \Lambda^0$. Since ψ_0 is a homomorphism, we have

$$S_v^* = \psi_{d(v)}\big(\chi_{Z(v)}\big)^* = \psi_0\big(\chi_{Z(v)}\big)^* = \psi_0(\chi_{Z(v)}{}^*) = \psi_0(\chi_{Z(v)}) = S_v.$$

Similarly,

$$S_v S_v = \psi_0\big(\chi_{Z(v)}\big)\psi_0\big(\chi_{Z(v)}\big) = \psi_0\big(\chi_{Z(v)}\chi_{Z(v)}\big) = \psi_0\big(\chi_{Z(v)}\big) = S_v.$$

So S_v is a projection. Now let $v, w \in \Lambda^0$. We have

$$S_v S_w = \psi_0\big(\chi_{Z(v)}\big)\psi_0\big(\chi_{Z(w)}\big) = \psi_0\big((\chi_{Z(v)})(\chi_{Z(w)})\big) = \delta_{w,v}\psi_0\big(\chi_{Z(v)}\big) = \delta_{w,v} S_v,$$

which implies that the collection $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections.

To check (TCK2), let $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$. We have

$$S_\lambda S_\mu = \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)\psi_{d(\mu)}\big(\chi_{Z(\mu)}\big) = \psi_{d(\lambda\mu)}\Big(\big(\chi_{Z(\lambda)}\big)\big(\chi_{Z(\mu)}\big)\Big).$$

The multiplication formula in $X(\Lambda^\infty)$ for $\chi_{Z(\lambda)} \in X_{d(\lambda)}(\Lambda^\infty)$ and $\chi_{Z(\mu)} \in X_{d(\mu)}(\Lambda^\infty)$ implies that

$$\begin{aligned}\Big(\big(\chi_{Z(\lambda)}\big)\big(\chi_{Z(\mu)}\big)\Big)(z) &= \chi_{Z(\lambda)}(z)\chi_{Z(\mu)}\big(\sigma^{d(\lambda)}(z)\big)\\ &= \chi_{Z(\lambda\mu)}(z).\end{aligned}$$

Then $S_\lambda S_\mu = \psi_{d(\lambda\mu)}\big(\chi_{Z(\lambda\mu)}\big) = S_{\lambda\mu}$.

To check (TCK3), let $\lambda \in \Lambda$. A routine calculation shows that

$$\begin{aligned}S_\lambda^* S_\lambda &= \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)^*\psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)\\ &= \psi_0\big(\langle \chi_{Z(\lambda)}, \chi_{Z(\lambda)}\rangle\big)\\ &= \chi_{Z(s(\lambda))} && \text{by (4.6)}\\ &= S_{s(\lambda)}.\end{aligned}$$

We will need (TCK5) for the proof of (TCK4). So we first check (TCK5). Lemma 4.1.6 says that it suffices to prove (TCK5) for $\mu, \nu \in \Lambda$ with $d(\mu) \wedge d(\nu) = 0$. For convenience, let $m := d(\nu)$ and $n := d(\mu)$. Let $\{\chi_{Z(\xi)}\}_{\xi \in \Lambda^{m+n}}$ be the partition of unity from lemma 4.2.6. Applying Proposition 3.1.2 to $\{\chi_{Z(\xi)}\}_{\xi \in \Lambda^{m+n}}$ gives

$$\begin{aligned}S_\mu^* S_\nu &= \psi_n(\chi_{Z(\mu)})^*\psi_m(\chi_{Z(\nu)})\\ &= \sum_{\xi \in \Lambda^m,\, \eta \in \Lambda^n} \psi_0\big(\langle \chi_{Z(\mu)}, \chi_{Z(\eta)} \circ \sigma^m\rangle\big)\psi_m(\chi_{Z(\xi)})\psi_n(\chi_{Z(\eta)})^*\psi_0\big(\langle \chi_{Z(\xi)} \circ \sigma^n, \chi_{Z(\nu)}\rangle\big)\end{aligned}$$

$$(4.7)\qquad = \sum_{\xi\in\Lambda^m,\,\eta\in\Lambda^n} \psi_m\big(\big\langle\chi_{Z(\mu)},\chi_{Z(\eta)}\circ\sigma^m\big\rangle\cdot\chi_{Z(\xi)}\big)\psi_n\big(\big\langle\chi_{Z(\xi)}\circ\sigma^n,\chi_{Z(\nu)}\big\rangle\cdot\chi_{Z(\eta)}\big)^*$$

We now consider a summand for fixed ξ and η. We have

$$\begin{aligned}
\Big(\big\langle\chi_{Z(\mu)},\chi_{Z(\eta)}\circ\sigma^m\big\rangle\cdot\chi_{Z(\xi)}\Big)(z) &= \big\langle\chi_{Z(\mu)},\chi_{Z(\eta)}\circ\sigma^m\big\rangle(z)\chi_{Z(\xi)}(z)\\
&= \chi_{Z(\xi)}(z)\sum_{\sigma^n(w)=z}\chi_{Z(\mu)}(w)\chi_{Z(\eta)}\big(\sigma^m(w)\big)\\
&= \begin{cases}1 & \text{if } z\in Z(\xi)\text{, and } \mu\xi=\alpha\eta \text{ for some } \alpha\in\Lambda^m\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases}\chi_{Z(\xi)} & \text{if } \mu\xi=\alpha\eta \text{ for some } \alpha\in\Lambda^m\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Similarly

$$\begin{aligned}
\Big(\big\langle\chi_{Z(\xi)}\circ\sigma^n,\chi_{Z(\nu)}\big\rangle\cdot\chi_{Z(\eta)}\Big)(z) &= \big\langle\chi_{Z(\xi)}\circ\sigma^n,\chi_{Z(\nu)}\big\rangle(z)\chi_{Z(\eta)}(z)\\
&= \chi_{Z(\eta)}(z)\sum_{\sigma^m(w)=z}\chi_{Z(\xi)}\big(\sigma^n(w)\big)\chi_{Z(\nu)}(w)\\
&= \begin{cases}1 & \text{if } z\in Z(\eta)\text{, and } \nu\eta=\beta\xi \text{ for some } \beta\in\Lambda^n\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases}\chi_{Z(\eta)} & \text{if } \nu\eta=\beta\xi \text{ for some } \beta\in\Lambda^n\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

It then follows that the ξ-η summand vanishes unless

$$\mu\xi=\alpha\eta \text{ and } \nu\eta=\beta\xi.$$

This means ξ and η must have the same source. Since Λ is 1-coaligned and $d(\xi)\wedge d(\eta)=0$ (note that $d(\xi)=m$ and $d(\eta)=n$), Lemma 4.2.4 implies that $\alpha=\nu$ and $\beta=\mu$. Thus the sum in (4.7) collapses to

$$\begin{aligned}
S_\mu^*S_\nu &= \sum_{(\xi,\eta)\in\Lambda^{\min}(\mu,\nu)}\psi_m\big(\chi_{Z(\xi)}\big)\psi_n\big(\chi_{Z(\eta)}\big)^*\\
&= \sum_{(\xi,\eta)\in\Lambda^{\min}(\mu,\nu)} S_\xi S_\eta^*,
\end{aligned}$$

which completes our proof of (TCK5).

To see (TCK4), let $v\in\Lambda^0$ and $n\in\mathbb{N}^k$. Suppose that $\lambda,\mu\in v\Lambda^n$ and $\lambda\neq\mu$. It follows from $d(\Lambda)=d(\mu)$ that $\Lambda^{\min}(\lambda,\mu)=\emptyset$. Now (TCK5) implies that

$$S_\lambda(S_\lambda^*S_\mu)S_\mu^*=0.$$

Thus $S_\lambda S_\lambda^* \perp S_\mu S_\mu^*$. It follows from [44, Corollary A.3] that $\sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*$ is a projection. Thus to check $S_v \geq \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*$, it suffices to prove

$$S_v\Big(\sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*\Big) = \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^* = \Big(\sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*\Big)S_v. \tag{4.8}$$

For the first equality, we have

$$\begin{aligned} S_v \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^* &= \sum_{\lambda\in v\Lambda^n} S_v S_\lambda S_\lambda^* \\ &= \sum_{\lambda\in v\Lambda^n} \psi_0\big(\chi_{Z(v)}\big)\psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)\psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)^*. \end{aligned}$$

Since $r(\lambda) = v$, a quick calculation shows that the left action of $\chi_{Z(v)}$ on $\chi_{Z(\lambda)}$ is $\chi_{Z(\lambda)}$. It then follows

$$S_v \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^* = \sum_{\lambda\in v\Lambda^n} \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)\psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)^* = \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*.$$

Similarly, the second equation in (4.8) follows from

$$\begin{aligned} \Big(\sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*\Big)S_v &= \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^* S_v \\ &= \sum_{\lambda\in v\Lambda^n} \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)\psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)^*\psi_0\big(\chi_{Z(v)}\big). \end{aligned}$$

Since $r(\lambda) = v$, we again have $\chi_{Z(v)} \cdot \chi_{Z(\lambda)} = \chi_{Z(\lambda)}$. Then

$$\Big(\sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*\Big)S_v = \sum_{\lambda\in v\Lambda^n} \psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)\psi_{d(\lambda)}\big(\chi_{Z(\lambda)}\big)^* = \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*.$$

We now have proved (TCK4) and therefore the collection $\{S_\lambda\}_{\lambda\in\Lambda}$ forms a Toeplitz-Cuntz-Krieger Λ-family in $\mathcal{NT}(X(\Lambda^\infty))$.

To see that the corresponding homomorphism π_S is injective, by [45, Theorem 8.1], it suffices to check

$$S_v \neq \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*$$

for all $v \in \Lambda^0$ and $n \in \mathbb{N}_+^k$. To do this, we use the Fock representation T of $X(\Lambda^\infty)$. Notice that the homomorphism $T_* : \mathcal{NT}(X(\Lambda^\infty)) \to \mathcal{L}(F(X(\Lambda^\infty)))$ satisfies

$$\begin{aligned} T_*\Big(S_v - \sum_{\lambda\in v\Lambda^n} S_\lambda S_\lambda^*\Big) &= T_*\Big(\psi_0(\chi_{Z(v)}) - \sum_{\lambda\in v\Lambda^n} \psi_{d(\lambda)}(\chi_{Z(\lambda)})\psi_{d(\lambda)}(\chi_{Z(\lambda)})^*\Big) \\ &= T_0(\chi_{Z(v)}) - \sum_{\lambda\in v\Lambda^n} T_{d(\lambda)}(\chi_{Z(\lambda)})T_{d(\lambda)}(\chi_{Z(\lambda)})^*. \end{aligned}$$

Now the adjoint formula (1.17) for the Fock representation says that $T_{d(\lambda)}(\chi_{Z(\lambda)})^*$ vanishes on the 0-summand in Fock module $F(X(\Lambda^\infty))$. Notice that Λ has no sources, and then the injectivity of T_0 implies that $T_0(\chi_{Z(v)}) \neq 0$. Thus

$$T_0(\chi_{Z(v)}) \neq \sum_{\lambda \in v\Lambda^n} T_{d(\lambda)}(\chi_{Z(\lambda)}) T_{d(\lambda)}(\chi_{Z(\lambda)})^*.$$

Another application of the injectivity of T_* gives $S_v \neq \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*$.

Finally, since the gauge actions in $\mathcal{T}C^*(\Lambda)$ and $\mathcal{NT}(X(\Lambda^\infty))$ satisfy $\gamma_{|_z}(s_\lambda) = z^{d(\lambda)} s_\lambda$ and $\gamma_z(\psi_m(x)) = z^m \psi_m(x)$, we have $\pi_S \circ \gamma_| = \gamma \circ \pi_S$. $\square$

Proof of (b). By Remark 4.1.5, we must check the conditions (TCK1)$-$(TCK3) and (CK). Since the quotient map q is a C^*-homomorphism, and the family $\{S_\lambda\}_{\lambda \in \Lambda}$ satisfies (TCK1)$-$(TCK3), so does $\{q \circ S_\lambda\}_{\lambda \in \Lambda}$. To check (CK), notice that $q \circ \psi$ is the universal Cuntz-Pimsner-covariant representation of $X(\Lambda^\infty)$. For convenience let $\rho := q \circ \psi$ (then the restriction ρ on each fibre X_n is $\rho_n = q \circ \psi_n$). Let $\mu \in \Lambda^n, n \in \mathbb{N}^k$. We first show that the left action of $\chi_{Z(\mu)}$ on the fibre X_n is by the finite rank operator $\Theta_{\chi_{Z(\mu)}, \chi_{Z(\mu)}}$.

To see this take $x \in X_n(\Lambda^\infty)$ and $z \in \Lambda^\infty$. We have

$$\begin{aligned}\Big(\Theta_{\chi_{Z(\mu)}, \chi_{Z(\mu)}}(x)\Big)(z) &= \Big(\chi_{Z(\mu)} \cdot \big\langle \chi_{Z(\mu)}, x \big\rangle\Big)(z) \\ &= \chi_{Z(\mu)}(z) \big\langle \chi_{Z(\mu)}, x \big\rangle (\sigma^n(z)) \\ &= \chi_{Z(\mu)}(z) \sum_{\sigma^n(w) = \sigma^n(z)} \overline{\chi_{Z(\mu)}(w)} x(w),\end{aligned}$$

and this vanishes unless $z, w \in Z(\mu)$. Since $\mu \in \Lambda^n$, $w, z \in Z(\mu)$, the equation $\sigma^n(w) = \sigma^n(z)$ has unique solution z and therefore the sum collapses to $\overline{\chi_{Z(\mu)}(z)} x(z)$. Thus

$$\Big(\Theta_{\chi_{Z(\mu)}, \chi_{Z(\mu)}}(x)\Big)(z) = \chi_{Z(\mu)}(z) x(z), \tag{4.9}$$

which equals the left action of $\chi_{Z(\mu)}$ on $x \in X_n$.

Next we check (CK). Let $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Then a routine calculation shows that

$$\begin{aligned}\sum_{\lambda \in v\Lambda^n} (q \circ S_\lambda)(q \circ S_\lambda)^* &= \sum_{\lambda \in v\Lambda^n} \rho_{d(\lambda)}(\chi_{Z(\lambda)}) \rho_{d(\lambda)}(\chi_{Z(\lambda)})^* \\ &= \sum_{\lambda \in v\Lambda^n} \rho^{(d(\lambda))}\big(\Theta_{\chi_{Z(\lambda)}, \chi_{Z(\lambda)}}\big) \\ &= \sum_{\lambda \in v\Lambda^n} \rho^{(d(\lambda))}\big(\varphi_{d(\lambda)}(\chi_{Z(\lambda)})\big) \qquad \text{by (4.9).}\end{aligned}$$

Since ρ is Cuntz-Pimsner-covariant,

$$\sum_{\lambda\in v\Lambda^n}(q\circ S_\lambda)(q\circ S_\lambda)^* = \sum_{\lambda\in v\Lambda^n}\rho_0(\chi_{Z(\lambda)}) = \rho_0\Big(\sum_{\lambda\in v\Lambda^n}\chi_{Z(\lambda)}\Big)$$
$$= q\circ\psi_0\Big(\chi_{Z(v)}\Big) = q\circ(S_v)$$

Thus (CK) holds and the collection $\{q\circ S_\lambda\}_{\lambda\in\Lambda}$ forms a Cuntz-Krieger Λ-family in $\mathcal{O}(X(\Lambda^\infty))$. This gives a homomorphism $\pi_{q\circ S}: C^*(\Lambda)\to\mathcal{O}(X(\Lambda^\infty))$.

Since the gauge actions in $C^*(\Lambda)$ and $\mathcal{O}(X(\Lambda^\infty))$ satisfy $\tilde{\gamma}_{|_z}(s_\lambda) = z^{d(\lambda)}s_\lambda$ and $\tilde{\gamma}_z(\rho_m(x)) = z^m\rho_m(x)$, we have $\pi_{q\circ S}\circ\tilde{\gamma}_| = \tilde{\gamma}\circ\pi_{q\circ S}$. Thus $\pi_{q\circ S}$ intertwines the gauge actions. Notice that Λ has no source. Since ρ_0 is injective (see for example [52, Lemma 3.15]), $\rho_0\Big(\chi_{Z(v)}\Big)\neq 0$ for all $v\in\Lambda^0$. Then the gauge-invariant uniqueness theorem (see [30, Theorem 3.4]) implies that $\pi_{q\circ S}$ is injective.

To show that $\pi_{q\circ S}$ is surjective, note that $\mathcal{O}(X(\Lambda^\infty))$ is generated by $\rho(X(\Lambda^\infty))$. We know from the Stone-Weierstrass theorem that the set $\{\chi_{Z(\lambda)}:\lambda\in\Lambda\}$ spans a dense $*$-subalgebra of $C(\Lambda^\infty)$. Since the norm of $X(\Lambda^\infty)$ is equivalent to $\|\cdot\|_\infty$ (see argument in the end of the Section 1.5), the elements $\{\chi_{Z(\lambda)}:\lambda\in\Lambda\}$ span a dense subspace of $X(\Lambda^\infty)$. Thus it is enough for us to show that $\rho_m(\chi_{Z(\mu)})$ lies in the range of $\pi_{q\circ S}$ for all $m,n\in\mathbb{N}^k$ and $\mu\in\Lambda^n$.

We first check this for $m=0$ and all $\mu\in\Lambda^n$. Since ρ is Cuntz-Pimsner-covariant, a routine calculation shows that

$$\begin{aligned}
\rho_0(\chi_{Z(\mu)}) &= \rho^{(d(\mu))}\big(\varphi_{d(\mu)}(\chi_{Z(\mu)})\big) \\
&= \rho^{(d(\mu))}\big(\Theta_{\chi_{Z(\mu)},\chi_{Z(\mu)}}\big) && \text{using (4.9)} \\
&= \rho_{d(\mu)}(\chi_{Z(\mu)})\rho_{d(\mu)}(\chi_{Z(\mu)})^* \\
&= (q\circ S_\mu)(q\circ S_\mu)^*, && (4.10)
\end{aligned}$$

which belongs to the range of $\pi_{q\circ S}$.

