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All the supersymmetric solutions of $N = 1, d = 5$ ungauged supergravity

Jorge Bellorín [1], *Patrick Meessen* [2], and *Tomás Ortín* [3]

Instituto de Física Teórica UAM/CSIC
Facultad de Ciencias C-XVI, C.U. Cantoblanco, E-28049-Madrid, Spain

Abstract

We classify the supersymmetric solutions of ungauged $N = 1$ $d = 5$ SUGRA coupled to vector multiplets and hypermultiplets. All the solutions can be seen as deformations of solutions with frozen hyperscalars. We show explicitly how the 5-dimensional Reissner-Nordström black hole is deformed when hyperscalars are living on $SO(4,1)/SO(4)$ are turned on, reducing its supersymmetry from 1/2 to 1/8. We also describe in the timelike and null cases the solutions that have one extra isometry and can be reduced to $N = 2, d = 4$ solutions. Our formulae allows the uplifting of certain $N = 2, d = 4$ black holes to $N = 1, d = 5$ black holes on KK monopoles or to pp-waves propagating along black strings.

[1]E-mail: `Jorge.Bellorin@uam.es`
[2]E-mail: `Partick.Meessen@cern.ch`
[3]E-mail: `Tomas.Ortin@cern.ch`

Contents

1 Introduction

With Ref. [1], Gauntlett *et al.* revolutionized the art of finding supersymmetric solutions, by extending the methods pioneered by Tod [2] and applying them to classify the supersymmetric solutions of minimal $N = 1$ $d = 5$ supergravity. Since then, there has been a renewed, vigorous and systematic effort in the literature to classify, or at least characterize, generic supersymmetric solutions of supergravity theories. In the framework of $N = 1, d = 5$ SUGRA the results of Ref. [1] were extended to the gauged case in Ref. [3], to include the coupling to an arbitrary number of vector multiplets in Ref. [4] and their Abelian gaugings were further considered in Refs. [5, 6][4]. In the framework of $N = 2, d = 4$ SUGRA the new methods allowed the extension of Tod's results to pure gauged $N = 2, d = 4$ SUGRA [9] and to ungauged $N = 2, d = 4$ SUGRA coupled to an arbitrary number of vector multiplets [10] and hypermultiplets [11]. The minimal $d = 6$ SUGRA was dealt with in Refs. [12, 13], some gaugings were considered in Ref. [14] and the coupling to hypermultiplets has been fully solved in Ref. [15]. Further works in other (higher) dimensions and number of supercharges or based on the alternative methods of spinorial geometry are Refs. [16, 17].

In this paper we will extend further the results obtained in ungauged $N = 1, d = 5$ SUGRA to include, on top of vector multiplets, hypermultiplets. This problem was considered before by Cacciatori, Celi and Zanon in Refs. [18, 19, 20], making progress towards a full solution of the problem which we present here.

Similar works in 4- and 6-dimensional SUGRAs with 8 supercharges ($N = 2, d = 4$ and $N = (1, 0), d = 6$) coupled to vector multiplets and hypermultiplets have been recently published [11, 15]. As the observant reader will see, there is a staggering similarity between the results found in those works and the ones presented here. The reason for this is simply because the hypermultiplets have a very characteristic, and minimal, way of coupling to the rest of the fields, a coupling that is roughly the same in the 3 theories with 8 supercharges, wherefore the resulting structures should be comparable.

Let us describe our results qualitatively: all the supersymmetric solutions can be seen as deformations of supersymmetric solutions with the same electric and magnetic charges but frozen hyperscalars (which is effectively the same as having only vector multiplets), which were classified in Ref. [3]. The effect of defrosting the hyperscalars is an electric and magnetic charge preserving deformation of those solutions; the deformations consist in a deformation of the base space in the timelike case and of the wavefront space in the null case. To be more precise, in the timelike case, the metrics of all the supersymmetric solutions have the general conformastationary form

$$ds^2 = f^2 (dt + \omega)^2 - f^{-1} h_{\underline{mn}} dx^m dx^n \,. \tag{1.1}$$

$h_{\underline{mn}}$ is the time-independent base space metric and when dealing with frozen hypermultiplets, it has to be hyper-Kähler. The metric, with $f = 1$ and $\omega = 0$ and vanishing matter fields is a supersymmetric solution by itself and can be seen as a background which is

[4] Previous work on these theories can be found in Refs. [7, 8].

excited when electric and magnetic charges are turned on. The functions f and ω are essentially determined by the electric and magnetic charges and satisfy covariant differential equations in the base space.

When the hyperscalars are turned on $h_{\underline{mn}}$ is no longer a hyper-Kähler manifold: the form of this metric is dictated by two requirements

1. The hyperscalars $q^X(x)$ are quaternionic maps[5] from the base space to the quaternionic-Kähler target manifold.

2. The anti-selfdual part of the spin connection of the base manifold has to be equal (up to gauge transformations) to the pullback of the $\mathfrak{su}(2)$ connection characterizing the quaternionic-Kähler target manifold.

These two conditions are interwoven but, as we will show in an explicit example, can be solved simultaneously.

Now, the metric, with $f = 1$ and $\omega = 0$, vanishing vector multiplets but unfrozen hyperscalars is a supersymmetric solution by itself and can be seen as a background which is excited when electric and magnetic charges are turned on. The functions f and ω satisfy the same covariant differential equations as before but in the new base space metric.

These solutions generically preserve only 1/8 of the available 8 supersymmetries.

