Induced Gravity and the Attractor Dynamics of Dark Energy/Dark Matter

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(Dated: December 9, 2010)

Attractor solutions that give dynamical reasons for dark energy to act like the cosmological constant, or behavior close to it, are interesting possibilities to explain cosmic acceleration. Coupling the scalar field to matter or to gravity enlarges the dynamical behavior; we consider both couplings together, which can ameliorate some problems for each individually. Such theories have also been proposed in a Higgs-like fashion to induce gravity and unify dark energy and dark matter origins. We explore restrictions on such theories due to their dynamical behavior compared to observations of the cosmic expansion. Quartic potentials in particular have viable stability properties and asymptotically approach general relativity.

I. INTRODUCTION

Over a decade ago two supernovae groups, the Supernova Cosmology Project and the High-Z Supernovae Search Team, provided evidence for an accelerated expansion of the Universe [1]. In recent years this discovery has gained more evidence from a variety of observations: further supernova data [2], measurements of the cosmic microwave background radiation [3] and galaxy surveys [4]. One possibility to explain this acceleration is to introduce a new component within the dynamics of General Relativity (GR), either as a uniform constant (cosmological constant) or as a scalar field evolving along a potential (as in the inflationary scenario). Since the current acceleration seems to be a unique phenomenon [5], at least since the time of primordial nucleosynthesis, this poses a fine tuning problem unless some dynamical attractor exists.

A different possibility is to look for acceleration as arising from modifications of gravity. In many cases this can be viewed as coupling scalar fields nonminimally to gravity, within the framework of scalar-tensor theories [6], an approach called extended quintessence [7]. Attractor mechanisms can work here as well [8–11], and also in the case where a scalar field is coupled to (dark) matter [12–16].

In the present work we investigate the influence of both couplings on the attractor dynamics. From a phenomeno-

logical point of view this enriches the phase space and also can help with problems that arise from one coupling or the other. By comparison with observations of the cosmic expansion behavior we can constrain the allowed parameter space. From a theoretical point of view several models can lend motivation to such a combination of couplings.

Induced gravity [17] is similar to standard scalar-tensor theories, but gravity is induced by a Higgs-like field. One motivation stems from Einstein's original ideas to incorporate Mach's principle into GR, by which the mass of a particle should originate from the interaction with all the particles of the universe, and so the interaction should be the gravitational one since it couples to all particles, i.e. to their masses or energies. To realize a stronger relationship with the material contents, Brans and Dicke [6] introduced their scalar-tensor theory of gravity, making the gravitational coupling, that is Newton's constant, a scalar function determined by the distribution of the cosmic content.

On the other hand, in modern particle physics the inertial mass is generated by the interaction with the Higgs field; the successful Higgs mechanism also lies precisely in the direction of Einstein's idea of producing mass by a gravitational interaction. One can show [18] that the Higgs field as source of the inertial mass of the elementary particles mediates a scalar gravitational interaction of Yukawa type between those particles that become massive as a consequence of the spontaneous symmetry breaking. Due to the equivalence principle, it seems natural to identify both approaches. For this reason, [19, 20] proposed a scalar-tensor theory of gravity where the Higgs field of elementary particles also plays simultaneously the role of a variable gravitational con-

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stant, instead of the scalar field introduced by Brans and Dicke.

Put another way, if there is an interaction between a scalar (Higgs) field and matter, and of course there exists coupling between matter and gravity, why not close the loop by incorporating a (non-minimal) coupling between the scalar field and gravity? Or conversely, if one explores scalar-tensor theories, and gravity couples with matter, why not include an explicit interaction between the scalar field and matter?

In this work we employ these ideas merely to motivate the couplings. For example, one might try to identify the dark energy (DE) with a Higgs-type (but not *the* Higgs) field that is coupled to some dark fermion sector. Accordingly, in our model, gravity, i.e. the Ricci scalar R, couples to a scalar field ϕ through the non-minimal coupling $\phi^2 R$. As the field evolves to its energy minimum the Higgs coupling might give rise to the mass of some dark fermion that would account for the dark matter (DM) of the model. The scalar field evolves to a constant, to generate the mass, and simultaneously generates Newton's constant through the non-minimal coupling to gravity. The resulting theory is an induced gravity in which GR is dynamically obtained through a Higgs mechanism from a scalar-tensor theory [19, 20].

The proposed Higgs mechanism is at present hypothetical, but phenomenologically interesting since it can account for the mass of the DM and since its field can act as a DE to accelerate the cosmological expansion. While many DM-DE interaction models aiming to unify the two quantities have difficulties in getting them simultaneously to exist and match observations, this can sometimes be made easier by adding a third element, such as inflation [21, 22] or, as in the present work, gravity. In any case, apart from the motivation, exploration of the dynamics of the matter- and gravity coupled system is of interest.

We begin by describing the general field equations in Sec. II, identifying the contributions of the gravity-scalar and scalar-matter couplings. This is then examined for the homogeneous and isotropic FRW universe background in Sec. III, including the evolution equations for the scalar field and matter. Section IV discusses the effective potential, illustrating it for a symmetry breaking form. In Sec. V we solve for the cosmic evolution of the field and matter, for general classes of potential and coupling, leading to constraints on the allowed parameter space from the cosmic expansion behavior.

II. FIELD EQUATIONS

The scalar-tensor theory Lagrangian used here is similar to the one studied in the past for inflationary dynamics [20]:

$$\mathcal{L} = \frac{\alpha}{16\pi}\phi^2 R + \frac{1}{2}\phi_{;\mu}\phi^{;\mu} - V(\phi) + \mathcal{L}_M \,, \qquad (1)$$

where ϕ is a real scalar field and α is a dimensionless parameter. We use the metric signature $(+---)$. Rewriting the Lagrangian as

$$\begin{aligned} \mathcal{L} &= \frac{1}{16\pi G_N}R + \frac{1}{16\pi G_N}\left(\frac{\phi^2}{v_{GR}^2} - 1\right)R \\ &+ \frac{1}{2}\phi_{;\mu}\phi^{;\mu} - V(\phi) + \mathcal{L}_M, \end{aligned} \qquad (2)$$

one sees that, formally, the Einstein-Hilbert action with the standard Newton's constant G_N corresponds to $\phi^2 \to v_{GR}^2 \equiv 1/(\alpha G_N)$. Note however that even if on average $\phi = v_{GR}$, the theory will still be distinct from GR because of perturbations in the field.