Now let $m\neq 0$ and take $\mu\in\Lambda^n$. Notice that $\chi_{Z(\mu)} = \sum_{\nu\in s(\mu)\Lambda^m}\chi_{Z(\mu\nu)}$. Each ν-summand is the pointwise multiplication of $\chi_{Z\left(\mu\nu(0,m)\right)}$ and $\chi_{Z\left(\mu\nu(m,m+n)\right)}\circ\sigma^m$. This is exactly the right action of $\chi_{Z\left(\mu\nu(m,m+n)\right)}$ on $\chi_{Z\left(\mu\nu(0,m)\right)}\in X_m(\Lambda^\infty)$. It follows

$$\begin{aligned}
\rho_m(\chi_{Z(\mu)}) &= \rho_m\Big(\sum_{\nu\in s(\mu)\Lambda^m}\chi_{Z\left(\mu\nu(0,m)\right)}\cdot\chi_{Z\left(\mu\nu(m,m+n)\right)}\Big) \\
&= \sum_{\nu\in s(\mu)\Lambda^m}\rho_m\Big(\chi_{Z\left(\mu\nu(0,m)\right)}\Big)\rho_0\Big(\chi_{Z\left(\mu\nu(m,m+n)\right)}\Big)
\end{aligned}$$

$$= \sum_{\nu \in s(\mu)\Lambda^m} (q \circ S_{\mu\nu(0,m)})\rho_0\Big(\chi_{Z\big(\mu\nu(m,m+n)\big)}\Big),$$

which lies in the range of $\pi_{q\circ S}$ by (4.10), as required. $\square$

4.3 KMS states on the Toeplitz algebras

In this section we want to see the relationship between KMS states of the C^*-algebras $\mathcal{T}C^*(\Lambda)$ and $\mathcal{NT}(X(\Lambda^\infty))$. The KMS states of $\mathcal{T}C^*(\Lambda)$ is described thoroughly in [27, Theorem 6.1]. We apply Theorem 3.3.1 to characterise KMS states of $\mathcal{NT}(X(\Lambda^\infty))$. It follows from [1, Proposition 7.3] that for the shift maps $\sigma^{e_i}(1 \leq i \leq k)$ on Λ^∞, each β_{c_i} in Theorem 3.3.1 is exactly $\ln\rho(A_i)$ used in [27, Theorem 6.1]. Thus the range of possible inverse temperatures studied in Theorem 3.3.1 is the same as that of [27, Theorem 6.1]. Now when we view $\mathcal{T}C^*(\Lambda)$ as a C^*-subalgebra of $\mathcal{NT}(X(\Lambda^\infty))$, restricting KMS states of $\mathcal{NT}(X(\Lambda^\infty))$ gives KMS states of $\mathcal{T}C^*(\Lambda)$ with the same inverse temperature. We expect from our results in [1, Corollary 7.6] to see that for the common inverse temperatures described in Theorem 3.3.1 and [27, Theorem 6.1] all KMS states of $\mathcal{T}C^*(\Lambda)$ arise as restrictions of KMS states of $\mathcal{NT}(X(\Lambda^\infty))$. We achieve this objective in Proposition 4.3.3.

We keep our notation in Theorem 3.3.1 to emphasise the parallels with [27, Theorem 6.1]. Then we have a clash when we try to use both descriptions at the same time. So we write δ for the measure ε in Theorem 3.3.1, and choose ε for the vectors in $[1,\infty)^{\Lambda^0}$ appearing in [27, Theorem 6.1]. We also choose α for the action of $\mathcal{NT}(X(\Lambda^\infty))$ and write $\alpha_|$ for the action of $\mathcal{T}C^*(\Lambda)$. Otherwise, we use the notation of Theorem 3.3.1.

Proposition 4.3.1. *Suppose that Λ is a finite 1-coaligned k-graph with no sources and no sinks. Let A_i be the vertex matrices of Λ. Suppose that $r \in (0,\infty)^k$ satisfies $\beta r_i > \ln\rho(A_i)$ for $1 \leq i \leq k$. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X(\Lambda^\infty))$ and $\alpha_| : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda))$ be given in terms of the gauge actions by $\alpha_t = \gamma_{e^{itr}}$ and $\alpha_{|_t} = \gamma_{|_{e^{itr}}}$. Let δ be a finite regular Borel measure on Λ^∞ such that $\int f_\beta\, d\delta = 1$. Define $\varepsilon = (\varepsilon_v) \in [0,\infty)^{\Lambda^0}$ by $\varepsilon_v = \delta(Z(v))$ and take $y = (y_v) \in [0,\infty)^{\Lambda^0}$ as in [27, Theorem 6.1]. Then $y \cdot \varepsilon = 1$, and the restriction of the state ϕ_δ of Theorem 3.3.1 to $(\mathcal{T}C^*(\Lambda), \alpha_|)$ is the state ϕ_ε of [27, Theorem 6.1].*

Proof. We first compute the function $f_\beta \in C(\Lambda^\infty)$. For $z \in \Lambda^\infty$, we have

$$f_\beta(z) = \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}|\sigma^{-n}(z)|$$

$$= \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}|\Lambda^n r(z)|$$

$$= \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}\Big(\sum_{v\in\Lambda^0}|\Lambda^n v|\chi_{Z(v)}(z)\Big).$$

Recall that $y_v = \sum_{\mu\in\Lambda v} e^{-\beta r\cdot d(\mu)}$. By applying the Tunelli theorem, we have

$$1 = \int f_\beta\, d\delta = \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}\sum_{v\in\Lambda^0}|\Lambda^n v|\delta(Z(v)) = \sum_{v\in\Lambda^0} y_v\varepsilon_v = y\cdot\varepsilon. \tag{4.11}$$

To see that ϕ_δ restricts to ϕ_ε, it suffices to compute both of them on the elements $S_\lambda S_\nu^*$. Equation (3.43) together with [27, (6.1)] imply that $\phi_\delta(S_\lambda S_\nu^*) = 0 = \phi_\varepsilon(S_\lambda S_\nu^*)$ for $d(\lambda)\neq d(\nu)$. So we assume $d(\lambda) = d(\nu) = p$ say. It then follows from (3.43) that

$$\phi_\delta(S_\lambda S_\nu^*) = \phi_\delta\Big(\psi_p(Z(\lambda))\psi_p(Z(\nu))^*\Big) = e^{-\beta r\cdot p}\int \big\langle \chi_{Z(\nu)}, \chi_{Z(\lambda)}\big\rangle d\mu, \tag{4.12}$$

where $\mu = \sum_{n\in N^k} e^{-\beta r\cdot n} R^n\delta$. Applying the inner product formula in the fibre X_p, we have

$$\big\langle \chi_{Z(\nu)}, \chi_{Z(\lambda)}\big\rangle(z) = \sum_{\sigma^p(w)=z}\overline{\chi_{Z(\nu)}(w)}\chi_{Z(\lambda)}(w) = \delta_{\lambda,\nu}\sum_{\sigma^p(w)=z}\chi_{Z(\lambda)}(w).$$

It then follows that $\langle \chi_{Z(\nu)}, \chi_{Z(\lambda)}\rangle = \delta_{\lambda,\nu}\chi_{Z(s(\lambda))}$. Putting this in (4.12), we have

$$\phi_\delta(S_\lambda S_\nu^*) = \delta_{\lambda,\nu} e^{-\beta r\cdot p}\mu\big(Z(s(\lambda))\big) \tag{4.13}$$

Next we compute $\mu(Z(v))$ for $v\in\Lambda^0$. Notice that for each n, we have

$$(R^n\delta)(Z(v)) = \int \chi_{Z(v)}\, d(R^n\delta)(z) = \int \sum_{\sigma^n(w)=z}\chi_{Z(v)}(w)\, d\delta(z).$$

We also have

$$\sum_{\sigma^n(w)=z}\chi_{Z(v)}(w) = |v\Lambda^n r(z)| = A^n(v, r(z)) = \sum_{u\in\Lambda^0} A^n(v,u)\chi_{Z(u)}(z).$$

Thus

$$(R^n\delta)(Z(v)) = \int \sum_{u\in\Lambda^0} A^n(v,u)\chi_{Z(u)}(z)\, d\delta(z) = \sum_{u\in\Lambda^0} A^n(v,u)\delta(Z(u)),$$

and

$$\mu(Z(v)) = \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}\sum_{u\in\Lambda^0} A^n(v,u)\delta(Z(v))$$

$$= \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}\sum_{u\in\Lambda^0} A^n(v,u)\varepsilon_v$$

$$= \sum_{n\in\mathbb{N}^k} e^{-\beta r\cdot n}(A^n\varepsilon)_v = \Big(\prod_{i=1}^k (1-e^{-\beta r_i}A_i)^{-1}\varepsilon\Big)_v.$$

Now we put this into (4.13), and write down

$$\phi_\delta(S_\lambda S_\nu^*) = \delta_{\lambda,\nu}e^{-\beta r\cdot p}\Big(\prod_{i=1}^k (1-e^{-\beta r_i}A_i)^{-1}\varepsilon\Big)_{s(\lambda)}, \tag{4.14}$$

which in the notation of [27, Theorem 6.1] is $\delta_{\lambda,\nu}e^{-\beta r\cdot p}m_{s(\lambda)}$. Now [27, (6.1)] implies that $\phi_\delta(S_\lambda S_\nu^*) = \phi_\varepsilon(S_\lambda S_\nu^*)$, as required.

□

Corollary 4.3.2. *Suppose that Λ is a finite 1-coaligned k-graph with no sources and no sinks. Let A_i be the vertex matrices of Λ. Suppose that $r \in (0,\infty)^k$ satisfies $\beta r_i > \ln\rho(A_i)$ for $1 \le i \le k$ and let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X(\Lambda^\infty))$ and $\alpha_| : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda))$ be given in terms of the gauge actions by $\alpha_t = \gamma_{e^{itr}}$ and $\alpha_|{}_t = \gamma|_{e^{itr}}$ Suppose that δ_1, δ_2 are regular Borel measures on Λ^∞ satisfying $\int f_\beta\, d\delta_i = 1$. Then $\phi_{\delta_1}|_{\mathcal{T}C^*(\Lambda)} = \phi_{\delta_2}|_{\mathcal{T}C^*(\Lambda)}$ if and only if $\delta_1(Z(v)) = \delta_2(Z(v))$ for all $v \in \Lambda^0$.*

Proof. Let δ_1, δ_2 be two regular Borel measures on Λ^∞ such that $\int f_\beta\, d\delta_i = 1$. Suppose $\phi_{\delta_1}|_{\mathcal{T}C^*(\Lambda)} = \phi_{\delta_2}|_{\mathcal{T}C^*(\Lambda)}$. Proposition 4.3.1 implies that for the corresponding $\varepsilon_i \in [0,\infty)^{\Lambda^0}$ (where $\varepsilon_i(v) = \delta_i(Z(v))$ for all $v \in \Lambda^0$) we have $\phi_{\varepsilon_1} = \phi_{\varepsilon_2}$. Now the injectivity of the map $\varepsilon \mapsto \phi_\varepsilon$ from [27, Theorem 6.1(c)] gives $\varepsilon_1 = \varepsilon_2$. But this says precisely that δ_1, δ_2 agree on each $Z(v)$.

For the other direction, let $\delta_1(Z(v)) = \delta_2(Z(v))$ for all $v \in \Lambda^0$. Then the corresponding ε_i are equal, and the formula (4.14) implies that $\phi_{\delta_1}, \phi_{\delta_2}$ agree on $\mathcal{T}C^*(\Lambda)$. □

Proposition 4.3.3. *Suppose that Λ is a finite 1-coaligned k-graph with no sources and no sinks. Let A_i be the vertex matrices of Λ. Suppose that $r \in (0,\infty)^k$ satisfies $\beta r_i > \ln\rho(A_i)$ for $1 \le i \le k$ and let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X(\Lambda^\infty))$ and $\alpha_| : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda))$ be given in terms of the gauge actions by $\alpha_t = \gamma_{e^{itr}}$ and $\alpha_|{}_t = \gamma|_{e^{itr}}$. Then every KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha_|)$ is the restriction of a KMS$_\beta$ state of $\mathcal{NT}(X(\Lambda^\infty), \alpha)$.*

Before starting the proof, we first describe a standard way of construction of measures on Λ^∞. We need the notion of *inverse limit* (see for example [9, Section 1, 2]):

Let P be a directed partially ordered set. An *inverse system of compact spaces* $(\{Y_p\}, \{r_{p,q}\})_{p,q\in P}$ consists of a family $\{Y_p\}_{p\in P}$ of compact spaces such that for any $p, q \in P$, $p \le q$ there exists a surjection $r_{p,q} : Y_q \to Y_p$ such that

(a) $r_{p,p} : Y_p \to Y_p$ is the identity map, and

(b) $r_{p,q} \circ r_{q,s} = r_{p,s}$ whenever $p \leq q \leq s$ and $p, q, s \in P$.

The *inverse limit*

$$\varprojlim(Y_p, r_{p,q})$$

is the set of all collections $\{y_p : y_p \in Y_p, p \in P\}$ such that for $p \leq q$, $r_{p,q}(y_q) = y_p$. It follows that for each $y_p \in Y_p$ there exists $y \in \varprojlim(Y_p, r_{p,q})$ with pth coordinate y_p. Thus we can define the *canonical maps* $\pi_p : \varprojlim(Y_p, r_{p,q}) \to Y_p$, by $\pi_p(y) = y_p$.

The next lemma shows how we can construct measures on the inverse limits.

Lemma 4.3.4 ([25, Lemma 5.2]). *Let P be a directed partially ordered set with the smallest element 0. For $p, q \in P$, let Y_p be a compact space and $r_{p,q} : Y_q \to Y_p$ be a surjection. Let $\varprojlim(Y_p, r_{p,q})$ be the inverse limit of the system $(\{Y_p\}, \{r_{p,q}\})_{p,q\in P}$ and let π_p be the canonical map from $\varprojlim(Y_p, r_{p,q})$ to Y_p. Suppose that we have Borel measure δ_p on Y_p such that δ_0 is finite and*

$$\int (f \circ r_{p,q})\, d\delta_q = \int f\, d\delta_p \text{ for } p \leq q \text{ and } f \in C(Y_p). \tag{4.15}$$

Then there is a unique finite Borel measure δ on $\varprojlim(Y_p, r_{p,q})$ such that

$$\int (f \circ \pi_p)\, d\delta = \int f\, d\delta_p \text{ for } f \in C(X_p).$$

Remark 4.3.5. Given a finite k-graph Λ, let $D := (1, \dots, 1)$ and $M := \{lD : l \in \mathbb{N}\}$. For each $m, n \in M$ such that $m \leq n$, define $r_{m,n} : \Lambda^n \to \Lambda^m$ by $r_{m,n}(\lambda) = \lambda(0, m)$. Clearly M is a directed partially ordered set, and each $r_{m,n}$ is a surjection. The argument of [30, Remark 2.2] shows that, by factorisation property, Λ^∞ can be viewed as the inverse limit of the system $(\{\Lambda^m\}, \{r_{m,n}\})_{m,n\in M}$.

Proof of Proposition 4.3.3. Suppose ϕ is a KMS$_\beta$ state of $(\mathcal{T}C^*(\Lambda), \alpha)$. Then [27, Theorem 6.1(c)] implies that there is a vector $\varepsilon \in [0, \infty)^{\Lambda^0}$ such that $y \cdot \varepsilon = 1$ and $\phi = \phi_\varepsilon$. If δ is a measure on Λ^∞ such that $\delta(Z(v)) = \varepsilon_v$ for all $v \in \Lambda^0$ and $\int f_\beta\, d\delta = 1$, then Proposition 4.3.1 implies that $\phi_\delta|_{\mathcal{T}C^*(\Lambda)} = \phi_\varepsilon$. So it suffices to show that there is such a measure.

To see this, we view Λ^∞ as the inverse limit described in Remark 4.3.5, and then we apply Lemma 4.3.4. So we must construct a sequence of measures δ_m on Λ^m satisfying (4.15). Let D be as in Remark 4.3.5. We recursively choose weights $\{w_\eta : \eta \in \Lambda \text{ with } d(\eta) = lD \text{ for some } l \geq 1\}$ such that

$$\sum_{\lambda \in v\Lambda^D} w_\lambda = \varepsilon_v,$$

and

$$\sum_{\lambda \in s(\mu)\Lambda^D} w_{\mu\lambda} = w_\mu, \tag{4.16}$$

for all $v \in \Lambda^0$ and $\mu \in \Lambda^{lD}$ $(l \geq 1)$. Then we set $\delta_0 := \varepsilon$ and $\delta_m(\mu) = w_\mu$ for all $\mu \in \Lambda^{lD}$.

Next we check (4.15) for these measures. Let $m \in M$. Since the characteristic functions of singletons span $C(\Lambda^m)$, it is enough to prove (4.15) for $f = \chi_{\{\mu\}}$ and $\mu \in \Lambda^m$. First notice that

$$\chi_{\{\mu\}} \circ r_{m,m+D} = \sum_{\lambda \in s(\mu)\Lambda^D} \chi_{\{\mu\lambda\}}.$$

Then we have

$$\begin{aligned}
\int \chi_{\{\mu\}} \circ r_{m,m+D}\, d\delta_{m+D} &= \int \sum_{\lambda \in s(\mu)\Lambda^D} \chi_{\{\mu\lambda\}}\, d\delta_{m+D} \\
&= \sum_{\lambda \in s(\mu)\Lambda^D} \delta_{m+D}(\mu\lambda) \\
&= \delta_m(\mu) \qquad\qquad \text{using (4.16)} \\
&= \int \chi_{\{\mu\}}\, d\delta_m. \qquad (4.17)
\end{aligned}$$

Since for each $n \in M$ with $m \leq n$, we have

$$r_{m,n} = r_{m,m+D} \circ r_{m+D,m+2D} \circ \cdots \circ r_{n-D,n},$$

applying the calculation (4.17) finitely many times gives

$$\int \chi_{\{\mu\}} \circ r_{m,m+n}\, d\delta_{m+n} = \int \chi_{\{\mu\}}\, d\delta_m.$$

This is precisely (4.15).