In the null case, the metric is generically of the form

$$ds^2 = 2f du(dv + H du + \omega) - f^{-2}\gamma_{\underline{rs}} dx^r dx^s \,, \tag{1.2}$$

where $r, s = 1, 2, 3$ and all functions are v-independent. The functions f and H and the 1-form ω depend on the electric and magnetic charges and satisfy differential equations in the background of the 3-dimensional wavefront metric $\gamma_{\underline{rs}}$. When the hyperscalars are frozen, this metric is flat; when they are turned on, the 3-dimensional metric is determined by exactly the same two conditions that the base space of supersymmetric solutions of $N = 2, d = 4$ SUGRA coupled to hypermultiplets satisfy, namely

1. The hyperscalars must satisfy

$$\partial_r q^X \ f_X{}^{iA} \ \sigma^r{}_i{}^j \ = \ 0 \,. \tag{1.3}$$

2. The spin connection of the 3-dimensional metric must be equal (up to gauge transformations) to the pull-back of the the $\mathfrak{su}(2)$ connection that characterizes the quaternionic-Kähler target manifold.

This suggests a relation with the 4-dimensional solutions. We thus consider the particular case in which the metric has an additional isometry and is, in particular, u-independent. It is not difficult to see that in general the solutions of the null case describe pp-waves propagating along a string. Solutions which are u-independent can be compactified along the

[5] Please see the discussion after Eq. (4.30) for more information about the notion of quaternionic maps.

direction in which the wave propagates, *i.e.* along the string and give solutions belonging to the 4-dimensional timelike class, *i.e.* black hole-type solutions.

This set of 5-dimensional solutions and their reductions are presented here for the first time and allow an uplifting of 4-dimensional black-hole-type solutions (with or without hypermultiplets) to $d = 5$ dimensions different from the one considered in Refs. [21, 22, 23, 24, 25, 26, 27]. There, 4-dimensional black holes were uplifted to 4-dimensional black holes in a KK monopole background. Here we are dealing with the electric-magnetic dual uplift since the simplest 5-dimensional pp-wave and the Sorkin-Gross-Perry KK monopole [28] are related by dimensional reduction to $d = 4$ dimensions and 4-dimensional electric-magnetic duality, the 4-dimensional solution being the so-called "KK black hole", which in this simple case is singular. This relation is known in the general case under the name of "r-map", whence the r-map will relate these new string-pp-wave upliftings[6] to the known black hole-KK monopole upliftings.

This uplift may be more convenient to understand the black hole solutions from a higher-dimensional point of view since they are direct realizations of the D1-D5-W model. It may shed light on Mathur's conjecture [30, 31] on the realization of D1-D5-W microstates as supergravity solutions [32].

For the sake of completeness we have also worked out the timelike case with one additional isometry as, with frozen hyperscalars, all of the interesting solutions (supersymmetric rotating black holes and black rings [33]) seem to belong to this class [1, 34, 4]. The base space manifold is now a generalization of the Gibbons-Hawking instanton metric [35]. The Gibbons-Hawking instanton metric is the most general 4-dimensional hyper-Kähler metric with one isometry and can be used as a base space metric $h_{\underline{mn}}$ in absence of hyperscalars. It has the form

$$ds^2_{(4)} = H^{-1}(dz+\chi)^2 + H\delta_{\underline{rs}}dx^r dx^s\,, \quad r,s = 1,2,3\,, \tag{1.4}$$

where H is a function harmonic on 3-dimensional Euclidean space.

In presence of unfrozen hyperscalars the metric to be considered is

$$ds^2_{(4)} = H^{-1}(dz+\chi)^2 + H\gamma_{\underline{rs}}dx^r dx^s\,, \quad r,s = 1,2,3\,, \tag{1.5}$$

where the spin connection of the 3-dimensional metric $\gamma_{\underline{rs}}$ has to be equal (up to gauge transformations) to the pullback of the $\mathfrak{su}(2)$ connection of the hyperscalar manifold.

This paper is organized as follows: in Section 2 we describe ungauged $N = 1, d = 5$ supergravity coupled to vector multiplets and hypermultiplets. In Section 3 we derive the integrability conditions (KSIs) of the Killing spinor equations (KSEs), that relate the equations of motion of the fields for supersymmetric configurations, which will allow us to minimize the number of independent equations that need to be solved. In Section 4 we proceed to find the supersymmetric configurations and solutions, both in the timelike, (Section 4.1,) and in the null (Section 4.3) classes. An explicit example of the timelike class

[6] A particular case of this kind of uplifting was also observed in Ref. [29], although the 5-dimensional solutions were interpreted as rotating strings.

with unfrozen hyperscalars is given in Section 4.2 and the general subclasses of solutions that generically have one additional isometry are given in Sections 4.2.1 (timelike case) and Section 4.3.3 (null case). Section 5 contains our conclusions and final thoughts. Appendix A contains our conventions on gamma matrices, spinors, spinor bilinears and real special geometry. Appendix B contains a brief introduction to quaternionic-Kähler manifolds. Finally, Appendices C and D contain the necessary geometric data for the 5-dimensional metrics that appear in this paper.

2 Matter-coupled, ungauged $N = 1, d = 5$ supergravity

In this section we describe briefly the supergravity theories we will be working with: $N = 1, d = 5$ (minimal) ungauged supergravity coupled to n_v vector multiplets and n_h hypermultiplets[7].

The supergravity multiplet consists of the graviton $e^a{}_\mu$, the graviphoton A_μ and the gravitino ψ^i_μ. The gravitino and the rest of spinors in the theory are pairs of symplectic-Majorana spinors $i = 1, 2$ as explained in Appendix A.1.

Each of the n_v vector multiplets, labeled by $x = 1, \cdots, n_v$ consists of one real vector field A^x_μ, a real scalar ϕ^x and a gaugino λ^{xi}. The scalars ϕ^x, parametrize a Riemannian manifold which we call "the scalar manifold". The full theory is formally invariant under an $SO(n_v + 1)$ symmetry that mixes the matter vectors $A^x{}_\mu$ with the supergravity vector $A_\mu \equiv A^0{}_\mu$ and so it is convenient to treat all the vector fields on the same footing denoting them by $A^I{}_\mu$ $I = 0, \cdots, n_v$. The symmetry that rotates the vectors acts on the scalars as well and, to make it manifest one defines $n_v + 1$ functions of the physical scalars $h^I(\phi)$. These functions satisfy the constraint

$$C_{IJK}h^I h^J h^K = 1\,, \tag{2.1}$$

where C_{IJK} is a fully symmetric real constant tensor which characterizes completely the couplings in the vectorial sector. In particular it determines the metric of the scalar manifold $g_{xy}(\phi)$ on the target of ϕ^x, the couplings between scalars and vector fields $a_{IJ}(\phi)$ and the coupling constants of the vector field Chern-Simons terms. The relations between these fields are given in the Appendix A.3.