Varying Eq. (1), one obtains the field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= -\frac{8\pi}{\alpha\phi^2}\left[T_{\mu\nu} + V(\phi)g_{\mu\nu}\right] \\ &- \frac{8\pi}{\alpha\phi^2}\left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\phi_{;\lambda}\phi^{;\lambda}g_{\mu\nu}\right] \\ &- \frac{1}{\phi^2}\left[(\phi^2)_{;\mu;\nu} - (\phi^2)^{;\lambda}_{\ ;\lambda}g_{\mu\nu}\right] \,, \end{aligned} \qquad (3)$$

where a semicolon stands for a covariant derivative and $T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\partial\sqrt{-g}\mathcal{L}_M}{\partial g^{\mu\nu}}$. For the dark matter (DM) Lagrangian we begin with a general "Yukawa" coupling, $\mathcal{L}_M = \bar{\psi}\left(i\gamma^\mu\partial_\mu - f(\phi)\right)\psi$, similar to in [13, 15], and later explore some particular forms. The dark matter thus has a ϕ dependent mass $f(\phi)$.

The scalar field fulfills the generalized Klein-Gordon equation

$$\phi^{;\lambda}_{\ ;\lambda} + \frac{\partial V}{\partial\phi} - \frac{\alpha}{8\pi}R\phi = \frac{\partial\mathcal{L}_M}{\partial\phi} \,. \qquad (4)$$

Note the term involving αR arises from the non-minimal coupling to gravity, and the right hand side comes from the coupling to dark matter.

Taking the trace of Eq. (3) and substituting it into Eq. (4) to remove R, one obtains

$$\phi^{2;\lambda}_{\ ;\lambda} + \frac{2}{1 + \frac{3\alpha}{4\pi}}\left(\phi\frac{\partial V}{\partial\phi} - 4V(\phi) - T - \phi\frac{\partial\mathcal{L}_M}{\partial\phi}\right) = 0. \qquad (5)$$

The first term in the parentheses is the normal GR term for an uncoupled scalar field (although with an altered prefactor), the $T+4V$ of the second and third terms stem from the nonminimal coupling to gravity (specifically the trace of the first line of Eq. (3), and the last term is due to the DM-DE interaction.

This equation can be recast as:

$$\frac{1}{2\phi}\phi^{2;\lambda}_{\ ;\lambda} + V'_{\text{eff}}(\phi) = 0, \qquad (6)$$

where a prime stands for partial derivative with respect to ϕ. The effective potential is

$$V_{\text{eff}}(\phi) \equiv \frac{1}{1 + \frac{3\alpha}{4\pi}}\left[V(\phi) - \mathcal{L}_M(\phi) - \int^\phi d\varphi \frac{4V(\varphi) + T(\varphi)}{\varphi}\right], \qquad (7)$$

from which the effective mass of the Higgs particle will be identified later on. We write the argument φ explicitly for $\mathcal{L}_M$ to remind that only the partial derivative with respect to the field is relevant. If $V(\phi) = V_0\phi^4$, i.e. there is no intrinsic mass, then one sees that the potential terms cancel out and the effective mass is completely determined by the trace T from the nonminimal gravity coupling and the DM-DE interaction term $\mathcal{L}_M$.

It is convenient to define the energy momentum tensors associated with the different contributions to the right hand side of Eq. (3):

$$T^{\mu}{}_{\nu}{}^{(\phi)} \equiv V(\phi)\delta^{\mu}{}_{\nu} + \phi^{;\mu}\phi_{;\nu} - \frac{1}{2}\phi^{;\lambda}\phi_{;\lambda}\delta^{\mu}{}_{\nu} \quad (8)$$

$$T^{\mu}{}_{\nu}{}^{(R\phi)} \equiv \frac{\alpha}{8\pi}\left[(\phi^2)_{;\nu;\lambda}g^{\lambda\mu} - (\phi^2)^{;\lambda}{}_{;\lambda}\delta^{\mu}{}_{\nu}\right], \quad (9)$$

where the first is the standard scalar field contribution and the second stems from the nonminimal coupling to gravity. Again, note that the GR limit in which $T^{\mu}{}_{\nu}{}^{(R\phi)}$ vanishes is not $\alpha = 0$ but $\phi \rightarrow$ const. Due to the couplings, the components of the energy momentum tensor are not individually conserved but the total energy momentum tensor is:

$$\left\{-\frac{8\pi}{\alpha\phi^2}\left[T^{\mu}{}_{\nu} + T^{\mu}{}_{\nu}{}^{(\phi)} + T^{\mu}{}_{\nu}{}^{(R\phi)}\right]\right\}_{;\mu} = 0. \quad (10)$$

The conservation equation for matter can be derived from this (using the equation of motion (4) for ϕ and the Einstein equations (3)), giving

$$T^{\mu\nu}{}_{;\nu} = -\phi^{;\mu}\frac{\partial \mathcal{L}_M}{\partial \phi}. \quad (11)$$

We will use this equation in the following to determine the evolution of the matter component.

III. FRW COSMOLOGY EQUATIONS

To study the background evolution of the universe we use the FRW metric

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right), \quad (12)$$

and assume that the DM fluid is given by $T_{\mu\nu} = \rho\, u_\mu u_\nu$ with the energy density $\rho = nf(\phi)$, where n is the number density, and $u_\mu \equiv dx_\mu/ds = \delta^0_\mu$ the comoving 4-velocity. The energy momentum tensor behaves as a pressureless dust perfect fluid because we assume the dark matter to consist of non-relativistic fermions. The equivalence between the description in terms of the spinor field lagrangian as specified in the previous section on the one hand, and the "dust" description on the other hand is demonstrated in [23] for a standard linear coupling ($f \propto \phi$). There, it is shown how a fermion field interacting with a scalar field ϕ, with action

$$S = \int d^4\sqrt{-g}(i\bar{\psi}\gamma^\mu\partial_\mu\psi - y(\phi - \phi_*)\bar{\psi}\psi), \quad (13)$$

where y and ϕ_* are constant, is equivalent to a model of a classical gas of pointlike particles, with action

$$S = -\sum_i \int y(\phi - \phi_*)ds_i, \quad (14)$$

in the limiting situation where the fermions' de Broglie wavelengths are much smaller than the characteristic length scale of variation of the ϕ field. Following the steps of that demonstration one can see that it is valid to replace the factor $y(\phi - \phi_*)$ by an arbitrary function of ϕ, see also Ref. [24]. In our case we start with a fermion field with a coupling $f(\phi)\bar{\psi}\psi$. Then, a valid effective action is

$$S = -\sum_i \int f(\phi)ds_i = -\int d^4x\sqrt{-g}\, f(\phi)n(x), \quad (15)$$

where

$$n = \sum_i \int ds_i \frac{\delta^4(x - x(s_i))}{\sqrt{-g}}. \quad (16)$$

This is exactly the action of a dust fluid with energy density $\rho = f(\phi)n$.

