A Composite Fermion Approach to the Ultracold Dilute Fermi Gas

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It is argued that the recently observed Fermi liquids in strongly interacting ultracold Fermi gases are adiabatically connected to a projected Fermi gas. This conclusion is reached by constructing a set of Jastrow wavefunctions, following Tan's observations on the structure of the physical Hilbert space [Annals of Physics **323**, 2952 (2008)]. The Jastrow projection merely implements the Bethe-Peierls condition on the BCS and Fermi gas wavefunctions. This procedure provides a simple picture of the emergence of Fermi polarons as composite fermions in the normal state of the highly polarized gas. It is also shown that the projected BCS wavefunction can be written as a condensate of pairs of composite fermions (or Fermi polarons). A Hamiltonian for the composite fermions is derived. Within a mean-field theory, it is shown that the ground state and excitations of this Hamiltonian are those of a non-interacting Fermi gas although they are described by Jastrow-Slater wavefunctions.

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I. INTRODUCTION

Recently, several groups [1–3] have performed accurate measurements of the thermodynamics in the unitary regime of an ultracold dilute Fermi gas. The measurements have been found consistent with Ho's [4] universal thermodynamics hypothesis. However, the group at ENS [2] also found that the equation of state of the *unpolarized* unitary gas can be fitted using Fermi liquid theory (FLT) in a temperature range from $T_c/\mu \simeq 0.32$ to $T/\mu \simeq 0.8$ (T_c being the transition temperature to the superfluid state and μ the chemical potential). Based on this analysis, the ENS group has claimed that the normal state of the unitary Fermi gas is a *weakly* correlated Fermi liquid [2]. These results appear to be in contradiction with the recent spectroscopic observations of the JILA group [5]. The latter support the existence of a pseudogap regime [7], in agreement with a number of theoretical claims [7, 9] (but not others [10]). Besides the unpolarized gas, Fermi liquid behavior has also been predicted [11], and observed in a much less controversial way [2, 12, 13], in the normal phase of the highly polarized Fermi gas. In a large region of the phase diagram, this polaron liquid also seems to behave as a weakly correlated Fermi liquid.

Theoretically, it is not clear at all how a weakly correlated Fermi liquid can emerge in the unitary limit above the critical temperature. At least naïvely, this Fermi liquid cannot be adiabatically connected to the Hartree-Fock solution at weak coupling. The reason is that the ground state and excitations of the latter are Slater determinants. Such wavefunctions do not obey the Bethe-Peierls condition, which, as emphasized recently by Tan [22] (see also [26]), defines the *physical* Hilbert space of the system. Whether this condition can be fully implemented within diagrammatic perturbation theory is also unclear, and may be one of the reasons why some theories fail to reproduce the experiments [2][45]. So far, several approaches have attempted to provide a quantitative description of the normal phase through sophisticated diagrammatic calculations (see *e.g.* Refs. [7, 9, 10, 15–19] and references therein). Although these analyses yield a wealth of quantitative predictions, in the present author's opinion, the picture provided by them is usually not very transparent. On the other hand, the picture in terms of wavefunctions provided by variational methods [29, 31, 32] as well as fixed-node diffusion Monte Carlo [33, 39] is physically more transparent. However, these methods mostly deal with the ground state [46], and therefore *a priori* they cannot be applied to understand the properties of the normal state at finite temperature.

Thus, it would desirable to have clearer theoretical picture of the emergence of Fermi liquid behavior, especially in the crossover region. Here it is argued that a more transparent (although at this point less quantitative) picture can be obtained by incorporating from the scratch Tan's observation [22] about the *physical* Hilbert space of the system. The latter is defined as the set of states obeying the Bethe-Peierls condition (see Eq. (4) below). By projecting the BCS wavefunction onto this space by means of a Jastrow factor, it is shown below that it can be written as a condensate of pairs of composite fermions. These composite fermions are related to the Fermi polarons introduced by Lobo et al. [11] to describe the normal state of the highly polarized gas. We show that, in general, the composite fermions (or polarons) are the elementary excitations of a projected Fermi gas. They correspond to the original fermions dressed by a density cloud of opposite-spin fermions. By deriving a Hamiltonian that acts on unprojected states, it is shown that the Jastrow-Slater wavefunctions are the eigenstates of a mean-field approximation to such Hamiltonian. Furthermore, it is found that, at the mean-field level, the polarons behave as non-interacting Fermi gas for all values of the scattering length and spin polarization. The interactions between them are thus described by corrections to the mean-field theory. Nevertheless, although it may be coincidental, we argue that this mean-field theory may provide a qualitative explanation for the emergence of a weakly correlated Fermi liquid in the crossover regime. Furthermore, these results also reveal interesting connections of this problem with the theory of the fractional quantum Hall effect and the physics of the Gutzwiller projection of Hubbard models [36, 37, 43].

The rest of the article is organized as follows: In the next section, we briefly review the mounting experimental and theoretical evidence for Fermi liquid behavior in strongly interacting dilute Fermi gases. The properties of the superfluid grond state are discussed in Sect III in terms of a projected BCS wavefunction. We use the structure of this wavefunction to motivate the introduction of a set of composite Fermi fields, which as shown in Sect. IV are related to the Fermi polarons. In the same section, the normal state is considered and we argue that it has the form of a Jastrow-Slater determinant. In Sect. V, this state is derived from a mean-field theory. Finally, in section VI we discuss some possible extensions to this work and provide our conclusions.

II. BRIEF REVIEW OF THE RELEVANT EXPERIMENTS AND THEORY

The findings of the ENS group described in the Introduction appear to be in contradiction with the recent angle-resolved photoemission spectroscopy measurements (ARPES) of JILA group [5]. Performing ARPES [6] above T_c and near unitarity, this group observed a prominent BCS-like feature in the spectral function, which shows a downturn consistent with a large gap in the single-particle excitation spectrum, for $k \simeq k_F$ (k_F is the Fermi wavenumber). This was observed at temperatures where essentially no condensed pairs pairs were found using a different technique. These experimental results, together with some numerical evidence [14], come in support of the theories [7, 9] claiming that the system enters a pseudogap regime above T_c. According to these theories, such a regime is characterized by the existence of preformed Cooper pairs, which have not yet undergone Bose-Einstein condensation. Because the electrons remain paired even above T_c, the density of states at the Fermi surface should be strongly suppressed. Therefore, it is at least naïvely expected that the existence of the preformed pairs should be also reflected in the thermodynamics as deviations from FLT [21], as observed in the hole-doped cuprate materials [20], which also exhibit a mysterious pseudogap phase [8].