Each of the n_h hypermultiplets consists of four real scalar-fields (*hyperscalars*) q^X, $X = 1, \cdots, 4n_h$ and two spinor fields (*hyperinos*) ζ^A, $A = 1, \ldots, 2n_h$. The index i associated to the symplectic-Majorana condition is embedded into the index A. The hyperscalars q^X parametrize a quaternionic-Kähler manifold, described in Appendix B, that we will refer to as the *hypervariety*. In particular we observe that the connection of quaternionic-Kähler manifolds can be decomposed in an $\mathfrak{sp}(1) \simeq \mathfrak{su}(2)$ and an $\mathfrak{sp}(n_h)$ component whose pullback to spacetime will act on objects with index i and A, respectively.

[7]We follow essentially the notation and conventions of Ref. [37] with some minor changes to adapt them to those in Refs. [38, 39]. The changes are explained in Appendix A. The original references on matter-coupled $N = 1, d = 5$ SUGRA are [40] and [41]. The origin of these theories from compactifications of 11-dimensional supergravity on Calabi-Yau 3-folds was studied in Ref. [42].

The bosonic part of the action is

$$
\begin{aligned}
S &= \int d^5x\sqrt{g}\,\Big\{R + \tfrac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y + \tfrac{1}{2}g_{XY}\partial_\mu q^X\partial^\mu q^Y \\
&\qquad -\tfrac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu} + \tfrac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha\Big\}\,.
\end{aligned}
\tag{2.2}
$$

Observe that the hyperscalars do not couple to any of the fields in the vector multiplets and couple to the supergravity multiplet only through the metric. This is similar to what happens in $N = 2, d = 4$ theories and will have similar consequences.

We use the following notation for the equations of motion

$$
\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{g}}\frac{\delta S}{\delta e^a{}_\mu}\,,\quad \mathcal{E}_x \equiv -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta\phi^x}\,,\quad \mathcal{E}_X \equiv -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta q^X}\,,\quad \mathcal{E}_I{}^\mu \equiv \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A^I{}_\mu}\,, \tag{2.3}
$$

which are given by

$$
\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} - \tfrac{1}{2}a_{IJ}\left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\right) + \tfrac{1}{2}g_{xy}\left(\partial_\mu\phi^x\partial_\nu\phi^y - \tfrac{1}{2}g_{\mu\nu}\partial_\rho\phi^x\partial^\rho\phi^y\right) \\
&\quad +\tfrac{1}{2}g_{XY}\left(\partial_\mu q^X\partial_\nu q^Y - \tfrac{1}{2}g_{\mu\nu}\partial_\rho q^X\partial^\rho q^Y\right)\,,
\end{aligned}
\tag{2.4}
$$

$$
g^{xy}\mathcal{E}_y = \mathfrak{D}_\mu\partial^\mu\phi^x + \tfrac{1}{4}g^{xy}\partial_y a_{IJ}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\,, \tag{2.5}
$$

$$
g^{XY}\mathcal{E}_Y = \mathfrak{D}_\mu\partial^\mu q^X\,, \tag{2.6}
$$

$$
\mathcal{E}_I{}^\mu = \nabla_\nu(a_{IJ}F^{J\,\nu\mu}) + \tfrac{1}{4\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}F^J{}_{\nu\rho}F^J{}_{\sigma\alpha}\,. \tag{2.7}
$$

To these definitions we add the following notation for the Bianchi identities of the vector fields:

$$
\mathcal{B}^I{}_{\mu\nu\rho} \equiv 3\nabla_{[\mu}F^I{}_{\nu\rho]}\,. \tag{2.8}
$$

In these equations $\mathfrak{D}_\mu$ is the covariant derivative in the spacetime and in the corresponding scalar manifold. Then, Eq. (2.6) states that q is a harmonic map from spacetime to the hypervariety.

The supersymmetry transformation rules for the fermionic fields, evaluated on vanishing fermions, are

$$\delta_\epsilon \psi^i_\mu = \mathsf{D}_\mu \epsilon^i - \tfrac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta} (\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha}\gamma_\beta)\, \epsilon^i \,, \tag{2.9}$$

$$\delta_\epsilon \lambda^{ix} = \tfrac{1}{2} \left(\partial\!\!\!/ \phi^x - \tfrac{1}{2} h^x_I \; F\!\!\!/^I \right) \epsilon^i \,, \tag{2.10}$$

$$\delta_\epsilon \zeta^A = \tfrac{1}{2} f_X{}^{iA} \; \partial\!\!\!/ q^X \epsilon_i \,, \tag{2.11}$$

where D_μ is the Lorentz- and $SU(2)$-covariant derivative

$$\mathsf{D}_\mu \epsilon^i \equiv \nabla_\mu \epsilon^i + \epsilon^j \mathsf{A}_j{}^i{}_\mu \,, \tag{2.12}$$

and the $\mathfrak{su}(2)$ connection is the pullback of the $\mathfrak{su}(2)$ connection of the hypervariety:

$$\mathsf{A}^r{}_\mu \equiv \partial_\mu q^X \; \omega_X{}^r \,, \qquad \mathsf{A}_j{}^i = i\mathsf{A}^r \; \sigma^r{}_j{}^i \,. \tag{2.13}$$

Observe that the hyperscalars only appear in the gravitino's and gauginos' supersymmetry transformation rules precisely through the $\mathfrak{su}(2)$ connection.