Now Lemma 4.3.4 implies that there is a unique measure δ on Λ^∞ such that

$$\int \chi_{\{v\}} \circ \pi_0\, d\delta = \int \chi_{\{v\}}\, d\delta_0 \text{ for } v \in \Lambda^0.$$

Notice that $\int \chi_{\{v\}} \circ \pi_0\, d\delta = \delta(Z(v))$ and $\int \chi_{\{v\}}\, d\delta_0 = \delta_0(v) = \varepsilon_v$. It also follows from the calculation (4.11) that $\int f_\beta\, d\delta = y \cdot \varepsilon = 1$. Thus δ has required properties. $\square$

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Appendix A

Realising the universal Nica-covariant representation as a doubly commuting representation

In this appendix we use our results from previous chapters and show that the universal Nica-covariant representation ψ satisfies the doubly commuting relation [54, Lemma 3.9 (i)]. We first need to understand the notations.

Let A be a C^*-algebra and Y be a right Hilbert A–A bimodule. Suppose that π is representation of A on $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

Let $Y \odot \mathcal{H}$ be the algebraic tensor product of Y and $\mathcal{H}$. It follows from [46, Proposition 2.6] that the formula

$$\big(y \odot r \big| y' \odot r'\big) = \big(r \big| \pi(\langle y, y'\rangle) r'\big) \text{ for } y \odot r, y' \odot r' \in Y \odot \mathcal{H},$$

defines a semi-definite inner product on $Y \odot \mathcal{H}$. Notice that

$$\begin{aligned}\big(y \cdot a \odot r - y \odot \pi(a) r \big| y' \odot r'\big) &= \big(r \big| \pi(\langle y \cdot a, y'\rangle) r'\big) - \big(\pi(a) r \big| \pi(\langle y, y'\rangle) r'\big)\\ &= \big(r \big| \pi(\langle y \cdot a, y'\rangle) r'\big) - \big(r \big| \pi(a^* \langle y, y'\rangle) r'\big)\\ &= \big(r \big| \pi(\langle y \cdot a, y'\rangle) r'\big) - \big(r \big| \pi(\langle y \cdot a, y'\rangle) r'\big)\\ &= 0.\end{aligned}$$

Now let $Y \otimes_\pi \mathcal{H}$ be the completion[1] of $Y \odot \mathcal{H}$ with respect to this semi-definite inner product (see [46, Lemma 2.16]). Since the completing process requires modding out element of length 0, in the completion we have $y \cdot a \otimes r - y \otimes \pi(a) r$.

[1]Completion with respect to semi-definite inner products are sometimes called Hausdorff completion (for example [54, page 92]).

Let $S \in \mathcal{L}(Y)$ and $U \in \pi(A)'$. A similar proof to that of [46, Proposition 2.66] shows that there is a well-defined bound operator $S \otimes U$ on $Y \otimes_\pi \mathcal{H}$ such that

$$S \otimes U(y \otimes r) = S(y) \otimes U(r) \text{ for } y \otimes r \in Y \otimes_\pi \mathcal{H}.$$

Muhly and Solel showed in [38, Lemma 3.4– 3.6] that there is a well-defined map $\widetilde{\pi} : Y \otimes_\pi \mathcal{H} \to \mathcal{H}$ such that

$$\widetilde{\pi}(y \otimes r) = \pi(y)r \text{ for all } y \otimes r \in Y \otimes_\pi \mathcal{H}.$$

The map $\widetilde{\pi}$ is called a *contraction.*

Let X be a product system of right Hilbert A–A bimodules over $\mathbb{N}^k$ and θ be a Toeplitz representation of X on $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. It follows that for each $m \in \mathbb{N}^k$ and fibre X_m, there is a contraction $\tilde{\theta}_m : X_m \otimes_{\theta_0} \mathcal{H} \to \mathcal{H}$ such that $\widetilde{\theta_m}(x \otimes r) = \theta_m(y)r$ for all $x \otimes r \in X_m \otimes_{\theta_0} \mathcal{H}$.

Let $\mathrm{I}_m, \mathrm{I}_\mathcal{H}$ be the identity maps on X_m and $\mathcal{H}$ respectively. Suppose that t_{e_i,e_j} is the flip map between fibres X_{e_i} and X_{e_j} as in Lemma 3.1.1. A representation θ is *doubly commuting representation* if for every $1 \leq i \neq j \leq k$, we have

$$\widetilde{\theta}^*_{e_j}\widetilde{\theta}_{e_i} = (\mathrm{I}_{e_j} \otimes \widetilde{\theta}_{e_i})(t_{e_i,e_j} \otimes \mathrm{I}_H)(\mathrm{I}_{e_i} \otimes \widetilde{\theta}^*_{e_j}).$$

Suppose θ is a doubly commuting representation and let $t_{m,n}$ be the flip map between fibres X_m and X_n. Write $\mathbb{N}^k_+$ for non zero elements of $\mathbb{N}^k$, and suppose $m, n \in \mathbb{N}^k_+$ satisfying $m \wedge n = 0$. [54, Lemma 3.9(i)] implies that

$$(\mathrm{I}_n \otimes \widetilde{\theta}_m)(t_{m,n} \otimes \mathrm{I}_H)(\mathrm{I}_m \otimes \widetilde{\theta}_n^{\,*}) = \widetilde{\theta}_n^{\,*}\widetilde{\theta}_m. \tag{A.1}$$

Now we want to show that the universal Nica-covariant representation ψ satisfies (A.1). It follows from [54, Remark 3.12] that we can consider ψ as a representation on a C^*-algebra $\mathcal{H}$.

Proposition A.0.6. *Let $h_1, \ldots, h_k$ be $*$-commuting and surjective local homeomorphisms on a compact Hausdorff space Z and let X be the associated product system as in Corollary 2.1.2. Take $m, n \in \mathbb{N}^k_+$ such that $m \wedge n = 0$. Then*

$$(\mathrm{I}_n \otimes \widetilde{\psi}_m)(t_{m,n} \otimes \mathrm{I}_H)(\mathrm{I}_m \otimes \widetilde{\psi_n}^{*}) = \widetilde{\psi_n}^{*}\widetilde{\psi}_m \tag{A.2}$$

We first need to calculate the adjoint $\widetilde{\psi_n}^{*} : \mathcal{H} \to X_n \otimes_{\psi_0} \mathcal{H}$. The next lemma gives a formula for $\widetilde{\psi_n}^{*}$ in terms of a general Parseval frame of X_n.

Lemma A.0.7. *Let $\{\eta_j\}_{j=0}^{d}$ be a Parseval frame for the fibre X_n. Then*

$$\widetilde{\psi_n}^*(r) = \sum_{j=0}^{d} \eta_j \otimes \psi_n(\eta_j)^* r \text{ for } r \in \mathcal{H}. \tag{A.3}$$

Proof. Fix $r \in \mathcal{H}$ and let $y \otimes s \in X_n \otimes_{\psi_0} \mathcal{H}$. We compute:

$$\begin{aligned}\Big(\sum_{j=0}^{d} \eta_j \otimes \psi_n(\eta_j)^* r \Big| y \otimes s\Big) &= \sum_{j=0}^{d} \Big((\psi_n(\eta_j)^* r \Big| \psi_0\langle \eta_j, y\rangle s\Big) \\ &= \Big(r \Big| \sum_{j=0}^{d} \psi_n(\eta_j)\big(\psi_0\langle \eta_j, y\rangle s\big)\Big) \\ &= \Big(r \Big| \sum_{j=0}^{d} \psi_n\big(\eta_j \cdot \langle \eta_j, y\rangle\big) s\Big) \\ &= \Big(r \Big| \psi_n(y) s\Big).\end{aligned}$$

This is precisely $\big(r \big| \widetilde{\psi}_n(y \otimes s)\big)$. Thus $\widetilde{\psi_n}^*(r) = \sum_{j=0}^{d} \eta_j \otimes \psi_n(\eta_j)^* r$. □

Proof of Proposition A.0.6. Let $x \otimes r \in X_m \otimes_{\psi_0} \mathcal{H}$. We evaluate both sides of (A.2) on $x \otimes r$. To do this we will need to have Parseval frames for fibres X_m, X_n. Let $\{\rho_i\}_{i=0}^{d}$ be a partition of unity such that $h^m|_{\operatorname{supp}\rho_i}, h^n|_{\operatorname{supp}\rho_i}$ are injective and suppose that $\tau_i := \sqrt{\rho_i}$. Notice that $\{\tau_i\}_{i=0}^{d}$ forms a Parseval frame for both fibres X_m, X_n. Also since $m \wedge n = 0$, $\{\tau_i \circ h^n\}_{i=0}^{d}$ and $\{\tau_i \circ h^m\}_{i=0}^{d}$ are Parseval frame for the fibres X_m, X_n, respectively.

We start computing the left-hand side of (A.2) by applying the adjoint formula (A.3) with Parseval frame $\{\tau_j\}_{j=0}^{d} \subset X_n$. For convenience, set

$$\dagger := (\mathrm{I}_n \otimes \widetilde{\psi}_m)(t_{m,n} \otimes \mathrm{I}_H)(\mathrm{I}_m \otimes \widetilde{\psi_n}^*).$$

We have

$$\dagger(x \otimes r) = (\mathrm{I}_n \otimes \widetilde{\psi}_m)(t_{m,n} \otimes \mathrm{I}_H)\big(x \otimes \sum_{j=0}^{d} \tau_j \otimes \psi_n(\tau_j)^* r\big).$$

Writing the reconstruction formula for the Parseval frame $\{\tau_i \circ h^n\}_{i=0}^{d} \subset X_m$ gives

$$\begin{aligned}\dagger(x \otimes r) &= \sum_{j=0}^{d} (\mathrm{I}_n \otimes \widetilde{\psi}_m)(t_{m,n} \otimes \mathrm{I}_H)\Big(\Big(\sum_{i=0}^{d} \tau_i \circ h^n \cdot \big\langle \tau_i \circ h^n, x\big\rangle\Big) \otimes \tau_j \otimes \psi_n(\tau_j)^* r\Big) \\ &= \sum_{0 \le i,j \le d} (\mathrm{I}_n \otimes \widetilde{\psi}_m)(t_{m,n} \otimes \mathrm{I}_H)\Big(\tau_i \circ h^n \otimes \big\langle \tau_i \circ h^n, x\big\rangle \cdot \tau_j \otimes \psi_n(\tau_j)^* r\Big).\end{aligned}$$

Applying the reconstruction formula for the Parseval frame $\{\tau_l\}_{l=0}^d \subset X_n$, we have

$$\dagger(x\otimes r) = \sum_{0\le i,j\le d} (\mathrm{I}_n \otimes \widetilde{\psi}_m)(t_{m,n}\otimes \mathrm{I}_H)\Big(\tau_i\circ h^n \otimes \Big(\sum_{l=0}^d \tau_l\cdot\Big\langle \tau_l, \langle \tau_i\circ h^n, x\rangle\cdot\tau_j\Big\rangle\Big)\otimes\psi_n(\tau_j)^* r\Big).$$

Now we continue by using the flip map (3.5)

$$\begin{aligned}
\dagger(x\otimes r) &= \sum_{0\le i,j,l\le d} (\mathrm{I}_n \otimes \widetilde{\psi}_m)(t_{m,n}\otimes \mathrm{I}_H)\Big(\tau_i\circ h^n \otimes \tau_l \otimes \psi_0\Big(\Big\langle \tau_l, \langle \tau_i\circ h^n, x\rangle\cdot\tau_j\Big\rangle\Big)\psi_n(\tau_j)^* r\Big)\\
&= \sum_{0\le i,j,l\le d} (\mathrm{I}_n \otimes \widetilde{\psi}_m)\Big(\tau_l\circ h^m \otimes \tau_i \otimes \psi_n\Big(\tau_j\cdot\Big\langle \langle \tau_i\circ h^n, x\rangle\cdot\tau_j, \tau_l\Big\rangle\Big)^* r\Big)\\
&= \sum_{0\le i,l\le d} (\mathrm{I}_n \otimes \widetilde{\psi}_m)\Big(\tau_l\circ h^m \otimes \tau_i \otimes \psi_n\Big(\sum_{j=0}^d \tau_j\cdot\Big\langle \tau_j, \langle x, \tau_i\circ h^n\rangle\cdot\tau_l\Big\rangle\Big)^* r\Big).
\end{aligned}$$

The reconstruction formula for frame $\{\tau_j\}_{j=0}^d \subset X_n$ implies that

$$\dagger(x\otimes r) = \sum_{0\le i,l\le d} \tau_l\circ h^m \otimes \psi_m(\tau_i)\psi_n\big(\langle x, \tau_i\circ h^n\rangle\cdot\tau_l\big)^* r. \tag{A.4}$$

Next we compute the right-hand side (A.2) by applying the adjoint formula (A.3) with the Parseval frame $\{\tau_l\circ h^m\}_{l=0}^d \subset X_n$.

$$\widetilde{\psi_n}^*\widetilde{\psi}_m(x\otimes r) = \widetilde{\psi_n}^*\psi_m(x)r = \sum_{l=0}^d \tau_l\circ h^m\otimes\psi_n(\tau_l\circ h^m)^*\psi_m(x)r.$$

Applying our formula (3.9) implies that

$$\begin{aligned}
\widetilde{\psi_n}^*\widetilde{\psi}_m(x\otimes r) &= \sum_{l=0}^d \tau_l\circ h^m\otimes\Big(\sum_{0\le i,j\le d}\psi_m\big(\langle\tau_l\circ h^m, \tau_j\circ h^m\rangle\cdot\tau_i\big)\psi_n\big(\langle x, \tau_i\circ h^n\rangle\cdot\tau_j\big)^*\Big) r\\
&= \sum_{0\le i,j,l\le d} \tau_l\circ h^m\otimes\psi_0\big(\langle\tau_l\circ h^m, \tau_j\circ h^m\rangle\big)\psi_m(\tau_i)\psi_n\big(\langle x, \tau_i\circ h^n\rangle\cdot\tau_j\big)^* r\\
&= \sum_{0\le i,j,l\le d} \tau_l\circ h^m\cdot\langle\tau_l\circ h^m, \tau_j\circ h^m\rangle\otimes\psi_m(\tau_i)\psi_n\big(\langle x, \tau_i\circ h^n\rangle\cdot\tau_j\big)^* r.
\end{aligned}$$

Now applying reconstruction formula for the Parseval frame $\{\tau_l\circ h^m\}_{l=0}^d \subset X_n$ gives

$$\widetilde{\psi_n}^*\widetilde{\psi}_m(x\otimes r) = \sum_{0\le i,j\le d} \tau_j\circ h^m\otimes\psi_m(\tau_i)\psi_n\big(\langle x, \tau_i\circ h^n\rangle\cdot\tau_j\big)^* r. \tag{A.5}$$

Comparing (A.5) and (A.4) completes our proof of (A.2). $\square$

Appendix B

KMS states on C^*-algebras associated to a local homeomorphism

In this appendix we provide our result about KMS states on dynamical systems associated to a single local homeomorphism. This work is published in *Internat. J. Math. Vol. 25, No. 7 (2014)* 1450066 (28 pages).

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KMS states on C^*-algebras associated to local homeomorphisms

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For every Hilbert bimodule over a C^*-algebra, there are natural gauge actions of the circle on the associated Toeplitz algebra and Cuntz–Pimsner algebra, and hence natural dynamics obtained by lifting these gauge actions to actions of the real line. We study the KMS states of these dynamics for a family of bimodules associated to local homeomorphisms on compact spaces. For inverse temperatures larger than a certain critical value, we find a large simplex of KMS states on the Toeplitz algebra, and we show that all KMS states on the Cuntz–Pimsner algebra have inverse temperature at most this critical value. We illustrate our results by considering the backward shift on the one-sided path space of a finite graph, where we can use recent results about KMS states on graph algebras to see what happens below the critical value. Our results about KMS states on the Cuntz–Pimsner algebra of the shift show that recent constraints on the range of inverse temperatures obtained by Thomsen are sharp.

Keywords: Toeplitz algebra; Cuntz–Pimsner algebra; gauge action; KMS state.

Mathematics Subject Classification 2010: 46L35

1. Introduction

We consider actions α of the real line $\mathbb{R}$ by automorphisms of a C^*-algebra A. When α describes the time evolution in a model of a physical system, the states of the system are given by positive functionals of norm 1. The equilibrium states are the states on A that satisfy a commutation relation called the KMS condition. This condition makes sense for every dynamical system of the form $(A, \mathbb{R}, \alpha)$, irrespective of its origin, and studying the KMS states of such systems often yields interesting information. This is certainly the case, for example, for the number-theoretic Hecke algebra of Bost and Connes [2] and its generalizations [21, 22], for systems involving gauge actions on graph algebras [8, 11, 18, 15], and for systems associated to local homeomorphisms of the sort arising in topological dynamics [34, 35].

Many of the systems studied in the papers mentioned above, and especially those associated to directed graphs, have natural analogs involving Toeplitz algebras in which crucial defining equations are relaxed to inequalities. Work of Exel, Laca and Neshveyev [11, 23] has shown that there is often a much richer supply of KMS states on these Toeplitz algebras, and this has been extended in recent years to various systems arising in number theory [25, 24, 6]. These papers contain detailed constructions of the KMS states on the various Toeplitz algebras, and re-examination of the techniques has led to similar constructions in a wide range of examples, including graph algebras [15, 16]. In this paper, we use similar techniques to construct KMS states on systems of interest in topological dynamics.

We consider a surjective local homeomorphism $h : Z \to Z$ on a compact Hausdorff space Z, and an associated C^*-algebra that has been variously described as an Exel crossed product [10], a groupoid algebra [34], or as both a groupoid algebra and a Cuntz–Pimsner algebra [7] (for a precise statement, see [17, Theorem 3.3]). Here, we view it as the C^*-algebra $\mathcal{O}(X(E))$ of a topological graph E, and then we use the graph-based formalism of Katsura [19] in calculations. The algebra $\mathcal{O}(X(E))$ carries a canonical gauge action of the circle $\mathbb{T}$, which we lift to an action α of $\mathbb{R}$. We are interested in the KMS states on $(\mathcal{O}(X(E)), \mathbb{R}, \alpha)$ and its Toeplitz analog $(\mathcal{T}(X(E)), \mathbb{R}, \alpha)$.