On the other hand, if the normal phase of the unpolarized gas is a *standard* Fermi liquid, the spectroscopic properties near the Fermi surface must be accounted for by the Landau quasi-particle (LQP) picture. In the framework of FLT, the existence of LQPs is related to the existence of a set of (approximate) eigenstates, $|\mathrm{qp}(N+1,\mathbf{k},\sigma)\rangle$, with total particle number $N+1$ and carrying momentum $\hbar\mathbf{k}$ and spin projection σ, such that:

$$\lim_{|\mathbf{k}|\to k_F} |\langle\Psi_0(N)|\psi_\sigma(\mathbf{k})|\mathrm{qp}(N+1,\mathbf{k},\sigma)\rangle|^2 = Z \qquad (1)$$

where $|\Psi_0(N)\rangle$ is the ground state of the system containing N particles, $\psi^\dagger_\sigma(\mathbf{k})$ is the Fermi operator that creates an atom carrying momentum $\hbar\mathbf{k}$ and spin σ, and the real number Z ($0 < Z \leq 1$) is the quasi-particle weight. Slightly away from the Fermi surface, the LQPs acquire a finite lifetime, which grows with the minimun of T^{-2} or $(\varepsilon-\mu)^{-2}$, where ε is the quasi-particle excitation energy.

The ENS group found the effective mass of the quasiparticles (m^*) in the upolarized gas to be close to the *bare* atomic mass (m), $m^* \simeq 1.13\,m$. Assuming that we are dealing with LQPs and taking into account that the quasi-particle weight $Z \sim m/m^*$, it follows that $Z \sim 1$. Thus, the density of states at the Fermi energy, which is proportional to Z, should be rather close to that of the non-interaccting gas in spite of the strong interactions characteristic of the unitary regime. In the case of ^{3}He, which is not a dilute system but where the atoms also interact via a short range potential, the LQP effective mass is such that $2.9 \leq m^*/m < 5.7$, depending on the pressure. Therefore, when compared to ^{3}He, the unpolarized gas at unitarity appears to be a weakly correlated Fermi liquid. On the other hand, the downturn observed by the JILA group would be consistent with a strong reduction of the density of states near the Fermi surface or pseudogap, which seems hard to fit into the picture put forward by the ENS group.

Some clues to better understand the puzzling experimental situation described above may come from recent lattice Monte Carlo calculations [21]. This method (unlike diffusion Monte Carlo) can access finite temperatures, although calculations become more cumbersome as $T = 0$ is approached. Overcoming these difficulties, Bulgac and coworkers [21] computed the thermodynamic properties of the gas as a function of the temperature in the crossover regime (for $-2 < k_F a_s < 0.5$, a_s being the s-wave scattering length) down to $T \simeq 0.1\,T_F$ (where T_F is the Fermi temperature of a non-interacting gas of the same density). At unitarity, it was found that the unpolarized unitary gas exhibits free Fermi gas-like behavior for $T \gtrsim 0.23T_F$. Moreover, in a narrow range, between $T_c \lesssim T \lesssim 0.23T_F$ ($T_c \simeq 0.15T_F$), the system behaves neither as a Fermi liquid nor as a superfluid. Below T_c, the energy behaves as that of a superfluid, which is accounted for by the joint contributions of the Anderson-Bogoliubov mode and the Bogoliubov quasi-particles. These numerical results are in good agreement with the data obtained by the ENS group for $T/\mu \gtrsim 0.4$ [2], which are also the most accurately fitted by FLT. Therefore, the results are also consistent with the scenario (in qualitative agreement with some diagrammatic approaches [9, 10]) where the pseudogap regime exists (if at all) only in a temperature window that is rather narrow even at unitarity (but may become broader on the BEC side of the crossover).

Above the upper limit of this window, the system would exhibit FLT-like thermodynamics.

In the case of a strongly polarized Fermi gas at unitarity, the Fermi liquid state can be understood as a liquid of Fermi 'polarons'. The latter are the result of dressing the fermions of the minority (say, $\uparrow$) spin component with a cloud of fermions in the majority spin ($\downarrow$) component. At unitarity, it is found [11, 29–31] that the Fermi polaron has the following dispersion $\epsilon^{p}_{\uparrow}(\mathbf{k}) = E_b + \frac{\hbar^2 \mathbf{k}^2}{2m^*}$, where $m^* \simeq 1.17\, m$ and $E_b/E_{F\downarrow} \simeq -0.64$, where $E_{F\downarrow}$ is the Fermi energy of the majority component, as determined experimentally in Ref. [12] ($m^* \simeq 1.20\, m$ according to [2]). In this case again, the effective mass of the polaron seems to be rather close to the free atom mass, in spite of the strong attractive interactions at unitarity. Further calculations using a T-matrix approximation also found small deviations ($\sim 50\%$ at most) from the bare mass for $-1 < (k_F a_s)^{-1} \lesssim 0.5$ [32]. Moreover, experimentally the ENS group found that the interactions between the polarons in the polarized gas appear to be negligible at unitarity [2].

III. THE SUPERFLUID STATE

In the theory of the BCS-BEC crossover, it is believed that the major features of the ground state are captured by the BCS wavefunction [24–26]:

$$|\Phi_{\rm BCS}\rangle = \left[\sum_{\mathbf{k}} \varphi(\mathbf{k}) \psi^{\dagger}_{\uparrow}(\mathbf{k}) \psi^{\dagger}_{\downarrow}(-\mathbf{k})\right]^{N/2} \quad (2)$$

$$= \left[\int d\mathbf{x} d\mathbf{y}\, \varphi(\mathbf{x}-\mathbf{y}) \psi^{\dagger}_{\uparrow}(\mathbf{x}) \psi^{\dagger}_{\downarrow}(\mathbf{y})\right]^{N/2} |0\rangle. \quad (3)$$

According to this theory, on the BCS side of the crossover (where the scattering length $a_s < 0$), atoms form Cooper pairs described by the pair wavefunction $\varphi(\mathbf{r}) = \frac{1}{\Omega}\sum_{\mathbf{k}} \varphi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}$ (here Ω is the volume of the system). The pairs form a Bose-Einstein condensate in which the center of mass state at $\mathbf{k} = \mathbf{0}$ is macroscopically ocupied [47]. As the scattering length a_s is tuned by means of a Feshbach resonance towards unitarity (where $k_F a_s \to -\infty$), the size of the pairs shrinks until a two-body bound state (a molecular dimer) forms. Further on, as a_s continues to grow positive, the dimers become more tightly bound on the BEC side of the crossover.