Finally, the supersymmetry transformation rules of the bosonic fields are

$$\delta_\epsilon e^a{}_\mu = \tfrac{i}{2} \bar{\epsilon}_i \gamma^a \psi^i_\mu \,, \tag{2.14}$$

$$\delta_\epsilon A^I{}_\mu = -\tfrac{i\sqrt{3}}{2} h^I \bar{\epsilon}_i \psi^i_\mu + \tfrac{i}{2} h^I_x \bar{\epsilon}_i \gamma_\mu \lambda^{x\,i} \,, \tag{2.15}$$

$$\delta_\epsilon \phi^x = \tfrac{i}{2} \bar{\epsilon}_i \lambda^{x\,i} \,, \tag{2.16}$$

$$\delta_\epsilon q^X = -i f_{iA}{}^X \bar{\epsilon}^i \zeta^A \,. \tag{2.17}$$

3 KSIs and integrability conditions

The bosons' supersymmetry transformation rules lead to the following KSIs [43, 44] associated to the gravitino, gauginos and hyperinos *resp.*:

$$\left(\mathcal{E}_\mu{}^\nu \gamma_\nu + \tfrac{\sqrt{3}}{2} h^I \mathcal{E}_{I\,\mu} \right) \epsilon^i = 0 \,, \tag{3.1}$$

$$\left(\mathcal{E}_x - h^I_x \; \mathcal{E}\!\!\!/_I \right) \epsilon^i = 0 \,, \tag{3.2}$$

$$f_{iA}{}^X \mathcal{E}_X \epsilon^i = 0 \,. \tag{3.3}$$

It is an implicit assumption, used to derive the KSIs, that the Bianchi identities are satisfied. This affects, in particular, the first two KSIs, where the vector field equations appears. It is, therefore, useful to derive them from the integrability conditions of the KSEs, even if the derivation requires much more work, because in this case, contrary to what happens in $N = 2, d = 4$ theories [10], there is no electric-magnetic symmetry indicating in what combination the Bianchi identities should accompany the Maxwell equations.

The integrability condition of the KSE associated to the gravitino supersymmetry transformation gives

$$\begin{aligned} 4\gamma^\nu \mathsf{D}_{[\mu}\delta_\epsilon \psi^i_{\nu]} &= \Big\{ \left(\mathcal{E}_\mu{}^\sigma - \tfrac{1}{3} g_\mu{}^\sigma \, \mathcal{E}_\rho{}^\rho\right)\gamma_\sigma \\ &\quad + \tfrac{1}{4\sqrt{3}} h^I \left[\gamma_\mu \left(\not{\mathcal{E}}_I + \tfrac{1}{6} a_{IJ} \not{\mathcal{B}}^J\right) + 3\left(\not{\mathcal{E}}_I + \tfrac{1}{6} a_{IJ} \not{\mathcal{B}}^J\right)\gamma_\mu\right]\Big\} \epsilon^i = 0\,. \end{aligned} \tag{3.4}$$

To obtain this equation we need to use Eqs. (B.11)-(B.13), with $\nu = -1$ as to ensure the correct normalization of the hyperscalars' energy-momentum tensor. It is a well-known result that manifolds with the opposite sign of ν cannot be coupled to supergravity and here we are just recovering this result.

Acting with γ^μ from the left, we get

$$\left[\mathcal{E}_\rho{}^\rho + \tfrac{\sqrt{3}}{2} h^I (\not{\mathcal{E}}_I - \tfrac{1}{3} a_{IJ} \not{\mathcal{B}}^J)\right] \epsilon^i = 0\,, \tag{3.5}$$

which can be used to eliminate $\mathcal{E}_\rho{}^\rho$ from the integrability equation:

$$\left[\left(\mathcal{E}_\mu{}^\sigma + \tfrac{\sqrt{3}}{2} h_I {}^\star\mathcal{B}^I{}_\mu{}^\sigma\right)\gamma_\sigma + \tfrac{\sqrt{3}}{2} h^I \mathcal{E}_{I\,\mu}\right] \epsilon^i = 0\,. \tag{3.6}$$

On the other hand, from the gauginos' supersymmetry transformation rule we get

$$2\,\not{\mathfrak{D}}\delta_\epsilon \lambda^{ix} = \left[\mathcal{E}_x - h^I_x \left(\not{\mathcal{E}}_I + \tfrac{1}{6} a_{IJ} \not{\mathcal{B}}^J\right)\right] \epsilon^i = 0\,. \tag{3.7}$$

Eqs. (3.6) and (3.7) are the modifications to the two KSIs Eq. (3.1) and Eq. (3.2) that we were seeking for.

Let us now obtain tensorial equations form the spinorial KSIs: acting with $i\bar{\epsilon}_i \gamma_\rho$ from the left on Eq. (3.6) and taking into account the properties of the spinor bilinears discussed in Appendix A.2, we get

$$f\left(\mathcal{E}_{\mu\rho} + \tfrac{\sqrt{3}}{2} h_I {}^\star\mathcal{B}^I{}_{\mu\rho}\right) + \tfrac{\sqrt{3}}{2} h^I \mathcal{E}_{I\,\mu} V_\rho = 0\,, \tag{3.8}$$

whose symmetric and antisymmetric parts give independent equations.

Doing the same on Eqs. (3.7) and (3.3), we get

$$\mathcal{E}_x V^\rho - f h^I_x \mathcal{E}_I{}^\rho = 0\,, \tag{3.9}$$

$$\mathcal{E}_X V^\rho = 0\,. \tag{3.10}$$

Finally, acting with $i\bar{\epsilon}_i$ on Eqs. (3.6), (3.7) and (3.3) from the left we get respectively

$$\left(\mathcal{E}_{\mu\rho} + \tfrac{\sqrt{3}}{2} h_I {}^\star\mathcal{B}^I{}_{\mu\rho}\right) V^\rho + \tfrac{\sqrt{3}}{2} f h^I \mathcal{E}_{I\,\mu} \;=\; 0\,, \tag{3.11}$$

$$f\mathcal{E}_x - h^I_x \mathcal{E}_{I\,\rho} V^\rho \;=\; 0\,, \tag{3.12}$$

$$\mathcal{E}_X f \;=\; 0\,. \tag{3.13}$$

which can be obtained from Eqs. (3.8)-(3.10) only in the timelike $f \neq 0$ case.