Several authors have shown that there is a bijection between the KMS states on $(\mathcal{O}(X(E)), \mathbb{R}, \alpha)$ and the probability measures on Z that satisfy an invariance relation (for example, [30, Theorem 3.3; 10, Theorem 9.6; 34, Theorem 6.2]). To find KMS states, one then has to find invariant measures, and existence has been demonstrated using a functional-analytic analog of the Perron–Frobenius theory (for example, in [30; 34, Sec. 6.2]). Here, we show that, for β larger than a critical value β_c, there is a bijection between the KMS_β states on $(\mathcal{T}(X(E)), \mathbb{R}, \alpha)$ and the probability measures on Z which satisfy an inequality that we call the *subinvariance relation*. We then describe a construction of all the measures satisfying the subinvariance relation, and give a spatial construction of the corresponding KMS states. Putting these constructions together gives a parametrization of the KMS_β states of $(\mathcal{T}(X(E)), \mathbb{R}, \alpha)$ by a concretely-described simplex of measures on Z for every $\beta > \beta_c$ (Theorem 5.1).

Our critical value β_c is an exponential bound for the number of preimages of points under iteration of the map h, and has previously appeared in the dynamics literature (for example, [12, 34]). In particular, Thomsen has shown that β_c is an upper bound for the inverse temperatures of KMS states on $\mathcal{O}(X(E))$ [34, Theorem 6.8]. So it seems likely that our results on $\mathcal{T}(X(E))$ are sharp. At β_c, we can show by taking limits of states on $\mathcal{T}(X(E))$ that there exist KMS_{β_c} states on $(\mathcal{O}(X(E)), \alpha)$ (Theorem 6.1).

Our approach is inspired by the analysis of KMS states on the Toeplitz–Cuntz–Krieger algebra $\mathcal{T}C^*(E)$ of a finite directed graph E in [15]. The usual description of $C^*(E)$ and $\mathcal{T}C^*(E)$ using a graph correspondence over the finite-dimensional algebra $C(E^0)$ [31, Sec. 8] does not quite fit our present analysis, though there are striking similarities. However, we can also realize $C^*(E)$ in the present setup as the Cuntz–Pimsner algebra $\mathcal{O}(X(E^\infty))$ associated to the shift σ on the infinite-path space E^∞ [5, Theorem 5.1]. We can therefore test our results by reconciling them with the known results for $C^*(E)$. When E is irreducible in the sense that its vertex matrix A is irreducible, there is a unique KMS state on $(C^*(E), \alpha)$, and its inverse temperature is given in terms of the spectral radius of A by $\beta = \ln \rho(A)$. We confirm that, for the local homeomorphism $\sigma : E^\infty \to E^\infty$, our β_c is indeed $\ln \rho(A)$ (Proposition 7.3).

Our computation of β_c for shifts works for arbitrary matrices of nonnegative integers, so we also consider the reducible case, where there is an interesting variety of examples [16]. In [34, Theorem 6.8], Thomsen also provides a lower bound for the set of possible inverse temperatures of KMS states of $C^*(E)$. The examples in [16] show that Thomsen's bounds are sharp, and that many values in between can be attained as well (see Sec. 8). Thus we think that graph algebras could provide an interesting supply of fresh examples for the study of KMS states in dynamics. This should be true also for the study of KMS states on Toeplitz algebras, although there is a curious wrinkle: the Toeplitz algebra $\mathcal{T}C^*(E)$ embeds in $\mathcal{T}(X(E^\infty))$, but as a proper subalgebra (see Proposition 7.1). Nevertheless, our new results are again compatible with those of [15, 16], and indeed every KMS state of $(\mathcal{T}C^*(E), \alpha)$ is the restriction of a KMS state of $(\mathcal{T}(X(E^\infty)), \alpha)$ (Corollary 7.6).

We begin with a short section on notation and conventions. We then look for a characterization of KMS states which will allow us to recognize them easily. This characterization could be of independent interest, because it works for the Toeplitz algebras of quite general Hilbert bimodules (Proposition 3.1). In Sec. 4, we discuss our subinvariance relation, which involves a measure-theoretic analog of a Ruelle operator. Importantly, we describe all solutions of this subinvariance relation (Proposition 4.2). In Sec. 5, we prove our main theorem about KMS states on the Toeplitz algebra, and then in Sec. 6 we discuss KMS states at the critical inverse temperature. Sections 7 and 8 contain our results about shifts on the path spaces of graphs.

2. Notation and Conventions

2.1. *Toeplitz algebras of Hilbert bimodules*

Suppose that X is a *Hilbert bimodule* over a C^*-algebra A, by which we mean that X is a right Hilbert A-module X with a left action of A implemented by a homomorphism $\varphi : A \to \mathcal{L}(X)$ (in other words, X is a correspondence over A). For $m \geq 0$, we write $X^{\otimes m}$ for the internal tensor product $X \otimes_A X \otimes_A \cdots \otimes_A X$ of m copies of X, which is also a Hilbert bimodule over A. A *representation* (ψ, π) of a Hilbert bimodule in a C^*-algebra C consists of a linear map $\psi : X \to C$ and a homomorphism $\pi : A \to C$ such that

$$\psi(a \cdot x \cdot b) = \pi(a)\psi(x)\pi(b) \quad \text{and} \quad \pi(\langle x, y\rangle) = \psi(x)^*\psi(y)$$

for every $x, y \in X$ and $a, b \in B$. For each $m \geq 1$, there is a representation $(\psi^{\otimes m}, \pi)$ of $X^{\otimes m}$ such that

$$\psi^{\otimes m}(x_1 \otimes_A x_2 \otimes_A \cdots \otimes_A x_m) = \psi(x_1)\psi(x_2)\cdots\psi(x_m).$$

For $m = 0$, we set $X^{\otimes 0} := A$ and $\psi^{\otimes 0} := \pi$.

The *Toeplitz algebra* $\mathcal{T}(X)$ is generated by a universal representation of X, which in this paper we always denote by (ψ, π). Proposition 1.3 of [14] says that there is such an algebra $\mathcal{T}(X)$, and that it carries a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{T}(X)$ characterized by $\gamma_z(\psi(x)) = z\psi(x)$ and $\gamma_z(\pi(a)) = \pi(a)$. By [14, Lemma 2.4], we have

$$\mathcal{T}(X) = \overline{\operatorname{span}}\{\psi^{\otimes m}(x)\psi^{\otimes n}(y)^* : m, n \in \mathbb{N}\}.$$

If (θ, ρ) is a representation of X in a C^*-algebra C, we write $\theta \times \rho$ for the representation of $\mathcal{T}(X)$ in C such that $(\theta \times \rho) \circ \psi = \theta$ and $(\theta \times \rho) \circ \pi = \rho$.

For $x, y \in X$, we write $\Theta_{x,y}$ for the adjointable operator on X given by $\Theta_{x,y}(z) = x \cdot \langle y, z\rangle$, and $\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X\} \subset \mathcal{L}(X)$. The representation (ψ, π) induces a homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}(X) \to \mathcal{T}(X)$ such that $(\psi, \pi)^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$. The *Cuntz–Pimsner algebra* $\mathcal{O}(X)$ of [29] is then the quotient of $\mathcal{T}(X)$ by the ideal generated by

$$\big\{\pi(a) - (\psi, \pi)^{(1)}(\varphi(a)) : a \in A \text{ satisfies } \varphi(a) \in \mathcal{K}(X)\big\}.$$

(Other definitions of the Cuntz–Pimsner algebra have been used in the literature, but for the bimodules considered here we have $\phi(A) \subset \mathcal{K}(X)$, and all the definitions give the same algebra.)

2.2. *Measures*

We will construct KMS states from Borel measures on compact Hausdorff spaces Z. All the measures we consider are regular Borel measures and are positive in the sense that they take values in $[0, \infty)$; indeed, they are all finite measures and hence are automatically regular (by [13, Theorem 7.8], for example). We write $M(Z)_+$ for the set of finite Borel measures on Z. Some of our measures will be defined by integrals, or as linear functionals on $C(Z)$, from which the Riesz representation theorem [13, Corollary 7.6] gives us an (automatically regular) Borel measure. For us, a *probability measure* is simply a Borel measure with total mass 1.

2.3. *Topological graphs*

A *topological graph* $E = (E^0, E^1, r, s)$ consists of two locally compact Hausdorff spaces, a continuous map $r : E^1 \to E^0$ and a local homeomorphism $s : E^1 \to E^0$. For paths in E, we use the convention of [31], so that a path of length 2, for example, is a pair ef with $e, f \in E^1$ and $s(e) = r(f)$. We mention this because in his first paper [19], Katsura used a different convention, and one has to be careful when consulting the literature because there are other conventions out there. Each such graph E has a Hilbert bimodule $X(E)$ described in [31, Chap. 9]. It is usually a completion of $C_c(E^1)$, but here the spaces E^0 and E^1 are always compact, and then no completion is necessary because the norm on $X(E)$ is equivalent (as a vector-space norm) to the usual supremum norm on $C(E^1) = X(E)$. For reference, we recall that the module actions are given by $(a \cdot x \cdot b)(z) = a(r(z))x(z)b(s(z))$ for $a, b \in C(E^0)$ and the inner product by $\langle x, y\rangle(z) = \sum_{s(w)=z} \overline{x(w)}y(w)$.

2.4. *KMS states*

We use the same conventions for KMS states as other recent papers, such as [25, 26, 15], for example. Suppose that $(A, \mathbb{R}, \alpha)$ is a C^*-algebraic dynamical system. An element a of A is *analytic* if $t \mapsto \alpha_t(a)$ is the restriction of an entire function $z \mapsto \alpha_z(a)$ on $\mathbb{C}$. A state ϕ of $(A, \mathbb{R}, \alpha)$ is a *KMS state with inverse temperature* β (or a KMS$_\beta$ state) if $\phi(ab) = \phi(b\alpha_{i\beta}(a))$ for all analytic elements a, b. Crucially, it suffices to check this condition for a, b in a family $\mathcal{F}$ of analytic elements which span a dense subspace of B, and it is usually easy to find a good supply of such elements.

3. A Characterization of KMS States

The following result is similar to [15, Proposition 2.1(a); 27, Proposition 4.1], but is substantially more general. (We have learned that Mitch Hawkins has independently proved a similar result for the bimodules $X(E)$ of topological graphs.)

Proposition 3.1. *Suppose that X is a Hilbert bimodule over a C^*-algebra A, and $\alpha : \mathbb{R} \to \operatorname{Aut} A$ is given in terms of the gauge action γ by $\alpha_t = \gamma_{e^{it}}$. Suppose $\beta > 0$ and ϕ is a state on $\mathcal{T}(X)$. Then ϕ is a KMS$_\beta$ state of $(\mathcal{T}(X), \alpha)$ if and only if $\phi \circ \pi$ is a trace on A and*

$$\phi(\psi^{\otimes l}(x)\psi^{\otimes m}(y)^*) = \begin{cases} 0 & \text{if } m \neq l, \\ e^{-\beta m}\phi \circ \pi(\langle y, x\rangle_A) & \text{if } m = l. \end{cases} \tag{3.1}$$

Proof. First suppose that ϕ is a KMS$_\beta$ state. For $a \in A$, $\alpha_t(\pi(a)) = \pi(a)$ for all $t \in \mathbb{R}$, and hence for all $t \in \mathbb{C}$. Thus the KMS relation says that $\phi \circ \pi$ is a trace. Two applications of the KMS relation give

$$\begin{aligned} \phi(\psi^{\otimes l}(x)\psi^{\otimes m}(y)^*) &= \phi(\psi^{\otimes m}(y)^*\alpha_{i\beta}(\psi^{\otimes l}(x))) \\ &= e^{-\beta l}\phi(\psi^{\otimes m}(y)^*\psi^{\otimes l}(x)) \\ &= e^{-\beta(l-m)}\phi(\psi^{\otimes l}(x)\psi^{\otimes m}(y)^*), \end{aligned}$$

which because $\beta > 0$ implies that both sides vanish for $m \neq l$. Now for $m = l$, the Toeplitz relation for (ψ, π) implies that

$$\phi\big(\psi^{\otimes m}(x)\psi^{\otimes m}(y)^*\big) = e^{-\beta m}\phi\big(\psi^{\otimes m}(y)^*\psi^{\otimes m}(x)\big) = e^{-\beta m}\phi\big(\pi(\langle y, x\rangle_A)\big),$$

and ϕ satisfies (3.1).

Next we suppose that $\phi \circ \pi$ is a trace and that ϕ satisfies (3.1). It suffices for us to prove that

$$\phi(bc) = e^{-\beta(l-m)}\phi(cb) \tag{3.2}$$

for $b = \psi^{\otimes l}(x)\psi^{\otimes m}(y)^*$ and $c = \psi^{\otimes n}(s)\psi^{\otimes p}(t)^*$, where x, y, s and t are elementary tensors. (When b and/or c lie in $\pi(A)$, this is relatively straightforward because $\phi\circ\pi$ is a trace and α fixes $\pi(A)$.) Formula (3.1) implies that both sides of (3.2) vanish unless $l+n = m+p$, and hence we assume this from now on. We also assume that $m \leq n$. To see that this suffices, suppose that we have dealt with the case $m \leq n$, and consider $m > n$. Then $\overline{\phi(a)} = \phi(a^*)$ implies that

$$\overline{\phi(bc)} = \phi(c^*b^*) = \phi\big(\psi^{\otimes p}(t)\psi^{\otimes n}(s)^*\psi^{\otimes m}(y)\psi^{\otimes l}(x)^*\big),$$

and we are back in the other case. Thus

$$\begin{aligned}\overline{\phi(bc)} &= e^{-\beta(p-n)}\phi\big(\psi^{\otimes m}(y)\psi^{\otimes l}(x)^*\psi^{\otimes p}(t)\psi^{\otimes n}(s)^*\big)\\ &= e^{-\beta(p-n)}\phi(b^*c^*) = \overline{e^{-\beta(p-n)}\phi(cb)};\end{aligned}$$

since $l+n = m+p$, we have $p-n = l-m$, and we have (3.2). So it does suffice to prove (3.2) when $m \leq n$.

So we assume that $l+n = m+p$ and $m \leq n$. Then we also have $p \geq l$. Since we are dealing with elementary tensors, we may write $s = s' \otimes s'' \in X^{\otimes m} \otimes X^{\otimes(n-m)}$ and $t = t' \otimes t'' \in X^{\otimes l} \otimes X^{\otimes(p-l)}$. (If $m = n$ then $p = l$ and we can dispense with this step.) Now we compute, remembering that $p = l + (n-m)$:

$$\begin{aligned}\phi(bc) &= \phi\big(\psi^{\otimes l}(x)\psi^{\otimes m}(y)^*\psi^{\otimes m}(s')\psi^{\otimes(n-m)}(s'')\psi^{\otimes p}(t)^*\big)\\ &= \phi\big(\psi^{\otimes l}(x)\pi(\langle y, s'\rangle)\psi^{\otimes(n-m)}(s'')\psi^{\otimes p}(t)^*\big)\\ &= \phi\big(\psi^{\otimes l}(x)\psi^{\otimes(n-m)}(\langle y, s'\rangle \cdot s'')\psi^{\otimes p}(t)^*\big)\\ &= e^{-\beta p}\phi\circ\pi\big(\big\langle t'\otimes t'', x\otimes(\langle y, s'\rangle\cdot s'')\big\rangle\big) \quad \text{(using (3.1))}\\ &= e^{-\beta p}\phi\circ\pi\big(\big\langle t'', \langle t', x\rangle\cdot(\langle y, s'\rangle \cdot s'')\big\rangle\big).\end{aligned}$$

A similar computation (but using the slightly less obvious identity $\psi(\xi)^*\pi(a) = \psi(a^*\cdot\xi)^*$) gives:

$$\begin{aligned}\phi(cb) &= \phi\big(\psi^{\otimes n}(s)\psi^{\otimes(p-l)}(t'')^*\psi^{\otimes l}(t')^*\psi^{\otimes l}(x)\psi^{\otimes m}(y)^*\big)\\ &= \phi\big(\psi^{\otimes n}(s)\psi^{\otimes(p-l)}(t'')^*\pi(\langle t', x\rangle)\psi^{\otimes m}(y)^*\big)\\ &= \phi\big(\psi^{\otimes n}(s)\psi^{\otimes(p-l)}(\langle x, t'\rangle\cdot t'')^*\psi^{\otimes m}(y)^*\big)\\ &= e^{-\beta n}\phi\circ\pi\big(\big\langle y\otimes(\langle x, t'\rangle\cdot t''), s'\otimes s''\big\rangle\big)\\ &= e^{-\beta n}\phi\circ\pi\big(\big\langle\langle x, t'\rangle\cdot t'', \langle y, s'\rangle\cdot s''\big\rangle\big).\end{aligned}$$

Since the left action is by adjointable operators, we have

$$\big\langle\langle x, t'\rangle\cdot t'', \langle y, s'\rangle\cdot s''\big\rangle = \big\langle t'', \langle t', x\rangle\cdot(\langle y, s'\rangle\cdot s'')\big\rangle,$$

and we deduce from our two calculations that $e^{\beta p}\phi(bc) = e^{\beta n}\phi(cb)$. Since $n-p = m-l$, this is precisely (3.2). ◻

4. KMS States and the Subinvariance Relation

Suppose ν is a finite regular Borel measure on a compact Hausdorff space Z and $h : Z \to Z$ is a surjective local homeomorphism. Define $f : C(Z) \to \mathbb{C}$ by

$$f(a) = \int \sum_{h(w)=z} a(w)\, d\nu(z) \quad \text{for } a \in C(Z).$$

Then f is a positive linear functional on $C(Z)$, and hence the Riesz representation theorem (for example, [13, Theorem 7.2]) says there is a unique finite regular Borel measure $R\nu$ on Z such that

$$\int a\, d(R\nu) = f(a) = \int \sum_{h(w)=z} a(w)\, d\nu(z) \quad \text{for } a \in C(Z). \tag{4.1}$$

The operation R on measures is affine and positive, and satisfies $\|R\nu\| \leq c_1\|\nu\|$ for the dual norm on $C(Z)^*$, where $c_1 := \max_{z\in Z} |h^{-1}(z)|$. Similar operations appear throughout the analysis of KMS states in dynamics (for example, in [34, Theorem 6.2]), and are sometimes described as "Ruelle operators".

