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Quantum quench dynamics of the sine-Gordon model in some solvable limits

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Abstract. With regard to the thermalization problem in isolated quantum systems, we investigate the dynamics following a quantum quench of the sine-Gordon model (sGM) in the Luther–Emery and the semiclassical limits. We consider the quench from the gapped to the gapless phase, as well as the reverse one. By obtaining analytic expressions for the one- and two-point correlation functions of the order parameter operator at zero-temperature, the manifestations of integrability in the absence of thermalization in the sGM are studied. It is shown that correlations in the long-time regime after the quench are well described by a generalized Gibbs ensemble. We also consider the case where the system is initially in contact with a reservoir at finite temperature. The possible relevance of our results to current and future experiments with ultracold atomic systems is also considered.

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1. Introduction

Recent experiments in the field of ultracold atomic gases have spurred much interest in the thermalization dynamics of isolated quantum systems [1]–[28]. So far, the latter were considered mere idealizations of real systems, as most of the many-particle systems of interest for quantum statistical mechanics, such as solids and quantum fluids, are strongly coupled to their environments. However, the creation of large ensembles of ultracold atoms with highly controllable properties, which remain fully quantum coherent for relatively long times (compared to the typical duration of an experiment), has completely changed this perception. This has also raised concerns about the mechanisms of thermalization in these systems, and even, in some recent experiment [6], lack of thermalization has been observed.

The problem of thermalization in isolated quantum systems can be posed as the study of the dynamics of a system following a *quantum quench*, that is to say, the study of the response of a system to the change of a control parameter of the Hamiltonian over a timescale that is much shorter than any other relevant timescale of the system, so that the sudden approximation can be applied. Therefore, it is assumed that, for $t < 0$, the Hamiltonian is $H_{\rm i}$ and the system is in a given eigenstate of it, $|\Phi_{\rm i}\rangle$. At $t = 0$ the Hamiltonian is changed to $H_{\rm f}$ and thus for $t > 0$ the system evolves unitarily in isolation according to the dynamics dictated by $H_{\rm f}$.

The quantum quench can also be used to describe the evolution of a system that has been prepared in a given state that is *not* an eigenstate of the Hamiltonian. Thus, the question that naturally arises is whether a set of sufficiently interesting observables of the system reaches some form of stationary state that can described by a standard Gibbs ensemble. In such a case, we would speak of thermalization to a standard statistical ensemble (microcanonical, canonical, or grand-canonical). However, this may not be the case as it turns out that for several integrable models [11, 14, 15, 19, 26] the long-time behavior of certain observables can instead be obtained from a generalized Gibbs ensemble characterized by a different temperature for each eigenmode of the Hamiltonian $H_{\rm f}$ [11, 14], [26]–[28]. The non-standard or generalized Gibbs ensembles follow from maximizing the von Neumann entropy with the constraints imposed by the set of integrals of motion larger than the total energy and particle number. As to the situation concerning nonintegrable systems, the issue of the thermalization dynamics is still not fully understood although some recent results [20, 21, 24] indicate that thermalization to a standard ensemble should eventually occur at sufficiently long times. However, the actual thermalization dynamics and how it depends on how close the system is to integrability are still very poorly understood. As far as one dimensional are concerned (for which the strong kinematic constraints usually cause the integrability to be more ubiquitous than in higher dimensions), numerical simulations have yielded conflicting results concerning the existence of thermalization [12, 13, 23]. Indeed, in a recent work, Rigol and coworkers [29, 30] have also pointed out that the statistics of the constituent particles may also play an important role in determining the thermalization dynamics.

Furthermore, in addition to studying the quench dynamics starting from a pure state, it will also be interesting to consider quantum quenches where the system is initially prepared in a thermal state by being at $t < 0$ in contact with an energy reservoir at temperature T. After the quench, the system is isolated from the reservoir and evolves unitarily according to the dynamics dictated by different Hamiltonians $H_{\rm f}$. This is the simplest kind of mixed initial state one can consider and it allows us to analyze the effect on the quench dynamics of mixing (by means of the initial temperature) a fraction of the excited states with the ground state of $H_{\rm i}$.

In this work, we shall analyze the quench dynamics of the sine-Gordon model (sGM). This is an integrable field theory, but its exact solution is indeed quite difficult to deal with as the elementary excitations satisfy a rather non-standard algebra. Instead, we confine ourselves to two limits in which the model can be written as a quadratic Hamiltonian. In one of these two limits, the so-called Luther–Emery (LE) limit, using a trick called refermionization [31, 32], the model describes a system of one-dimensional (1D) massive (i.e. gapped) Dirac fermions. The fermions and their anti-particles (or holes, to use solid-state physics language) describe the solitonic and anti-solitonic excitations of the sGM, which, in the LE limit, happen to be non-interacting. In another limit, the model can be rather well approximated by a quadratic model of massive bosons. The latter describes a series of bound states of solitons and anti-solitons in the limit where interaction between them is strongly attractive. As explained elsewhere [11, 26], the quench dynamics of a quadratic Hamiltonian can be solved exactly via a time-dependent Bogoliubov transformation. We shall consider here two kinds of quenches, which correspond to the appearance and disappearance of the mass term (i.e. the gap) in the sGM. At zero temperature, we found that correlations that occur when the system is quenched from the gapped to the gapless phase at zero or low temperature exhibit an exponential decay with a correlation length/time fixed by the gap. This result is in agreement with the results of Calabrese and Cardy [9, 15], based on a mapping to a boundary conformal field theory.

At high temperatures, however, the correlation length (time) is determined by the temperature. At intermediate temperatures, the system will exhibit a crossover between the zero temperature (gap-dominated) and the high-temperature (temperature-dominated) regimes. On the other hand, correlations following a quench from the gapped to the gapless phase at the LE and the semi-classical limit exhibit somewhat different behavior, which may indicate a breakdown of the semiclassical approximation or a qualitative change in the behavior of correlations as one moves away from the LE limit.

This paper is organized as follows. In section 2, we introduce the (quantum) sine-Gordon model and its phases and describe the problem of quantum quenches in this model. In section 3 we consider quenches in the LE limit at zero temperature, whereas in section 4 we consider the quenches in the semiclassical limit. In section 5, we consider the effect of thermal fluctuations on the initial state assuming that contact with the energy reservoir is removed at the time the system is quenched and therefore it evolves in isolation subsequently. In section 6, we consider the issue of the long-time asymptotic behavior of correlations and expectation values, and show that it is given by a generalization of the Gibbs ensemble, as recently pointed out by Rigol *et al* [14]. Finally, in section 7, we discuss some possible experimental consequences of this work. We provide a summary of the conclusions of this work in section 8. The results for the correlations at zero temperature reported in this paper were previously available as part of our unpublished preprint [28]. Since our preprint appeared, other authors have also considered the thermalization dynamics in the sGM. In particular (see this special issue), Sabio and Kehrein [33] used a flow equation method. Fioretto and Mussardo used [34] form factor methods to tackle quantum quenches in the sGM, finding strong evidence that the long-time behavior of a local operator is described by a generalized Gibbs ensemble. Furthermore, using the time-dependent renormalization group [35, 36], Barmettler *et al* [37] have investigated the quench dynamics of the XXZ model, which in the continuum limit reduces to the sGM.

2. The sine-Gordon model

The sGM is described by the following Hamiltonian:

$$H_{\rm sG}(t) = H_0 - \frac{\hbar v g(t)}{\pi a_0^2} \int {\rm d}x \, \cos 2\phi, \tag{1}$$

$$H_0 = \frac{\hbar v}{2\pi} \int {\rm d}x \, : K^{-1} (\partial_x \phi)^2 + K \, (\partial_x \theta)^2 :, \tag{2}$$

where $: \ldots :$ stands for the normal order of the operators [31, 32, 38], a_0 is a short-distance cut-off and the phase and density fields, $\theta(x)$ and $\phi(x)$, are canonically conjugated in the sense that they obey $[\phi(x), \partial_{x'}\theta(x')] = {\rm i}\pi\delta(x - x')$. This model can be regarded as a perturbation of the Luttinger model (see e.g. [31, 32, 38] for reviews), which still yields an integrable model. In equilibrium the model is known to exhibit two phases, which, according to the renormalization group analysis [31, 32] and for infinitesimal and positive values of the coupling in front of the cosine term, roughly correspond to $K < 2$ (gapped phase) and $K \geqslant 2$ (gapless phase).

In order to study the non-equilibrium (quench) dynamics, we will consider two different types of quenches: the quench from the gapped to the gapless phase and the reverse process, the quench from the gapless to the gapped. In the first case, we assume that the *dimensionless* coupling $g(t)$ is suddenly turned on, i.e. $g(t) = g\,\theta(-t)$. With this choice, $H_{\rm i} = H_{\rm sG}(t \leqslant 0)$

is a Hamiltonian whose ground state exhibits a frequency gap, m, to all excitations, whereas $H_{\rm f} = H_{\rm sG}(t > 0)$ has gapless excitations. Conversely, in the second case, we consider that $g(t)$ is suddenly turned off, i.e. $g(t) = g\,\theta(t)$. In this case, the ground state of $H_{\rm i}$ is gapless, whereas the Hamiltonian performing the time evolution, $H_{\rm f}$, has gapped excitations. However, although both $H_{\rm i}$ and $H_{\rm f}$ define integrable field theories, for a general choice of the parameters K and g, the quench dynamics cannot be analyzed, in general, by the elementary methods of [11, 26]. Nevertheless, in two limits, the LE (which corresponds to $K = 1$, see section 3) and the semiclassical (that is, for $K \ll 1$, section 4) limits, it is possible to study the quench dynamics by methods similar to those of [26]. However, as explained above, the statistics of the elementary excitations happens to be different in these two limits.

3. The Luther–Emery limit

3.1. Introductory remarks

Let us start by considering the sGM, equation (3), for $K = 1$, which is the so-called Luther–Emery (LE) limit. It is convenient to introduce rescaled density and phase fields, which will be denoted as $\varphi(x) = K^{-1/2}\phi(x)$ and $\tilde{\varphi}(x) = K^{1/2}\theta(x)$. Thus, the Hamiltonian of equation (1) becomes

$$H_{\rm gG}(t) = \frac{\hbar v}{2\pi}\int {\rm d}x : (\partial_x\varphi)^2 + (\partial_x\tilde{\varphi})^2 : -\frac{\hbar v g(t)}{\pi a_0^2}\int {\rm d}x \cos\kappa\varphi, \tag{3}$$

where $\kappa = 2\sqrt{K}$. At the LE limit, $\kappa = 2$ (i.e. $K = 1$), and the model can be rewritten as a 1D model of *massive* Dirac fermions with mass by using the following bosonization formula for the Fermi field operators [31, 32]:

$$\psi_\alpha(x) = \frac{\eta_\alpha {\rm e}^{{\rm i}s_\alpha\pi/4}}{\sqrt{2\pi a}}\,{\rm e}^{{\rm i}s_\alpha\phi_\alpha(x)}, \tag{4}$$

where $s_r = -s_l = +1$ and the chiral fields $\phi_r(x) = \varphi(x) + \tilde{\varphi}(x)$ and $\phi_l(x) = \varphi(x) - \tilde{\varphi}(x)$. For computational convenience, we choose the Majorana fermions in equation (4) to be $\eta_r = \sigma_x$ and $\eta_l = {\rm i}\sigma_y$, where σ_x and σ_y are the familiar Pauli matrices. In addition, we note that the gradient terms in equation (3) can be written as the kinetic energy of free massless Dirac fermions in one dimension [31, 32, 38]:

$$H_0 = -{\rm i}\hbar v\int {\rm d}x : \psi_r^\dagger(x)\partial_x\psi_r(x) - \psi_l^\dagger(x)\partial_x\psi_l(x) : . \tag{5}$$