Summarizing, in the timelike case, defining the unimodular timelike vector $v^\mu \equiv V^\mu/f$, we have

$$\mathcal{E}^{\mu\nu} \;=\; -\tfrac{\sqrt{3}}{2} h^I \mathcal{E}_I{}^{(\mu} v^{\nu)}\,, \tag{3.14}$$

$$h_I {}^\star\mathcal{B}^{I\,\mu\nu} \;=\; -h^I \mathcal{E}_I{}^{[\mu} v^{\nu]}\,, \tag{3.15}$$

$$\mathcal{E}_x \;=\; h^I_x \mathcal{E}_{I\rho} v^\rho\,, \tag{3.16}$$

$$\mathcal{E}_X \;=\; 0\,, \tag{3.17}$$

which imply that all the supersymmetric configurations automatically solve the equation of motion of the hyperscalars and that, if the Maxwell equations are satisfied, then the Einstein and scalar equations and the projections $h_I \mathcal{B}^I$ of the Bianchi identities are also satisfied. Therefore, in the timelike case, the necessary and sufficient condition for a supersymmetric configuration to also be a solution of the theory, is that it must solve the Maxwell equations and the Bianchi identities. Observe that, contrary to the 4-dimensional cases in which only one component of the Maxwell equations and Bianchi identities (the time component) need to be checked because the rest are automatically satisfied, in this 5-dimensional case we need to check all the components of the Maxwell equations and of the Bianchi identities.

In the null ($f = 0$) case, we get, renaming V^μ as l^μ

$$\mathcal{E}_{\mu\rho} l^\rho \;=\; -\tfrac{\sqrt{3}}{2} h_I {}^\star\mathcal{B}^I{}_{\mu\rho} l^\rho\,, \tag{3.18}$$

$$h^I \mathcal{E}_{I\,\mu} \;=\; 0\,, \tag{3.19}$$

$$h^I_x \mathcal{E}_{I\,\rho} l^\rho \;=\; 0\,, \tag{3.20}$$

$$\mathcal{E}_x = 0\,, \tag{3.21}$$

$$\mathcal{E}_X = 0\,, \tag{3.22}$$

which imply that the scalar and hyperscalars equations are automatically satisfied and so are certain projections of the Maxwell and Einstein equations.

4 Supersymmetric configurations and solutions

In this section we will follow the procedure of Ref. [1] to obtain supersymmetric configurations of supergravity, which consists in deriving equations for all the bilinears that can be constructed from the Killing spinors. These equations contain the lion's part of the information contained in the KSEs and can be used to constrain the form of the bosonic fields. These constraints are necessary conditions for the configurations to be supersymmetric and subsequently one has to prove that they are also sufficient (or find the missing conditions, as will happen in the null case). Finally one has to impose the equations of motion on the supersymmetric configurations in order to have classical supersymmetric solutions. The KSIs, derived in the previous section, simplify this task since only a small number of equations of motion are independent for supersymmetric configurations.

As we remarked in section 2, the hyperscalars appear only implicitly in the gravitino and gauginos supersymmetry transformations through the pullback of the $\mathfrak{su}(2)$ connection. The equations we are going to obtain for the fields in the supergravity and vector multiplets are, therefore, formally identical to the case without hypermultiplets considered in Ref. [5], but containing implicitly the $\mathfrak{su}(2)$ connection and its consequences. This is similar to what happens in the coupling of $N = 2, d = 4$ theories to hypermultiplets considered only recently in Ref. [11]

Our goal is to find all the field configurations for which the KSEs

$$\left\{\mathsf{D}_\mu - \tfrac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta}\,(\gamma_{\mu\alpha\beta} - 4g_{\mu\alpha}\gamma_\beta)\right\}\epsilon^i = 0\,, \tag{4.1}$$

$$\left(\not\partial\phi^x - \tfrac{1}{2} h_I^x\ \not F^I\right)\epsilon^i = 0\,, \tag{4.2}$$

$$f_X{}^{iA}\,\not\partial q^X \epsilon_i = 0\,, \tag{4.3}$$

admit at least one solution ϵ^i. We are going to assume its existence and we are going to derive necessary conditions for this to happen. These conditions will arise as consistency conditions of the equations satisfied by the tensors that can be constructed as bilinears of the Killing spinor whose existence was assumed from the onset.

As explained in Appendix (A.2), the tensor-bilinears that can be constructed from a symplectic-Majorana spinor are a scalar f, a vector V and three 2-forms Φ^r. f and V are $SU(2)$-singlets whereas the Φs form an $SU(2)$-triplet.

The fact that the Killing spinor satisfies Eq. (4.1) leads to the following differential equations for the bilinears:

$$df = \tfrac{1}{\sqrt{3}} h_I i_V F^I \,, \tag{4.4}$$

$$\nabla_{(\mu} V_{\nu)} = 0\,, \tag{4.5}$$

$$dV = -\tfrac{2}{\sqrt{3}} f h_I F^I - \tfrac{1}{\sqrt{3}} h_I {}^\star \left(F^I \wedge V\right) \,, \tag{4.6}$$

$$\mathsf{D}_\alpha \Phi^r{}_{\beta\gamma} = -\tfrac{1}{\sqrt{3}} h_I F^{I\,\rho\sigma} \left(g_{\rho[\beta} {}^\star\Phi^r{}_{\gamma]\alpha\sigma} - g_{\rho\alpha} {}^\star\Phi^r{}_{\beta\gamma\sigma} - \tfrac{1}{2} g_{\alpha[\beta} {}^\star\Phi^r{}_{\gamma]\rho\sigma}\right) \,, \tag{4.7}$$

where

$$\mathsf{D}_\alpha \Phi^r{}_{\beta\gamma} = \nabla_\alpha \Phi^r{}_{\beta\gamma} + 2\varepsilon^{rst} \mathsf{A}^s{}_\alpha \Phi^t{}_{\beta\gamma} \,. \tag{4.8}$$

These equations are formally identical to those obtained in Ref. [5] but now the covariant derivative that acts on the triplet of 2-forms is an $SU(2)$-covariant derivative.