Proposition 4.1. *Suppose that $h : Z \to Z$ is a surjective local homeomorphism on a compact Hausdorff space Z. Let E be the topological graph (Z, Z, id, h) and $X(E)$ the graph correspondence. Define $\alpha : \mathbb{R} \to \mathrm{Aut}\,\mathcal{T}(X(E))$ in terms of the gauge action by $\alpha_t = \gamma_{e^{it}}$. Suppose that ϕ is a KMS$_\beta$ state on $(\mathcal{T}(X(E)), \alpha)$, and μ is the probability measure on Z such that $\phi(\pi(a)) = \int a\, d\mu$ for all $a \in C(Z)$. Then the measure $R\mu$ satisfies*

$$\int a\, d(R\mu) \leq e^{\beta} \int a\, d\mu \quad \textit{for all positive } a \textit{ in } C(Z). \tag{4.2}$$

Proof. Suppose that $a \in C(Z)$ and $a \geq 0$. We begin by writing the integrand $\sum_{h(w)=z} a(w)$ in (4.1) in terms of the inner product in $X(E)$. Let $\{U_i\}_{i=0}^{k}$ be an open cover of Z such that $h|_{U_i}$ is injective, and choose a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$. Define $\xi_i \in X(E)$ by $\xi_i = \sqrt{\rho_i}$. Then

$$\begin{aligned}\sum_{h(w)=z} a(w) &= \sum_{h(w)=z} \sum_{i=0}^{k} \xi_i(w)^2 a(w) = \sum_{i=0}^{k} \sum_{h(w)=z} \xi_i(w)^2 a(w)\\ &= \sum_{i=0}^{k} \sum_{h(w)=z} \overline{\xi_i(w)}(a \cdot \xi_i)(w) = \sum_{i=0}^{k} \langle \xi_i, a \cdot \xi_i\rangle(z).\end{aligned}$$

Thus

$$\begin{aligned}\int a\, d(R\mu) &= \int \sum_{h(w)=z} a(w)\, d\mu(z) = \int \sum_{i=0}^{k} \langle \xi_i, a \cdot \xi_i\rangle(z)\, d\mu(z)\\ &= \phi\left(\pi\left(\sum_{i=0}^{k} \langle \xi_i, a \cdot \xi_i\rangle\right)\right) = \sum_{i=0}^{k} \phi(\psi(\xi_i)^*\psi(a \cdot \xi_i)).\end{aligned}$$

Now, since ϕ is a KMS$_\beta$ state, we have

$$\int a\,d(R\mu) = \sum_{i=0}^{k} e^{\beta}\phi(\psi(a\cdot\xi_i)\psi(\xi_i)^*). \tag{4.3}$$

Our next task is to compare the operator $\sum_{i=0}^{k}\psi(a\cdot\xi_i)\psi(\xi_i)^*$ appearing on the right-hand side of (4.3) with $\pi(a)$. For this, we use the Fock representation (T,φ_∞) of $\mathcal{T}(X(E))$ from [14, Example 1.4]. As a right A-module, $F(X(E))$ is the Hilbert module direct sum $\bigoplus_{n=0}^{\infty} X(E)^{\otimes n}$, with the left action of A by diagonal operators giving a homomorphism $\varphi_\infty : A \to \mathcal{L}(F(X(E)))$. The homomorphism $T : X(E) \to \mathcal{L}(F(X(E)))$ sends $x \in X(E)$ to the creation operator $T(x) : y \mapsto x \otimes_A y$, and $T \times \varphi_\infty$ is an injection on $\mathcal{T}(X(E))$ [14, Corollary 2.2].

Let $n \geq 1$ and $x = x_1 \otimes \cdots \otimes x_n \in X(E)^{\otimes n}$. Then

$$\begin{aligned}\sum_{i=0}^{k} T(a\cdot\xi_i)T(\xi_i)^*(x) &= \sum_{i=0}^{k} T(a\cdot\xi_i)\big(\langle\xi_i, x_1\rangle\cdot x_2\otimes\cdots\otimes x_n\big)\\ &= \sum_{i=0}^{k}\big(a\cdot\xi_i\cdot\langle\xi_i,x_1\rangle\big)\otimes x_2\otimes\cdots\otimes x_n.\end{aligned}$$

Since $h|_{U_i}$ is injective and $\operatorname{supp}\xi_i \subset U_i$, we have

$$\begin{aligned}(\xi_i\cdot\langle\xi_i,x_1\rangle)(z) &= \xi_i(z)\langle\xi_i,x_1\rangle(h(z))\\ &= \xi_i(z)\sum_{h(w)=h(z)}\overline{\xi_i(w)}x_1(w) = \xi_i(z)^2x_1(z).\end{aligned}$$

Thus

$$\sum_{i=0}^{k} T(a\cdot\xi_i)T(\xi_i)^*(x) = \sum_{i=0}^{k} a\cdot(\xi_i^2x_1\otimes x_2\otimes\cdots\otimes x_n) = a\cdot x = \varphi_\infty(a)(x).$$

Thus $\sum_{i=0}^{k} T(a\cdot\xi_i)T(\xi_i)^* = \varphi_\infty(a)$ as operators on $X(E)^{\otimes n}$ for $n \geq 1$. Since each $T(a\cdot\xi_i)T(\xi_i)^*$ vanishes on $C(Z) = X(E)^{\otimes 0}$ and a is positive, we have

$$\sum_{i=0}^{k} T(a\cdot\xi_i)T(\xi_i)^* \leq \varphi_\infty(a) \quad \text{in } \mathcal{L}(F(X(E)));$$

since the homomorphism $T\times\varphi_\infty$ is faithful, we deduce that

$$\sum_{i=0}^{k}\psi(a\cdot\xi_i)\psi(\xi_i)^* \leq \pi(a) \quad \text{in } \mathcal{T}(X(E)). \tag{4.4}$$

To finish off, we apply ϕ to (4.4):

$$\phi\left(\sum_{i=0}^{k}\psi(a\cdot\xi_i)\psi(\xi_i)^*\right) \leq \phi(\pi(a)) = \int a\,d\mu.$$

On the other hand, (4.3) implies that

$$\phi\left(\sum_{i=0}^{k} \psi(a\cdot\xi_i)\psi(\xi_i)^*\right) = e^{-\beta}\int a\,d(R\mu),$$

and the result follows from the last two displays. ☐

When Z is a finite set and A is a nonnegative matrix, μ is a vector in $[0,\infty)^Z$, and the relation (4.2) in the form $A\mu \le e^{\beta}\mu$ says that μ is a subinvariant vector for A in the sense of Perron–Frobenius theory [33, Chap. 1]. Subinvariant vectors played an important role in the analysis of KMS states on the Toeplitz algebras of graphs in [15, Sec. 2], and (4.2) will play a similar role in our analysis. So we shall refer to (4.2) as the *subinvariance relation*.

We now show how to construct the probability measures which satisfy the subinvariance relation. Proposition 4.2 is an analog for our operation R on measures of [15, Theorem 3.1(a)], which is about the subinvariance relation for the vertex matrix of a finite directed graph. Here, the powers R^n are defined inductively by $R^{n+1}\nu = R(R^n\nu)$, and then we have

$$\int a\,d(R^n\nu) = \int \sum_{h^n(w)=z} a(w)\,d\nu(z) \quad \text{for } a\in C(Z). \tag{4.5}$$

Proposition 4.2. *Suppose that $h : Z\to Z$ is a surjective local homeomorphism on a compact Hausdorff space Z. Let*

$$\beta_c := \limsup_{n\to\infty}\left(n^{-1}\ln\left(\max_{z\in Z}|h^{-n}(z)|\right)\right), \tag{4.6}$$

and suppose that $\beta > \beta_c$.

(a) *The series $\sum_{n=0}^{\infty} e^{-\beta n}|h^{-n}(z)|$ converges uniformly for $z\in Z$ to a continuous function $f_\beta(z)$, which satisfies*

$$f_\beta(z) - \sum_{h(w)=z} e^{-\beta} f_\beta(w) = 1 \quad \textit{for all } z\in Z. \tag{4.7}$$

(b) *Suppose that ε is a finite regular Borel measure on Z. Then the series $\sum_{n=0}^{\infty} e^{-\beta n}R^n\varepsilon$ converges in norm in the dual space $C(Z)^*$ with sum μ, say. Then μ satisfies the subinvariance relation (4.2), and we have $\varepsilon = \mu - e^{-\beta}R\mu$. Then μ is a probability measure if and only if $\int f_\beta\,d\varepsilon = 1$.*

(c) *Suppose that μ is a probability measure which satisfies the subinvariance relation (4.2). Then $\varepsilon = \mu - e^{-\beta}R\mu$ is a finite regular Borel measure satisfying $\int f_\beta\,d\varepsilon = 1$, and we have $\mu = \sum_{n=0}^{\infty} e^{-\beta n}R^n\varepsilon$.*

Remark 4.3. Part (b) applies when $\epsilon = 0$, and gives $\mu = 0$. However, it is implicit in part (c) that ϵ is not zero (because $\int f_\beta \, d\varepsilon = 1$), and hence $\mu \neq e^{-\beta} R\mu$. Thus part (c) implies that the invariance relation $R\mu = e^{\beta}\mu$ has no solutions[a] for $\beta > \beta_c$.

Proof of Proposition 4.2. We first claim that there exist $\delta > 0$ and $K \in \mathbb{N}$ such that

$$m \geq K \Rightarrow e^{-\beta m}|h^{-m}(z)| < e^{-\delta m} \quad \text{for all } z \in Z. \tag{4.8}$$

Write $c_n := \max_{z \in Z} |h^{-n}(z)|$, so that $\beta > \beta_c$ means $\beta > \limsup n^{-1} \ln c_n$. Then for large n, we have $\beta > \sup_{m \geq n} m^{-1} \ln c_m$. Thus there exist $\delta > 0$ and K such that

$$\begin{aligned} m \geq K \Rightarrow \beta - \delta > m^{-1} \ln c_m &\Rightarrow c_m < e^{\beta m - \delta m} \\ &\Rightarrow e^{-\beta m}|h^{-m}(z)| < e^{-\delta m} \quad \text{for all } z \in Z. \end{aligned}$$

This proves our claim.

Take δ as in (4.8). Then comparing the series $\sum e^{-\beta n}|h^{-n}(z)|$ with $\sum e^{-\delta n}$ shows that the series $\sum_{n=0}^{\infty} e^{-\beta n}|h^{-n}(z)|$ converges uniformly for $z \in Z$. Since h is a local homeomorphism on a compact space, each $z \mapsto |h^{-1}(z)|$ is locally constant (by [5, Lemma 2.2], for example), and hence continuous. Thus $f_\beta(z) := \sum_{n=0}^{\infty} e^{-\beta n}|h^{-n}(z)|$ is the uniform limit of a sequence of continuous functions, and is therefore continuous. To see (4.7), we note that because all the series converge absolutely, we can interchange the order of sums in the following calculation:

$$\begin{aligned} f_\beta(z) - \sum_{h(w)=z} & e^{-\beta} f_\beta(w) \\ &= \sum_{n=0}^{\infty} e^{-\beta n}|h^{-n}(z)| - \sum_{h(w)=z} e^{-\beta} \left(\sum_{m=0}^{\infty} e^{-\beta m}|h^{-m}(w)| \right) \\ &= \sum_{n=0}^{\infty} e^{-\beta n}|h^{-n}(z)| - \sum_{m=0}^{\infty} e^{-\beta(m+1)} \left(\sum_{h(w)=z} |h^{-m}(w)| \right) \\ &= \sum_{n=0}^{\infty} e^{-\beta n}|h^{-n}(z)| - \sum_{m=0}^{\infty} e^{-\beta(m+1)}|h^{-(m+1)}(z)| \\ &= e^{-\beta 0}|h^{-0}(z)| = 1. \end{aligned}$$

We have now proved (a).

[a]The analog of Proposition 4.1 for the Cuntz–Pimsner algebra will say that the measure μ satisfies the invariance relation. Thus Proposition 4.2(c) will imply that there are no KMS$_\beta$ states on $\mathcal{O}(X(E))$ for $\beta > \beta_c$. This is consistent with [34, Theorem 6.8] and our Corollary 5.3.

Next, we look at the series in (b). Take δ, K satisfying (4.8). Then for $N > M \geq K$ and $g \in C(Z)$ we calculate using (4.5):

$$\begin{aligned}\left|\sum_{n=M+1}^{N} e^{-\beta n} \int g \, d(R^n \varepsilon)\right| &= \left|\sum_{n=M+1}^{N} e^{-\beta n} \int \sum_{h^n(w)=z} g(w)\, d\varepsilon(z)\right| \\ &\leq \sum_{n=M+1}^{N} e^{-\beta n} |h^{-n}(z)| \, \|\varepsilon\|_{C(Z)^*} \|g\|_\infty \\ &\leq \sum_{n=M+1}^{N} e^{-\delta n} \|\varepsilon\|_{C(Z)^*} \|g\|_\infty.\end{aligned}$$

Thus the series $\sum_{n=0}^{\infty} e^{-\beta n} R^n \varepsilon$ converges in the norm of $C(Z)^*$, as asserted in (b). Since the operation R is affine and norm-continuous on positive measures, the sum $\mu := \sum_{n=0}^{\infty} e^{-\beta n} R^n \varepsilon$ satisfies

$$\mu - e^{-\beta} R\mu = \sum_{n=0}^{\infty} e^{-\beta n} R^n \varepsilon - \sum_{n=0}^{\infty} e^{-\beta(n+1)} R^{n+1} \varepsilon = \varepsilon;$$

since ε is a (positive) measure, this implies that μ satisfies the subinvariance relation. The Riesz representation theorem implies that μ is a regular Borel measure, and

$$\begin{aligned}\mu(Z) = \sum_{n=0}^{\infty} e^{-\beta n} (R^n \varepsilon)(Z) &= \sum_{n=0}^{\infty} e^{-\beta n} \int 1 \, d(R^n \varepsilon) \\ &= \sum_{n=0}^{\infty} e^{-\beta n} \int |h^{-n}(z)| \, d\varepsilon(z),\end{aligned}$$

which by the monotone convergence theorem is $\int f_\beta \, d\varepsilon$. Thus μ is finite, and it is a probability measure if and only if $\int f_\beta \, d\varepsilon = 1$.

For part (c), we first note that the subinvariance relation implies that ε is a positive measure, and it is finite because μ is. Next we compute:

$$\begin{aligned}\int f_\beta \, d\varepsilon &= \int f_\beta \, d\mu - e^{-\beta} \int f_\beta \, d(R\mu) \\ &= \int f_\beta(z) \, d\mu(z) - e^{-\beta} \int \sum_{h(w)=z} f_\beta(w) \, d\mu(z) \\ &= \int \left(f_\beta(z) - \sum_{h(w)=z} e^{-\beta} f_\beta(w) \right) d\mu(z),\end{aligned}$$

which by (4.7) is $\mu(Z) = 1$. Finally, we have

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-\beta n} R^n \varepsilon &= \sum_{n=0}^{\infty} e^{-\beta n} R^n (\mu - e^{-\beta} R\mu) \\ &= \sum_{n=0}^{\infty} e^{-\beta n} R^n \mu - \sum_{n=0}^{\infty} e^{-\beta(n+1)} R^{n+1} \mu = \mu.\end{aligned}$$ □

5. KMS States on the Toeplitz Algebra

Our main theorem is the following analog of [15, Theorem 3.1].

Theorem 5.1. *Suppose that $h : Z \to Z$ is a surjective local homeomorphism on a compact Hausdorff space Z, E is the topological graph (Z, Z, id, h), and $X(E)$ is the graph correspondence. Define $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(X(E))$ in terms of the gauge action by $\alpha_t = \gamma_{e^{it}}$. Take β_c as in* (4.6)*, suppose that $\beta > \beta_c$, and let f_β be the function in Proposition* 4.2(a).

(a) *Suppose that ε is a finite regular Borel measure on Z such that $\int f_\beta \, d\varepsilon = 1$, and take $\mu = \sum_{n=0}^{\infty} e^{-\beta n} R^n \varepsilon$. Then there is a KMS$_\beta$ state ϕ_ε on $(\mathcal{T}(X(E)), \alpha)$ such that*

$$\phi_\varepsilon\big(\psi^{\otimes l}(x)\psi^{\otimes m}(y)^*\big) = \begin{cases} 0 & \text{if } l \neq m, \\ e^{-\beta m} \displaystyle\int \langle y, x\rangle \, d\mu & \text{if } l = m. \end{cases} \tag{5.1}$$

(b) *The map $\varepsilon \mapsto \phi_\varepsilon$ is an affine isomorphism of*

$$\Sigma_\beta := \left\{ \varepsilon \in M(Z)_+ : \int f_\beta \, d\varepsilon = 1 \right\}$$

onto the simplex of KMS$_\beta$ states of $(\mathcal{T}(X(E)), \alpha)$. The inverse takes a state ϕ to $\varepsilon := \mu - e^{-\beta} R\mu$, where μ is the probability measure such that $\phi(\pi(a)) = \int a \, d\mu$ for $a \in C(Z)$.

In the proof of this theorem, we will need to do some computations in the Toeplitz algebra, and the following observation will help.