Thus, the matter Lagrangian is proportional to $f(\phi)$, i.e. $\mathcal{L}_M = -T = -\rho \propto f(\phi)$. From the gravity field Eq. (3) one obtains the generalized Friedmann equations

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi}{3\alpha\phi^2}\left(\rho + V(\phi) + \frac{1}{2}\dot{\phi}^2 - \frac{3\alpha}{4\pi}H\phi\dot{\phi}\right), \quad (17)$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} = \frac{8\pi}{\alpha\phi^2}\left(V(\phi) - \frac{1}{2}\dot{\phi}^2\right) - \frac{2}{\phi}\left(\ddot{\phi} + 2H\dot{\phi} + \frac{\dot{\phi}^2}{\phi}\right), \quad (18)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter.

Rearranging the terms, the acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3\alpha\phi^2}[\rho - 2V(\phi)] - \left(1 + \frac{8\pi}{3\alpha}\right)\frac{\dot{\phi}^2}{\phi^2} - H\frac{\dot{\phi}}{\phi} - \frac{\ddot{\phi}}{\phi}. \quad (19)$$

The redundant Klein-Gordon equation (6) becomes

$$\ddot{\phi} + \frac{\dot{\phi}^2}{\phi} + 3H\dot{\phi} + V_{\text{eff}}' = 0, \quad (20)$$

where

$$V_{\text{eff}}(\phi) = \frac{1}{1 + \frac{3\alpha}{4\pi}}\left[V(\phi) + \rho(\phi) - \int^{\phi} d\varphi \frac{4V(\varphi) + \rho(\varphi)}{\varphi}\right]. \quad (21)$$

Since the field is coupled to matter (either employing a Higgs or other mechanism), the source term V_{eff}' involves

the density of the interacting DM fluid as well as the potential of the DE field.

The conservation equation, Eq. (11), yields

$$\dot{\rho} + 3H\rho = -\mathcal{L}_M'\dot{\phi} = \dot{\phi}\rho\frac{f'}{f} , \qquad (22)$$

so that the matter behaves the same way as in GR with a DM-DE interaction [13]. This equation can be directly integrated to give

$$\rho = \frac{n_0}{a^3} f(\phi), \qquad (23)$$

where n_0 is the DM number density at present. This simply tells us that DM particle number is conserved and the change in energy per comoving volume is purely due to the varying mass $f(\phi)$.

IV. EFFECTIVE POTENTIAL

The nonminimal coupling to gravity, and the coupling to matter, in addition to adding symmetry to the relations between the scalar field, matter, and gravity, also can create a nonzero vacuum expectation value (vev) – an effective cosmological constant – that can adiabatically evolve. The nonminimal gravity coupling can also reduce the driving term $V_{\rm eff}'$ (this coupling gives rise to the negative term in Eq. 21), slowing the field down. This slow roll can often alleviate instabilities in coupled matter perturbations [15].

We can examine these influences in terms of the effective potential of the theory, which alters the bare scalar field potential through the coupling to dark matter and to gravity. Using the Yukawa coupling $f \propto \phi$ in Eq. (21), the effective potential becomes

$$V_{\rm eff}(\phi) = \frac{1}{1 + \frac{3\alpha}{4\pi}} \left[V(\phi) - 4 \int^{\phi} d\varphi \, \frac{V(\varphi)}{\varphi} \right] . \qquad (24)$$

Note that the density terms in Eq. (21) – the interacting DM term $\mathcal{L}_M'$ and the trace term T stemming from the nonminimal coupling $\phi^2 R$ – cancel out since ρ is linearly proportional to ϕ. If furthermore $V(\phi) = V_0\phi^4$, then the effective potential vanishes and the dark energy field, even coupled, is massless and acquires infinite range. This is because in this case the theory has no explicit mass scale in it and is thus scale invariant.

For the Yukawa coupling, the effective potential does not run with the density, so it differs from – and is actually simpler than – what happens in the GR case with matter coupling. The effective mass is only determined by the potential terms. The field acts like quintessence with a $V_{\rm eff}(\phi)$ given by the full Eq. (24).

For a Landau-Ginzburg symmetry breaking form for the Higgs potential,

$$V(\phi) = \frac{\lambda_a}{24} \left(\phi^2 - \frac{6\mu^2}{\lambda_a} \right)^2 , \qquad (25)$$

where λ_a is a dimensionless constant, $\mu\sqrt{2}$ is the mass of the field at the potential minimum, and the Higgs ground state v, such that $V(v) = 0$, is given by

$$v^2 = \frac{6\mu^2}{\lambda_a} . \qquad (26)$$

In induced gravity, the Higgs potential $V(\phi)$ generates a time varying gravitational coupling (as in scalar-tensor theory)

$$G(\phi) = \frac{1}{\alpha\phi^2} \qquad (27)$$

as ϕ rolls from an initial state to its ground state and thus for a given field value determined by the potential, α needs to be chosen such that $G(\phi) \approx G_N$. For example, we might choose α such that $G(\phi_{\rm min}) = G_N$.

Relating this to the particle physics of the Higgs mechanism, one has

$$\alpha = 2\pi \left(g\frac{M_{Pl}}{M_b} \right)^2 , \qquad (28)$$

where $M_{Pl} \equiv 1/\sqrt{G_N} \approx 1.2 \times 10^{19}\,{\rm GeV}$ is the Planck mass, M_b the boson mass, and g a coupling constant. If we were to consider the standard model Higgs, one has $M_b = M_{\rm W} = 80\,{\rm GeV}$ for the W-boson and $g = 0.18$, therefore $\alpha \approx 10^{33}$. Such a value is huge and would not pass cosmological constraints on α discussed below. Therefore, we consider another Higgs-like particle with a much larger mass than that of the Higgs of the standard model of particle physics. The parameter α will need to be determined through cosmological observations.

The effective potential for the Landau-Ginzburg plus Yukawa coupling case is

$$V_{\rm eff} = \frac{1}{1 + \frac{3\alpha}{4\pi}} \frac{\lambda_a v^4}{12} \left[(\phi^2/v^2) - \ln(\phi^2/v^2) \right] , \qquad (29)$$

and has a minimum at $\phi = v$ at all times, at which the mass of Higgs particle is $m_H^2 = V_{\rm eff}''(\phi = v) = \lambda_a v^2/(3[1 + 3\alpha/(4\pi)])$.

We show the difference between the bare potential and the effective potentials in induced gravity and in GR with Yukawa coupling in Fig. 1. Since the matter density contribution cancels out in induced gravity, the effective potential is time independent, while the GR case is not. The induced gravity case gives a broad, nearly flat minimum. When the field is very slowly rolling, the modifications to the right hand side of Eq. (3) are small.