In an important development in the theory of the BCS-BEC crossover, Tan [22] has recently emphasized (see also [26]) the importance of the Bethe-Peierls condition (BPC) in defining the *physical* Hilbert space of a *dilute* ultracold Fermi gas. This condition states that the s-wave interactions, which are also described in terms of the Lee-Huang-Yang pseudo-potential [27] $V_{\rm int}(\mathbf{r}) = \frac{4\pi\hbar^2 a_s}{m}\delta(\mathbf{r})\partial_r\,(r\cdot)$, can be replaced by a boundary condition in the limit where the range of the potential goes to zero (*i.e.* $r_0 \to 0$). For the many-particle states, this means that, provided three and higher-body interactions can be neglected (that is, the gas is dilute and Efimov bound states are not present), any physical state of the system containing $N_{\uparrow}$ spin-up fermions and $N_{\downarrow}$ spin-down fermions exhibits the following behavior:

$$\lim_{|\mathbf{x}_i - \mathbf{y}_j| \to 0} \Psi_{\rm phys}(\{\mathbf{x}_k\}_{k=1}^{N_\uparrow}; \{\mathbf{y}_l\}_{l=1}^{N_\downarrow}) = \left(\frac{1}{|\mathbf{x}_i - \mathbf{y}_j|} - \frac{1}{a}\right) \times \tilde{\Psi}(\frac{\mathbf{x}_i + \mathbf{y}_j}{2}, \{\mathbf{x}_k\}_{k\neq i}, \{\mathbf{y}_l\}_{l\neq j}), \quad (4)$$

for any pair of opposite spin particles ($i = 1, \ldots, N_{\uparrow}$ and $j = 1, \ldots, N_{\downarrow}$). As Tan discovered, the BPC has important consequences for the short-distance correlation functions and the total energy of the states. In particular, he predicted [22] that the momentum distribution of any physical state Ψ behaves as $C_{\Psi}\, k^{-4}$ at large wavenumber, k, where Tan's contact $C_{\Psi} = \langle \Psi | \rho_{\uparrow}(\mathbf{r}) \rho_{\downarrow}(\mathbf{r}) | \Psi \rangle$. This prediction has been recently verified by the JILA group [23].

Tan also pointed out [22] that the BCS wavefunction, Eqs. (2,3), does not satisfy the BPC and therefore it is not in the physical space. This is so even if the behavior of the pair wavefunction, $\varphi(\mathbf{r})$, matches the solution of the two-body problem as $|\mathbf{r}| \to 0$. Indeed, by using the determinantal representation of the BCS wavefunction (cf. Eq. 8), it can be shown that the many-body wavefunction constructed from such a $\varphi(\mathbf{r})$ cannot be written as in Eq. (4), where $\tilde{\Psi}$ depends only on $(\mathbf{x}_i + \mathbf{y}_j)/2$ and the coordinates of the other particles but not on $\mathbf{x}_i - \mathbf{y}_j$. The BPC is also often imposed on the anomalous correlator [25], by requiring that

$$\langle \psi_{\uparrow}(\mathbf{k}) \psi_{\downarrow}(-\mathbf{k}) \rangle \sim \frac{4\pi}{k^2}, \quad (5)$$

for $k \gg k_F$ ($\frac{4\pi}{k^2}$ is the Fourier transform of the term $\frac{1}{r}$ in the BPC). This is akin to imposing the condition on an expectation value, which is a necessary but not sufficient requirement for Eq. (4) to hold. In fact, when the expectation value is taken over the BCS state, this is equivalent to imposing the condition on the pair wavefunction $\varphi(\mathbf{k})$ because $\langle \psi_{\uparrow}(\mathbf{k}) \psi_{\downarrow}(-\mathbf{k}) \rangle = \varphi(\mathbf{k})/(1 + |\varphi(\mathbf{k})|^2)$.

In spite of the shortcoming pointed by Tan, the BCS wavefunction still captures the long-distance correlations of particles in the ground state, as discussed above. In a sense, it is the simplest ansatz that describes a condensate of fermion pairs that turn into molecular dimers as the interaction is tuned across the Feshbach resonance. Therefore, we shall adopt the simple-minded approach that the shortcoming of BCS wavefunction can be fixed by projecting it onto the physical Hilbert space. Indeed, it has been recently shown [28] that, in the quantum Monte Carlo method, a good measure of the quality of a trial wavefunction is its overlap with the ground state. Since the latter is certainly in the physical Hilbert space, projection simply gets rid of the undesired components of the trial wavefunction that decrease the overlap and increase the variational energy. Thus, on a formal level, we can define a projector, $\mathcal{J}_{\rm BPC}(a_s)$, onto the space of states obeying Eq. (4) for a given value of the scattering length, a_s, so that, a variational sense, we choose to work with:

$$|\Psi_{\rm BCS}\rangle = \mathcal{J}_{\rm BPC}(a_s)|\Phi_{\rm BCS}\rangle. \quad (6)$$

Practically, the projection can be implemented in various ways, and the result should be independent of the details provided it is optimal in a variational sense. One method that is particularly popular in fixed-node Monte Carlo calculations is a Jastrow factor [33, 39]. Following those approaches, we write the wavefunction as (in what follows we collectively denote by $\mathbf{X} = \{\mathbf{x}_k\}$ the coordinates of the spin-up fermions and by $\mathbf{Y} = \{\mathbf{y}_l\}$ those of the spin-down fermions):

$$\Psi_{\rm BCS}(\mathbf{X},\mathbf{Y}) = \prod_{i,j=1}^{N/2} \chi_J(\mathbf{x}_i - \mathbf{y}_j)\, \Phi_{\rm BCS}(\mathbf{X},\mathbf{Y}), \quad (7)$$

$$\Phi_{\rm BCS}(\mathbf{X},\mathbf{Y}) = \det\left[\varphi(\mathbf{x}_i - \mathbf{y}_j)\right], \quad (8)$$

where we have written the unprojected BCS function, $\Phi_{\rm BCS}$ as a determinant [34] (for $N_\uparrow = N_\downarrow = N/2$). The above wavefunction should not be regarded as optimal as far as the calculation of the ground state energy is concerned, in so much as $\Phi_{\rm BCS}$ is also not the best (unprojected) variational state and many improvements are possible, such additional Jastrow factors, Feynman backflow, etc [33]. However, this wavefunction is sufficiently simple for the purpose of discussing the change of picture brought about by the projection. In order to fullfil (4) we demand that $\chi_J(\mathbf{r}) \sim \left(\frac{1}{r} - \frac{1}{a} + O(r)\right)$ and $\varphi(\mathbf{r}) = \text{const.} + O(r^2)$ for $r_0 \ll r \ll \min\left\{a_s, k_F^{-1}\right\}$. The Jastrow is only required to correct the short-distance behavior of the wavefunction, and therefore $\chi_J(\mathbf{r}) \to 1$ for $r \gg k_F^{-1}$. The detailed behavior of $\chi_J(\mathbf{r})$ between these two asymptotic limits must be determined variationally and it depends on the gas parameter $k_F a_s$ and the spin polarization. Moreover, since $\chi_J(\mathbf{r})$ is a zero-energy solution of the two-body problem for $r_0 \to 0$ (see next section), we expect that, deep into the BCS and BEC regime where $|k_F a_s| \lesssim 1$, it strongly deviates from unity only in rather small region, $r \lesssim |a_s|$.