Lemma 5.2. *For $n \geq 1$ we consider the topological graph $F_n = (Z, Z, \mathrm{id}, h^n)$. Then there is an isomorphism ρ_n of $X(E)^{\otimes n}$ onto $X(F_n)$ such that*

$$\rho_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n)(z) = x_1(z) x_2(h(z)) \cdots x_n(h^{n-1}(z)).$$

Proof. We prove this by induction on n. It is trivially true for $n = 1$ — indeed, we have $E = F_1$, and ρ_1 is the identity. Suppose that there is such an isomorphism ρ_n and define $\rho_{n+1}(x_1 \otimes x)(z) = x_1(z)\rho_n(x)(h(z))$. Routine calculations show that ρ_{n+1} is a bimodule homomorphism. We next show that ρ_{n+1} preserves the inner products. Let $x_1 \otimes x$ and $y_1 \otimes y$ be elementary tensors in $X(E) \otimes_{C(Z)} X(E)^{\otimes n}$. Then for $z \in Z$ we have

$$\begin{aligned}
&\big\langle \rho_{n+1}(x_1 \otimes x), \rho_{n+1}(y_1 \otimes y) \big\rangle(z) \\
&\quad = \sum_{h^{n+1}(w) = z} \overline{x_1(w)\rho_n(x)(h(w))} y_1(w) \rho_n(y)(h(w)) \\
&\quad = \sum_{h^n(v) = z} \sum_{h(w) = v} \overline{x_1(w)\rho_n(x)(h(w))} y_1(w) \rho_n(y)(h(w))
\end{aligned}$$

$$= \sum_{h^n(v)=z} \overline{\rho_n(x)(v)} \left(\sum_{h(w)=v} \overline{x_1(w)} y_1(w) \right) \rho_n(y)(v)$$

$$= \sum_{h^n(v)=z} \overline{\rho_n(x)(v)} \big(\langle x_1, y_1 \rangle \cdot \rho_n(y) \big)(v)$$

$$= \langle \rho_n(x), \langle x_1, y_1 \rangle \cdot \rho_n(y) \rangle (z)$$

$$= \langle x_1 \otimes \rho_n(x), y_1 \otimes \rho_n(y) \rangle (z).$$

Since the range of ρ_{n+1} contains $C(Z)$ (take $x = 1$), we deduce that ρ_{n+1} is an isomorphism of Hilbert bimodules. □

Proof of Theorem 5.1. We aim to construct the KMS state ϕ_ε using a representation[b] (θ, ρ) of $X(E)$ on $H_{\theta,\rho} := \bigoplus_{n=0}^\infty L^2(Z, R^n\varepsilon)$. We write elements of the direct sum as sequences $\xi = (\xi_n)$. For $a \in C(Z)$, we take ρ to be the direct sum of the representations ρ_n of $C(Z)$ on $L^2(Z, R^n\varepsilon)$ given by $(\rho_n(a)\xi_n)(z) = a(z)\xi_n(z)$. Next we claim that for each $x \in X$ there is a bounded operator $\theta(x)$ on $H_{\theta,\rho}$ such that

$$(\theta(x)\xi)_{n+1}(z) = x(z)\xi_n(h(z)) \quad \text{for } n \geq 0 \quad \text{and} \quad (\theta(x)\xi)_0 = 0.$$

To justify the claim, we take $\xi = (\xi_n) \in \bigoplus_{n=0}^\infty L^2(Z, R^n\varepsilon)$ and compute:

$$\begin{aligned}
\|\theta(x)\xi\|^2 &= \sum_{n=0}^\infty \|(\theta(x)\xi)_{n+1}\|^2 \\
&= \sum_{n=0}^\infty \int |x(z)|^2 |\xi_n(h(z))|^2 \, d(R^{n+1}\varepsilon)(z) \\
&\leq \sum_{n=0}^\infty \|x\|_\infty^2 \int \sum_{h(w)=z} |\xi_n(h(w))|^2 \, d(R^n\varepsilon)(z) \\
&= \sum_{n=0}^\infty \|x\|_\infty^2 \int \sum_{h(w)=z} |\xi_n(z)|^2 \, d(R^n\varepsilon)(z) \\
&\leq \sum_{n=0}^\infty \|x\|_\infty^2 c_1 \int |\xi_n(z)|^2 \, d(R^n\varepsilon)(z) \quad (\text{where } c_1 = \max_z |h^{-1}(z)|) \\
&= c_1 \|x\|_\infty^2 \|\xi\|^2.
\end{aligned}$$

A similar calculation shows that the adjoint $\theta(x)^*$ satisfies

$$(\theta(x)^*\eta)_n(z) = \sum_{h(w)=z} \overline{x(w)} \eta_{n+1}(w) \quad \text{for } \eta \in H_{\theta,\rho}. \tag{5.2}$$

[b]As in our previous papers, this construction was motivated by the one in the proof of [23, Theorem 2.1], which suggests that we should take a representation, here the representation M_ε of $A = C(Z)$ by multiplication operators on $L^2(Z, \varepsilon)$, and work in the induced representation $F(X(E))\text{-}\mathrm{Ind}_A^{\mathcal{T}} M_\varepsilon$ of $\mathcal{T} = \mathcal{T}(X(E))$, where $F(X(E))$ is the Fock bimodule. However, this requires many identifications, and it seems clearer to write down a concrete Hilbert space.

Next we claim that (θ, ρ) is a representation of $X(E)$. It is easy to check that $\theta(a \cdot x) = \rho(a)\theta(x)$, and almost as easy to see that $\theta(x \cdot a) = \theta(x)\rho(a)$: for $\xi = (\xi_n)$ we have $(\theta(x \cdot a)\xi)_0 = 0 = (\theta(x)(\rho(a)\xi))_0$, and for $n \geq 1$

$$\begin{aligned}(\theta(x \cdot a)\xi)_n(z) &= (x(z)a(h(z)))\xi_{n-1}(h(z)) = x(z)(\rho(a)\xi)_{n-1}(h(z)) \\ &= \big(\theta(x)(\rho(a)\xi)\big)_n(z).\end{aligned}$$

For $n \geq 0$, we have

$$\begin{aligned}(\rho(\langle x, y\rangle)\xi)_n(z) = \langle x, y\rangle(z)\xi_n(z) &= \sum_{h(w)=z} \overline{x(w)}y(w)\xi_n(z) \\ &= \sum_{h(w)=z} \overline{x(w)}y(w)\xi_n(h(w)) = \sum_{h(w)=z} \overline{x(w)}(\theta(y)\xi)_{n+1}(w) \\ &= (\theta(x)^*\theta(y)\xi)_n(z) \quad \text{(using (5.2))}.\end{aligned}$$

Now the universal property of $\mathcal{T}(X(E))$ gives a homomorphism $\theta \times \rho : \mathcal{T}(X(E)) \to B(H_{\theta,\rho})$ such that $(\theta \times \rho) \circ \psi = \theta$ and $(\theta \times \rho) \circ \pi = \rho$.

For each $k \geq 1$, we choose a finite partition $\{Z_{k,i} : 1 \leq i \leq I_k\}$ of Z by Borel sets such that h^k is one-to-one on each $Z_{k,i}$. We write also $I_0 = 1$ and $Z_{0,1} = Z$. Let $\chi_{k,i} = \chi_{Z_{k,i}}$, and define $\xi^{k,i} \in \bigoplus_{n=0}^\infty L^2(Z, R^n\varepsilon)$ by

$$\xi_n^{k,i} = \begin{cases} 0 & \text{if } n \neq k, \\ \chi_{k,i} & \text{if } n = k. \end{cases}$$

We aim to define our state $\phi_\varepsilon : \mathcal{T}(X(E)) \to \mathbb{C}$ by

$$\phi_\varepsilon(b) = \sum_{k=0}^\infty \sum_{i=1}^{I_k} e^{-\beta k}\big(\theta \times \rho(b)\xi^{k,i} \,|\, \xi^{k,i}\big) \quad \text{for } b \in \mathcal{T}(X(E)), \tag{5.3}$$

but of course we have to show that the series converges. It suffices to do this for positive b, and then since $b \leq \|b\|1$ it suffices to prove that the series for $\phi_\varepsilon(1)$ converges. Since for each k the $Z_{k,i}$ partition Z, we have

$$\sum_{k=0}^\infty \sum_{i=1}^{I_k} e^{-\beta k}\big(\chi_{Z_{k,i}} \,|\, \chi_{Z_{k,i}}\big) = \sum_{k=0}^\infty \sum_{i=1}^{I_k} e^{-\beta k} R^k\varepsilon(Z_{k,i}) = \sum_{k=0}^\infty e^{-\beta k} R^k\varepsilon(Z).$$

Proposition 4.2 implies that this converges with sum $\mu(Z) = 1$. Thus the formula (5.3) gives us a well-defined state on $\mathcal{T}(X(E))$.

We now prove that this state satisfies (5.1). So we take $x \in X(F_l) = X^{\otimes l}$, $y \in X(F_m) = X^{\otimes m}$ and $b = \psi^{\otimes l}(x)\psi^{\otimes m}(y)^*$. Since $\xi^{k,i}$ is zero in all except the kth summand of $\bigoplus_{n=0}^\infty L^2(Z, R^n\varepsilon)$,

$$\theta \times \rho(b)\xi^{k,i} = \theta^{\otimes l}(x)\theta^{\otimes m}(y)^*\xi^{k,i}$$

is zero in all but the $(k - m + l)$th summand. Thus

$$(\theta \times \rho(b)\xi^{k,i} \,|\, \xi^{k,i}) = 0 \quad \text{for all } k, i \text{ whenever } l \neq m,$$

and ϕ_ε certainly satisfies (5.1) when $l \neq m$. So we suppose that $l = m \geq 0$.

Next, note that $\theta^{\otimes m}(x)\theta^{\otimes m}(y)^*\xi^{k,i} = 0$ if $k < m$. For $k \geq m$, we know that h^k is injective on $Z_{k,i}$, and hence so is h^m. Thus $w, z \in Z_{k,i}$ and $h^m(w) = h^m(z)$ imply $w = z$, and

$$\begin{aligned}
&\big(\theta^{\otimes m}(x)\theta^{\otimes m}(y)^*\xi^{k,i} \,|\, \xi^{k,i}\big) \\
&\quad = \int \left(x(z) \sum_{h^m(w)=h^m(z)} \overline{y(w)}\chi_{k,i}(w) \right) \overline{\chi_{k,i}(z)}\, d(R^k\varepsilon)(z) \\
&\quad = \int x(z)\overline{y(z)\chi_{k,i}(z)}\, d(R^k\varepsilon)(z).
\end{aligned}$$

Since the $Z_{k,i}$ partition Z, summing over i gives

$$\sum_{i=1}^{I_k} \big(\theta \times \rho(\psi^{\otimes m}(x)\psi^{\otimes m}(y)^*)\xi^{k,i} \,|\, \xi^{k,i}\big) = \int x(z)\overline{y(z)}\, d(R^k\varepsilon)(z).$$

Thus from (4.5) and the formula for the inner product on $X(E)^{\otimes m} = X(F_m)$ we have

$$\begin{aligned}
\phi_\varepsilon\big(\psi^{\otimes m}(x)\psi^{\otimes m}(y)^*\big) &= \sum_{k=m}^{\infty} e^{-\beta k} \int x(z)\overline{y(z)}\, d(R^k\varepsilon)(z) \\
&= \sum_{k=m}^{\infty} e^{-\beta k} \int \sum_{h^m(w)=z} x(w)\overline{y(w)}\, d(R^{k-m}\varepsilon)(z) \\
&= \sum_{k=0}^{\infty} e^{-\beta(m+k)} \int \langle y, x\rangle(z)\, d(R^k\varepsilon)(z) \\
&= e^{-\beta m} \int \langle y, x\rangle\, d\left(\sum_{k=0}^{\infty} e^{-\beta k} R^k\varepsilon\right) \\
&\qquad \text{by Proposition 4.2(b)} \\
&= e^{-\beta m} \int \langle y, x\rangle\, d\mu. \qquad (5.4)
\end{aligned}$$

This is (5.1). Applying (5.1) with $m = 0$ shows that $\phi_\varepsilon(\pi(a)) = \int a\, d\mu$, which says that the last integral in (5.4) is $\phi_\varepsilon \circ \pi(\langle y, x\rangle)$. Thus ϕ_ε satisfies (3.1), and Proposition 3.1 implies that ϕ_ε is a KMS_β state. We have now proved part (a).

Now suppose that ϕ is a KMS_β state, and let μ be the probability measure such that $\phi \circ \pi(a) = \int a\, d\mu$ for $a \in C(Z)$. Then Proposition 4.1 implies that μ satisfies the subinvariance relation $R\mu \leq e^\beta\mu$, and hence Proposition 4.2(c) implies that $\varepsilon := \mu - e^{-\beta}R\mu$ is a positive measure which belongs to Σ_β and satisfies $(1 - e^{-\beta}R)^{-1}\varepsilon = \mu$. Thus formulas (3.1) and (5.1) imply that $\phi = \phi_\varepsilon$. This shows that $\varepsilon \mapsto \phi_\epsilon$ is surjective. Since applying the construction of this paragraph to the state ϕ_ε gives us $\epsilon = \mu - e^{-\beta}R\mu$ back, it also shows that $\varepsilon \mapsto \phi_\varepsilon$ is one-to-one.

Thus $\varepsilon \mapsto \phi_\varepsilon$ maps Σ_β onto the set of KMS_β states, and it is affine and continuous for the respective weak* topologies. So we have proved our theorem. $\square$

The next Corollary is contained in [34, Theorem 6.8] (here the function F of that theorem is identically 1 — see Remark 6.3), but the proof in [34] is quite different.

Corollary 5.3. *Take $h : Z \to Z$ and E as in Theorem* 5.1, *and define $\alpha : \mathbb{R} \to \operatorname{Aut}\mathcal{O}(X(E))$ in terms of the gauge action γ by $\alpha_t = \gamma_{e^{it}}$. If there is a KMS state of $(\mathcal{O}(X(E)), \alpha)$ with inverse temperature β, then $\beta \le \beta_c$.*

Proof. Suppose $\beta > \beta_c$ and there is a KMS_β state ϕ of $(\mathcal{O}(X(E)), \alpha)$. Denote by q the quotient map of $\mathcal{T}(X(E))$ onto $\mathcal{O}(X(E))$. Then $\phi \circ q$ is a KMS_β state of the system $(\mathcal{T}(X(E)), \alpha)$ considered in Theorem 5.1. Thus there is a measure ε on Z such that $\int f_\beta \, d\varepsilon = 1$ and $\phi \circ q = \phi_\varepsilon$. Notice in particular that $\varepsilon(Z) > 0$. We can find a finite open cover $\{U_j : 1 \le j \le I\}$ of Z by sets such that $h|_{U_j}$ is a homeomorphism, and we can find open sets $\{V_j : 1 \le j \le I\}$ which still cover Z but have $\overline{V_j} \subset U_j$ (see [32, Lemma 4.32], for example). Since $\varepsilon(Z) > 0$, there exists j such that $\varepsilon(V_j) > 0$. Now choose a function $f \in C_c(Z)$ such that $f(z) \neq 0$ for $z \in V_j$ and $\operatorname{supp} f \subset U_j$. Then the left action of $|f|^2 \in C(Z)$ on $X(E)$ is implemented by the finite-rank operator $\Theta_{f,f}$, and hence

$$\begin{aligned}\pi(|f|^2) - \psi(f)\psi(f)^* &= \pi(|f|^2) - (\psi, \pi)^{(1)}(\Theta_{f,f}) \\ &= \pi(|f|^2) - (\psi, \pi)^{(1)}(\varphi(|f|^2))\end{aligned}$$

belongs to the kernel of the quotient map q. But with μ as in Theorem 5.1(b), we have

$$\begin{aligned}\phi_\varepsilon(\pi(|f|^2) - \psi(f)\psi(f)^*) &= \int |f|^2 \, d\mu - e^{-\beta} \int \sum_{h(w)=z} |f|^2(w) \, d\mu \\ &= \int |f|^2 \, d(\mu - e^{-\beta} R\mu) = \int |f|^2 \, d\varepsilon > 0.\end{aligned}$$

Thus ϕ_ε does not vanish on $\ker q$, and we have a contradiction. Thus $\beta \le \beta_c$. □

Example 5.4. Suppose that $A \in M_d(\mathbb{Z})$ is an integer matrix with $N := |\det A| > 1$. Then there is a covering map $\sigma_A : \mathbb{T}^d \to \mathbb{T}^d$ such that $\sigma_A(e^{2\pi i x}) = e^{2\pi i A x}$ for $x \in \mathbb{R}^d$. The inverse image of each $z \in \mathbb{T}^d$ has N elements, and hence $|\sigma_A^{-n}(z)| = N^n$ for all z. Thus

$$\frac{1}{n} \ln \left(\max_{z \in \mathbb{T}^d} |\sigma_A^{-n}(z)| \right) = \frac{1}{n} \ln N^n = \ln N \quad \text{for all } n,$$

and $\beta_c = \ln N$. Suppose $\beta > \ln N$ and ν is a probability measure on $\mathbb{T}^d$. The function f_β is the constant function

$$f_\beta \equiv \sum_{n=0}^{\infty} e^{-\beta n} N^n = \frac{1}{1 - N e^{-\beta}},$$

and hence the measure $\varepsilon := (1 - Ne^{-\beta})\nu$ satisfies $\int f_\beta \, d\varepsilon = 1$. Thus with $E = (\mathbb{T}^d, \mathbb{T}^d, \mathrm{id}, \sigma_A)$, Theorem 5.1 gives a KMS$_\beta$ state ϕ_ε on $(\mathcal{T}(X(E)), \alpha)$ such that

$$\phi_\varepsilon\big(\psi^{\otimes k}(x)\psi^{\otimes l}(y)^*\big) = \delta_{k,l} e^{-\beta k} \sum_{j=0}^{\infty} e^{-\beta j} \int \langle y, x\rangle \, d(R^j \varepsilon) \tag{5.5}$$

for $x \in X^{\otimes k}$, $y \in X^{\otimes l}$. We claim that ϕ_ε is the KMS state $\psi_{\beta,\nu}$ described in [26, Proposition 6.1].