As far as the cosine operator is concerned, the bosonization formula, equation (4), implies that

$$\psi_r^\dagger(x)\psi_l(x) + \psi_l^\dagger(x)\psi_r(x) = \frac{\Gamma}{\pi a_0}\cos 2\varphi(x) \tag{6}$$

$$= \frac{\Gamma}{\pi} : \cos 2\varphi(x) : \tag{7}$$

where $\Gamma = {\rm i}\sigma_x\sigma_y$. This is almost the cosine term of the sGM in the LE limit (cf equation (3)), except for the presence of the operator Γ. However, we note that $\Gamma^2 = 1$ and that this operator also commutes with H_0 and with the operator in the left-hand side of equation (6). The first property implies that the eigenvalues of Γ are ± 1, whereas the second property

implies that $H_{\rm LE}(t) = H_0 + \hbar v g(t) \int {\rm d}x [\psi_r^\dagger(x)\psi_l(x) + \psi_l^\dagger(x)\psi_r(x)]$ and Γ can be diagonalized simultaneously. After choosing the eigenspace where $\Gamma = -1$, we obtain that

$$H_{\rm LE}(t) = -{\rm i}\hbar v \int {\rm d}x : \psi_r^\dagger(x)\partial_x\psi_r(x) - \psi_l^\dagger(x)\partial_x\psi_l(x) : \\ + \hbar v g(t) \int {\rm d}x \left[\psi_r^\dagger(x)\psi_l(x) + \psi_l^\dagger(x)\psi_r(x)\right], \tag{8}$$

which is equivalent to equation (3) when $\kappa = 2$.

To gain some insight into the phases described by the sGM, let us first consider the LE Hamiltonian $H_{\rm LE}$ in two (time-independent) situations: (i) $g(t) = 0$ (the gapless free fermion phase, which coincides with the Luttinger model for $K = 1$ [31, 32, 38]) and (ii) $g(t) = g > 0$, i.e. a time-independent constant (which corresponds to the gapped phase). In order to diagonalize the Hamiltonian, it is convenient to work in Fourier space and write the fermion field operator as

$$\psi_\alpha(x) = \frac{1}{L}\sum_p {\rm e}^{-a_0|p|}\,{\rm e}^{{\rm i}px}\,\psi_\alpha(p), \tag{9}$$

where $\alpha = r, l$. The limit where the cut-off $a_0 \to 0^+$ should be formally taken at the end of the calculations, but in some cases we shall not do it in order to regularize certain short-distance divergences of the quantum sGM. It will also be useful to introduce a spinor whose components are the right and left moving fields and which will make the notation more compact:

$$\Psi(p) = \begin{bmatrix}\psi_r(p)\\ \psi_l(p)\end{bmatrix}, \quad \mathcal{H}(p) = \begin{bmatrix}\hbar\omega_0(p) & 0\\ 0 & -\hbar\omega_0(p)\end{bmatrix}. \tag{10}$$

Thus the Hamiltonian for the gapless phase, H_0, reads

$$H_0 = \sum_p : \Psi^\dagger(p)\cdot\mathcal{H}_0(p)\cdot\Psi(p) :, \tag{11}$$

where $\omega_0(p) = vp$ is the fermion dispersion. However, the Hamiltonian of the gapped phase, corresponding to $g(t) = g > 0$, $H_{\rm LE}$, is not diagonal in terms of the right and left moving Fermi fields. In the compact spinor notation, it reads

$$H_{\rm LE} = \sum_p : \Psi^\dagger(p)\cdot\mathcal{H}(p)\cdot\Psi(p) :, \tag{12}$$

where

$$\mathcal{H}(p) = \begin{bmatrix}\hbar\omega_0(p) & \hbar m\\ \hbar m & -\hbar\omega_0(p)\end{bmatrix}, \tag{13}$$

being $m = vg$. Nevertheless, $H_{\rm LE}$ can be rendered diagonal by means of the following unitary transformation:

$$\tilde\Psi(p) = \begin{bmatrix}\psi_{\rm c}(p)\\ \psi_{\rm v}(p)\end{bmatrix} = \begin{bmatrix}\cos\theta(p) & \sin\theta(p)\\ -\sin\theta(p) & \cos\theta(p)\end{bmatrix}\begin{bmatrix}\psi_r(p)\\ \psi_l(p)\end{bmatrix}, \tag{14}$$

where

$$\tan 2\theta(p) = \frac{m}{\omega_0(p)}. \tag{15}$$

Thus the Hamiltonian of the gapped phase, in diagonal form, reads (after dropping an unimportant constant that amounts to the ground state energy)

$$H_{\rm LE} = \sum_p \hbar\omega(p) \left[: \psi_{\rm c}^\dagger(p)\psi_{\rm c}(p) - \psi_{\rm v}^\dagger(p)\psi_{\rm v}(p) : \right], \tag{16}$$

where $\omega(p) = \sqrt{\omega_0(p)^2 + m^2}$. We associate $\psi_{\rm v}^\dagger(p)$ ($\psi_{\rm c}^\dagger(p)$) with the creation operator for particles in the valence (conduction) band.

Before considering quantum quenches, let us briefly discuss some of the properties of the ground states of the Hamiltonians H_0 and $H_{\rm LE}$. In what follows, these states will be denoted as $|\Phi_0\rangle$ and $|\Phi\rangle$, respectively. As mentioned above, the spectrum of H_0 is gapless, and the fermion occupancies in the ground state $|\Phi_0\rangle$ are

$$n_r(p) = \langle\Phi_0|\psi_r^\dagger(p)\psi_r(p)|\Phi_0\rangle = \theta(-p), \tag{17}$$

$$n_l(p) = \langle\Phi_0|\psi_l^\dagger(p)\psi_l(p)|\Phi_0\rangle = \theta(p). \tag{18}$$

That is, all single-particle levels with negative momentum are filled. However, $H_{\rm LE}$ has a gapped spectrum and, therefore, when constructing its ground state, $|\Phi\rangle$, only the levels in the valence band (which have negative energy) are filled, whereas the levels in the conduction band remain empty

$$n_{\rm v}(p) = \langle\Phi|\psi_{\rm v}^\dagger(p)\psi_{\rm v}(p)|\Phi\rangle = 1, \tag{19}$$

$$n_{\rm c}(p) = \langle\Phi|\psi_{\rm c}^\dagger(p)\psi_{\rm c}(p)|\Phi\rangle = 0. \tag{20}$$

3.2. Quench from the gapped to the gapless phase

The first situation we shall consider is when $g(t) = g\theta(-t)$ in equation (8), so that the spectrum of the Hamiltonian abruptly changes from gapped to gapless (i.e. quantum critical). In the following, we denote $H_{\rm i} \equiv H_{\rm LE}(t<0) = H_{\rm LE}$ and $H_{\rm f} \equiv H_{\rm LE}(t>0) = H_0$. The time evolution of $\psi_{r,l}$ for $t>0$ is thus

$$\psi_{r,l}(p,t) = {\rm e}^{\pm {\rm i}\omega_0(p)t}\psi_{r,l}(p). \tag{21}$$

We first consider the zero temperature quench in this section, and postpone to section 5 the discussion of the more complicated finite-temperature case. In the zero temperature case the initial state $\rho_{\rm i} = |\Phi\rangle\langle\Phi|$. Note that, in this state, $\langle\Phi|\cos 2\phi(x)|\Phi\rangle = \langle\Phi|\cos 2\varphi(x)|\Phi\rangle = {\rm Re}\langle\Phi|{\rm e}^{-2{\rm i}\varphi(x)}|\Phi\rangle = -\langle\psi_r^\dagger(x)\psi_l(x)\rangle \neq 0$ (the minus sign stems from the eigenvalue of the operator $\Gamma = \eta_r\eta_l$), whereas in the ground state of H_0, $|\Phi_0\rangle$, the expectation value of the same operator vanishes. Therefore, it behaves like an order parameter in equilibrium, and we can expect that it exhibits interesting dynamics out of equilibrium. Indeed,

$$\mathcal{C}(t) = \langle{\rm e}^{-2{\rm i}\varphi(x,t)}\rangle = -\frac{1}{L}\sum_p \langle\psi_r^\dagger(p,t)\psi_l(p,t)\rangle = \frac{1}{2L}\sum_p {\rm e}^{-2{\rm i}\omega_0(p)t}\,\sin 2\theta(p), \tag{22}$$

where, to evaluate the expectation value, we have set $T=0$ and therefore $\langle\psi_r^\dagger(p)\psi_l(p)\rangle = -\frac{1}{2}\sin 2\theta(p)$, as follows from equations (14), (19) and (20). The above expression can be readily evaluated by recalling that $\sin 2\theta(p) = m/\omega(p)$, which yields, in the $L\to\infty$ limit,

$$\mathcal{C}(t) = m\int_0^{+\infty}\frac{{\rm d}p}{2\pi}\frac{\cos 2\omega_0(p)t}{\omega(p)} = \left(\frac{m}{2\pi v}\right)K_0(2mt) \simeq \frac{1}{4v}\sqrt{\frac{m}{\pi t}}{\rm e}^{-2mt}, \tag{23}$$

where K_0 is the modified Bessel function. Thus we see that the 'order parameter' $\langle \cos 2\varphi(0,t)\rangle$ decays exponentially at long times at $T=0$. The decay rate is proportional to the gap between the ground state (the initial state) and the first excited state of the initial Hamiltonian $H_\mathrm{i}=H$. The existence of this gap means, in particular, that correlations in the initial state between degrees of freedom of the system are exponentially suppressed beyond a distance of the order $\xi_\mathrm{c}\approx v/m$. Upon quenching, the evolution of the system is dictated by a critical Hamiltonian, H_0, that is, a Hamiltonian describing excitations that propagate ballistically along 'light cones' corresponding to the 'trajectories' $x\pm vt$. Thus, as discussed by Calabrese and Cardy [9, 15], the correlation length scale characterizing the initial state translates into an exponential decay in time of the order parameter at long times. This exponential decay is also found (for the same type of quench) in the semiclassical limit of the sGM (see section 4 below).