Next we note that, by introducing $J_i(\mathbf{x}_i;\mathbf{X}) = \prod_{j=1}^{N/2} \chi_J(\mathbf{x}_i - \mathbf{y}_j)$ and $J_j(\mathbf{Y};\mathbf{y}_j) = \prod_{i=1}^{N/2} \chi_J(\mathbf{x}_i - \mathbf{y}_j)$, it is possible to write the projected BCS state (7) as a determinant:

$$\Psi_{\rm BCS}(\mathbf{X},\mathbf{Y}) = \det\left[\varphi(\mathbf{x}_i - \mathbf{y}_j) J_i(\mathbf{x}_i;\mathbf{Y})\right] \quad (9)$$

$$= \det\left[\varphi(\mathbf{x}_i - \mathbf{y}_j) J_j(\mathbf{X};\mathbf{y}_j)\right] \quad (10)$$

$$= \det\left[\varphi(\mathbf{x}_i - \mathbf{y}_j) J_i^{1/2}(\mathbf{x}_i;\mathbf{Y}) J_j^{1/2}(\mathbf{X};\mathbf{y}_j)\right] \quad (11)$$

Therefore, we can regard the above wavefunction again as a BCS state, not of the original fermions, but of a system of 'composite fermions' instead, which are defined by attaching to each fermion a Jastrow factor. This can be seen more clearly by working in second quantization. By analogy to the theory of composite particles in the fractional quantum Hall effect [37, 38], we introduce two (hermitian) operators $U_\sigma(\mathbf{r})$ ($\sigma = \uparrow,\downarrow$) having the following property $U_\sigma(\mathbf{r})\psi^\dagger_{-\sigma}(\mathbf{s}) = \chi_J(\mathbf{r}-\mathbf{s})\psi^\dagger_{-\sigma}(\mathbf{s})U_\sigma(\mathbf{r})$, but otherwise commuting. Indeed, provided that $\chi_J(\mathbf{r})$ is a positive function, an explicit construction of these operators is $U_\sigma(\mathbf{r}) = \exp\left[\int d\mathbf{r}' \ln \chi_J\left(\mathbf{r}-\mathbf{r}'\right)\rho_{-\sigma}(\mathbf{r}')\right]$. Thus, if we define the following quasi-particle operator $\pi^\dagger_\sigma(\mathbf{r}) = \psi^\dagger_\sigma(\mathbf{r})U_\sigma(\mathbf{r})$, the above state can be written as a condensate of pairs of these quasi-particles:

$$|\Psi_{\rm BCS}\rangle = \left[\int d\mathbf{x}d\mathbf{y}\, \varphi(\mathbf{x}-\mathbf{y})\pi^\dagger_\uparrow(\mathbf{x})\pi^\dagger_\downarrow(\mathbf{y})\right]^{N/2} |0\rangle, \quad (12)$$

$$= \left[\sum_{\mathbf{k}} \varphi(\mathbf{k})\pi^\dagger_\uparrow(\mathbf{k})\pi^\dagger_\downarrow(-\mathbf{k})\right]^{N/2} |0\rangle. \quad (13)$$

It is also possible to define a quasi-hole operator $\eta_\sigma(\mathbf{r}) = U^{-1}_\sigma(\mathbf{r})\psi_\sigma(\mathbf{r})$. It can be shown that $\left\{\pi^\dagger_\sigma(\mathbf{r}), \pi^\dagger_{\sigma'}(\mathbf{r}')\right\} = \{\eta_\sigma(\mathbf{r}), \eta_{\sigma'}(\mathbf{r}')\} = 0$ and $\left\{\eta_\sigma(\mathbf{r}), \pi^\dagger_{\sigma'}(\mathbf{r}')\right\} = \delta_{\sigma,\sigma'}\delta(\mathbf{r}-\mathbf{r}')$. However, note that $\pi^\dagger_\sigma(\mathbf{r}) \neq [\eta_\sigma(\mathbf{r})]^\dagger$. Therefore, these operators should be handled with care as they result from a non-unitary transformation and the states created by them may turn out to be non-orthogonal (see Sect. IV for further discussion).

The wavefunction in (7) can be regarded as a simple model for the ground state of the superfluid state. The Jastrow factor introduces attractive correlations between fermions in different pairs because, at short distances, $\chi_J(\mathbf{r})$ matches the solution of the two-body problem which describes two-particle attraction for $a_s < 0$ and a shallow bound state for $a_s > 0$. These correlations can be regarded as a 'pairing frustration' mechanism. The mechanism is less effective deep into the BCS and BEC regimes, where $\chi_J(\mathbf{r})$ is short-ranged in the sense that it strongly deviates from unity only for $r \lesssim |a_s|$. Therefore, the projected wave function will have a large overlap (at finite N) with the unprojected BCS wavefunction. Nevertheless, in the deep BEC regime, the short range Jastrow correlations describe the interactions between the molecular dimers. On the other hand, in the crossover regime where $|k_F a_s| \gg 1$, the size of the pairs and the range of the Jastrow are both comparable to k_F^{-1}, the mean inter-particle distance. Therefore, pairing will be strongly frustrated, which will lead to a reduction of the superfluid density from the unprojected BCS state. As to the fermionic excitations (Bogoliubov particles), they can be obtained by projecting the Bogoliubov excitations of the unprojected state $|\Phi_{\rm BCS}\rangle$. However, since we are interested in the normal state, we shall not pursue this analysis here.

IV. THE NORMAL STATE

When obtained by variationally minimizing the energy of $|\Psi_{BCS}\rangle = \mathcal{J}_{\rm BPC}(a_s)|\Phi_{\rm BPC}\rangle$, the gap Δ that parametrizes the unprojected BCS state, $|\Phi_{\rm BCS}\rangle$, is a function of a_s, the mean density, and the spin polarization measured by *e.g.* $P = |N_\uparrow - N_\downarrow|$. Nevertheless, let us for a while consider Δ as independent of a_s and imagine that the gap collapses (*i.e.* $\Delta \to 0$) while keeping a_s constant. Physically, the collapse can be caused either by thermal fluctuations or by making P sufficiently large [48]. Naïvely, the state that results from such a limit would be $|\Phi_{\rm FS}\rangle = \prod_{|\mathbf{k}|<k_{F\uparrow}} \psi^\dagger_{\mathbf{k}\uparrow} \prod_{|\mathbf{k}|<k_{F\downarrow}} \psi^\dagger_{\mathbf{k}\downarrow}|0\rangle$. However, this state is unphysical because it does *not* satisfy the BPC, Eq. (4), for the given value of a_s. The correct state is $|\Psi_{\rm N}\rangle = \mathcal{J}_{\rm BPC}(a_s)|\Phi_{\rm FS}\rangle$. In terms of the quasi-particle operators, $\pi^\dagger_\sigma(\mathbf{k})$, the projected normal state reads:

$$|\Psi_{\rm N}\rangle = \prod_{|\mathbf{k}|<k_{F\uparrow}} \pi^\dagger_\uparrow(\mathbf{k}) \prod_{|\mathbf{k}|<k_{F\downarrow}} \pi^\dagger_\downarrow(\mathbf{k})|0\rangle. \quad (14)$$

In coordinate representation this state is the following Jastrow-Slater wavefunction ($|\mathbf{k}_\alpha| < k_{F\uparrow}, |\mathbf{k}_\beta| < k_{F\downarrow}$):

$$\Psi_{\rm N} = \prod_{i,j} \chi_J(\mathbf{x}_i - \mathbf{y}_j) \det\left[\phi_{\mathbf{k}_\alpha}(\mathbf{x}_i)\right] \det\left[\phi_{\mathbf{k}_\beta}(\mathbf{y}_j)\right]. \quad (15)$$

From this form, it can be shown that $\pi^\dagger_\sigma(\mathbf{k})|\Psi_{\rm N}\rangle = 0$ if $|\mathbf{k}| < k_{F\sigma}$ and $\eta_\sigma(\mathbf{k})|\Psi_{\rm N}\rangle = 0$ if $|\mathbf{k}| > k_{F\sigma}$. In words, the state has a well defined Fermi surface.