Eqs. (4.2) and (4.3) lead to algebraic equations for the tensor bilinears: contracting Eq. (4.2) with $i\bar{\epsilon}_i$ and $\sigma^r{}_i{}^j \bar{\epsilon}_j$ we get

$$\pounds_V \phi^x = 0\,, \tag{4.9}$$

$$h^x_I F^I_{\alpha\beta} \Phi^{r\,\alpha\beta} = 0\,, \tag{4.10}$$

and the contraction of Eq. (4.3) with $i\bar{\epsilon}_k$ yields

$$\pounds_V q^X = 0\,. \tag{4.11}$$

Contracting now Eq. (4.2) with $i\bar{\epsilon}_i \gamma^\mu$ and $\sigma^r{}_i{}^j \bar{\epsilon}_j \gamma^\mu$ we get

$$f d\phi^x = -h^x_I i_V F^I \,, \tag{4.12}$$

$$0 = \Phi^r{}_{\mu\nu} \partial^\nu \phi^x + \tfrac{1}{4} \varepsilon_{\mu\nu\alpha\beta\gamma} h^x_I F^{I\,\nu\alpha} \Phi^{r\,\beta\gamma} \,, \tag{4.13}$$

and, finally, operating on Eq. (4.3) with $\bar{\epsilon}_k \gamma^\mu$

$$f \partial_\mu q^X + \Phi^r{}_\mu{}^\nu \partial_\nu q^Y J^r{}_Y{}^X = 0\,, \tag{4.14}$$

where we have identified the complex structures of the target quaternionic-Kähler manifold,

$$J^r{}_Y{}^X = f_Y{}^{iA} \mathsf{J}^r{}_{iA}{}^{jB} f_{jB}{}^X \,. \tag{4.15}$$

Eq. (4.5) says that V is an isometry of the space-time metric. The differential equation of Φ^r (4.7) implies

$$d\Phi^r + 2\varepsilon^{rst}\mathsf{A}^s \wedge \Phi^t = 0\,, \tag{4.16}$$

i.e. the three 2-forms are covariantly closed respect to the induced $\mathfrak{su}(2)$ connection.

In order to make further progress, it is necessary to separate the timelike ($f \neq 0$) and null ($f = 0$) cases.

4.1 The timelike case

4.1.1 The equations for the bilinears

In this case the Killing vector V is a timelike, $V^2 = f^2 > 0$. We introduce an adapted time coordinate t: $V = \partial_t$. With this choice of coordinates the metric can be decomposed in the following way

$$ds^2 = f^2\,(dt+\omega)^2 - f^{-1}h_{\underline{mn}}dx^m dx^n\,, \tag{4.17}$$

where ω is a time-independent 1-form and $h_{\underline{mn}}$ is a time-independent Riemannian four-dimensional metric.[8] Eqs. (4.4),(4.9) and (4.11) imply that with our choice of coordinates the scalars f, ϕ^x and q^X are time-independent.

Following Ref. [1] we define the following decomposition

$$f d\omega = G^+ + G^-\,, \tag{4.18}$$

where G^+ and G^- are the selfdual and anti-selfdual parts respect to the metric h.

The Fierz identity Eq. (A.23) indicates that the Φ^rs have no time components and the Fierz identity Eq. (A.24) implies that they are anti-selfdual respect to the spatial metric h. Moreover, the identity Eq. (A.25) becomes

$$\Phi^r{}_m{}^n\Phi^s{}_n{}^p = -\delta^{rs}\delta_m{}^p + \varepsilon^{rst}\Phi^t{}_m{}^p\,, \tag{4.19}$$

where all operations on the spatial indices refers to the spatial metric h. This is the algebra of the imaginary unit quaternions, whence we may conclude that the spatial manifold is endowed with an *almost* quaternionic structure.

The next step is to obtain the form of the supersymmetric vector fields from Eqs. (4.4), (4.6), (4.10) and (4.12): these equations contain no explicit contributions from the hyper-scalars and, therefore lead to the same form of the vector fields found in Ref. [5], namely

$$F^I = -\sqrt{3}\{d\left[fh^I\,(dt+\omega)\right] + \Theta^I\}\,, \tag{4.20}$$

where the Θ^Is are spatial selfdual 2-forms and

[8] Appendix C contains a Vielbein basis and the non-vanishing components of the connection and Ricci tensor in that basis.

$$G^{+} = -\tfrac{3}{2} h_I \Theta^I \, . \tag{4.21}$$

From (4.7) information about the derivatives of the two-forms Φ^r can be extracted using the above expression for F^I: first, by introducing the spin connection of the metric given in Appendix C we may obtain the spatial components of the five-dimensional covariant derivative,

$$\nabla^{(5)}_m \Phi^r{}_{nq} = f^{3/2} \nabla_m \Phi_{nq} - \tfrac{2}{3} \left(\delta_{m[n} \partial_{p]} f^{3/2} \Phi^r{}_{pq} - \delta_{m[q} \partial_{p]} f^{3/2} \Phi^r{}_{pn} - \partial_m f^{3/2} \Phi^r{}_{nq} \right) \, , \tag{4.22}$$

where ∇_m is the covariant derivative of the four-dimensional spatial metric. On the right hand side of this expression all of the flat indices refers to the Vielbein $v_m{}^{\underline{i}}$. On the other hand, the spatial components of the equation (4.7) are