In this Ansatz the Higgs field, rolling along the effective potential, plays the role of DE and the particle masses produced by the Higgs mechanism provide the DM of the model. However, as we will see the dynamics does not generally favor such a form for the potential.
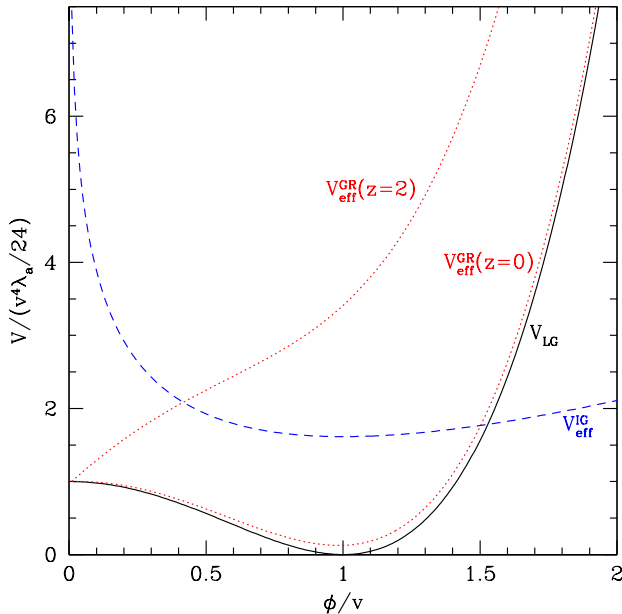
system becomes

$$\frac{dx}{dN} = -\frac{3}{2}xy^2 - \frac{3}{2}x - 3\sqrt{\beta}\,x^2 + \frac{3}{2}x^3$$
$$+ \frac{1 + \sqrt{\beta}\,x}{1 + \beta}\left(\sqrt{\frac{3}{2}}\lambda y^2 - C\right), \qquad (31)$$

$$\frac{dy}{dN} = y\left[-\sqrt{\frac{3}{2}}\lambda x + \frac{3}{2}\left(x^2 - y^2 + 1\right) - 2\sqrt{\beta}\,x\right.$$
$$\left. + \frac{\sqrt{\beta}}{1 + \beta}\left(\sqrt{\frac{3}{2}}\lambda y^2 - C\right)\right]. \qquad (32)$$

Here

$$C \equiv \frac{\kappa}{\sqrt{6}H^2}\left(\rho\frac{f'}{f} - \frac{4V + \rho}{\phi}\right) \qquad (33)$$

$$= \frac{\sqrt{\beta}}{2}\left[(F - 1)\,\Omega_m(N) - 4y^2\right], \qquad (34)$$

where $F \equiv d\ln f/d\ln\phi$. The quantity C encodes the information on the matter coupling (in the first term involving f) and the non-minimal gravitational coupling.

The fractional matter density comes directly from Eq. (17) as

$$\Omega_m(N) \equiv \frac{\kappa^2\rho}{3H^2} = 1 - \Omega_\phi$$
$$\Omega_\phi = x^2 + y^2 - 2\sqrt{\beta}\,x. \qquad (35)$$

To evolve the field ϕ appearing in λ or in $d\ln f/d\ln\phi$, one uses the auxiliary equation

$$\frac{d\ln\phi}{dN} = \sqrt{\beta}\,x. \qquad (36)$$

We define the effective equation of state for a component in terms of $\kappa^2\rho_i$ (since the sum of these quantities is conserved, see Eq. 10), by

$$w_i = -1 - \frac{1}{3}\frac{d\ln\kappa^2\rho_i}{dN}. \qquad (37)$$

This takes into account effective pressure terms due to the interactions. In particular, the matter equation of state gains an effective nonzero term, due to the non-minimal coupling to both the scalar field directly and to gravity through the $\phi^2 R$ term. One has

$$w_m = \frac{\sqrt{\beta}}{3}\left(2 - F\right)x. \qquad (38)$$

One has $w_m = 0$ only when the field is frozen ($x = 0$), which restores GR as discussed in Sec. II, or when $f \sim \phi^2$, which counteracts the κ^2 term in Eq. (37).

The equation of state for the scalar field is more involved:

$$w_\phi = \left[y^2[\beta(F - U) - 3] - (F - 2)x^3\sqrt{\beta}(\beta + 1)\right.$$
$$+ x^2(\beta[2(F - 2)\beta + 3F - 2] + 3) - \beta(F - 1)$$
$$\left. - x\sqrt{\beta}\left[y^2(F - 2)(\beta + 1) + \beta(2 + F) + 4 - F\right]\right]$$
$$/\left[3(\beta + 1)\left(x^2 - 2x\sqrt{\beta} + y^2\right)\right], \qquad (39)$$

FIG. 1: The Landau-Ginzburg potential in the uncoupled, GR case (solid curve), in the matter coupled but minimally gravity coupled (GR) case for two values of the redshift z (dotted lines), and in the nonminimally gravity coupled induced gravity (IG) case (dashed curve). The matter coupling is taken to be of the Yukawa form.

V. DYNAMICAL EQUATIONS

We now examine the dynamics of the dark energy, the effective equation of state of the dark matter, and the overall expansion behavior.

A. Variables and Equations of Motion

The dynamics of the scalar field, non-minimally coupled to gravity and to dark matter, are described by an autonomous system of equations which can be solved in a straightforward manner. One can define the following set of variables [25]:

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H} \qquad ; \qquad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}$$

$$\lambda \equiv -\frac{V'}{\kappa V} = -\sqrt{\frac{\beta}{6}}\frac{d\ln V}{d\ln\phi} \quad ; \quad N \equiv \ln a, \quad (30)$$

where $\kappa = \sqrt{6/\beta}/\phi$ is a function of the field, with $\beta \equiv 3\alpha/(4\pi)$.

Assuming spatial flatness, i.e. $k = 0$, the autonomous

which depends on the nonminimal gravity coupling through β, on the interacting DM-DE term (F), and on the potential ($U \equiv d\ln V/d\ln\phi$).

The total equation of state

$$w_{\text{tot}} \equiv -1 - \frac{1}{3}\frac{d\ln H^2}{dN} \qquad (40)$$

$$= w_m(a)\,\Omega_m(a) + w(a)\,\Omega_\phi(a) \qquad (41)$$

takes the same form with or without coupling to matter. The deceleration parameter $q = (1 + 3w_{\text{tot}})/2$ as usual.