In order to obtain a better insight into the excitations created by the quasi-particle operator $\pi_\sigma(\mathbf{k})$, let us focus on a state describing one spin-up fermion 'impurity' in a Fermi sea of $N_\downarrow$ fermions [49]:

$$\langle \mathbf{x}, \mathbf{Y}|\pi^\dagger_\uparrow(\mathbf{x}) \prod_{|\mathbf{k}_\alpha|<k_{F\downarrow}} \pi^\dagger_\downarrow(\mathbf{k}_\alpha)|0\rangle = \langle \mathbf{x}, \mathbf{Y}|\psi^\dagger_\uparrow(\mathbf{x}) U_\uparrow(\mathbf{x})$$
$$\times \prod_{|\mathbf{k}_\alpha|<k_{F\downarrow}} \psi^\dagger_\downarrow(\mathbf{k}_\alpha)|0\rangle = \prod_{j=1}^{N_\downarrow} \chi_J(\mathbf{x} - \mathbf{y}_i) \det\left[\phi_{\mathbf{k}_\alpha}(\mathbf{y}_j)\right], \quad (16)$$

where $\phi_{\mathbf{k}}(\mathbf{r}) = \Omega^{-1/2} e^{i\mathbf{k}\cdot\mathbf{r}}$ and $\langle \mathbf{x}, \mathbf{Y}| = \langle 0|\psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y}_1)\cdots\psi_\downarrow(\mathbf{y}_M)$. The above state describes a 'Fermi polaron' and it was first introduced in the analysis of the normal state of the highly spin-polarized Fermi gas at unitarity [11]. Indeed, by expanding the exponent of $U_\uparrow(\mathbf{x}) = \exp\left[\int d\mathbf{z} \ln \chi_J(\mathbf{x} - \mathbf{z})\rho_\downarrow(\mathbf{z})\right]$ to lowest order, the following state is obtained ($|\Phi^\downarrow_{FS}\rangle = \prod_{|\mathbf{k}_\alpha|<k_{F\downarrow}} \psi^\dagger_\downarrow(\mathbf{k}_\alpha)|0\rangle$, $|\mathbf{k}(\uparrow)\rangle = \psi^\dagger_\uparrow(\mathbf{k})|0\rangle$, $f_J(\mathbf{q}) = \int d\mathbf{s}\, \ln \chi_J(\mathbf{s})\, e^{-i\mathbf{q}\cdot\mathbf{r}}$):

$$|\pi(\mathbf{k}, \uparrow)\rangle \simeq |\mathbf{k}(\uparrow)\rangle|\Phi^\downarrow_{\rm FS}\rangle + \frac{1}{\Omega}\sum_{\mathbf{pq}} f_J(\mathbf{q})\, |\mathbf{k} - \mathbf{q}(\uparrow)\rangle$$
$$\times \psi^\dagger_\downarrow(\mathbf{p} + \mathbf{q})\psi_\downarrow(\mathbf{p})|\Phi^\downarrow_{\rm FS}\rangle, \quad (17)$$

which has been employed in simple variational approaches to the problem [29–31]. The above equations show that the composite fermionic excitations described by $\pi_\sigma(\mathbf{k})$ correspond to the original fermions dressed by a cloud of opposite spin fermions.

In general, for arbitrary numbers $N_\uparrow$ and $N_\downarrow$, the actual energy of $|\Psi_N\rangle$ must be obtained by variationally optimizing the Jastrow projector so that

$$E^N_0 = \frac{\langle \Psi_N|H|\Psi_N\rangle}{\langle \Psi_N|\Psi_N\rangle} \quad (18)$$
$$= \frac{\langle \Phi_{\rm FS}|\mathcal{J}_{\rm BPC}(a_s) H \mathcal{J}_{\rm BPC}(a_s)|\Phi_{\rm FS}\rangle}{\langle \Phi_{\rm FS}|\mathcal{J}^2_{\rm BPC}(a_s)|\Phi_{\rm FS}\rangle} \quad (19)$$

is minimum. For $N_\uparrow = N_\downarrow$, using the Jastrow-Slater state Ψ_N as the trial wavefunction of a fixed-node diffusion Monte Carlo calculation, it was found in Ref. [11] that the energy per particle is about 30% higher than the energy of the superfluid state. This shows that the normal state has a superfluid instability. The latter can be described as condensation of pairs of composite fermions, according to the discussion in Sect. III.

The excitation energy of a quasi-particle state is obtained by evaluating:

$$\epsilon^p_\sigma(\mathbf{k}) + E^N_0 = \frac{\langle \Psi_N|\pi_\sigma(\mathbf{k}) H \pi^\dagger_\sigma(\mathbf{k})|\Psi_N\rangle}{\langle \Psi_N|\pi_\sigma(\mathbf{k})\pi^\dagger_\sigma(\mathbf{k})|\Psi_N\rangle}$$
$$= \frac{\langle \Phi_{\rm FS}|\psi_\sigma(\mathbf{k})\mathcal{J}_{\rm BPC}(a_s) H \mathcal{J}_{\rm BPC}(a_s)\psi^\dagger_\sigma(\mathbf{k})|\Phi_{\rm FS}\rangle}{\langle \Phi_{\rm FS}|\psi_\sigma(\mathbf{k})\mathcal{J}^2_{\rm BPC}(a_s)\psi^\dagger_\sigma(\mathbf{k})|\Phi_{\rm FS}\rangle}, \quad (20)$$

(for quasi-hole states, $\pi^\dagger_\sigma(\mathbf{k})$ should be replaced by $\eta_\sigma(\mathbf{k})$). Note that the Jastrow factor in this calculation should be built from the same $\chi_J(\mathbf{r})$ function as for the ground state. The logic behind such an approximation is that the ground state correlations are not strongly modified by the creation of a few low-energy excitations.