Next we shall consider the (equal-time) two-point correlation function of the same operator, namely

$$\mathcal{G}(x,t)=\langle \mathrm{e}^{-2\mathrm{i}\varphi(x,t)}\mathrm{e}^{2\mathrm{i}\varphi(0,t)}\rangle. \tag{24}$$

Using the fermionic representation of $\mathrm{e}^{2\mathrm{i}\varphi(x,t)}$ and expanding in Fourier modes, we get

$$\mathcal{G}(x,t)=\frac{1}{L^2}\sum_{p_1,p_2,p_3,p_4}\mathrm{e}^{\mathrm{i}(p_1-p_2)x}\mathrm{e}^{-\mathrm{i}[\omega_0(p_1)+\omega_0(p_2)-\omega_0(p_3)-\omega_0(p_4)]t}\langle\psi_r^\dagger(p_1)\psi_l(p_2)\psi_l^\dagger(p_3)\psi_r(p_4)\rangle. \tag{25}$$

Applying Wick's theorem, there are three different contractions of the above four fermion expectation values, which can be evaluated using equations (14), (19) and (20). This yields the following contractions:

$$\langle\psi_r^\dagger(p)\psi_l^\dagger(p)\rangle=\langle\psi_r(p)\psi_l(p)\rangle=0 \tag{26}$$

$$\langle\psi_l^\dagger(p)\psi_r(p)\rangle=\langle\psi_r^\dagger(p)\psi_l(p)\rangle=-\tfrac{1}{2}\sin 2\theta(p), \tag{27}$$

$$\langle\psi_r^\dagger(p)\psi_r(p)\rangle=\sin^2\theta(p), \tag{28}$$

$$\langle\psi_l(p)\psi_l^\dagger(p)\rangle=\cos^2\theta(p). \tag{29}$$

Hence, in the thermodynamic limit ($L\to\infty$), we obtain

$$\mathcal{G}(x,t)=\mathcal{C}(x,t)\,\mathcal{C}(0,t)+|\frac{\mathrm{i}}{2}\delta(x)+\mathcal{H}(x,t)|^2, \tag{30}$$

where

$$\mathcal{H}(x,t)=\int_0^{+\infty}\frac{\mathrm{d}p}{2\pi}\sin px\frac{\omega_0(p)}{\omega(p)}\mathrm{e}^{-a_0p}=\frac{m}{2\pi v}K_1\left(\frac{m|x|}{v}\right) \tag{31}$$

in the limit $a_0\to 0$. Therefore, for $|x|\neq 0$,

$$\mathcal{G}(x,t)=\left(\frac{m}{2\pi v}\right)^2\left([K_0(2mt)]^2+\left[K_1\left(\frac{m|x|}{v}\right)\right]^2\right). \tag{32}$$

Let us examine the behavior of this correlation function in the asymptotic limit where $|x|\gg\xi_\mathrm{c}=v/m$ and $2vt\gg\xi_\mathrm{c}$. Since the Bessel functions decay exponentially for large values of their arguments, the leading term in $\mathcal{G}(x,t)$ depends on whether $t<|x|/2v$ or $t>|x|/2v$. Thus

$$\mathcal{G}(x,t)\simeq\begin{cases}\dfrac{m}{16\pi vx}\mathrm{e}^{-2m|x|/v}, & t>|x|/2v,\\[2ex] \dfrac{m}{32\pi v^2t}\mathrm{e}^{-4mt}, & t<|x|/2v.\end{cases} \tag{33}$$

These results are also in agreement with those obtained using a mapping to a boundary conformal field theory by Calabrese and Cardy for general quantum quenches from a non-critical to a critical state [9, 15].

3.3. *Quench from the gapless to the gapped phase*

We next consider the reverse situation of the one discussed in the previous subsection. In this case, we set $g(t) = g\theta(t)$ in equation (8), i.e. the initial state is critical and corresponds to the ground state of $H_\mathrm{i} = H_0$, whereas the time evolution is performed according to $H_\mathrm{f} = H_\mathrm{LE}$. We shall consider the same correlation functions as in the previous subsection and therefore it is convenient in this case to obtain the time evolution of the operators $\psi_r(p)$ and $\psi_l(p)$, whose action on the initial state is known (cf e.g. equations (17) and (18)). Once again, we first restrict ourselves to the $T = 0$ case and defer the discussion of finite temperature effects to section 5. We first note that the time-evolved Fermi operators can be related to the operators at $t = 0$ by means of the following (time-dependent) transformation:

$$\psi_r(p,t) = \mathrm{e}^{\mathrm{i}H_\mathrm{f}t}\psi_r(p)\mathrm{e}^{-\mathrm{i}H_\mathrm{f}t/\hbar} = f(p,t)\psi_r(p) + g^*(p,t)\psi_l(p), \tag{34}$$

$$\psi_l(p,t) = \mathrm{e}^{\mathrm{i}H_\mathrm{f}t}\psi_l(p)\mathrm{e}^{-\mathrm{i}H_\mathrm{f}t/\hbar} = g^*(p,t)\psi_r(p) + f^*(p,t)\psi_l(p), \tag{35}$$

where $f(p,t) = \cos\omega(p)t - \mathrm{i}\cos 2\theta(p)\sin\omega(p)t$ and $g(p,t) = \mathrm{i}\sin 2\theta(p)\sin\omega(p)t$. This transformation can be shown to respect the anti-commutation relations characteristic of Fermi statistics, and it is therefore a canonical transformation. Using equations (34) and (35), we can now compute the decay of the order parameter operator $\mathrm{e}^{2\mathrm{i}\varphi(x,t)}$. The calculation yields

$$\mathcal{C}(x,t) = -\frac{2}{L}\sum_{p>0}\mathrm{Re}\left[f^*(p,t)g(p,t)\right]. \tag{36}$$

In deriving the above expression, we have used that $f(-p,t) = f^*(p,t)$ and $g(-p,t) = g(p,t)$, which follow from $\cos 2\theta(-p) = -\cos 2\theta(p)$ because $\cos 2\theta(p) = \omega_0(p)/\omega(p)$. Thus, setting $\mathrm{Re}[f^*(p,t)g(p,t)] = -\cos 2\theta(p)\sin 2\theta(p)\sin^2\omega(p)t$ and taking $L \to +\infty$, we find that

$$\mathcal{C}(x,t) = 2\int_0^{+\infty}\frac{\mathrm{d}p}{2\pi}\frac{m\omega_0(p)}{[\omega(p)]^2}\mathrm{e}^{-a_0p}\sin^2\omega(p)t = A(ma_0) + \frac{m}{2\pi v}\mathrm{ci}(2mt), \tag{37}$$

where ci is the cosine integral function. The first term is a non-universal constant that depends on the short-distance cut-off a_0 introduced above (cf equation (9)). For long times this expression can be approximated by

$$\mathcal{C}(x,t) \simeq A(ma_0) + \frac{1}{4\pi vt}\sin 2mt + O(t^{-2}). \tag{38}$$

Hence we conclude that, when quenched from the critical (gapless) phase into the gapped phase, the order parameter exhibits an oscillatory decay towards a (non-universal) constant value, $A(ma_0)$.

Using similar methods the (equal-time) two-point correlation function, $\mathcal{G}(x,t) = \langle \mathrm{e}^{-2\mathrm{i}\varphi(x,t)}\mathrm{e}^{2\mathrm{i}\varphi(0,t)}\rangle$, can also be evaluated. The resulting expression can be cast in a form identical to equation (30) of subsection 3.2. In the thermodynamic limit, we find that, in the present case, the function $\mathcal{H}(x,t)$ takes the form

$$\mathcal{H}(x,t) = -\int_0^{\infty}\frac{\mathrm{d}p}{2\pi}\left\{-1 + [1-\cos 2\omega(p)t]\,\frac{m^2}{[\omega(p)]^2}\right\}\mathrm{e}^{-a_0p}\,\sin px. \tag{39}$$

We have not been able to obtain a closed analytical expression for this function at all times. However, in the $t \to +\infty$ limit, in which case the term in the integrand proportional to $\cos 2\omega(p)t$ oscillates very rapidly and therefore yields a vanishing contribution, an analytical expression can be obtained. Upon performing the momentum integral, we obtain the following result for large $|x|$ (after taking the limit $a_0 \to 0$):

$$\mathcal{H}(x, t \to \infty) \approx \frac{-4v^2}{2\pi m^2 |x|^3}. \tag{40}$$

Hence, we obtain the following asymptotic behavior of the two-point correlation for $t \to \infty$:

$$\lim_{t\to+\infty} \mathcal{G}(x, t) = [A(ma_0)]^2 + \frac{4v^4}{(2\pi)^2 m^4 x^6}. \tag{41}$$

This result is clearly different from the equilibrium behavior of the same correlation function in the gapped phase, where it decays exponentially to a constant [31, 32]. Instead, when the system is quenched from the gapless into the gapped phase, we find that both the order parameter and the two-point correlation function (equations (38) and (41)) decay algebraically to constant (non-universal) values.

4. The semiclassical limit

4.1. Introductory remarks

A good approximation to the sGM (cf equation (3)) can be obtained in the limit where $\kappa \ll 1$, which corresponds to the $K \ll 1$ limit in the original notation of equation (1). In this limit, we can expand the cosine term in (3) about one of its minima, e.g. $\varphi = 0$. Retaining only the leading quadratic term yields the following *quadratic* Hamiltonian for the boson field $\varphi(x)$:

$$H_{\mathrm{gG}} \simeq H_{\mathrm{sc}} = \frac{\hbar v}{2\pi} \int \mathrm{d}x \left[: (\partial_x \varphi(x))^2 + K \, (\partial_x \tilde{\varphi}(x))^2 : \right] + \frac{\hbar v g(t) \kappa^2}{2\pi a_0^{2-\kappa^2/2}} \int \mathrm{d}x \; : \varphi^2(x) : . \tag{42}$$

Within this approximation, the problem of studying a quantum quench in the sGM becomes akin to the general problem studied in [26]. To see this, let us first expand $\varphi(x)$ in Fourier modes:

$$\varphi(x) = \frac{\phi_0}{\sqrt{K}} + \mathrm{i}\frac{\pi x}{\sqrt{K} L} \delta N + \frac{1}{2} \sum_{q \neq 0} \left(\frac{2\pi v}{\omega_0(q) L} \right)^{1/2} \left[\mathrm{e}^{\mathrm{i}qx} b(q) + \mathrm{e}^{-\mathrm{i}qx} b^\dagger(q) \right], \tag{43}$$

where $\omega_0(q) = v|q|$; the b-operators introduced above obey the standard Heisenberg algebra:

$$\left[b(q), b^\dagger(q') \right] = \delta_{q,q'}, \tag{44}$$

commuting otherwise. The first two terms in equation (43) are the so-called zero modes, whose dynamics is only important at finite L. In what follows we restrict our attention to the thermodynamic limit ($L \to \infty$) and therefore neglect the dynamics of those zero modes. Introducing (43) into (42), the Hamiltonian takes the general form

$$H(t) = \sum_q \hbar \left[\omega_0(q) + m(q, t) \right] b^\dagger(q) b(q) + \frac{1}{2} \sum_q \hbar g(q, t) \left[b(q) b(-q) + b^\dagger(q) b^\dagger(-q) \right], \tag{45}$$

with the following identifications: $\omega_0(q) = v|q|$ and $g(q,t) = m(q,t) = 2vg(t)\kappa^2/|q|a_0^{2-\kappa^2/2}$. As in the study of the LE limit, we shall assume that $g(t) = g\theta(-t)$, which corresponds to a quench from the gapped to a gapless phase[4], or $g(t) = g\theta(t)$, which corresponds to a quench from the gapless to the gapped phase. Following the procedure described in the appendix of [26], the quench dynamics of this Hamiltonian can be solved by the following canonical transformation (indeed, the bosonic version of equations (34) and (35):

$$b(q,t) = f(q,t)\, b(q) + g^*(q,t)\, b^\dagger(-q), \tag{46}$$

where

$$f(q,t) = \cos\omega(q)t - \mathrm{i}\sin\omega(q)t\ \cosh 2\beta(q), \tag{47}$$

$$g(q,t) = \mathrm{i}\sin\omega(q)t\ \sinh 2\beta(q). \tag{48}$$

Introducing $m^2 = 4\, g v^2\kappa^2/a_0^{2-\kappa^2/2}$, which is the gap in the frequency spectrum of the gapped phase and setting $m(q) = g(q) = m^2/2\omega_0(q)$, the parameter $\beta(q)$ satisfies

$$\tanh 2\beta(q) = \frac{m^2/2}{\omega_0^2(q) + m^2/2}, \tag{49}$$

and the frequency

$$\omega(q) = \sqrt{\omega_0(q)^2 + m^2} \tag{50}$$

is the dispersion of the excitations in the gapped phase.