B. Critical points

Investigating the critical (or fixed) points of the dynamical system, we find the relation (for $y_c \neq 0$)

$$y_c^2 = 1 + \sqrt{\beta}\,\frac{U+2}{3}\,x_c + \left(1 + \beta\frac{U+2}{3}\right)x_c^2. \qquad (42)$$

We restrict to the cases of U and F being constant in this analysis; otherwise the equations are transcendental. The dark energy density and equation of state are

$$\Omega_{\phi,c} = 1 - \frac{\sqrt{\beta}}{3}(4-U)\,x_c + \frac{1}{3}\left[6 + \beta(2+U)\right]x_c^2 \quad(43)$$

$$w_{\phi,c} = -1 + \frac{\sqrt{\beta}}{3}(2-U)\,x_c. \qquad (44)$$

The fixed point x_c is given by a quadratic equation. The first solution is

$$x_{c1} = \sqrt{\beta}\,\frac{4-U}{6+\beta(2+U)} \qquad (45)$$

$$\Omega_{\phi,c1} = 1 \qquad (46)$$

$$1 + w_{\phi,c1} = \frac{\beta}{3}\frac{(2-U)(4-U)}{6+\beta(2+U)}. \qquad (47)$$

This is a stable fixed point over a wide range of parameters (see next section) and it is independent of the matter coupling $f(\phi)$. Note that we obtain a future de Sitter state ($w = -1$) for $V \sim \phi^2$ or $V \sim \phi^4$ asymptotic behavior. The $U = 4$ case represents an attractor to GR, since in this case $x_c = 0 = \dot\phi$. An $x_c = 0$ solution was also found in GR with no matter couplings [8, 24, 26], and with an exponential coupling [16], and also in induced gravity with no matter couplings [11, 27]. For $2 < U < 4$ the fixed point gives a phantom ($w_c < -1$) attractor; a phantom solution was obtained in Ref. [28] for $F = 0$ and a particular choice of U. The general attractor solution is illustrated in Fig. 2. Note that there is a wide range of U and α (and all of F, which does not enter) where the attractor gives acceleration, and in fact stays close to $w = -1$. In all cases $\alpha \to 0$ makes $w_{\phi,c1} \to -1$.

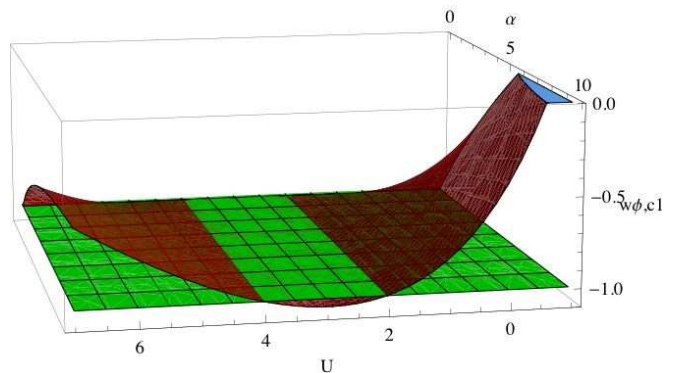The second solution (mostly a saddle point) does de-

FIG. 2: The dark energy equation of state for the first critical point is plotted as a function of the power law index U of the potential and the value α of the gravity coupling (the matter coupling does not affect this critical point). The value $w = -1$ is indicated by the green mesh plane. Note the attractor gives acceleration, and $w \approx -1$, over a wide range of values.

pend on $f(\phi)$ and is given by

$$x_{c2} = \frac{3}{(F-U)\sqrt{\beta}} \qquad (48)$$

$$\Omega_{\phi,c2} = \frac{18 + \beta(6 + 7U - 4F + F^2 - FU)}{\beta(F-U)^2} \qquad (49)$$

$$w_{\phi,c2} = \frac{2-F}{F-U}. \qquad (50)$$

Again, $w_c = -1$ when $V \sim \phi^2$ (except when $F = 2$ also, since for $F = U$ this solution does not exist). A solution within GR for an exponential potential and an exponential coupling function [12] also has a critical point depending on the equivalent of F and U.

Note that x^2, y^2, and Ω_m can all be greater than unity. This is possible because of the negative term $-H\phi\dot\phi$ in the modified Friedmann equation (17), which permits the effective dark energy density to go negative. This causes no physical problems within our Ansatz, as we demonstrate in Sec. V D.

Three further mathematical solutions exist for $y_c = 0$. These have $x_{c4,c5} = \sqrt{\beta} \pm \sqrt{1+\beta}$, which are mostly unstable and inaccessible, but of more interest is

$$x_{c3} = \frac{\sqrt{\beta}\,(1-F)}{3 + \beta(2+F)} \qquad (51)$$

$$\Omega_{\phi,c3} = \frac{\beta(F-1)[5 + F + 2\beta(2+F)]}{[3 + \beta(2+F)]^2} \qquad (52)$$

$$w_{\phi,c3} = \frac{\beta(F-1)(F-2)}{3\,[3 + \beta(2+F)]}. \qquad (53)$$

This solution is important because $y \approx 0$ corresponds to $\kappa^2 V/(3H^2) \approx 0$, which holds at early times. We indeed find a metastable attractor to this behavior at high redshift (basically, $dx/dN \approx 0$ but y keeps growing). Note that this high redshift attractor is generally $w_\phi = w_m$, i.e. a scaling solution. However, V (and y) are not actually zero, just small compared to the other densities,

so the case $F = 1$, which would give $x_c = 0$, breaks the condition $y_c \ll x_c$ and instead here the dynamics forces $w_\phi = -1$ and $w_m = 0$.

C. Stability

Carrying out a linear stability analysis, we find that the stability of the critical points generally depends on the values of α (i.e. β), U, and F, making analytic statements difficult. However, a reasonable rule of thumb is that for values of these variables not too positive or too negative, the first critical point is a stable node (both eigenvalues of the perturbation matrix negative) and the second critical point tends to be a saddle point (one eigenvalue positive, one negative). See [25] for a general discussion of stability analysis and classification.

We show the 3-dimensional eigenvalue surfaces for the first critical point in Fig. 3, for three different values for α. The region of stability exists for a broad interval of U and F, including "natural" values. As α decreases, the area of stability grows, as seen in the sequence of the three plots. For $\alpha < 0.1$ the system is stable for any $U \in [-20, 20]$, and $F \in [-10, 10]$.

By contrast, the second critical point is a saddle point, as seen in Fig. 4. One eigenvalue is always positive and the other negative, independent of the chosen U and F. As α decreases, the eigenvalue sheets tend to pull further away from the zero plane. Since the qualitative nature of one positive eigenvalue, one negative eigenvalue does not change, we show two values of α together in the figure. Also note that the eigenvalues diverge for $F = U$, as expected from Eq. (48).

The third critical point is also a saddle point for $U \in [-10, 20]$ and $F \in [-10, 10]$, but it has a stability window for $\alpha = 1$ in the region $U \in [-10, -20]$ and $F \in [-1, -5]$. As $\alpha \to 0$ the sheets flatten, one below and above the $\mu = 0$ plane, making this a saddle point for any U and F. Finally, the fourth and fifth fixed points are mostly unstable, with small regions of stability (for big $|F|$ and $|U|$) for $\alpha \geq 0.5$, but these solutions are anyway uninteresting since they are nonaccelerating and require $y = 0 = V$ while $\Omega_\phi = 1$, hence $\phi = 0$ and vanishing gravity.

A key point is that $U = 4$ always has a stable attractor to the first critical point, with $w = -1$ and $\dot{\phi} = 0$, for all values of α and F (even $F \neq$ constant, as we discuss in the next section). So for a quartic potential there is an attractor leading to ΛCDM and a restoration of GR.