$$\nabla^{(5)}_m \Phi^r{}_{nq} + 2 f^{3/2} \varepsilon^{rst} \mathsf{A}^s{}_m \Phi^t{}_{nq} = -\tfrac{1}{\sqrt{3}} f h_I F^{I\,p0} \left(\delta_{p[n} \Phi^r{}_{q]m} - \delta_{pm} \Phi^r{}_{nq} - \delta_{m[n} \Phi^r{}_{q]p} \right) \tag{4.23}$$

where we have used the fact that Φ^r are spatial, anti-selfdual 2-forms. Now from Eq. (4.20) we read

$$h_I F^{I\,p0} = \sqrt{3} f^{-1/2} \partial_p f \tag{4.24}$$

and by comparing Eqs. (4.22) and (4.23) we find that the 2-forms Φ^r are $SU(2)$- and Lorentz-covariantly constant over the 4-dimensional spatial manifold:

$$\nabla_m \Phi^r{}_{np} + 2 \varepsilon^{rst} \mathsf{A}^s{}_m \Phi^t{}_{np} \; = \; \partial_m \Phi^r{}_{np} - 2 \xi_{m[n|}{}^q \Phi^r{}_{q|p]} + 2 \varepsilon^{rst} \mathsf{A}^s{}_m \Phi^t{}_{np} \; = \; 0 \, , \tag{4.25}$$

Here ξ is the standard spin connection of the 4-dimensional spatial manifold.

Had the base space not been 4-dimensional, the conclusion would have been that we are dealing with a quaternionic-Kähler manifold. But in four dimensions the above equation, taken at face value, is rather void: given a Vierbein we can construct a kosher quaternionic structure by inducing the one from $\mathbb{R}^4$ and then the unique A solving Eq. (4.25), is given by

$$\mathsf{A}^r_m \; = \; \tfrac{1}{16} \, \varepsilon^{rts} \; \Phi^{t\,n}_{p} \; \nabla_m \Phi^{s\,p}_{n} \, . \tag{4.26}$$

In the case at hand, however, said arbitrariness is nothing but an illusion since the connection A is the one induced from an $\mathfrak{sp}(1)$ connection on a quaternionic-Kähler manifold and is therefore not to be chosen but to be deduced. At this point one can then already appreciate the interwoven nature of the problem: Since the quaternionic structure on the 4-dimensional space is basically known, Eq. (4.25) determines, part of, the spin connection in terms of the pull-back of an $\mathfrak{sp}(1)$ connection. This pull-back, however, is defined by means of a harmonic map satisfying Eq. (4.14), which presupposes knowing the Vierbein, and hence also the spin connection.

A 'trivial' solution to the requirement that the hyperscalars form a harmonic map satisfying Eq. (4.14), is to take them to be constant: Eq. (4.25) then states that Φ defines a covariantly constant hypercomplex structure, so that the 4-dimensional manifold has to be hyper-Kähler, and we recover the results of [1, 5]. As is well-known the holonomy of a 4-dimensional hyper-Kähler space is $\mathfrak{su}(2) \subset \mathfrak{so}(4)$, and in a suitable frame the spin connection can be taken to be selfdual. The technical reason why the spin connection can be taken to be selfdual lies in the fact that the Φs are anti-selfdual and that the split into anti- and selfdual components corresponds to the Lie algebraic split $\mathfrak{so}(4) \cong \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$; if we then take the Φs to be induced from the ones on $\mathbb{R}^4$, called J, and denote the projection of the spin connection onto $\mathfrak{su}(2)_\pm$ by $\xi^\pm$, then Eq (4.25) can be expressed as $[\xi^-_m, \mathsf{J}^r] = 0$, which immediately implies $\xi^- = 0$.

In the general case there will still be no constraint on ξ^+, but we can solve equation (4.25) to give

$$\xi^-_{m\ n}{}^q \;=\; -\vec{\mathsf{A}}_m \;\cdot\; \vec{\mathsf{J}}_n{}^q \;, \tag{4.27}$$

where as above, we made use of the quaternionic structure induced from flat space.

In the above we were able to match things up without much ado, since the relevant $\mathfrak{su}(2)$s both acted in the vector representation. When considering the Killing spinor equation, however, the representations do not add up that nicely, and one finds that a necessary condition for having unbroken supersymmetry is that the generators of $\mathfrak{su}(2)$ and $\mathfrak{su}(2)_-$ should have identical actions on the Killing spinors, *i.e.*

$$\epsilon^j \; i\sigma^r{}_j{}^i \;=\; \tfrac{1}{4}\; \mathsf{J}^r{}_{mn}\gamma^{mn}\; \epsilon^i \;, \tag{4.28}$$

and these conditions will appear as projectors $\Pi^{r\,+}{}_i{}^j$ acting on the Killing spinors, where

$$\Pi^{r\,\pm}{}_i{}^j \;=\; \tfrac{1}{2}\left[\;\delta \;\pm\; \tfrac{i}{4}\; \not{\mathsf{J}}^{(r)}\sigma^{(r)}\right]_i{}^j \,. \tag{4.29}$$

In principle we only need to impose one such constraint for each non-trivial component A^r.

The last constraint on the bosonic fields comes from Eq. (4.14). In the timelike case this equation is purely spatial and in 4-dimensional notation reads

$$\partial_m q^X \;=\; \Phi^r{}_m{}^n\; \partial_n q^Y\; J^r{}_Y{}^X \,. \tag{4.30}$$

This condition implies that q is what Ref. [45] calls a *quaternionic map*. In said reference it is shown that a quaternionic map between hyper-Kähler manifolds implies that the map is harmonic, *i.e.* it solves

$$\mathfrak{D}_\mu \partial^\mu\; q^X = 0 \,. \tag{4.31}$$

Here, however, we are not dealing with maps between hyper-Kähler manifolds, yet the KSIs state that q is automatically harmonic. The question then is: Apart from being quaternionic, what properties must q satisfy in order to be harmonic?