4.2. *Quench from the gapped to the gapless phase*

Let us begin by discussing the situation where $g(t) = g\theta(-t)$. In this case, the initial state is the ground state of the following Hamiltonian (we omit the zero-mode part henceforth):

$$H_\mathrm{i} = H_\mathrm{sc} = \sum_{q\neq 0} \hbar\omega(q)\, a^\dagger(q) a(q), \tag{51}$$

where the operators $a(q)$ and $a^\dagger(q)$ are bosonic operators related to $b(q)$ and $b^\dagger(q)$ by means of the following canonical transformation:

$$a(q) = \cosh\beta(q)\, b(q) + \sinh\beta(q)\, b^\dagger(-q), \tag{52}$$

with $\beta(q)$ satisfying equation (49). At $t = 0$ the Hamiltonian abruptly changes to $H_\mathrm{f} = H_0$, which is diagonal in the $b(q)$ and $b^\dagger(q)$ basis, namely

$$H_0 = \sum_{q\neq 0} \hbar\omega_0(q)\, b^\dagger(q) b(q). \tag{53}$$

In this case the evolution of the expectation value of the order parameter operator $\mathrm{e}^{-2\mathrm{i}\phi(x)} = \mathrm{e}^{-\mathrm{i}\kappa\varphi(x)}$ or its correlation functions can be obtained from a knowledge of the two-point (equal time) correlation function out of equilibrium for the boson field $\varphi(x)$, i.e. $\mathcal{F}(x,t) = \langle\varphi(x,t)\varphi(0,t)\rangle - \langle\varphi^2(0,t)\rangle$, where the expectation value is taken over the ground state of H_sc but the time evolution is dictated by H_0. To compute this object, we first insert into the

[4] This case was studied earlier by Calabrese and Cardy in [15], although not as a limit of the sGM, and therefore they considered the quench dynamics of different observables.

expectation value the Fourier expansion of $\varphi(x)$, equation (43), and use equation (46). Thus, we arrive at

$$\langle\varphi(x,t)\varphi(0,t)\rangle = -\frac{1}{4}\sum_{q\neq 0}\left(\frac{2\pi v}{\omega_0(q)L}\right) \times\Big[\sinh 2\beta(q)\cos(qx-2\omega_0(q)t) - e^{iqx}\sinh^2\beta(q) - e^{-iqx}\cosh^2\beta(q)\Big]. \quad (54)$$

Using this result, let us consider the behavior of the order parameter following the quench. Taking into account that $\langle e^{-2i\phi(0,t)}\rangle = \langle e^{-i\kappa\varphi(0,t)}\rangle = e^{-\frac{\kappa^2}{2}\langle\varphi^2(0,t)\rangle}$, we see that $\langle\varphi^2(0,t)\rangle$ must be evaluated in closed form using equation (54). Before performing any manipulation of this expression, it is convenient to subtract the constant $\langle\varphi^2(0,0)\rangle$, which is formally infinite (i.e. it depends on the short distance cut-off, a_0). Thus, taking the thermodynamic limit where $L\to\infty$, we obtain

$$\langle\varphi^2(0,t)\rangle - \langle\varphi^2(0,0)\rangle = \frac{1}{2}\int_0^{+\infty}\frac{d(qv)}{\omega(q)}\left[\left(\frac{\omega(q)}{\omega_0(q)}\right)^2 - 1\right]\sin^2\omega_0(q)t. \quad (55)$$

Inserting the expressions for $\omega(p)$ and $\omega_0(p)$ in the above equation, we obtain

$$\langle\varphi^2(0,t)\rangle - \langle\varphi^2(0,0)\rangle = -f(2mt/\hbar), \quad (56)$$

where $f(z)$ is defined as

$$f(z) = 1 + \frac{1}{2}G^{21}_{13}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}0 & 3/2\\ & 1\\ & 1/2\end{matrix}\right), \quad (57)$$

G^{21}_{13} being the Meijer G function [39]. Using the asymptotic expansion for this function, $f(z)\approx 1-\frac{\pi|z|}{2}$; hence, the long-time behavior of the order parameter is

$$\langle e^{-2i\phi(0,t)}\rangle = \langle e^{-2i\kappa\varphi(0,t)}\rangle = \langle e^{2i\phi(0,0)}\rangle e^{\kappa^2/8(1-\pi mt)}. \quad (58)$$

We next examine the behavior of the two-point correlation function of the same (order-parameter) operator,

$$\mathcal{G}(x,t) = \langle e^{2i\phi(x,t)}e^{-2i\phi(0,t)}\rangle = e^{\frac{\kappa^2}{2}\mathcal{F}(x,t)}, \quad (59)$$

where we have defined $\mathcal{F}(x,t) = \langle\varphi(x,t)\varphi(0,t)\rangle - \langle\varphi^2(0,t)\rangle$. At zero temperature, with the help of equation (54), we find that

$$\mathcal{F}(x,t) - \mathcal{F}(x,0) = -\frac{m^2}{4}\int_0^{\infty}\frac{d(vq)}{[\omega_0(q)]^2\omega(q)}(1-\cos qx)(1-\cos 2\omega_0(q)t), \quad (60)$$

where

$$\mathcal{F}(x,0) = -\frac{1}{2}\int_0^{\infty}\frac{d(vq)}{\omega(q)}(1-\cos qx)\,e^{-qa_0}, \quad (61)$$

with a_0 being the short-distance cutoff. Evaluating the integrals

$$\mathcal{F}(x,t) - \mathcal{F}(x,0) = f(mx/v) + f(2mt) - \frac{f[m(x/v+2t)] + f[m(x/v-2t)]}{2}, \quad (62)$$

where $f(z)$ has been defined in equation (57). Thus, asymptotically (for $\max\{|x|/2v, t\}\gg m^{-1}$),

$$\mathcal{G}(x,t) = e^{\kappa^2}\mathcal{G}(x,0)\times\begin{cases} e^{-\kappa^2\pi m|x|/2v}, & \text{for} \quad t>|x|/2v,\\ e^{-\kappa^2\pi mt}, & \text{for} \quad t<|x|/2v,\end{cases} \quad (63)$$

where $\mathcal{G}(x,0)$ describes the correlations in the initial (gapped ground) state and exhibits the following asymptotic behavior:

$$\mathcal{G}(x,0) \simeq B(a_0)\left(1-\kappa^2\frac{\pi v}{mx}\mathrm{e}^{-m|x|/v}\right), \tag{64}$$

where $B(a_0)$ is a non-universal prefactor. Thus we see that the asymptotic form of $\mathcal{G}(x,t)$ (equation (63)), as well as that of the order parameter, equation (58), has the same form as the results found limit the LE limit, and also agree with the results of Calabrese and Cardy [9, 15] based on a mapping to boundary conformal field theory.

4.3. Quench from the gapless to the gapped phase

In this case, the system finds itself initially in the ground state of $H_\mathrm{i} = H_0$, and suddenly (at $t=0$) the Hamiltonian is changed to $H_\mathrm{f} = H_\mathrm{sc}$. For this situation, it is convenient to obtain the evolution of the observables from the time-dependent canonical transformation of equation (46), where $\beta(q)$ and $\omega(q)$ are given by equations (49) and (50), respectively. In this case,

$$\langle\varphi(x,t)\varphi(0,t)\rangle = \frac{1}{4}\sum_{q\neq 0}\left(\frac{2\pi v}{\omega(q)L}\right)\Big[\sinh 2\beta(q)\cos\left(qx-2\omega(q)t\right) + \mathrm{e}^{\mathrm{i}qx}\sinh^2\beta(q)+\mathrm{e}^{-\mathrm{i}qx}\cosh^2\beta(q)\Big]. \tag{65}$$

As in the previous subsection, $\langle \mathrm{e}^{-2\mathrm{i}\phi(x)}\rangle = \mathrm{e}^{-\frac{\kappa^2}{2}\langle\varphi^2(0,t)\rangle}$, and using equation (65), we find that

$$\langle\varphi^2(0,t)\rangle - \langle\varphi^2(0,0)\rangle = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}(vq)}{\omega_0(q)}\left[\left(\frac{\omega_0(q)}{\omega(q)}\right)^2-1\right]\sin^2\omega(q)t. \tag{66}$$

Note, interestingly, that this result can be obtained from equation (55) by exchanging $\omega_0(q)$ and $\omega(q)$. However, when evaluating the integral we find that $\langle\varphi^2(0,t)\rangle = +\infty$, for all $t>0$, due to the presence of infrared divergences that are not cured by the existence of a gap in the spectrum of $H_\mathrm{f} = H_\mathrm{sc}$. Thus, we conclude that $\langle \mathrm{e}^{-2\mathrm{i}\phi(x)}\rangle = \mathrm{e}^{-\frac{\kappa^2}{2}\langle\varphi^2(0,t)\rangle}$ vanishes at all $t>0$.

The above result for the evolution of the order parameter seems to indicate that the system apparently remains critical after the quench. This conclusion is also supported by the behavior of the two-point correlation function of the operator $\mathrm{e}^{2\mathrm{i}\phi(x,t)}$: let $\mathcal{G}(x,t) = \langle \mathrm{e}^{2\mathrm{i}\phi(x,t)}\mathrm{e}^{-2\mathrm{i}\phi(0,t)}\rangle = \mathrm{e}^{\kappa^2\mathcal{F}(x,t)}$, where $\mathcal{F}(x,t) = \langle\varphi(x,t)\varphi(0,t)\rangle - \langle\varphi^2(0,t)\rangle$. Using equations (46) and (43), we arrive at the following result (at zero temperature, and for $L\to+\infty$):

$$\mathcal{F}(x,t)-\mathcal{F}(x,0) = \frac{m^2}{4}\int_0^\infty \frac{\mathrm{d}(vq)}{\omega_0(q)\left[\omega(q)\right]^2}(1-\cos qx)\left(1-\cos 2\omega(q)t\right), \tag{67}$$

where

$$\mathcal{F}(x,0) = -\int_0^\infty \frac{\mathrm{d}(vq)}{2\omega_0(q)}(1-\cos qx)\mathrm{e}^{-qa_0}. \tag{68}$$

To illustrate the above point about the apparent 'criticality' of the (asymptotic) non-equilibrium state, we can analyze the behavior of the two-point correlation function, $\mathcal{G}(x,t)$, in two limiting

cases, for $t = 0$ and $t \to +\infty$. At $t = 0$, the correlation function, as obtained from equation (68), reads

$$\mathcal{G}(x, 0) = A(a_0) \left(\frac{a_0}{x}\right)^{2\kappa^2}, \tag{69}$$

where $A(a_0)$ depends on the short-distance cut-off a_0. Thus, the correlations are power-law because the initial state is critical. In the limit where $t \to +\infty$, the part of the integral in equation (67) containing the term $\cos 2\omega(q)t$ oscillates very rapidly and upon integration, averages to zero. The remaining integral can be done with the help of tables [39], yielding

$$\mathcal{F}(x, t \to \infty) - \mathcal{F}(x, 0) = -\frac{\sqrt{\pi}}{2} G^{22}_{04}\left(\frac{m^2x^2}{4v^2}\,\middle|\,\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\;\begin{matrix} 0 & 1/2 \end{matrix}\right), \tag{70}$$

where G^{22}_{04} is a Meijer function. Using the asymptotic behavior of the Meijer function [39], we obtain

$$\lim_{t\to+\infty} \mathcal{G}(x, t) = B(a_0) \left(\frac{2v}{mx}\right)^{\kappa^2}, \tag{71}$$

with $B(a_0)$ being a non-universal constant. Thus, although initially the system is critical and therefore correlations at equilibrium decay as a power law with exponent $2\kappa^2$, when the system is quenched into a gapped phase (where equilibrium correlations exhibit an exponential decay characterized by a correlation length $\xi_c \approx v/m$), the correlations remain power law, within the semiclassical approximation. The exponent turns out to be smaller, equal to κ^2, which is half of the exponent in the initial (gapless) state. In other words, within this approximation, it seems that the system keeps memory of its initial state, and behaves as if it was critical *also* after the quench. This behavior seems somewhat different from the results obtained for the same type of quench in the LE limit, where both the order parameter and the correlations for $t + \infty$ approach a constant value, $A(ma_0)$ (unless the non-universal amplitude $A(ma_0) = 0$, which seems to require some fine-tuning). It is not clear at the moment whether the differences found here between the LE and the semi-classical limits are due to a breakdown of the quasi-classical approximation, which neglects the existence of solitons and anti-solitons in the spectrum of the sGM, or due to a qualitative change in the dynamics as one moves away from the LE limit. To clarify this issue will require further investigation with more sophisticated methods.