D. Dynamical History

Turning now to the full dynamics, some numerical solutions are illustrated in Fig. 5. At high redshift, where H^2 is dominated by the matter density, y is small and the field quickly forgets the initial conditions and goes to the scaling solution of the third critical point (except when
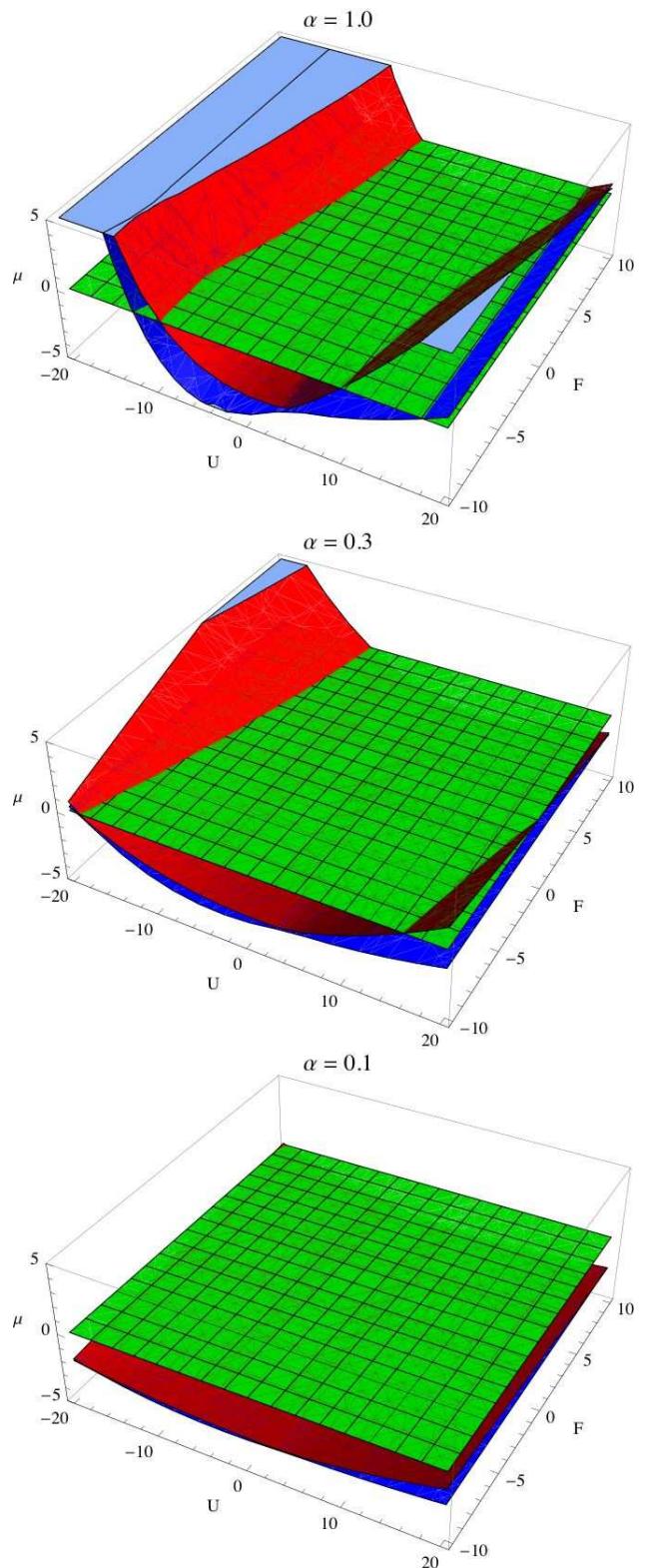

FIG. 3: Surfaces for the two eigenvalues μ_1 (upper red) and μ_2 (lower blue) of the first critical point. The stability region for the critical point is where both eigenvalues are negative, i.e. below the (green horizontal) $\mu = 0$ plane, and is a function of the power law index U of the potential, F of the matter coupling, and the value α of the gravity coupling. The three panels shows how the surfaces change with α: the smaller α, the more stable the system.
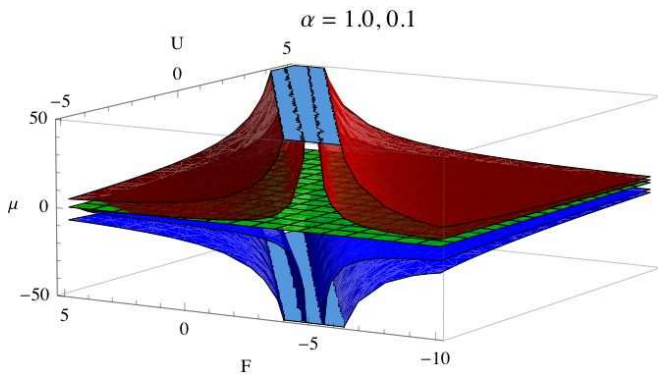
FIG. 4: Surfaces for the two eigenvalues μ_1 (upper red) and μ_2 (lower blue) of the second critical point. Since one eigenvalue is positive, this critical point is not stable. As α decreases, the sheets pull away from the zero plane.

$F = 1$, then $w_\phi = -1$). As the contribution of the dark energy potential energy becomes relatively more important, it then approaches the asymptotic attractor of the first critical point.

We define the present ($a = 1$) by when $\Omega_\phi = 0.72$. For clarity we show in Fig. 5 the results for a value of the gravity coupling $\alpha = 1$; for small α the deviation from $w(a) = -1$ will scale roughly as α. The high redshift metastable attractor and the asymptotic future stable attractor behaviors can clearly be seen. Note that for the Yukawa coupling ($F = 1$) the dynamics stays close to $w = -1$. For $F < 1$, the $-H\phi\dot\phi$ term can drive the effective dark energy density through zero, making w go to $\pm\infty$. This has no physical pathology, since the dark energy density is merely an effective quantity and as we will soon see the matter and total equations of state are well behaved. In the future, the attractor solutions for $V \sim \phi^2$ and $V \sim \phi^4$, i.e. $U = 2$ and 4, are the de Sitter state $w = -1$. The latter case represents the attractor to GR.

Examining the future attractor solution more closely, we see from Eq. (47) that within the range $2 < U < 4$, the attractor is to a phantom state $w < -1$, while outside this range it is to $w > -1$ (unless U gets extremely negative). In general, reasonable inverse power law potentials do not provide $w \approx -1$ and would be disfavored by observations. Figure 6 shows the full behaviors for a variety of U values, fixing the matter coupling to the Yukawa form $f \sim \phi$ ($F = 1$).