V. MEAN FIELD THEORY APPROACH

Although it is possible to deal with the composite fermions (CFs) numerically using the Jastrow-Slater wavefunctions introduced above, it can be also useful to derive an effective Hamiltonian that describes them. Indeed, we show in the following that the normal state $|\Psi_{\rm N}\rangle$ is the ground state of a certain mean-field (MF) Hamiltonian. Let us first derive the Hamiltonian for the unprojected states by considering the many-body Schrödinger equation for the physical states:

$$i\hbar\partial_t \Psi_{\rm phys}(\mathbf{X}, \mathbf{Y}, t) = H\Psi_{\rm phys}(\mathbf{X}, \mathbf{Y}, t) \quad (21)$$

where we make no particular assumption about the form of $\Psi_{\rm phys}(\mathbf{X}, \mathbf{Y}, t)$. The Hamiltonian,

$$H = \sum_{i=1}^{N_\uparrow} \frac{\mathbf{p}^2_{i\uparrow}}{2m} + \sum_{j=1}^{N_\downarrow} \frac{\mathbf{p}^2_{j\downarrow}}{2m} + \sum_{i,j} V(\mathbf{x}_i - \mathbf{y}_i). \quad (22)$$

In the above expression $\mathbf{p}_{i\uparrow}$ ($\mathbf{p}_{j\downarrow}$) is the momentum operator of the i-th (j-th) spin-up (spin-down) particle, $\mathbf{x}_i$ ($\mathbf{y}_j$) its position operator, and $V(\mathbf{x} - \mathbf{y})$ is a short-range two-body potential, which in the limit $r_0 \to 0$

can be replaced by the BPC, Eq. (4). Upon introducing $\Psi_{\rm phys}(\mathbf{X},\mathbf{Y},t) = \mathcal{J}_{\rm BPC}(a_s)\Phi_{\rm CF}(\mathbf{X},\mathbf{Y},t) = \prod_{i,j}\chi_J(\mathbf{x}-\mathbf{y}_j)\Phi_{\rm CF}(\mathbf{X},\mathbf{Y},t)$ into (21), we arrive at:

$$i\hbar\mathcal{J}_{\rm BPC}(a_s)\partial_t\Phi_{\rm CF} = \mathcal{J}_{\rm BPC}(a_s)H_{\rm CF}\Phi_{\rm CF}, \tag{23}$$

where

$$\begin{aligned} H_{\rm CF} &= \sum_{i=1}^{N_\uparrow}\frac{\left(\mathbf{p}_{i\uparrow}+\mathbf{v}_\uparrow(\mathbf{x}_i)\right)^2}{2m} + \sum_{j=1}^{M_\downarrow}\frac{\left(\mathbf{p}_{j\downarrow}+\mathbf{v}_\downarrow(\mathbf{y}_j)\right)^2}{2m} \\ &\quad + \sum_{i,j} V(\mathbf{x}_i-\mathbf{y}_j), \end{aligned} \tag{24}$$

the vectors $\mathbf{v}_\sigma(\mathbf{r}) = -is_\sigma\,\hbar\int d\mathbf{s}\,\frac{\nabla_{\mathbf{r}}\chi_J(\mathbf{r}-\mathbf{s})}{\chi_J(\mathbf{r}-\mathbf{s})}\,\rho_{-\sigma}(\mathbf{s})$, where $\rho_\sigma(\mathbf{r}) = \sum_k \delta(\mathbf{r}-\mathbf{r}_k)$ is the density operator of the CFs carrying spin σ and $s_\uparrow = -s_\downarrow = +1$. In retrospect, it may seem that we have gained very little by the previous transformation. The CF Hamiltonian, Eq. (24), looks even more complicated that the original one, Eq. (22). However, it can be shown that the presence of the interaction potential in (24) is somewhat redundant: Upon expanding the kinetic energy term operator, the following term appears:

$$\begin{aligned} \sum_{i,j}\frac{1}{\chi_J(\mathbf{x}_i-\mathbf{y}_j)}\Bigg[&-\frac{\hbar^2}{2m}\left(\nabla^2_{\mathbf{x}_i}+\nabla^2_{\mathbf{y}_j}\right) \\ &+V(\mathbf{x}_i-\mathbf{y}_j)\Bigg]\chi_J(\mathbf{x}_i-\mathbf{y}_j), \end{aligned} \tag{25}$$

which vanishes because $\chi_J(\mathbf{r})$ obeys:

$$\left[-\frac{\hbar^2}{m}\nabla^2_{\mathbf{r}} + V(\mathbf{r})\right]\chi_J(\mathbf{r}) = 0. \tag{26}$$

Hence,

$$\begin{aligned} H_{\rm CF} &= \sum_{i=1}^{N_\uparrow}\left[\frac{\mathbf{p}^2_{i\uparrow}}{2m} + \frac{1}{m}\mathbf{v}_\uparrow(\mathbf{x}_i)\cdot\mathbf{p}_{i\uparrow}\right] \\ &\quad + \sum_{j=1}^{N_\downarrow}\left[\frac{\mathbf{p}^2_{j\downarrow}}{2m} + \frac{1}{m}\mathbf{v}_\downarrow(\mathbf{y}_j)\cdot\mathbf{p}_{j\downarrow}\right] \\ &\quad + \frac{1}{2m}\sum_{i=1}^{N_\uparrow}\left[\mathbf{v}^2_\uparrow(\mathbf{x}_i)+\xi_\uparrow(\mathbf{x}_i)\right] \\ &\quad + \frac{1}{2m}\sum_{j=1}^{N_\downarrow}\left[\mathbf{v}^2_\downarrow(\mathbf{y}_j)+\xi_\downarrow(\mathbf{y}_j)\right], \end{aligned} \tag{27}$$

where the scalar potential $\xi_\sigma(\mathbf{r}) = \hbar^2\int d\mathbf{s}\left[\frac{\nabla_{\mathbf{r}}\chi_J(\mathbf{r}-\mathbf{s})}{\chi_J(\mathbf{r}-\mathbf{s})}\right]^2\rho_{-\sigma}(\mathbf{s})$. This Hamiltonian describe a system of CF interacting via *short-range* vector $v_\sigma(\mathbf{r})$ and scalar $\xi_\sigma(\mathbf{r})$ potentials. The potentials depend on the density distribution of the opposite spin species. The CF Hamiltonian still looks quite complicated, but it is well defined in the unitary limit where $|a_s|\to\infty$.

The above Hamiltonian, $H_{\rm CF}$, greatly simplifies upon performing a mean-field approximation. The approximation is similar in spirit to the mean-field theories of composite particles in the fractional quantum Hall effect [36–38]. Let us replace the density operator $\rho_{-\sigma}(\mathbf{s})$ of the CFs by its expectation value in the expressions for $\mathbf{v}_\sigma(\mathbf{r})$ and $\xi_\sigma(\mathbf{r})$. Thus, $\mathbf{v}_\sigma(\mathbf{r}) = -i\hbar s_\sigma\int d\mathbf{s}\,\ln\chi_J(\mathbf{r}-\mathbf{s})\nabla_{\mathbf{s}}\rho_{-\sigma}(\mathbf{s}) = 0$ and $\xi_\sigma(\mathbf{r}) = \text{const}$. Dropping the constant term, the mean-field Hamiltonian reads:

$$H^{\rm MF}_{\rm CF} = \sum_{i=1}^{N_\uparrow}\frac{\mathbf{p}^2_{i\uparrow}}{2m} + \sum_{j=1}^{N_\downarrow}\frac{\mathbf{p}^2_{j\downarrow}}{2m}. \tag{28}$$

In other words, the CFs become non-interacting and their ground state wavefunction is the Slater determinant $\Phi_{\rm FS}(\mathbf{X},\mathbf{Y})$, that is, the CF 'Fermi sea'. The ground state for the actual fermions is $|\Psi_N\rangle = \mathcal{J}_{BPC}(a_s)|\Phi_{\rm FS}\rangle$, which was introduced in Sect. IV rather heuristically. Furthermore, the excitations of this state are obtained from projecting the excitations of non-interacting CF states, which are also Slater determinants. In second quantization, these excitations correspond to the quasi-particles (created by $\pi^\dagger_\sigma(\mathbf{k})$ with $|\mathbf{k}| > k_{F\sigma}$, cf. Eq. 29) and the quasi-holes (created by $\eta_\sigma(\mathbf{k})$ with $|\mathbf{k}| < k_{F\sigma}$, cf. Eq. 30) introduced in Sect. III, and which were related to the Fermi polarons in Sect. IV.