5. Dynamics in the Luther–Emery limit at finite temperatures

In this section, we shall consider that the initial state of the system corresponds to the thermal mixed state, which describes a sGM system in contact with an energy reservoir (i.e. the canonical ensemble) at a temperature $T = \beta^{-1}$. The state is thus mathematically described by a density operator $\rho_i = e^{-H_i/T}/Z_i$. We shall assume that the coupling to the thermal bath is turned off at $t = 0$, and the system subsequently evolves unitarily in isolation according to H_f. We shall consider only the LE limit of the sGM, where exact results can be obtained at all temperatures within the sGM model. The latter is not true in the semiclassical limit discussed above because this approximation only captures the breather part of the spectrum and not the solitonic part. We shall therefore not consider finite temperature quenches in this limit here.

5.1. Quench from massive to massless

Consider first the quench from the gapped to the gapless phase. The initial (gapped) Hamiltonian $H_\mathrm{i} = H_\mathrm{LE}$ is thus diagonal in the valence and conduction fermion basis and therefore immediately follows

$$\langle \psi_\mathrm{v}^\dagger(p)\psi_\mathrm{v}(p')\rangle = f_\mathrm{F}[-\omega(p)]\delta_{pp'}, \tag{72}$$

$$\langle \psi_\mathrm{c}^\dagger(p)\psi_\mathrm{c}(p')\rangle = f_\mathrm{F}[\omega(p)]\delta_{pp'}, \tag{73}$$

$$\langle \psi_\mathrm{v}^\dagger(p)\psi_\mathrm{c}(p')\rangle = 0. \tag{74}$$

Here f_F is the Fermi factor and

$$\langle \cdots \rangle = \mathrm{Tr}\{\mathrm{e}^{-\beta H_\mathrm{i}} \cdots\}. \tag{75}$$

The final Hamiltonian $H_\mathrm{f} = H_0$; thus, using equations (72)–(74), we can compute the finite temperature versions of equations (27)–(29):

$$\langle \psi_R^\dagger(p)\psi_R(p)\rangle = \cos^2\theta_p f_\mathrm{F}[\omega(p)] + \sin^2\theta_p f_\mathrm{F}[-\omega(p)], \tag{76}$$

$$\langle \psi_L^\dagger(p)\psi_L(p)\rangle = \sin^2\theta_p f_\mathrm{F}[\omega(p)] + \cos^2\theta_p f_\mathrm{F}[-\omega(p)], \tag{77}$$

$$\langle \psi_R^\dagger(p)\psi_L(p)\rangle = \langle \psi_L^\dagger(p)\psi_R(p)\rangle = -\frac{1}{2}\sin 2\theta_p \tanh\frac{\beta\omega(p)}{2}. \tag{78}$$

Note that these averages automatically vanish if the fermion operators are evaluated at different values of p because of momentum conservation. The time evolution is again dictated by $H_\mathrm{f} = H_0$ as in the zero-temperature case.

Let us next consider some interesting observables. We begin with the order parameter, which reads

$$\mathcal{C}(t;\beta) = m\int_0^\infty \frac{\mathrm{d}p}{2\pi}\frac{\cos 2\omega_0(p)t}{\omega(p)}\tanh\frac{\hbar\beta\omega(p)}{2}. \tag{79}$$

This integral can be transformed into an infinite sum by expanding $\tanh\beta\varepsilon/2$ in powers of $\mathrm{e}^{-\beta\varepsilon}$ and integrating term by term. We thus obtain the following low-temperature expansion:

$$\mathcal{C}(t;\beta) = \frac{m}{2\pi v}\sum_{n\in\mathbb{Z}}(-1)^n K_0\left[\sqrt{(2mt)^2 + (n\hbar\beta m)^2}\right]. \tag{80}$$

Since the function K_0 decays exponentially for large values of its argument, from this expression we see that when the temperature is decreased, less terms are needed to approximate the sum. In particular, at zero temperature ($\beta\to\infty$), only the $n = 0$ term contributes and we recover the zero temperature result of equation (23). At finite but low temperatures, the asymptotic behavior at long times is an exponential decay where the characteristic time decay is fixed by the gap. However, at higher temperatures, more terms contribute to the sum, whereas the alternating sign leads to some partial cancellations. As a result, we expect a faster decay in time of the order parameter. To further analyze the long-time behavior in this regime, we shall use an identity of Bessel functions (see equation (A.1)) which allows us to obtain the following high-temperature expansion:

$$\mathcal{C}(t;\beta) = \frac{m}{2v}\sum_{l\in\mathbb{Z}}\frac{\exp\left[-\frac{2t}{\hbar\beta}\sqrt{(\hbar\beta m)^2 + \pi^2(2l+1)^2}\right]}{\sqrt{(\hbar\beta m)^2 + \pi^2(2l+1)^2}}. \tag{81}$$

Note that, for this expansion, the higher the temperature, the smaller the number of terms that need to be retained to accurately approximate the sum. In particular, in the infinite temperature limit, $\beta m\hbar \ll 1$, only the $l = 0$ terms contribute and thus the decay in time is exponential:

$$\mathcal{C}(t;\beta) \approx \frac{m}{2\pi v}\exp[-t/\tau_c], \quad \hbar\beta m \ll 1, \tag{82}$$

but now the characteristic decay time τ_c is fixed by the inverse temperature:

$$\tau_c = \frac{\hbar\beta}{2\pi}. \tag{83}$$

Thus, to summarize, the asymptotic behavior of the order parameter following a quench from the gap to the gapless phase is described by an exponential decay both at very low and very high temperatures. At very low temperatures, the characteristic decay time is given by the frequency gap, m, but as the temperature of the initial state is increased, the characteristic time is determined by the temperature. The behavior for intermediate temperatures is a crossover between these two types of exponentially decaying behavior. Moreover, it is also worth mentioning that the exponential decay at large temperatures is characteristic of a critical theory at finite temperatures. Indeed, in the sGM, we expect that, as the temperature is raised well above the gap energy scale, $\hbar m$, the properties of the system will become indistinguishable from those of a critical system.

As for the finite-temperature correlation function, it can be again cast in the same form as the zero temperature case, equation (30), with the function $\mathcal{H}(x,t;\beta)$ having the following $t \to +\infty$ limit:

$$\lim_{t\to+\infty}\mathcal{H}(x,t;\beta) = \int_0^\infty \frac{\mathrm{d}p}{2\pi}\frac{\omega_0(p)}{\omega(p)}\left[\tanh\frac{\hbar\beta\omega(p)}{2}\right]\sin px. \tag{84}$$

Using the same technique as for the order parameter, we evaluate this function as

$$\lim_{t\to+\infty}\mathcal{H}(x,t;\beta) = \frac{mx}{2\pi v^2}\sum_{n=1}^{\infty}(-1)^n\frac{K_1\left(m\sqrt{x^2/v^2+n^2\hbar^2\beta^2}\right)}{\sqrt{x^2/v^2+n^2\hbar^2\beta^2}}. \tag{85}$$

The reasoning is the same as before: at very low temperatures, only the term with $n = 0$ contributes, and we recover the zero temperature expression. However, as the temperature increases, more terms become important. We can use the dual expression equation (A.2), which when applied to equation (85) yields

$$\lim_{t\to+\infty}\mathcal{H}(x,t;\beta) = \frac{1}{2v\hbar\beta}\sum_{l\in\mathbb{Z}}\exp\left[-\frac{x}{v\hbar\beta}\sqrt{(m\hbar\beta)^2+\pi^2(2l+1)^2}\right]. \tag{86}$$

This function decays exponentially for long distances (only the $l = 0$ term contributes at high temperatures), with a characteristic correlation length that is fixed by the temperature of the initial state:

$$\lim_{t\to+\infty}\mathcal{H}(x,t;\beta) \simeq \frac{1}{2v\hbar\beta}\mathrm{e}^{-|x|/\xi}, \quad \xi = \frac{v\hbar\beta}{\pi}. \tag{87}$$

Introducing these results in (the finite temperature equivalent of) equation (30), we obtain an expression whose asymptotic behavior again depends on whether $|x| > 2vt$ or $|x| < 2vt$ (with correlation time/lengths given by m or T depending on the temperature range). Thus, the correlations at finite temperature also will exhibit the so-called 'light-cone' effect [9, 15].

5.2. Quench from the gapless to the gapped phase

We next consider that the system is quenched from the gapless phase to the gapped phase. Thus we need to assume that the system was initially described by $H_{\rm i} = H_{\rm LE}$ and in contact with an energy reservoir at temperature $T = \beta^{-1}$. The following expectation values (understood over the initial thermal ensemble) will be required in the calculations to follow:

$$\langle \psi_R^\dagger(p)\psi_R(p)\rangle = f_{\rm F}[\omega_0(p)], \tag{88}$$

$$\langle \psi_L^\dagger(p)\psi_L(p)\rangle = f_{\rm F}[-\omega_0(p)], \tag{89}$$

$$\langle \psi_R^\dagger(p)\psi_L(p)\rangle = \langle \psi_L^\dagger(p)\psi_R(p)\rangle = 0 \tag{90}$$

and hence

$$\langle \psi_{\rm v}^\dagger(p)\psi_{\rm v}(p)\rangle = \frac{1}{2}\left(1+\cos 2\theta_p \tanh\frac{\beta\omega_0(p)}{2}\right), \tag{91}$$

$$\langle \psi_{\rm c}^\dagger(p)\psi_{\rm c}(p)\rangle = \frac{1}{2}\left(1-\cos 2\theta_p \tanh\frac{\beta\omega_0(p)}{2}\right). \tag{92}$$

Thus, the time evolution of the order parameter can be obtained and reads

$$\mathcal{C}(t;\beta) = 2\int_0^\infty \frac{{\rm d}p}{2\pi}\frac{m\omega_0(p)}{\omega(p)^2}\tanh\frac{\hbar\beta\omega_0(p)}{2}{\rm e}^{-a_0 p}\sin^2\omega(p)t. \tag{93}$$

As $t\to+\infty$ this function approaches a non-universal constant that depends on the energy cut-off a_0 and the temperature. At high temperatures,

$$\mathcal{C}(t,\beta) = A(ma_0,\beta) + \hbar\beta\frac{\sqrt{\pi m}}{16\pi v t^{3/2}}\sin\left(2mt+\frac{\pi}{4}\right). \tag{94}$$

Thus, after the sudden quench at $t=0$ from a high-temperature state in the critical regime to the gapped phase, the order parameter shows an oscillatory decay towards a constant value. However, the exponent of the decaying law is different from the decaying exponent in the case of a quench from a zero (or low) temperature state. The whole picture is the following: the order parameter exhibits an oscillatory decaying low as $t^{-3/2}$ for times smaller than a time scale fixed by the temperature, where there is a crossover to a $t^{-1/2}$ behavior characteristic of low temperatures.