Regarding the early time behavior, as stated the dark energy equation of state becomes undefined when the dark energy density passes through zero, as can occur for $F < 1$. Since the dark energy density vanishes, however, this has no physical effects. In particular, the matter equation of state w_m and total equation of state $w_{\rm tot}$ exhibit no sign of this occurrence. In Fig. 7 we see that at high redshift w_m goes to its metastable attractor solution (third critical point), and because of scaling $w_{\rm tot} = w_m$. In the Yukawa coupling case ($F = 1$), $w_m = 0$, as it is
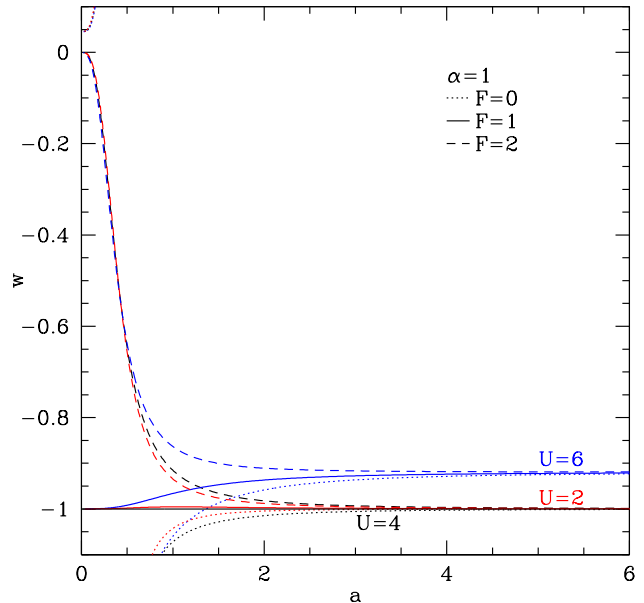

FIG. 5: The dark energy equation of state is plotted for three power law indices of the potential and the matter coupling. All solutions asymptote to the stable fixed point $w_{c1}(U)$, independent of F. Note both quadratic and quartic potentials ($U = 2, 4$) reach the de Sitter state $w = -1$. The deviations of $w(a)$ from -1 scale linearly with α for small α.

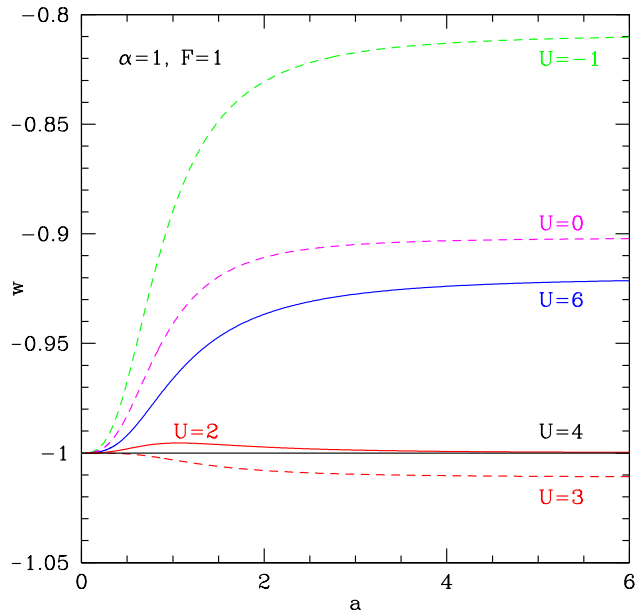
FIG. 6: The evolution of the dark energy equation of state is shown for various power law potentials $V \sim \phi^U$, with Yukawa coupling $f \sim \phi$ and gravity coupling $\alpha = 1$. The attractor values $1 + w_c$ scale with α for small α. Note that ϕ^2 and ϕ^4 potentials give $w = -1$ attractors, while power law indices in between give phantom attractors.

for the $F = 2$ case always as well. Both w_m and w_{tot} then smoothly evolve toward the future stable attractor solution. Both w_m and $1 + w_{\text{tot}}$ scale linearly with α for small α. For $F = 1$, all quantities are well behaved and follow ΛCDM until recently.
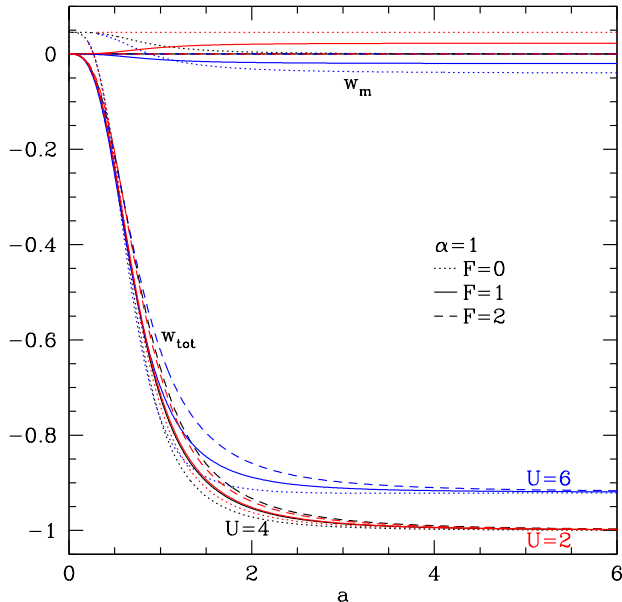

FIG. 7: The matter equation of state w_m and total equation of state w_{tot} are plotted for the same potential and matter coupling cases as in Fig. 5. The asymptotic future values of w_m and $1 + w_{\text{tot}}$ scale linearly with α for small α. For Yukawa coupling ($F = 1$), until recently w_m was zero and w_{tot} followed the standard ΛCDM history (solid black line, also for $U = 4$, $F = 1$) as well.

When α is small, the equations of state of matter, dark energy, and the total energy are nearly the same as for ΛCDM, regardless of F and U. This is demonstrated in Fig. 8 for $\alpha = 0.1$.

A small value of α corresponds to ϕ greater than the Planck energy to induce G_N (see Eq. 27). However, there is no actual energy density that we are taking to be greater than the Planck scale, so it is not clear there is a problem in treating this as a low energy effective theory. In fact, our theory does not have an explicit Planck scale in it so it is not even clear what the cutoff energy is above which our theory is expected to break down.