Nevertheless, it must be noted that the quasi-particle and quasi-hole states resulting from the projection are not necessarily orthogonal. For single quasi-particle/quasi-hole states, spin and momentum conservation require that $\langle\Psi_N|\pi_\sigma(\mathbf{k})\pi^\dagger_{\sigma'}(\mathbf{p})|\Psi_N\rangle = 0$ only if $\mathbf{k}\neq\mathbf{p}$ and/or $\sigma\neq\sigma'$. However, states containing different number of excitations but carrying the same total momentum and spin are not orthogonal. The non orthogonality stems from the non-unitary transformation involving the operators $U_\sigma(\mathbf{r})$ introduced in Sect. III, which is needed to carry out the projection. In spite of their lack of orthogonality, the quasi-particle and quasi-hole states are left eigenstates of $\mathcal{J}_{\rm BPC}(a_s)H^{\rm MF}_{\rm CF}$ and therefore are stationary states at the mean-field level. However, because of projection and the mean-field approximation, $\mathcal{J}_{\rm BPC}(a_s)H^{\rm MF}_{\rm CF}$ is not a hermitian operator. This feature makes it difficult to set up a perturbative approach to analyze the effect of corrections to the mean-field theory. Similar difficulties have been found in the approach to the fractional Hall effect described in [38]. In standard Landau FLT, the quasi-particles are not eigenstates of the system but narrow superpositions of the latter. This leads to similar difficulties when defining quasi-particle operators in the standard theory of Fermi liquids [35]. In the present mean-field theory, the matrix elements for quasi-particle decay vanish and therefore the quasi-particle and quasi-holes become stationary states. The non-orthogonality, however, can regarded as a remnant of the fact that they are not true eigenstates of the system.

Although it may be coincidental as it results from a rather uncontrolled approximation, the lack of renormalization of the quasi-particle mass is in qualitative agreement with the experimental evidence around unitary. In this regard, the ENS group found that [2] $m^* \simeq 1.13m$. The lattice Monte Carlo calculations of [21] are also in agreement with an internal energy that, up to an overall shift, is the same as for a non-interacting Fermi gas. If we rely on perturbation theory at weak coupling ($|k_F a_s| \ll 1$), for $N_\uparrow = N_\downarrow$, it is found that $m^*/m = 1 + \frac{8}{15\pi^2}(7\ln 2 - 1)(k_F a_s)^2 + O\left((k_F a_s)^2\right) \simeq 1 + 0.208\,(k_F a_s)^2$ (see *e.g.* Ref. [41], pag. 148). This formula yields about a 20% correction for $k_F a_s = -1$, outside its validity range, and which would be about 10% higher than the value of the effective mass determined by the ENS group at unitarity. In the highly polarized gas at unitarity $m^* \simeq 1.17\,m$ [2, 12] and $m^* \simeq 1.20\,m$ from the thermodynamic measuremnts of Ref. [2]. Furthermore, theoretical calculations of the Fermi polaron effective mass [11, 19, 30, 32] also found that m^* does not strongly renormalize from the bare atom mass in a large region of the crossover regime where the normal phase is found to be stable. Thus, phenomenologically, these small deviations from the bare mass can be thus regarded as measure of the strength of the fluctuations beyond the mean-field approximation. The potential fluctuations can be related to those of the operator $\nabla\rho_\sigma(\mathbf{r})$ within the range of the Jastrow factor $\sim \min\{|a_s|, k_F^{-1}\}$. In particular, when a molecular bound state forms, the mean-field theory will break down. This is because it assumes that the density around a given CF is constant and its fluctuations are negligible. But this is no longer true when the bound state forms. In other words, such a bound state cannot be described by a projecting a Slater determinant of CFs. Thus, ultimately, the justification for the mean-field approximation is directly related to the accuracy of the description of the normal state by the Jastrow Slater wavefunctions. The latter have been shown to be good trial wavefunctions for the importance sampling in fixed-node diffusion Monte Carlo calculations [33, 39, 40], which means that they are able to capture the nodal surface of the normal state.

The thermodynamics of the normal state at the mean-field level is the same as for the non-interacting Fermi gas because thermodynamic functions only depend on the spectrum and their degeneracies and not on the wavefunctions. Thus, Fermi liquid theory trivially applies to this system. To see this, we first recall that the ground state $|\Psi_N\rangle$ has a well defined Fermi surface. Furthermore, the quasi-particle and quasi-hole excitations are fermions that carry the same quantum numbers as the original fermions. This is because they result from projecting non-interacting Fermi gas states:

$$\pi^\dagger_\sigma(\mathbf{k})|\Psi_N\rangle = \mathcal{J}_{\mathrm{BPC}}(a_s)\psi^\dagger_\sigma(\mathbf{k})|\Phi_{\mathrm{FS}}\rangle, \tag{29}$$

$$\eta_\sigma(\mathbf{k})|\Psi_N\rangle = \mathcal{J}_{\mathrm{BPC}}(a_s)\psi_\sigma(\mathbf{k})|\Phi_{\mathrm{FS}}\rangle. \tag{30}$$

In general, projection can be used as a formal device to establish a one to one correspondence between the states of the non-interacting gas of CFs and the excitations of the real system. Just as in standard Landau FLT, this allows to introduce the distribution function of quasi-particles (or quasi-holes) $n_\sigma(\mathbf{k})$. The Landau free energy functional within the mean-field approximation reads:

$$F^{MF} = F_0 + \sum_{\mathbf{k},\sigma}\left[\epsilon_\sigma(\mathbf{k}) - \mu_\sigma\right]\,\delta n_\sigma(\mathbf{k}) \tag{31}$$

where $\delta n_\sigma(\mathbf{k}) = n_\sigma(\mathbf{k}) - n^0_\sigma(\mathbf{k})$, and $n^0_\sigma(\mathbf{k}) = \theta(k_{F\sigma} - k)$ and F_0 is the ground state free energy. The Landau parameters vanish at the mean-field level. Experimentally, however, it is found that $F^s_1 \neq 0$ ($F^s_1 = 0.39$ for the unitary unpolarized gas [2]), which by virtue of Galilean invariance, accounts for the deviations of the effective mass from the bare mass. Furthermore, $F^s_0 = -0.42$ was determined by the ENS group [2] by fitting the FLT equation of state of the unpolarized gas at unitarity. Thus, interactions between the composite Fermions which are described by the corrections to the mean-field theory should account for the Landau parameters.