The algebra $\mathcal{T}(M_L)$ in [26] is associated to an Exel system $(C(\mathbb{T}^d), \sigma_A^*, L)$, in which σ_A^* is the endomorphism $f \mapsto f \circ \sigma_A$ and L is a "transfer operator" defined by $L(f)(z) = N^{-1} \sum_{\sigma_A(w)=z} f(w)$. The bimodule M_L is a copy of $C(\mathbb{T}^d)$ with operations $a \cdot m \cdot b = am\sigma_A^*(b)$ and inner product $\langle m, n\rangle = L(m^* n)$. The map $m \mapsto N^{-1/2} m$ is an isomorphism of M_L onto $X(E)$, and this isomorphism induces isomorphisms of $\mathcal{T}(M_L)$ onto $\mathcal{T}(X(E))$ and of the system $(\mathcal{T}(M_L), \sigma)$ in [26] onto our $(\mathcal{T}(X(E)), \alpha)$. In the presentation of $\mathcal{T}(M_L)$ used in [26], we need to consider elements $\{u_m v^k : m \in \mathbb{Z}^d,\ k \in \mathbb{N}\}$; such an element $u_m v^k$ lies in $\psi^{\otimes k}(M_L^{\otimes k})$. The isomorphism of $M_L^{\otimes k}$ onto $X(E)^{\otimes k} = X(F_k)$ takes $u_m v^k$ to the function $N^{-k/2}\gamma_m : z \mapsto N^{-k/2} z^m$, and the inner product on $X(F_k)$ is given in terms of L by $\langle y, x\rangle = N^k L^k(\overline{y} x)$. For $a \in C(\mathbb{T}^d)$, we have

$$\int a \, d(R^j \varepsilon) = \int \sum_{\sigma_A^j(w)=z} a(w) \, d\varepsilon(z) = \int N^j L^j(a)(z) \, d\varepsilon(z).$$

Putting this into (5.5) gives

$$\begin{aligned}
\phi_\varepsilon(u_m v^k v^{*l} u_n^*) &= \delta_{k,l} e^{-\beta k} \sum_{j=0}^{\infty} e^{-\beta j} \int N^j L^j\big(N^k L^k\big(\overline{N^{-k/2}\gamma_n} N^{-k/2}\gamma_m\big)\big) \, d\varepsilon \\
&= \delta_{k,l} \sum_{j=k}^{\infty} e^{-\beta j} N^{j-k} \int L^j(\gamma_{m-n}) \, d\varepsilon.
\end{aligned}$$

The calculation in the third paragraph of the proof of [26, Proposition 3.1] (applied to A^j rather than A), shows that with $B := A^t$ we have

$$L^j(\gamma_{m-n}) = \begin{cases} 0 & \text{unless } m - n \in B^j \mathbb{Z}^d, \\ \gamma_{B^{-j}(m-n)} & \text{if } m - n \in B^j \mathbb{Z}^d. \end{cases}$$

Thus

$$\begin{aligned}
\phi_\varepsilon(u_m v^k v^{*l} u_n^*) &= \delta_{k,l} \sum_{\{j \geq k \,:\, m-n \in B^j \mathbb{Z}^d\}} e^{-\beta j} N^{j-k} \int \gamma_{B^{-j}(m-n)} \, d\varepsilon \\
&= \delta_{k,l} \sum_{\{j \geq k \,:\, m-n \in B^j \mathbb{Z}^d\}} e^{-\beta j} N^{j-k} \int z^{B^{-j}(m-n)} (1 - Ne^{-\beta}) \, d\nu(z).
\end{aligned}$$

Thus ϕ_ε is the state $\psi_{\beta,\nu}$ described in [26, Proposition 6.1], as claimed.

6. KMS States at the Critical Inverse Temperature

Theorem 6.1. *Suppose that $h : Z \to Z$ is a surjective local homeomorphism on a compact Hausdorff space Z, E is the topological graph (Z, Z, id, h), and $X(E)$ is the graph correspondence. Define $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(X(E))$ and $\bar{\alpha} : \mathbb{R} \to \operatorname{Aut} \mathcal{O}(X(E))$ in terms of the gauge actions by $\alpha_t = \gamma_{e^{it}}$ and $\bar{\alpha}_t = \bar{\gamma}_{e^{it}}$. Take β_c as in* (4.6)*. Then there exists a KMS$_{\beta_c}$ state on $(\mathcal{T}(X(E)), \alpha)$, and at least one such state factors through a KMS$_{\beta_c}$ state of $(\mathcal{O}(X(E), \bar{\alpha}))$.*

For the proof we need a variant on [25, Lemma 10.3; 15, Lemma 2.2], where the generating sets P were required to consist of projections. We thank the referee for providing this one, which is much stronger than we need.

Lemma 6.2. *Suppose $(A, \mathbb{R}, \alpha)$ is a dynamical system, and J is an ideal in A generated by a set P of positive elements which are fixed by α. If ϕ is a KMS$_\beta$ state of (A, α) and $\phi(p) = 0$ for all $p \in P$, then ϕ factors through a state of A/J.*

Proof. Consider $p \in P$, and let a, b be analytic elements for α. Since elements of the form apb span a dense subspace of J, it suffices to show that $\phi(apb) = 0$. Since ap is analytic for α with $\alpha_{i\beta}(ap) = \alpha_{i\beta}(a)p$, the KMS condition and the Cauchy–Schwarz inequality give

$$\begin{aligned} 0 \le |\phi(apb)|^2 &= |\phi(b\alpha_{i\beta}(a)p)|^2 \\ &\le \phi(b\alpha_{i\beta}(a)\alpha_{i\beta}(a)^*b^*)\phi(p^2) \\ &\le \phi(b\alpha_{i\beta}(a)\alpha_{i\beta}(a)^*b^*)\|p\|\phi(p) = 0, \end{aligned}$$

and hence $\phi(apb) = 0$, as required. □

Proof of Theorem 6.1. Choose a decreasing sequence $\{\beta_n\}$ such that $\beta_n \to \beta_c$ and a probability measure ν on Z. Then $K_n := \int f_{\beta_n}\, d\nu$ belongs to $[1, \infty)$, and $\varepsilon_n := K_n^{-1}\nu$ satisfies $\int f_{\beta_n}\, d\varepsilon_n = 1$. Thus for each n, Theorem 5.1 gives us a KMS$_{\beta_n}$ state ϕ_{ε_n} on $(\mathcal{T}(X(E)), \alpha)$. By passing to a subsequence, we may assume that $\{\phi_{\varepsilon_n}\}$ converges in the weak* topology to a state ϕ, and [3, Proposition 5.3.23] implies that ϕ is a KMS$_{\beta_c}$ state.

To find a KMS$_{\beta_c}$ state which factors through $\mathcal{O}(X(E))$, we apply the construction of the previous paragraph to a particular sequence of measures ε_n. Since each $z \mapsto |h^{-n}(z)|$ is continuous [5, Lemma 2.2], Proposition 2.3 of [12] implies[c] that there exists $p \in Z$ such that

$$|h^{-n}(p)| \ge e^{n\beta_c} \quad \text{for all } n \in \mathbb{N}. \tag{6.1}$$

Now we let δ_p be the unit point mass at p, and take $\varepsilon_n := f_{\beta_n}(p)^{-1}\delta_p$. The argument of the first paragraph yields a KMS$_{\beta_c}$ state ϕ on $(\mathcal{T}(X(E)), \alpha)$ which is a weak* limit of the KMS$_{\beta_n}$ states ϕ_{ε_n}.

[c] Strictly speaking, [12] requires throughout that their space is metric, but their argument for this proposition does not seem to use this.

Next we choose a partition of unity $\{\rho_i : 1 \le i \le k\}$ for Z such that h is injective on each $\operatorname{supp}\rho_i$, and take $\xi_i := \sqrt{\rho_i} \in X(E)$ as in the proof of Proposition 4.1. Temporarily, we write ϕ_A for the homomorphism of $A = C(Z)$ into $\mathcal{L}(X(E))$ given by the left action. A calculation like the one in the second paragraph of the proof of Proposition 4.1 shows that for every $a \in A$, $\phi_A(a)$ is the finite-rank operator $\sum_{i=1}^k \Theta_{a\cdot\xi_i,\xi_i}$. Thus the kernel of the quotient map $q : \mathcal{T}(X(E)) \to \mathcal{O}(X(E))$ is generated by the elements

$$\begin{aligned}\pi(a) - (\psi,\pi)^{(1)}\left(\sum_{i=1}^k \Theta_{a\cdot\xi_i,\xi_i}\right) &= \pi(a) - \sum_{i=1}^k \psi(a\cdot\xi_i)\psi(\xi_i)^* \\ &= \pi(a)\left(1 - \sum_{i=1}^k \psi(\xi_i)\psi(\xi_i)^*\right),\end{aligned}$$

and hence also by the single element $1 - \sum_{i=1}^k \psi(\xi_i)\psi(\xi_i)^*$. Equation 4.4 implies that this single generator is positive in $\mathcal{T}(X(E))$, so if we can show that $\phi\big(\sum_{i=1}^k \psi(\xi_i)\psi(\xi_i)^*\big) = 1$, then it will follow from Lemma 6.2 that ϕ factors through $\mathcal{O}(X(E))$.

We therefore calculate $\phi\big(\sum_{i=1}^k \psi(\xi_i)\psi(\xi_i)^*\big)$. We write μ_n for the measure $\sum_{j=0}^\infty e^{-\beta_n j} R^j\varepsilon_n$ of Theorem 5.1(b). Then (5.1) implies that

$$\begin{aligned}\phi\left(\sum_{i=1}^k \psi(\xi_i)\psi(\xi_i)^*\right) &= \lim_{n\to\infty}\sum_{i=1}^k \phi_{\varepsilon_n}\big(\psi(\xi_i)\psi(\xi_i)^*\big) \\ &= \lim_{n\to\infty} e^{-\beta_n}\int \sum_{i=1}^k \langle \xi_i,\xi_i\rangle\, d\mu_n. \end{aligned}\tag{6.2}$$

Since h is injective on each $\operatorname{supp}\xi_i$, we have

$$\begin{aligned}\sum_{i=1}^k \langle\xi_i,\xi_i\rangle(z) &= \sum_{i=1}^k \sum_{h(w)=z} \overline{\xi_i(w)}\xi_i(w) = \sum_{h(w)=z}\sum_{i=1}^k |\xi_i(w)|^2 \\ &= \sum_{h(w)=z} 1 = |h^{-1}(z)|.\end{aligned}$$

Thus

$$\begin{aligned}e^{-\beta_n}\int \sum_{i=1}^k \langle\xi_i,\xi_i\rangle\, d\mu_n &= e^{-\beta_n}\int |h^{-1}(z)|\, d\mu_n(z) \\ &= \sum_{j=0}^\infty e^{-\beta_n}e^{-\beta_n j}\int |h^{-1}(z)|\, d(R^j\varepsilon_n)(z) \\ &= \sum_{j=0}^\infty e^{-\beta_n(j+1)}\int \sum_{h^j(w)=z} |h^{-1}(w)|\, d\varepsilon_n(z).\end{aligned}$$

Since ε_n is a point mass, we have

$$e^{-\beta_n}\int\sum_{i=1}^{k}\langle\xi_i,\xi_i\rangle\,d\mu_n=\sum_{j=0}^{\infty}e^{-\beta_n(j+1)}|h^{-(j+1)}(p)|f_{\beta_n}(p)^{-1}$$
$$=\sum_{j=1}^{\infty}e^{-\beta_n j}|h^{-j}(p)|f_{\beta_n}(p)^{-1}.$$

Since $f_{\beta_n}(p)=\sum_{j=0}^{\infty}e^{-\beta_n j}|h^{-j}(p)|$, we deduce that

$$e^{-\beta_n}\int\sum_{i=1}^{k}\langle\xi_i,\xi_i\rangle\,d\mu_n=\frac{f_{\beta_n}(p)-1}{f_{\beta_n}(p)}. \tag{6.3}$$

We now need to take the limit of (6.3) as $n\to\infty$. Since we chose the point p to satisfy (6.1), we have

$$f_{\beta_n}(p)=\sum_{j=0}^{\infty}e^{-\beta_n j}|h^{-j}(p)|\geq\sum_{j=0}^{\infty}\left(e^{-(\beta_n-\beta_c)}\right)^j.$$

Since $e^{-(\beta_n-\beta_c)}\to 1$ as $n\to\infty$, for fixed J we have

$$\sum_{j=0}^{J}\left(e^{-(\beta_n-\beta_c)}\right)^j\to J+1\quad\text{as } n\to\infty,$$

and $f_{\beta_n}(p)\to\infty$ as $n\to\infty$. Thus (6.3) converges to 1 as $n\to\infty$, and (6.2) implies that

$$\phi\left(\sum_{i=1}^{k}\psi(\xi_i)\psi(\xi_i)^*\right)=1,$$

as required. □

Remark 6.3. Theorem 6.1, and in particular the existence of KMS states on $(\mathcal{O}(X(E)),\bar{\alpha})$ at the inverse temperature β_c, overlaps with work of Thomsen [34]. His results concern KMS states on the C^*-algebra of a Deaconu–Renault groupoid, but his Theorem 3.1 identifies his reduced groupoid algebra $C_r^*(\Gamma_h)$ as an Exel crossed product $D\rtimes_{\alpha,L}\mathbb{N}$. In our setting, where the space Z is compact Hausdorff, his D is $C(Z)$, his endomorphism α is given by $\alpha(f)=f\circ h$, and his transfer operator L is given by $L(f)(z)=|h^{-1}(z)|^{-1}\sum_{h(w)=z}f(w)$; Thomsen's Exel crossed product is the Cuntz–Pimsner bimodule of a Hilbert bimodule M_L [4, Proposition 3.10]. The bimodule is not quite the same as our $X(E)$, but the map $U: X(E)\to M_L$ given by $(Uf)(z)=|h^{-1}(h(z))|^{1/2}f(z)$ is an isomorphism of $X(E)$ onto M_L (see [5, Sec. 6]). So our $\mathcal{O}(X(E))$ is naturally isomorphic to the C^*-algebra $C_r^*(\Gamma_h)$ in [34]. This isomorphism carries the gauge action $\gamma:\mathbb{T}\to\operatorname{Aut}\mathcal{O}(X(E))$ into the gauge action τ used in [34, Sec. 6], and hence our action $\bar{\alpha}$ is the action α^F of [34] for the function $F\equiv 1$ (see the top of [34, p. 414]).

For $F\equiv 1$, the sequences $A_F^{\phi}(k)$ and $B_F^{\phi}(k)$ in [34, Sec. 6] are given by $A_F^{\phi}(k)=k=B_F^{\phi}(k)$, and hence the numbers $A_F^{\phi}=\lim_{k\to\infty}k^{-1}A_F^{\phi}(k)$ and B_F^{ϕ} are both 1.

The number $h_m(\phi)$ in [34, Sec. 6] is our β_c. Thus [34, Theorem 6.12] implies that our system $(\mathcal{O}(X(E)), \bar{\alpha})$ has a KMS_{β_c} state. Our approach through $\mathcal{T}(X(E))$ seems quite different.

7. The Shift on the Path Space of a Graph

In this section we consider a finite directed graph $E = (E^0, E^1, r, s)$ with no sinks or sources. In the conventions of [31], we write E^∞ for the set of infinite paths $z = z_1 z_2 \cdots$ with $s(z_i) = r(z_{i+1})$. The cylinder sets

$$Z(\mu) = \{z \in E^\infty : z_i = \mu_i \text{ for } i \leq |\mu|\}$$

form a basis of compact open sets for a compact Hausdorff topology on E^∞. The shift $\sigma : E^\infty \to E^\infty$ is defined by $\sigma(z) = z_2 z_3 \cdots$. Then σ is a local homeomorphism — indeed, for each edge $e \in E^1$, σ is a homeomorphism of $Z(e)$ onto $Z(s(e))$ — and is a surjection if and only if E has no sinks. Shifts on path spaces were used extensively in the early papers on graph algebras, and in particular in the construction of the groupoid model [20]. Here, we shall use them to illustrate our results and those of Thomsen [34].

We consider the topological graph $(E^\infty, E^\infty, \mathrm{id}, \sigma)$, and write $X(E^\infty)$ for the associated Hilbert bimodule over $C(E^\infty)$. The Cuntz–Pimsner algebra $\mathcal{O}(X(E^\infty))$ is isomorphic to the graph C^*-algebra $C^*(E)$ (this is essentially a result from [5] — see the end of the proof below). The relationship between the Toeplitz algebra $\mathcal{T}(X(E^\infty))$ and the Toeplitz algebra $\mathcal{T}C^*(E)$ is more complicated.

Proposition 7.1. *Suppose that E is a finite directed graph. Then the elements $S_e := \psi(\chi_{Z(e)})$ and $P_v := \pi(\chi_{Z(v)})$ of $\mathcal{T}(X(E^\infty))$ form a Toeplitz–Cuntz–Krieger family. The corresponding homomorphism $\pi_{S,P}$ of $\mathcal{T}C^*(E)$ into $\mathcal{T}(X(E^\infty))$ is injective, and $q \circ \pi$ factors through an isomorphism of $C^*(E)$ onto $\mathcal{O}(X(E^\infty))$. Both isomorphisms intertwine the respective gauge actions of $\mathbb{T}$.*

Proof. Since the $\chi_{Z(v)}$ are mutually orthogonal projections in $C(E^\infty)$, the $\{P_v : v \in E^0\}$ are mutually orthogonal projections in $\mathcal{T}(X(E^\infty))$. For $e, f \in E^1$, we have

$$S_e^* S_f = \psi(\chi_{Z(e)})^* \psi(\chi_{Z(f)}) = \pi\big(\langle \chi_{Z(e)}, \chi_{Z(f)} \rangle\big).$$

A calculation shows that $\langle \chi_{Z(e)}, \chi_{Z(f)} \rangle$ vanishes unless $e = f$, and then equals $\chi_{Z(s(e))}$; this implies that $S_e^* S_e = P_{s(e)}$, and that the range projections $S_e S_e^*$ and $S_f S_f^*$ are mutually orthogonal. Since the left action satisfies $\chi_{Z(v)} \cdot \chi_{Z(e)} = \chi_{Z(e)}$ when $v = r(e)$, we have $P_v S_e S_e^* = S_e S_e^*$ when $v = r(e)$, and $P_v \geq \sum_{r(e)=v} S_e S_e^*$. Thus (S, P) is a Toeplitz–Cuntz–Krieger family. Since the adjoints $\psi(x)^*$ vanish on the 0-summand in the Fock module and the representation π is faithful there, $P_v \neq \sum_{r(e)=v} S_e S_e^*$ as operators on the Fock module $F(X(E^\infty))$. Thus [14, Corollary 4.2] implies that $\pi_{S,P}$ is faithful. Since the gauge actions satisfy $\gamma_z(s_e) = z s_e$ and $\gamma_z(\psi(f)) = z\psi(f)$, we have $\pi_{S,P} \circ \gamma = \gamma \circ \pi_{S,P}$.