Another issue is how to apply standard scalar-tensor theory limits from the Solar system, e.g. on the Jordan-Brans-Dicke parameter ω_{JBD}. Without the matter coupling and with $U = 0$, one would say that $\omega_{JBD} = 2\pi/\alpha$ and so these constraints [29] require $\alpha < 10^{-4}$. In Ref. [27] it is shown a compatibility of the model $U = 4$, $F = 0$ with current cosmological parameters respecting Solar system constraints. However, the matter coupling might alter that conclusion. Ref. [30] considers the differences between matter coupled and nonminimally gravity
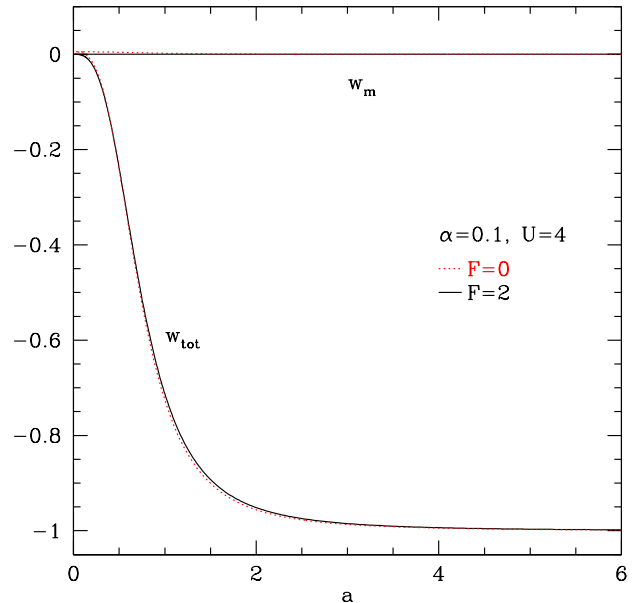

FIG. 8: As the $U = 4$ case of Fig. 7, but for $\alpha = 0.1$. Deviations of w_m from zero, and w_{tot} from the ΛCDM behavior are less than 10^{-2}.

coupled models, but so far no analysis has been carried out including both effects. The equations for the field perturbations with all the couplings are quite complicated; we leave that analysis for future work and here only consider effects on the cosmic expansion.

We can consider going beyond constant F and U. In the case of an exponential matter coupling, the logarithmic derivative F is not constant. Figure 9 shows $w(a)$ for such a case, where $f \sim e^{b\phi}$, so $F = b\phi$ (here the field values are all in units of the Planck mass). The dynamics is now dependent on the initial value of the field ϕ_i, and resembles that of the constant F case with $F = b\phi_i$ (a better approximation is evaluating this at a ϕ partway between ϕ_i and a later time value). In particular this applies to the high redshift attractor. However, note that the late time attractor given by Eq. (47) remains the same, independent of F and ϕ_i.

One can also consider potentials where the logarithmic derivative U is not constant. Figure 10 illustrates $w(a)$ for a potential of Landau-Ginzburg form (25). In this case, with varying U, the dynamics is quite different from the constant U case, and no attractor solution is apparent.

VI. CONCLUSIONS

The dynamics of a scalar field model incorporating both nonminimal coupling to gravity and coupling to matter is rich. The relations to induced gravity through a Higgs mechanism and the symmetrical ideas of coupling
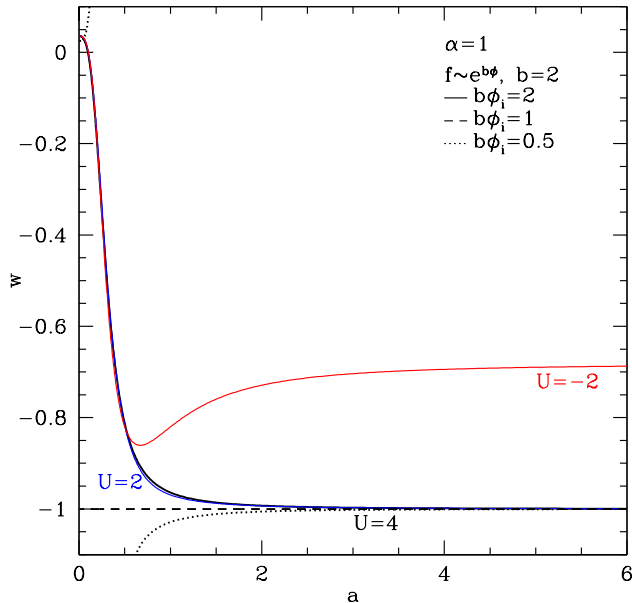
FIG. 9: Constant U potentials with exponential coupling $f \sim e^{b\phi}$, shown for $b = 2$. For $U = 4$ (thick, black curves), the cases for three initial field values ϕ_i are shown; for $U = 2$ and $U = -2$ only the $b\phi_i = 2$ case is plotted. The $w(a)$ dynamics depends on ϕ_i (although it is quite similar to a corresponding constant F) but the final attractor depends on neither f nor ϕ_i.
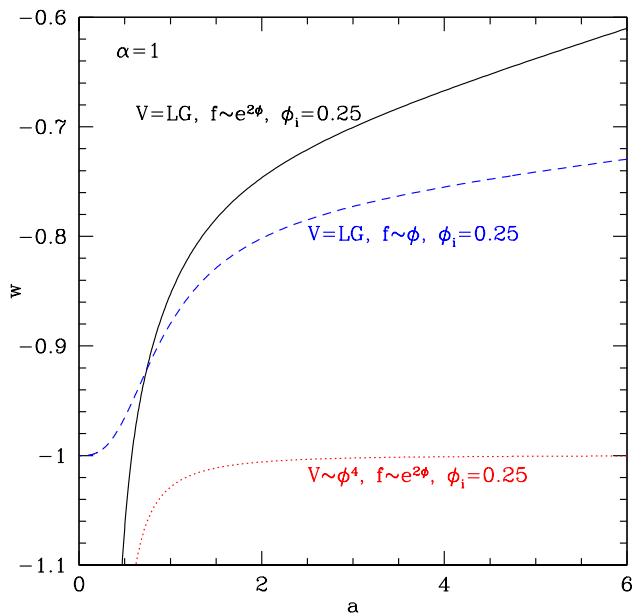

FIG. 10: Comparison of the Landau-Ginzburg (LG) form, with either constant F or non power law f, to the ϕ^4 potential shows significant differences. In particular, the non power law potential does not exhibit attractor behavior.

around the circle of scalar field, matter, and gravity are interesting. Furthermore, adding both couplings can relieve some problems of either individual coupling.

A set of attractor solutions for the dynamics is found, with a stable attractor to a dark energy dominated universe with w either equal to or near -1. High redshift behavior is standard ΛCDM for a Yukawa matter coupling (better behaved than if there were no coupling), and a scaling solution otherwise. Throughout cosmic history the matter and total equations of state can also be acceptably near ΛCDM behavior, for α not much larger than unity, depending on the couplings.

A simple quartic potential $V \sim \phi^4$ appears viable (again better behaved than if the potential were constant), with a stable de Sitter attractor for all values of gravity and matter couplings. Gravity is asymptotically restored to general relativity (at least as far as the background behavior is concerned). A quadratic potential also has a de Sitter attractor.

We identified the roles of the various couplings in the evolution equations and the effective potential. The slowing of field dynamics due to the gravity coupling likely alleviates problems with coupled matter instabilities, through the adiabatic mechanism discussed by [15], and the coupling to matter may help issues with gravity tests, but the system of perturbation equations becomes quite complicated due to the two extra scalar field couplings and we leave that to future work. Here we concentrated solely on the field and expansion dynamics, which in itself can constrain the parameter space, while pointing the way to interesting attractor behaviors and the possibility of dynamically establishing $w \approx -1$.

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