On the other hand, although the projection procedure allows us to establish a one to one correspondence with the non-interacting Fermi gas states, it does not quite correspond to the process of adiabatic continuity envisaged by Landau (see *e.g.* Ref. [35], pag. 2 and following). In the case of ^{3}He, the Jastrow Slater wavefunctions describe a standard Fermi liquid. However, in this case, unlike the case of ^{3}He where the Jastrow factor is introduced to obtain a better variational description of the ground state, the Jastrow factor introduced above simply implements the BPC, that is, the projection onto the physical Hilbert space. It is therefore not just a convenience, but a physical need. Thus, when dealing with the quasi-particle and quasi-hole excitations, we are effectively projecting non-interacting states [42] (cf. Eqs. 29,30). It can be argued that, by using the device of varying the scattering length so that $a_s \to 0$, the correspondence with the particle and hole states of the non-interacting Fermi gas becomes more apparent. Such a procedure does not warranty that the overlaps:

$$Z_{\mathrm{qp}} = \lim_{k\to k_F^+}\langle\Psi_{\mathrm{N}}|\psi_\sigma(\mathbf{k})\pi^\dagger_\sigma(\mathbf{k})|\Psi_{\mathrm{N}}\rangle, \tag{32}$$

$$Z_{\mathrm{qh}} = \lim_{k\to k_F^-}\langle\Psi_{\mathrm{N}}|\psi^\dagger_\sigma(\mathbf{k})\eta_\sigma(\mathbf{k})|\Psi_{\mathrm{N}}\rangle \tag{33}$$

are in general equal, as required by *standard* FLT theory, or even remain finite for all values of a_s in the thermodynamic limit. If they vanished, the quasi-particles and quasi-holes, which are the true elementary excitations of the system, will not behave as Landau quasi-particles [42, 43]. The deep reason why Z_{qp} and Z_{qh} may turn out to be different or even vanish is the following: When adding one fermion to the ground state $|\Psi_N\rangle$ (or in general to any physical state), the resulting state is not in the physical Hilbert space. This is because the other particles cannot immediately adapt to the newcomer. However, this conclusion does not apply to the state that results from removing one fermion. Such a state belongs the physical space but also contains the 'correlation hole' left by the removed particle. The latter situation is relevant to spectroscopic measurements, such as those carried out by the JILA group [5]. Therefore, there is a clear asymmetry between the states with one particle more and one particle less. Investigating these issues is beyond the scope of the present work and will be done elsewhere [44].

VI. CONCLUSSIONS

In previous sections, we have described a wavefunction-based approach to study the properties of the ultracold dilute Fermi gas. The basic idea is that projection onto the physical Hilbert space of wave functions obeying the Bethe-Peierls condition must be implemented from the scratch. The need for projection stems from the short range of the interaction and diluteness of the gas. This implies that two-body encounters are described by the two-body wavefunction, which is assumed to reach its asymptotic value before other collisions can take place. This physics is encapsulated in the Bethe-Peiers condition, which in this work has been implemented by means of a Jastrow factor. It is interesting to note that the situation is reminiscent of the composite Fermion approach of the fractional quantum Hall effect [36–38]. It is also related to Anderson's 'hidden Fermi liquid theory' [36, 42] of the Gutzwiller projected Hubbard model. The common thread between these ideas is that appropriate projection of a Fermi gas defined in an unphysical Hilbert space can account for Fermi-liquid like behavior in systems with no obvious small parameter for a perturbative analysis to be carried out. This is the case of the ultracold dilute Fermi gas in the crossover regime.

Furthermore, as far as the superfluid state is concerned, in Sect. III we have argued that projection automatically fixes an important shortcoming of the BCS wavefunction pointed by Tan [22]. By considering that the gap of the (unprojected) BCS state collapses, the same principle leads to a projected Fermi gas, which is postulated as the most natural candidate for the normal state of the system, even in the strongly interacting limit. This state is described by Jastrow-Slater wavefunction, which is also shown to be the ground state of a mean-field theory. The mean field theory also exhibits non-interacting quasi-particle and quasi-hole excitations, which at low temperatures yield Fermi liquid thermodynamics. The quasi-particles are related to the Fermi polarons, which have been found to describe the normal state of a highly polarized Fermi gas at unitarity [11, 12]. We have also shown that the superfluid ground state can be understood as a condensate of pairs of composite Fermions or Fermi polarons.

A rather direct way of approaching the composite fermion theory is to investigate numericaly the properties low-lying excited states of the normal and superfluid state state. This has been done already to a large extent for ground state properties using the fixed-node diffusion Monte Carlo method [33, 39, 40]. However, it is also possible to employ the same Jastrow projector to analyze excitations of the system as in Jain's approach to the theory of composite fermions of the fractional quantum Hall effect [36]. For example, a better understanding of the quasi-particle interactions can be obtained through a Monte Carlo calculation of the energies of two quasi-particle (two quasi-hole, o quasi-particle and quasi-hole) excitations, which will allow to estimate the Landau parameters. Furthermore, analytical progress may be also possible by studying corrections to the mean-field approach discussed above. However, it may be necessary to find more convenient ways of implementing the projection, *e.g.* by defining a constrained functional integral. This would make it possible to develop a field theoretical approach to deal with the composite fermion or Fermi polaron liquids, and systematically study corrections to the mean-field theory. As argued in previous sections, we expect the composite fermions to be much weakly interacting than the original fermionic atoms, as part of the interactions have been transfered to the Jastrow factor. If successful, such efforts will take the theory from the present qualitative level to one where quantitative predictions can be made and compared directly with the experiments. Hopefully, progress on this problem will be reported elsewhere [44].

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[45] The Bethe-Peierls condition can be implemented in few-body correlation functions (such as the pair correlation function) [25]. Such a requirement is much weaker than directly imposing the condition on the wavefunction. See Sect. III for an example dealing with the BCS wavefunction.
[46] An important exception is the calculation of the dispersion of on Fermi polaron in the highly polarized gas at unitarity [11]
[47] Indeed, this feature, namely the lack of finite momentum pairs in $|\Phi_{\rm BCS}\rangle$, makes it unsuitable to describe collective excitations such as the Anderson-Bogoliubov mode, which is the Goldstone mode appropriate to a neutral superfluid. However, we shall be concerned here with the fate of individual fermionic excitations, and therefore the BCS wavefunction will be sufficient to the purpose of our discussion
[48] If the gap collapses due to thermal fluctuations, the state of the system is a mixed state. However, we adopt the point of view that such a mixed state can be constructed from excitations of the normal state. It is also not implied the gap will collapse in a continuous transition as the spin polarization grows.
[49] Note that projection onto the physical space is not needed for fully polarized states.