Concerning the two-point correlation function, it can again be recast as in equation (30) with the function $\mathcal{H}(x,t;\beta)$ being given by

$$\mathcal{H}(x,t;\beta) = -\int_0^\infty \frac{{\rm d}p}{2\pi}\sin px\left\{-1+[1-\cos 2\omega(p)t]\frac{m^2}{\omega(p)^2}\right\}\tanh\frac{\hbar\beta\omega_0(p)}{2}{\rm e}^{-a_0 p}. \tag{95}$$

At long times, the cosine within the integral oscillates very rapidly, yielding a vanishing contribution. The remaining integral can be evaluated using the Cauchy theorem, resulting in an infinite sum over positive odd Matsubara frequencies. The sum can be performed, yielding

$$\mathcal{H}(x,t\to\infty) = -\frac{m}{4v}\frac{1}{\pi}{\rm e}^{-\pi x/v\hbar\beta}\left[\Phi\left({\rm e}^{-2\pi x/v\hbar\beta},1,\frac{1}{2}+\frac{\hbar\beta m}{2\pi}\right)-\Phi\left({\rm e}^{-2\pi x/v\hbar\beta},1,\frac{1}{2}-\frac{\hbar\beta m}{2\pi}\right)\right]$$
$$+\frac{1}{2\hbar\beta v}{\rm cosech}\left(\frac{\pi x}{\hbar v\beta}\right)-\frac{m}{4v}{\rm e}^{-mx/v}\tan\frac{\hbar\beta m}{2}, \tag{96}$$

where $\Phi(x, y, z)$ is the Lerch function [39]. Likewise, the long-distance behavior is dominated by the lowest (Matsubara) frequency term:

$$\mathcal{H}(x, t \to \infty) \approx -\frac{1}{v\hbar\beta}\left[\frac{\pi^2}{(\hbar\beta m)^2 - \pi^2}\right] e^{-\pi v x/\hbar\beta} - \frac{m}{4v} e^{-mx/v} \tan\frac{\hbar\beta m}{2}. \quad (97)$$

At long distances, this function decays exponentially, but two competing length scales appear: v/m and $v\hbar\beta/\pi$. The largest sets the characteristic length of the decay.

6. Long-time dynamics and the generalized Gibbs ensemble

It was recently pointed out by Rigol *et al* [14] that, at least for certain observables like the momentum distribution or the density, the asymptotic (long-time) behavior of an integrable system following a quantum quench can be described by adopting the maximum entropy (also called 'subjective') approach to statistical mechanics, pioneered by Jaynes [40, 41]. Within this approach, the equilibrium state of a system is described by a density matrix that extremizes the von Neumann entropy, $S = -\mathrm{Tr}\rho \ln \rho$, subject to all possible constraints provided by the integrals of motion of the Hamiltonian of the system. In the case of an integrable system, if $\{I_m\}$ is a set of certain (but not all of the possible) independent integrals of motion of the system, this procedure leads to a 'generalized' Gibbs ensemble, described by the following density matrix:

$$\rho_{\mathrm{gG}} = \frac{1}{Z_{\mathrm{gG}}} e^{-\sum_m \lambda_m I_m}, \quad (98)$$

where $Z_{\mathrm{gG}} = \mathrm{Tr}\, e^{-\sum_m \lambda_m I_m}$. The values of the Lagrange multipliers, λ_m, must be determined from the condition that

$$\langle I_m \rangle_{\mathrm{gG}} = \mathrm{Tr}\,[\rho_0 I_m] = \langle I_m \rangle, \quad (99)$$

where ρ_0 describes the initial state of the system, and $\langle \cdots \rangle_{\mathrm{gG}}$ stands for the average taken over the generalized Gibbs ensemble, equation (98). Although $\rho_i = |\Phi(t=0)\rangle\langle\Phi(t=0)|$ in the case of a pure state, as was first used in [14], nothing prevents us from taking ρ_0 to be an arbitrary mixed state and, in particular, a thermal state characterized by an absolute temperature T. In this case, the Lagrange multipliers will depend on T or any other parameter that defines the initial state.

Rigol *et al* numerically tested the above conjecture by studying the quench dynamics of a 1D lattice gas of hard-core bosons (see [14] for more details). One of us showed analytically [11] that correlations of the Luttinger model also relax to averages taken over this ensemble. This result for the Luttinger model was extended to include finite temperature fluctuations in the initial state in [26]. The question that naturally arises then is whether the family of integrable models studied in this work (see equation (45) and their fermionic equivalences of equation (12) and (13)) relax in agreement with the mentioned conjecture. In other words, does the average $\langle O \rangle(t)$ at long times relax to the value $\langle O \rangle_{\mathrm{gG}} = \mathrm{Tr}\,\rho_{\mathrm{gG}}\, O$, for any of the correlation functions considered previously? In what follows, we shall address this question by analyzing quantum quenches in the sGM at zero temperature. The generalization at finite temperatures should be straightforward, as discussed in [26].

6.1. Quench from the gapped to the gapless phase in the Luther–Emery limit

In this case, the type evolution of the system is performed by H_0 (cf equation (11)), which is diagonal in the operators $n_\alpha(p) =: \psi_\alpha^\dagger(p)\psi_\alpha(p) :$ $(\alpha = r, l)$. Thus, the generalized Gibbs ensemble is defined by the following set of integrals of motion $I_m \to I_\alpha(p) = n_\alpha(p)$. We see immediately that the fact that this ensemble is diagonal in $n_\alpha^F(p)$ means that the order parameter, $\langle e^{-2i\varphi(x)}\rangle_{gG} = \langle \psi_r(x)\psi_l(x)\rangle_{gG} = 0$, which agrees with the $t \to +\infty$ limit of the order parameter, was shown in section 3 to exhibit an exponential decay to zero. However, the two-point correlator of $e^{2i\varphi(x)}$ has a non-vanishing limit for $t \to +\infty$. Thus, our main concern here will be the calculation of the correlation function:

$$\langle e^{-2i\varphi(x)} e^{2i\varphi(0)}\rangle_{gG} = \langle \psi_r^\dagger(x)\psi_l(x)\psi_l^\dagger(0)\psi_r(0)\rangle_{gG} \tag{100}$$

$$= \sum_{p_1,p_2,p_3,p_4} \frac{e^{i(p_1-p_2)x}}{L^2} \langle \psi_r^\dagger(p_1)\psi_l(p_2)\psi_l^\dagger(p_3)\psi_r(p_4)\rangle_{gG}. \tag{101}$$

Since the ensemble is diagonal in the chirality index, α, as well as the momentum, p, an evaluation of the above expression can be carried out by noting that

$$\langle \psi_\alpha^\dagger(p)\psi_\alpha(p')\rangle_{gG} = \frac{\mathrm{Tr}\, e^{-\sum_{p,\alpha'} \lambda_{\alpha'}(p) I_{\alpha'}(p)} \psi_\alpha^\dagger(p)\psi_\alpha(p)}{\mathrm{Tr}\, e^{-\sum_{p,\alpha'} \lambda_{\alpha'}(p) I_{\alpha'}(p)}} = \frac{\delta_{p,p'}}{e^{\lambda_\alpha(p)} + 1}, \tag{102}$$

where the Lagrange multipliers $\lambda(q)$ can be related to the values of the same expectation values in the initial states by imposing their conservation, that is,

$$\langle \psi_l^\dagger(p)\psi_l(p)\rangle_{gG} = \frac{1}{e^{\lambda_l(p)} + 1} = \langle \psi_l^\dagger(p)\psi_l(p)\rangle = \cos^2\theta(p), \tag{103}$$

$$\langle \psi_r^\dagger(p)\psi_r(p)\rangle_{gG} = \frac{1}{e^{\lambda_r(p)} + 1} = \langle \psi_r^\dagger(p)\psi_r(p)\rangle = \sin^2\theta(p). \tag{104}$$

Hence,

$$\langle \psi_r^\dagger(p_1)\psi_l(p_2)\psi_l^\dagger(p_3)\psi_r(p_4)\rangle_{gG} = \langle \psi_r^\dagger(p_1)\psi_r(p_4)\rangle_{gG} \times \langle \psi_l(p_2)\psi_l^\dagger(p_3)\rangle_{gG} \tag{105}$$

$$= \delta_{p_1,p_4}\delta_{p_2,p_3} \sin^2\theta(p_1)\left(1 - \cos^2\theta(p_2)\right) \tag{106}$$

$$= \delta_{p_1,p_4}\delta_{p_2,p_3} \sin^2\theta(p_1)\sin^2\theta(p_2). \tag{107}$$

Introducing the last expression into equation (101) yields

$$\langle e^{-2i\varphi(x)} e^{2i\varphi(0)}\rangle_{gG} = \left| \frac{1}{L} \sum_p e^{ipx} \sin^2\theta(p) \right|^2 \tag{108}$$

and using that $\sin^2\theta(p) = (1 - \cos^2 2\theta(p))/2$ and $\cos 2\theta(p) = \omega_0(p)/\sqrt{\omega_0^2(p) + m^2}$, we find that (for $x \neq 0$)

$$\langle e^{-2i\varphi(x)} e^{2i\varphi(0)}\rangle_{gG} = \left(\frac{m}{2\pi v}\right)^2 \left[K_1\left(\frac{m|x|}{v}\right)\right]^2, \tag{109}$$

which is equal to the $t \to +\infty$ limit of equation (32).

6.2. Quench from the gapless to the gapped phase in the Luther–Emery limit

In this case the initial state is the gapless ground state of H_0, equation (11), whereas the Hamiltonian that performs the time evolution has a gap in the spectrum and it is diagonal in the basis of the $\psi_v(p)$ and $\psi_c(p)$ Fermi operators (cf equation (16)). Therefore, the conserved

quantities are

$$I_v(p) = n_v(p) = \psi_v^\dagger(p)\psi_v(p), \tag{110}$$

$$I_c(p) = n_c(p) = \psi_c^\dagger(p)\psi_c(p). \tag{111}$$

The associated Lagrange multipliers (at zero temperature), $\lambda_v(p)$ and $\lambda_c(p)$, can be obtained by equating $\langle I_{v,c}(p)\rangle_{\rm gG} = \langle\Psi(0)|I_{v,c}(p)|\Psi(0)\rangle$. This yields

$$\langle I_v(p)\rangle_{\rm gG} = \frac{1}{{\rm e}^{\lambda_v(p)}+1} = \vartheta(-p)\sin^2\theta(p) + \vartheta(p)\cos^2\theta(p), \tag{112}$$

$$\langle I_c(p)\rangle_{\rm gG} = \frac{1}{{\rm e}^{\lambda_c(p)}+1} = \vartheta(-p)\cos^2\theta(p) + \vartheta(p)\sin^2\theta(p), \tag{113}$$

where $\vartheta(p)$ denotes the step function. Using these expressions we next proceed to compute the expectation values of the following observables:

6.2.1. Order parameter. We start by computing the order parameter

$$\langle {\rm e}^{-2{\rm i}\varphi(x)}\rangle_{\rm gG} = \langle\psi_R^\dagger(x)\psi_L(x)\rangle = \frac{1}{2L}\sum_p \sin 2\theta(p)\left[\langle I_c(p)\rangle_{\rm gG} - \langle I_v(p)\rangle_{\rm gG}\right], \tag{114}$$

and by using equations (112) and (113),

$$\langle {\rm e}^{-2{\rm i}\varphi(x)}\rangle_{\rm gG} = -\frac{1}{L}\sum_{p>0}\sin 2\theta(p)\cos 2\theta(p) = -\int_0^\infty \frac{{\rm d}p}{2\pi}\,\frac{m\omega_0(p)}{\omega^2(p)}{\rm e}^{-pa_0} = A(ma_0), \tag{115}$$

where we have assumed that $\cos 2\theta_{-p} = -\cos 2\theta_p$; $A(ma_0)$ is the non-universal constant introduced in section 4.3. This result agrees with the one obtained in section 4.3 for the order parameter in the limit $t \to +\infty$.