The left action of $\chi_{Z(\mu)}$ in $X(E^\infty)$ is the finite rank operator $\Theta_{\chi_{Z(\mu)},\chi_{Z(\mu)}}$, and hence we have

$$\begin{aligned} q\circ\pi(\chi_{Z(\mu)}) &= q\circ(\pi,\psi)^{(1)}(\Theta_{\chi_{Z(\mu)},\chi_{Z(\mu)}}) \\ &= q\big(\psi^{\otimes|\mu|}(\chi_{Z(\mu)})\psi^{\otimes|\mu|}(\chi_{Z(\mu)})^*\big) \\ &= q(S_\mu S_\mu^*). \end{aligned} \tag{7.1}$$

Thus every $q\circ\pi(\chi_{Z(\mu)})$ belongs to $C^*(q(S_e),q(P_v))$, and $q\circ\pi(C(E^\infty))$ is contained in $C^*(q(S_e),q(P_v))$. Since $\chi_{Z(v)} = \sum_{r(e)=v}\chi_{Z(e)}$ in $C(E^\infty)$, the calculation (7.1) shows that $(q\circ S, q\circ P)$ is a Cuntz–Krieger family in $\mathcal{O}(X(E))$, and the induced homomorphism $\pi_{q\circ S,q\circ P}: C^*(E)\to\mathcal{O}(X(E^\infty))$ carries the action studied in [15] to the one we use here. This homomorphism intertwines the gauge actions, and an application of the gauge-invariant uniqueness theorem shows that $\pi_{q\circ S,q\circ P}$ is an isomorphism of $C^*(E)$ onto $\mathcal{O}(X(E^\infty))$. (The details are in [5, Theorem 5.1], modulo some scaling factors which come in because the inner product in [5] is defined using a transfer operator L which has been normalized so that $L(1)=1$ (see the discussion in [5, Sec. 9]). With our conventions, $L(1)$ would be the function $z\mapsto|\sigma^{-1}(z)|$. Theorem 5.1 of [5] extends an earlier theorem of Exel for Cuntz–Krieger algebras [9, Theorem 6.2].) □

Remark 7.2. While Proposition 7.1 implies that the Toeplitz algebra $\mathcal{T}(X(E^\infty))$ contains a faithful copy of $\mathcal{T}C^*(E)$, Corollary 7.5 implies that $\mathcal{T}(X(E^\infty))$ is substantially larger than $\mathcal{T}C^*(E)$: for example, there seems to be no way to get $\pi(\chi_{Z(\mu)})$ in $C^*(S,P)$.

Since the injections of Proposition 7.1 intertwine the gauge actions, they also intertwine the dynamics studied in [15] with those studied here (and there seems little danger in calling them all α). Thus applying our results to the local homeomorphism σ gives us KMS states on $(\mathcal{T}C^*(E),\alpha)$ and $(C^*(E),\alpha)$, and we should check that our results are compatible with those of [15].

When E is strongly connected, the system $(C^*(E),\alpha)$ has a unique KMS state, and its inverse temperature is the natural logarithm of the spectral radius $\rho(A)$ of the vertex matrix A of E [15, Theorem 4.3] (see also [8, 18]). So Theorem 6.1 implies that, for strongly connected E, our critical inverse temperature β_c must be $\ln\rho(A)$. Of course, we should be able to see this directly, and in fact it is true for all finite directed graphs. (The restriction to graphs with cycles in the next proposition merely excludes the trivial cases in which E^∞ is empty and $\rho(A)=0$.)

Proposition 7.3. *Suppose that E is a finite directed graph with at least one cycle. Let A denote the vertex matrix of E, and let σ denote the shift on the infinite-path space E^∞. Then*

$$\frac{1}{N}\ln\Big(\max_{z\in E^\infty}|\sigma^{-N}(z)|\Big)\to\ln\rho(A)\quad \textit{as } N\to\infty.$$

Proof. (Again, we thank the referee for providing this elegant proof.) We first claim that for any $E^0 \times E^0$ matrix B, the operator norm on $\ell^2(E^0)$ is bounded by

$$\max_{v,w\in E^0} |B(v,w)| \le \|B\| \le |E^0|^{3/2} \max_{v,w\in E^0} |B(v,w)|.$$

Indeed, the left-hand inequality is easy, and the right-hand one follows quickly from estimates using the inequalities $\|x\|_2 \le \|x\|_1 \le |E^0|^{1/2}\|x\|_2$ relating the ℓ^2 and ℓ^1 norms.

Now we use that

$$\max_{z\in E^\infty} |\sigma^{-N}(z)| = \max_{w\in E^0} \sum_{v\in E^0} A^N(v,w)$$

to estimate

$$\begin{aligned} |E^0|^{-3/2}\|A^N\| &\le \max_{v,w\in E^0} A^N(v,w) \le \max_{w\in E^0} \sum_{v\in E^0} A^N(v,w) \\ &= \max_{z\in E^\infty} |\sigma^{-N}(z)| \le |E^0| \max_{v,w\in E^0} A^N(v,w) \le |E^0|\,\|A^N\|. \end{aligned}$$

From this we get

$$\frac{1}{N}\ln\left(|E^0|^{-3/2}\|A^N\|\right) \le \frac{1}{N}\ln\left(\max_{z\in E^\infty} |\sigma^{-N}(z)|\right) \le \frac{1}{N}\ln\left(|E^0|\,\|A^N\|\right),$$

and the result follows from the spectral radius formula. □

Proposition 7.3 implies that, for the shifts σ on E^∞, the range $\beta > \beta_c$ in Theorem 5.1 is the same as the range $\beta > \ln\rho(A)$ in [15, Theorem 3.1]. When we view $\mathcal{T}C^*(E)$ as a C^*-subalgebra of $\mathcal{T}(X(E^\infty))$, restricting KMS states of $(\mathcal{T}(X(E^\infty)),\alpha)$ gives KMS states of $(\mathcal{T}C^*(E),\alpha)$ with the same inverse temperature. Since we know from [15, Theorem 3.1] exactly what the KMS states of $(\mathcal{T}C^*(E),\alpha)$ are, it is natural to ask which ones arise as the restrictions of states of $(\mathcal{T}(X(E^\infty)),\alpha)$.

We chose notation in Sec. 5 to emphasize the parallels with [15, Sec. 3], and hence we have a clash when we try to use both descriptions at the same time. So we write δ for the measure ε in Theorem 5.1, and keep ε for the vectors in $[1,\infty)^{E^0}$ appearing in [15, Theorem 3.1]. Otherwise we keep the notation of Theorem 5.1.

Proposition 7.4. *Suppose that E is a finite directed graph with at least one cycle, and A is the vertex matrix of E. Suppose that $\beta > \ln\rho(A)$, and that δ is a regular Borel measure on E^∞ satisfying $\int f_\beta\, d\delta = 1$. Define $\varepsilon = (\varepsilon_v) \in [0,\infty)^{E^0}$ by $\varepsilon_v = \delta(Z(v))$. Take $y = (y_v) \in [1,\infty)^{E^0}$ as in* [15, Theorem 3.1]. *Then $y\cdot\varepsilon = 1$, and the restriction of the KMS$_\beta$ state ϕ_δ of Theorem* 5.1 *to $(\mathcal{T}C^*(E),\alpha)$ is the state ϕ_ε of* [15, Theorem 3.1].

Proof. We begin by computing the function $f_\beta \in C(E^\infty)$. For $z \in E^\infty$, we have

$$\begin{aligned} f_\beta(z) &= \sum_{n=0}^{\infty} e^{-\beta n}|\sigma^{-n}(z)| = \sum_{n=0}^{\infty} e^{-\beta n}|E^n r(z)| \\ &= \sum_{n=0}^{\infty} e^{-\beta n}\left(\sum_{v\in E^0} |E^n v|\chi_{Z(v)}(z)\right). \end{aligned}$$

Since $y_v = \sum_{\mu\in E^*v} e^{-\beta|\mu|}$, an application of the monotone convergence theorem shows that

$$1 = \int f_\beta \, d\delta = \sum_{n=0}^{\infty} e^{-\beta n} \sum_{v\in E^0} |E^n v|\delta(Z(v)) = \sum_{v\in E^0} y_v\varepsilon_v = y\cdot\varepsilon. \tag{7.2}$$

To see that ϕ_δ restricts to ϕ_ε, it suffices to compute them both on elements $S_\lambda S_\nu^*$. Since $S_\lambda = \psi^{\otimes|\lambda|}(\chi_{Z(\lambda)})$ belongs to $X(E^\infty)^{\otimes|\lambda|}$, Eqs. (5.1) and [15, (3.1)] imply that $\phi_\delta(S_\lambda S_\nu^*) = 0 = \phi_\varepsilon(S_\lambda S_\nu^*)$ when $|\lambda| \neq |\nu|$. So we suppose $|\lambda| = |\nu| = n$, say. Then (5.1) implies that

$$\phi_\delta(S_\lambda S_\nu^*) = e^{-\beta n}\int \left\langle \chi_{Z(\nu)}, \chi_{Z(\lambda)}\right\rangle d\mu,$$

where $\mu = \sum_{k=0}^{\infty} e^{-\beta k}R^k\delta$. Viewing $X(E^\infty)^{\otimes n}$ as $X(F_n)$, as in Lemma 5.2, we can compute

$$\langle \chi_{Z(\nu)}, \chi_{Z(\lambda)}\rangle(z) = \sum_{\sigma^n(w)=z} \overline{\chi_{Z(\nu)}(w)}\chi_{Z(\lambda)}(w) = \delta_{\lambda,\nu}\sum_{\sigma^n(w)=z}\chi_{Z(\lambda)}(w),$$

and deduce that $\langle \chi_{Z(\nu)}, \chi_{Z(\lambda)}\rangle = \delta_{\lambda,\nu}\chi_{Z(s(\lambda))}$. Thus

$$\phi_\delta(S_\lambda S_\nu^*) = \delta_{\lambda,\nu}e^{-\beta n}\mu(Z(s(\lambda))). \tag{7.3}$$

So we want to compute $\mu(Z(v))$ for $v \in E^0$. For each k, we have

$$(R^k\delta)(Z(v)) = \int \chi_{Z(v)}\, d(R^k\delta)(z) = \int \sum_{\sigma^k(w)=z} \chi_{Z(v)}(w)\, d\delta(z).$$

We have

$$\sum_{\sigma^k(w)=z} \chi_{Z(v)}(w) = |vE^k r(z)| = A^k(v, r(z)) = \sum_{u\in E^0} A^k(v,u)\chi_{Z(u)}(z).$$

Thus

$$(R^k\delta)(Z(v)) = \int \sum_{u\in E^0} A^k(v,u)\chi_{Z(u)}\, d\delta = \sum_{u\in E^0} A^k(v,u)\delta(Z(u)),$$

and

$$\begin{aligned}\mu(Z(v)) &= \sum_{k=0}^{\infty} e^{-\beta k} \sum_{u\in E^0} A^k(v,u)\delta(Z(v)) \\ &= \sum_{k=0}^{\infty} e^{-\beta k}(A^n\varepsilon)_v = \big((1-e^{-\beta}A)^{-1}\varepsilon\big)_v.\end{aligned}$$

Now we go back to (7.3), and write down

$$\phi_\delta(S_\lambda S_\nu^*) = \delta_{\lambda,\nu} e^{-\beta n}\big((1-e^{-\beta}A)^{-1}\varepsilon\big)_{s(\lambda)}, \tag{7.4}$$

which in the notation of [15, Theorem 3.1(b)] is $\delta_{\lambda,\nu}e^{-\beta n}m_{s(\lambda)}$. It follows from this and [15, (3.1)] that $\phi_\delta(S_\lambda S_\nu^*) = \phi_\varepsilon(S_\lambda S_\nu^*)$, as required. $\square$

Proposition 7.4 implies that the system $(\mathcal{T}(X(E^\infty)),\alpha)$ has many more KMS states than $(\mathcal{T}C^*(E),\alpha)$.

Corollary 7.5. *Suppose that $\beta > \ln\rho(A)$, and that δ_1, δ_2 are regular Borel measures on E^∞ satisfying $\int f_\beta\, d\delta_i = 1$. Then $\phi_{\delta_1}|_{\mathcal{T}C^*(E)} = \phi_{\delta_2}|_{\mathcal{T}C^*(E)}$ if and only if $\delta_1(Z(v)) = \delta_2(Z(v))$ for all $v\in E^0$.*

Proof. Suppose that δ_1 and δ_2 are as described, and $\phi_{\delta_1}|_{\mathcal{T}C^*(E)} = \phi_{\delta_2}|_{\mathcal{T}C^*(E)}$. Then Proposition 7.4 implies that corresponding ε_i have $\phi_{\epsilon_1} = \phi_{\epsilon_2}$, and the injectivity of the map $\varepsilon \mapsto \phi_\varepsilon$ from [15, Theorem 3.1(c)] says that $\epsilon_1 = \epsilon_2$. But this says precisely that δ_1 and δ_2 agree on each $Z(v)$.

On the other hand, if $\delta_1(Z(v)) = \delta_2(Z(v))$ for all $v \in E^0$, then the corresponding ε_i are equal, and the formula (7.4) implies that ϕ_{δ_1} and ϕ_{δ_2} agree on $\mathcal{T}C^*(E)$. $\square$

Corollary 7.6. *Suppose that $\beta > \ln\rho(A)$. Then every KMS$_\beta$ state of $(\mathcal{T}C^*(E),\alpha)$ is the restriction of a KMS$_\beta$ state of $(\mathcal{T}(X(E^\infty)),\alpha)$.*

Proof. Suppose that ϕ is a KMS$_\beta$ state on $(\mathcal{T}C^*(E),\alpha)$. Then [15, Theorem 3.1] implies that there is a vector $\varepsilon \in [1,\infty)^{E^0}$ such that $y\cdot\varepsilon = 1$ and $\phi = \phi_\varepsilon$. If δ is a measure on E^∞ such that $\delta(Z(v)) = \varepsilon_v$ for all $v\in E^0$ and $\int f_\beta\, d\delta = 1$, then Proposition 7.4 implies that $\phi_\delta|_{\mathcal{T}C^*(E)} = \phi_\varepsilon$. So it suffices to show that there is such a measure δ.

We can construct measures on E^∞ by viewing it as an inverse limit $\varprojlim(E^n, r_n)$, where $r_n : E^{n+1} \to E^n$ takes $\nu = \nu_1\nu_2\cdots\nu_n\nu_{n+1}$ to $\nu_1\nu_2\cdots\nu_n$. Then any family of measures δ_n on E^n such that $\delta_{n+1}(Z(\nu)\cap E_{n+1}) = \delta_n(Z(\nu))$ for $|\nu| = n$ gives a measure δ on E^∞ such that $\delta(Z(\nu)) = \delta_n(Z(\nu))$ for $|\nu| = n$ (see, for example, [1, Lemma 6.1]). We can construct such a sequence by taking $\delta_0 = \varepsilon$, inductively choosing weights w_e such that $\sum_{r(e)=v} w_e = \varepsilon_v$, recursively choosing $\{w_{\nu e}\in[0,\infty) : \nu e \in E^{n+1}\}$ such that $\sum_{r(e)=s(\nu)} w_{\nu e} = w_\nu$, and setting $\delta_{n+1}(\nu e) = w_{\nu e}$. Now the calculation (7.2) shows that $\int f_\beta\, d\delta = y\cdot\varepsilon = 1$, and hence δ has the required properties. $\square$

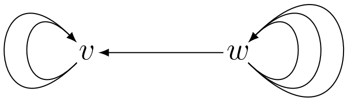
8. KMS States Below the Critical Inverse Temperature

In Remark 6.3, we showed that our critical inverse temperature β_c is the same as the one found by Thomsen [34]. He only considers states of the Cuntz–Pimsner system $(\mathcal{O}(X(E)), \bar{\alpha})$, and we agree that this system has no KMS_β states with $\beta > \beta_c$. However, he leaves open the possibility that there are KMS_β states with $\beta < \beta_c$. Indeed, he considers also the number

$$\beta_l := \limsup_{N\to\infty} \Big(N^{-1} \ln \Big(\min_{z\in Z} |h^{-N}(z)| \Big) \Big), \tag{8.1}$$

and then [34, Theorem 6.8] implies that the KMS states of $(\mathcal{O}(X(E^\infty)), \bar{\alpha})$ all have inverse temperatures in the interval $[\beta_l, \beta_c]$. Since $(\mathcal{O}(X(E^\infty)), \bar{\alpha}) = (C^*(E), \alpha)$, we can use examples from [16] to see that Thomsen's bounds are best possible.

More precisely, consider the dumbbell graphs



with m loops at vertex v and n loops at vertex w. (So in the above picture, we have $m = 2$ and $n = 3$. This graph was discussed in [16, Example 6.2], and the one with $m = 3$ and $n = 2$ in [16, Example 6.1].) The vertex matrix A of such a graph E is upper triangular and has spectrum $\{m, n\}$. For $m \geq n$, the system $(C^*(E), \alpha)$ has a single $\mathrm{KMS}_{\ln m}$ state, and this is the only KMS state.

Now we suppose that $m < n$. Then $\rho(A) = n$, and $(C^*(E), \alpha)$ has two KMS states. The first is denoted by $\psi_{\{w\}}$ in [16], and has inverse temperature $\ln n$. The second factors through the quotient map of $C^*(E)$ onto the C^*-algebra of the graph with vertex v and m loops, which is a Cuntz–algebra $\mathcal{O}_m$. It has inverse temperature $\ln m$. For this graph, we have $\beta_c = \ln \rho(A) = \ln n$. To compute β_l, we let $z \in E^\infty$. Then

$$|\sigma^{-N}(z)| = |E^N r(z)| = \begin{cases} m^N & \text{if } r(z) = v, \\ n^N + \displaystyle\sum_{j=0}^{N-1} n^j m^{N-1-j} & \text{if } r(z) = w. \end{cases}$$

Since $m < n$, the minimum is attained when $r(z) = v$, and $\min_{z\in E^\infty} |\sigma^{-N}(z)| = m^n$, giving $\beta_l = \ln m$. Thus for this graph, the possible inverse temperatures are precisely the end-points of Thomsen's interval.

Remark 8.1. By adding appropriate strongly connected components between w and v in this last example, we can construct examples for which there are KMS states with inverse temperatures between β_l and β_c. However, there are number-theoretic constraints on the possible inverse temperatures (see [28], [16, Sec. 7]).

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