6.2.2. Two-point correlation function. We next consider the two-point correlator of the order parameter, namely

$$\langle {\rm e}^{2{\rm i}\varphi(x)}{\rm e}^{2{\rm i}\varphi(0)}\rangle_{\rm gG} = \frac{1}{L^2}\sum_{p_1,p_2,p_3,p_4} {\rm e}^{{\rm i}(p_1-p_2)x}\langle\psi_r^\dagger(p_1)\psi_l(p_2)\psi_l^\dagger(p_3)\psi_r(p_4)\rangle_{\rm gG}. \tag{116}$$

The calculation of the average in this case is slightly more involved, but it can be performed by resorting to a factorization akin to Wick's theorem. This is applicable only in the thermodynamic limit, as it neglects terms in which the four momenta of the above expectation value are equal. These terms yield contributions of $O(1/L)$ compared to the others. By factorizing as dictated by Wick's theorem, the only non-vanishing terms are

$$\begin{aligned}\langle\psi_r^\dagger(p_1)\psi_l(p_2)\psi_l^\dagger(p_3)\psi_r(p_4)\rangle_{\rm gG} = &-\delta_{p_1p_4}\delta_{p_2p_3}\langle\psi_r^\dagger(p_1)\psi_r(p_4)\rangle_{\rm gG}\langle\psi_l^\dagger(p_3)\psi_l(p_2)\rangle_{gG}\\ &+\delta_{p_1p_2}\delta_{p_4p_3}\langle\psi_r^\dagger(p_1)\psi_l(p_2)\rangle_{\rm gG}\langle\psi_l^\dagger(p_3)\psi_r(p_4)\rangle_{\rm gG}.\end{aligned} \tag{117}$$

By using

$$\langle\psi_r^\dagger(p)\psi_r(p)\rangle_{\rm gG} = \tfrac{1}{2}\vartheta(p)\sin^2 2\theta(p) + \vartheta(-p)\left(1-\tfrac{1}{2}\sin^2 2\theta(p)\right), \tag{118}$$

$$\langle\psi_l^\dagger(p)\psi_l(p)\rangle_{\rm gG} = \tfrac{1}{2}\vartheta(-p)\sin^2 2\theta(p) + \vartheta(p)\left(1-\tfrac{1}{2}\sin^2 2\theta(p)\right), \tag{119}$$

$$\langle\psi_l^\dagger(p)\psi_r(p)\rangle_{\rm gG} = \langle\psi_r^\dagger(p)\psi_l(p)\rangle_{\rm gG} = -\tfrac{1}{2}\sin 2\theta(p)\cos 2\theta(p)\,{\rm sign}(p), \tag{120}$$

the average over the generalized Gibbs ensemble of the four Fermi fields on the right-hand side of equation (116) can be computed and yields the following expression for the two-point correlation function (up to terms of $O(1/L^2)$):

$$\langle e^{2i\varphi(x)}e^{-2i\varphi(0)}\rangle_{gG} = \left|-\frac{1}{L}\sum_{p>0}\sin 2\theta(p)\cos 2\theta(p)\right|^2 + \left|\frac{1}{L}\sum_{p} e^{ipx}\left[\frac{1}{2}\vartheta(p)\sin^2 2\theta(p) + \vartheta(-p)\left(1-\frac{1}{2}\sin^2 2\theta_p\right)\right]\right|^2 \tag{121}$$

The first term in the rhs of the above expression is just $\langle e^{2i\varphi(x)}\rangle_{gG}\langle e^{-2i\varphi(0)}\rangle_{gG}$ (cf equation (115)), whereas the first term in the right-hand side can be written as

$$\left|\frac{1}{L}\sum_{p} e^{ipx}\left[\theta(-p)+\frac{1}{2}\mathrm{sign}(p)\sin^2 2\theta_p\right]\right|^2 = \left|\frac{1}{L}\sum_{p>0} e^{-ipx} + \frac{i}{L}\sum_{p>0}\sin px\,\frac{m^2}{\omega^2(p)}\right|^2 \tag{122}$$

$$= \left|\frac{1}{L}\sum_{p>0} e^{-ipx} + \lim_{t\to+\infty}\mathcal{H}(x,t)\right|^2, \tag{123}$$

that is, it coincides with the $t\to+\infty$ limit of the second term in the right-hand side of equation (30) in section 3.3 (the function $\mathcal{H}(x,t)$ is defined in equation (39)).

6.3. Quench from the gapless to the gapped phase in the semi-classical limit

In this case the Hamiltonian performing the time evolution is gapless (H_0) and thus diagonal in the b-operators. Hence, the conserved quantities are

$$I(q) = b^\dagger(q)b(q). \tag{124}$$

The Lagrange multipliers of the corresponding generalized Gibbs density matrix are fixed from the condition

$$\langle I(q)\rangle_{gG} = \frac{1}{e^{\lambda(q)}-1} = \langle\Phi(0)|b^\dagger(q)b(q)|\Phi(0)\rangle \tag{125}$$

$$= \sinh^2\beta(q), \tag{126}$$

where $\beta(q)$ is defined by equation (49). Hence, using this result, we next proceed to compute the order parameter and the two-point correlation function. We first note that the order parameter vanishes in the generalized Gibbs ensemble since $\langle e^{-2i\phi(x)}\rangle_{gG} = e^{-2i\langle\phi^2(0)\rangle_{gG}}$ and $\langle\phi^2(0)\rangle_{gG} = \frac{\kappa^2}{4}\langle\varphi^2(0)\rangle$ is divergent in the $L\to+\infty$ limit (see below). This agrees with the result found in section 4.2, where it was found that the order parameter decays exponentially in time. Thus, in what follows, we shall be concerned with the two-point correlation function.

6.3.1. Two-point correlation function. Since $\langle e^{-2i\phi(x)}e^{2i\phi(0)}\rangle_{gG} = e^{-\frac{\kappa^2}{2}C^{gG}(x)}$, where $C^{gG}(x) = \langle\varphi(x)\varphi(0)\rangle_{gG} - \langle\varphi^2(0)\rangle_{gG}$. In order to obtain this correlator, we introduce the Fourier expansion of $\varphi(x)$ (ignoring the zero-mode part)

$$\varphi(x) = \frac{1}{2}\sum_{q\neq0}\left(\frac{2\pi v}{\omega_0(q)L}\right)^{1/2} e^{iqx}[b(q)+b^\dagger(-q)], \tag{127}$$

into the expectation value, and using (126) to evaluate the averages in the generalized Gibbs ensemble, we find that, in the thermodynamic limit,

$$\langle\varphi(x)\varphi(0)\rangle_{\rm gG} = \int_0^\infty \frac{{\rm d}(vq)}{4\omega_0(q)}\,\cos qx\,\cosh 2\beta(q), \tag{128}$$

and therefore

$$\mathcal{C}^{\rm gG}(x) = \langle\varphi(x)\varphi(0)\rangle_{\rm gG} - \langle\varphi^2(0)\rangle_{\rm gG} = -\int_0^\infty \frac{{\rm d}(vq)}{\omega_0(q)}\,\cosh 2\beta(q)\,(1-\cos qx) \tag{129}$$

$$= \mathcal{C}(x,0) - \frac{m^2}{4}\int_0^{+\infty} \frac{{\rm d}(vq)}{\omega(q)\,[\omega_0(q)]^2}(1-\cos qx), \tag{130}$$

where $\mathcal{C}(x,0) \equiv \mathcal{C}(x,t=0)$ is defined in equation (61). Comparing the last result with equation (60) in the limit where $t \to +\infty$, we found that they are identical.

6.4. *Quench from the gapless to a gapped phase*

In this case, the Hamiltonian that performs the time evolution is gapped, whereas the initial state is gapless. Thus, in contrast to the previous case, the Hamiltonian that performs the evolution is diagonal in the a-operators and therefore the conserved quantities are $I(q) = a^\dagger(q)a(q)$. The corresponding Lagranges (at zero temperature) are fixed from the condition:

$$\langle I(q)\rangle_{\rm gG} = \frac{1}{{\rm e}^{\lambda(q)}-1} = \langle\Phi(0)|a^\dagger(q)a(q)|\Phi(0)\rangle \tag{131}$$

$$= \sinh^2\beta(q), \tag{132}$$

where $\beta(q)$ is given by equation (49).

In order to obtain the one- and two-point correlation functions of ${\rm e}^{2{\rm i}\phi(x)} = {\rm e}^{2{\rm i}\kappa\varphi(x)}$, we first need to write the $\varphi(x)$ field in terms of the a-operators. Using the canonical transformation equation (52),

$$\varphi(x) = \frac{1}{2}\sum_{q\neq 0}\left(\frac{2\pi}{\omega(q)L}\right)^{1/2} {\rm e}^{{\rm i}qx}[a(q) + a^\dagger(-q)]. \tag{133}$$

Hence, since $\langle{\rm e}^{-2{\rm i}\phi(x)}\rangle_{\rm gG} = \langle{\rm e}^{-{\rm i}\kappa\varphi(x)}\rangle_{\rm gG} = {\rm e}^{-\frac{\kappa^2}{2}\langle\varphi^2(0)\rangle_{\rm gG}}$ and $\langle\varphi^2(0)\rangle_{\rm gG}$ is logarithmically divergent in the thermodynamic limit (see the expressions below), we find that $\langle{\rm e}^{-{\rm i}\kappa\varphi(x)}\rangle_{\rm gG} = 0$. This result is in agreement with the one found in section 4.3 for the order parameter.

6.4.1. Two-point correlation function. Next we consider the two-point correlation function of the same operator, namely $\langle{\rm e}^{-2{\rm i}\phi(x)}{\rm e}^{2{\rm i}\phi(0)}\rangle_{\rm gG} = {\rm e}^{-(\kappa^2/2)\mathcal{C}^{\rm gG}(x)}$, where $\mathcal{C}^{\rm gG}(x) = \langle\varphi(x)\varphi(0)\rangle_{\rm gG} - \langle\varphi^2(0)\rangle_{\rm gG}$. We first obtain

$$\langle\varphi(x)\varphi(0)\rangle_{\rm gG} = \int_0^\infty \frac{{\rm d}(vq)}{4\omega(q)}\,\cos qx\,\cosh 2\beta(q). \tag{134}$$

Hence,

$$\mathcal{C}^{\rm gG}(x) = -\int_0^\infty \frac{{\rm d}(vq)}{2\omega(q)}\,\cosh\beta(q)\,(1-\cos qx) \tag{135}$$

$$= \mathcal{C}(x,0) + \frac{m^2}{4}\int_0^{+\infty} \frac{{\rm d}(vq)}{\omega_0(q)[\omega(q)]^2}(1-\cos qx), \tag{136}$$

where $\mathcal{C}(x,0)$ is defined in equation (68). The latter result agrees with equation (67) in the $t \to +\infty$ limit.

7. Relevance to experiments

As mentioned in the introduction, ultracold atomic systems are the ideal arena to study the quench dynamics of isolated quantum many-body systems. This is because they can be treated, to a large extent, as entirely isolated systems. Furthermore, since this work is concerned with the quench dynamics of a specific 1D model, the quantum sGM, it is also worth emphasizing that the properties of these systems are highly tunable and, in particular, so is their effective dimensionality. Thus, there are already a number of experimental realizations of 1D systems (see e.g. [42]–[44] and references therein), and in particular, there are also experiments where non-equilibrium dynamics has been probed in one dimension, e.g. [6, 45]. Thus, there is some chance that some of the results obtained above may be relevant to current or future quench experiments with ultracold atoms. However, since the sGM considered in the previous sections is nothing but an effective (low-energy) description of certain 1D systems, any comparison must be done with great care, as there is no fundamental reason why the low-energy effective theory should capture the essentials of the (highly non-equilibrium) quench dynamics. This is to be contrasted with the equilibrium dynamics, where renormalization group arguments show that the sGM is indeed sufficient to describe the (universal) properties of certain 1D physical systems. There is, in fact, a lot of evidence, both analytical and numerical, accumulated over the years, for the latter fact. However, we are not in a comparable situation in the case of non-equilibrium dynamics and thus future studies should try to address this question more carefully.

With the above caveat, let us proceed to mention a few situations where the sGM is applicable, at least as a good description of the equilibrium state of a system that can be realized with ultracold atomic systems. There are basically two kinds of systems, depending on the interpretation of the order parameter operator, $e^{-2i\phi(x)}$. The first instance is a 1D Bose gas moving in a periodic potential, where the sGM is the effective field-theory description of the Mott insulator-to-superfluid (MI to SF) transition in 1D [31, 46]. In this case, the order parameter field is the periodic component of the boson density. A quantum quench from the gapped (gapless) to the gapless (gapped) in this system can be realized by suddenly turning on (off) the periodic potential applied to the 1D gas. The evolution of the 1D density could be monitored by performing *in situ* measurements, and the two-point correlations by measuring, at different times, the (instantaneous) structure factor using Bragg spectroscopy.

In the second instance, the order parameter field is interpreted as the (relative) phase of two [47] (or more [48, 49]) 1D Bose gases coupled by Josephson coupling of two 1D Bose gases. Thus, in this setup, a quench experiment [16] from the gapped (gapless) to the gapless (gapped) phase would correspond to suddenly switching on (off) the (Josephson) tunneling, which can be achieved by controlling the (optical or magnetic trapping) potentials that confine the atoms to 1D. This should be done with care, ensuring that the atoms remain in the 1D regime both in the initial and final states, that is, ensuring that e.g. the potential trapping the atoms transversally is always sufficiently tight. The evolution of the relative phase can be monitored by analyzing the interference fringes at different times.

8. Conclusions

To sum up, we have investigated the time evolution of two-point correlation functions and the order parameter after a quantum quench of the relevant operator term in the sGM. We considered two different kinds of quenches: a quench from a gapped phase to a gapless phase and vice versa. In addition to the initial pure state, we considered an initial mixed state coupled to an energy

reservoir at finite temperature, which is suddenly disconnected at the same time that the quench is performed. In order to compute correlation functions, we studied two limits in which the Hamiltonian renders quadratic in terms of either fermion or boson operators, and the dynamics can be solved exactly: the LE and the semiclassical limits. In the quench from the gapped to the gapless phase, the order parameter decays exponentially for long times to the ground state value, with a timescale fixed by the gap. In turn, the correlation function exhibits a light-cone effect: it decays exponentially with time with a timescale fixed by the gap until it relaxes to a value that decreases exponentially with distance with correlation length fixed also by the mass. These results are valid for both the LE and the semiclassical limits, and agree with the results obtained in [9, 15] . For an initial state at high temperatures, these decays are also exponential (and the light-cone effect is preserved), but the characteristic time and length is set by the temperature. In between, a crossover connects these two limiting cases.

A general statement for the long-time dynamics when the term that opens the gap is suddenly turned on is elusive, since in the semiclassical approximation the long-time limit of the correlation function exhibits power-law decay with distance, and the order parameter vanishes, these features being characteristic of a critical state. On the other hand, the results in the LE limit indicate a relaxation to a non-universal constant value for long times, signaling an ordered state. The latter behavior at the LE limit is also present at finite temperature. Whether these differences are due to an artifact introduced by the semiclassical approximation or to a very special behavior that occurs at the solvable point needs to be further clarified.

We have shown that the long-time behavior of correlation functions and the order parameter in the different types of quenches can be obtained from the generalized Gibbs ensemble [14] in which the conservation of a certain set of independent integrals of motion is fixed as a constraint for maximization of the statistical entropy. Finally, the relevance of the quantum quench dynamics in the sGM to cold atomic gases was discussed. The superfluid–Mott insulator transition appears to be the most appropriate scenario to observe the described effects.

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Appendix. Some identities involving the Bessel functions

In this appendix, we prove the identities

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\phi} K_0\left(\alpha\sqrt{\mu^2+n^2}\right) = \pi \sum_{l\in\mathbb{Z}} \frac{\exp\left[-\mu\sqrt{\alpha^2+(2\pi l+\phi)^2}\right]}{\sqrt{\alpha^2+(2\pi l+\phi)^2}}, \tag{A.1}$$

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\phi} \frac{\alpha}{\sqrt{\mu^2+n^2}} K_1\left(\alpha\sqrt{\mu^2+n^2}\right) = \frac{\pi}{\mu} \sum_{l\in\mathbb{Z}} \exp\left[-\mu\sqrt{\alpha^2+(2\pi l+\phi)^2}\right], \tag{A.2}$$

where $\phi \in [0, 2\pi)$. Using the standard integral representation

$$K_0(\alpha z) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \frac{\cos x\alpha}{x^2+z^2}, \tag{A.3}$$

we have

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m\phi} K_0\left(\alpha\sqrt{\mu^2+n^2}\right) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\phi} \int \mathrm{d}k_x \frac{\mathrm{e}^{\mathrm{i}k_x\alpha}}{\sqrt{k_x^2+\mu^2+n^2}} \tag{A.4}$$

$$= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \int \mathrm{d}k_x \int \mathrm{d}k_y \frac{\mathrm{e}^{\mathrm{i}k_x\alpha+\mathrm{i}n\phi}}{k_x^2+k_y^2+n^2+\mu^2} \tag{A.5}$$

$$= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \int \mathrm{d}k_x \int \mathrm{d}k_y \int \mathrm{d}k_z \delta(k_z-n) \frac{\mathrm{e}^{\mathrm{i}k_x\alpha+\mathrm{i}k_z\phi}}{k_x^2+k_y^2+k_z^2+\mu^2}. \tag{A.6}$$

Next we employ the Poisson summation technique in the form $\sum_{n\in\mathbb{Z}} \delta(k_z-n) = \sum_{l\in\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}lk_z}$, and thus

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m\phi} K_0\left(\alpha\sqrt{\mu^2+n^2}\right) = \frac{1}{2\pi}\sum_{l\in\mathbb{Z}} \int \mathrm{d}k_x \int \mathrm{d}k_y \int \mathrm{d}k_z \frac{\mathrm{e}^{\mathrm{i}k_x\alpha+\mathrm{i}k_z\phi+2\pi\mathrm{i}lk_z}}{k_x^2+k_y^2+k_z^2+\mu^2} \tag{A.7}$$

$$= \frac{1}{2\pi}\sum_{l\in\mathbb{Z}} \int \mathrm{d}^3k \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_l}}{k^2+\mu^2}, \tag{A.8}$$

where we introduced the notation

$$r_x = \alpha, \tag{A.9}$$

$$r_y = 0, \tag{A.10}$$

$$r_z = \phi + 2\pi l. \tag{A.11}$$

Using the standard expression for the three-dimensional bosonic propagator, we finally obtain

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m\phi} K_0\left(\alpha\sqrt{\mu^2+n^2}\right) = \pi\sum_{l\in\mathbb{Z}} \frac{\mathrm{e}^{-\mu r_l}}{r_l}, \tag{A.12}$$

where $r_l = |\mathbf{r}_l|$. From this equation the first identity equation (A.1) immediately follows.

To prove the second identity equation (A.2), we do integration on both sides of equation (A.1):

$$\int \mathrm{d}\alpha\alpha \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\phi} K_0\left(\alpha\sqrt{\mu^2+n^2}\right) = -\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\phi} \frac{\alpha}{\sqrt{\mu^2+n^2}} K_1\left(\alpha\sqrt{\mu^2+n^2}\right) \tag{A.13}$$

$$= \int \mathrm{d}\alpha\alpha\pi \sum_{l\in\mathbb{Z}} \frac{\exp\left[-\mu\sqrt{\alpha^2+(2\pi l+\phi)^2}\right]}{\sqrt{\alpha^2+(2\pi l+\phi)^2}} \tag{A.14}$$

$$= -\frac{\pi}{\mu}\sum_{l\in\mathbb{Z}} \exp\left[-\mu\sqrt{\alpha^2+(2\pi l+\phi)^2}\right] + C'(\phi). \tag{A.15}$$

Thus, from equations (A.13) and (A.15),

$$\sum_{n\in\mathbb{Z}} e^{in\phi}\frac{\alpha}{\sqrt{\mu^2+n^2}}K_1\left(\alpha\sqrt{\mu^2+n^2}\right)=\frac{\pi}{\mu}\sum_{l\in\mathbb{Z}}\exp\left[-\mu\sqrt{\alpha^2+(2\pi l+\phi)^2}\right]+C(\phi). \tag{A.16}$$

$C(\phi)$ is determined from the behavior of $K_1(z)$ for $z\to 0$:

$$K_1(z)\sim\frac{1}{z}. \tag{A.17}$$

Thus, in the limit $\alpha\to 0$, on the one hand,

$$\lim_{\alpha\to 0}\sum_{n\in\mathbb{Z}} e^{in\phi}\frac{\alpha}{\sqrt{\mu^2+n^2}}K_1\left(\alpha\sqrt{\mu^2+n^2}\right)=\sum_{n\in\mathbb{Z}}\frac{e^{in\phi}}{\mu^2+n^2}=\frac{\pi}{\mu}\frac{\cosh\mu(\pi-\phi)}{\sinh\mu\pi}. \tag{A.18}$$

and, on the other,

$$\lim_{\alpha\to 0}\frac{\pi}{\mu}\sum_{l\in\mathbb{Z}}\exp\left[-\mu\sqrt{\alpha^2+(2\pi l+\phi)^2}\right]=\frac{\pi}{\mu}\sum_{l\in\mathbb{Z}}\exp\left[-\mu\,|2\pi l+\phi|\right]=\frac{\pi}{\mu}\frac{\cosh\mu(\pi-\phi)}{\sinh\mu\pi}. \tag{A.19}$$

Therefore, both limits coincide and $C(\phi)=0$, and equation (A.16) reduces to the second identity.

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