Let us be a bit more general and consider the situation in which the $\mathfrak{sp}(1)$ connection A appearing in Eq. (4.25) is *not* the pull-back of the $\mathfrak{sp}(1)$ connection, denoted B, defined on the hypervariety. By then differentiating Eq. (4.30), using Eqs. (4.25) and the formulas in App. (B), we obtain

$$\begin{aligned}\mathfrak{D}_m \partial_n q^X &= -2\varepsilon^{str} \left[\mathsf{A}^s{}_n - \partial_n q^Z \, \mathsf{B}^s{}_Z\right] \Phi^t{}_m{}^p \partial_p q^Y \, J^r{}_Y{}^X \\ &\quad + \Phi^r{}_n{}^p \, \mathfrak{D}_m \partial_p q^y \, \vec{J}^r{}_Y{}^X \, .\end{aligned} \tag{4.32}$$

Contracting the free indices, we find that

$$\mathfrak{D}_m \partial^m q^X = 2\varepsilon^{str} \left[\mathsf{A}^s{}_m - \partial_m q^Z \, \mathsf{B}^s{}_Z\right] \Phi^{t\ nm} \partial_n q^Y J^r{}_Y{}^X \, . \tag{4.33}$$

In our case, we have $\mathsf{A} = dq \cdot \mathsf{B}$ whence the fact that q is a quaternionic map, by itself, implies that it is harmonic.

Therefore, supersymmetric configurations of the hyperscalars consist of quaternionic maps q such that the $\mathfrak{su}(2)_-$ connection of the 4-dimensional space manifold is canceled by the pullback of the one on the hypervariety.

In the next section we are going to check whether the conditions that we have derived on the fields are sufficient to have unbroken supersymmetry, *i.e.* identically solve the KSEs.

4.1.2 Solving the Killing spinor equations

We begin with Eq. (4.2), from the gaugino supersymmetry transformation. After use of the expression of the vectorial fields Eq. (4.20), it can be put in the form

$$\left(2 \, \partial\!\!\!/ \phi^x - \tfrac{\sqrt{3}}{2} \, \not{\Theta}^I\right) R^- \epsilon^i = 0 \, , \tag{4.34}$$

where we have defined the projectors $R^\pm$

$$R^\pm \equiv \tfrac{1}{2}\left(1 \pm \gamma^0\right) \, . \tag{4.35}$$

Obviously, this equation can always be solved by imposing the projection

$$R^- \epsilon^i = 0 \, , \tag{4.36}$$

which is equivalent to a chirality condition on the spinors over the spatial manifold due to the relation $\gamma^0 = \gamma^{1234}$. R^+ and R^- have rank 2 and therefore this projection breaks/preserves 1/2 of the original supersymmetries.

Now we analyze Eq. (4.3), from the hyperinos supersymmetry transformations. Using Eq. (4.30) we can rewrite it in the form

$$f_X{}^{jA} \, \partial\!\!\!/ q^X \left[3\delta_j{}^i + \tfrac{i}{4} \sum_r \not{J}^{(r)} \sigma^{(r)}{}_j{}^i \gamma_0\right] \epsilon_i - \gamma_m \mathsf{J}^r{}_{mn} \partial_n q^Y J^r{}_Y{}^X f_X{}^{iA} R^- \epsilon_i = 0 \, , \tag{4.37}$$

which can be solved by imposing the projection Eq. (4.36) and

$$\Pi^{r+}{}_j{}^i \epsilon^j = 0\,, \tag{4.38}$$

where the $\Pi^{r\pm}{}_j{}^i$s are the objects defined in Eq. (4.29). The $\Pi^{r+}{}_j{}^i$ satisfy the algebra

$$\Pi^{r\,+}\Pi^{s\,+} \;=\; \tfrac{1}{2}\Pi^{r\,+} \;+\; \tfrac{1}{2}\Pi^{s\,+} \;-\; \tfrac{1}{2}\varepsilon^{rst}\Pi^{t\,+} \;-\; \tfrac{1}{4}\,\delta^{rs}R^{-}\,, \tag{4.39}$$

and are idempotent (and, therefore, projectors) only in the subspace of spinors satisfying the projection Eq. (4.36).

Observe that, in principle, we need to impose the three projections $r = 1, 2, 3$ on the Killing spinors. The above algebra shows that only two of them are independent and it is easy to see that they preserve only 1/4 of the supersymmetries preserved by the projection Eq. (4.36), *i.e.* only 1/8 of the supersymmetries is generically preserved in presence of non-trivial hyperscalars.

We turn now to Eq. (4.1) from the gravitino supersymmetry transformation. We consider separately the timelike and spacelike components of this equation. By using the spin connection of the five-dimensional metric Eqs. (C.4) and the expression of the vector fields Eq. (4.20), the timelike component takes the form

$$\partial_0\epsilon^i + \left[2\,\not{\partial} f^{1/2} - \tfrac{1}{4}f\left(1-\tfrac{1}{3}\gamma^0\right)\not{G}^{+} - \tfrac{1}{4}f\;\not{G}^{-}\right]R^{-}\epsilon^i = 0\,, \tag{4.40}$$

which is automatically solved by time-independent Killing spinors satisfying the projection Eq. (4.36).

The space-like components of Eq. (4.1) take, after use of Eq. (4.36), the form

$$\nabla_m\eta^i \;+\; \eta^j \mathsf{A}_{m\,j}{}^i \;=\; 0\,, \qquad \eta^i \equiv f^{-1/2}\epsilon^i\,. \tag{4.41}$$

To solve this equation, the quaternionic nature of the 4-dimensional spatial manifold comes to our rescue: in the special Vierbein basis and $SU(2)$ gauge in which Eq. (4.27) holds, the 2-forms $\Phi^r{}_{mn}$ are the constants $\mathsf{J}^r{}_{mn}$. Using this splitting, the above equation takes the form

$$\nabla^{+}_m\eta^i \;+\; i\;\mathsf{A}^r{}_m\left(\sigma^r{}_j{}^i + \tfrac{i}{4}\;\not{\mathsf{J}}^r\delta_j{}^i\right)\eta^j = 0\,, \qquad \nabla^{+}_m\eta^i \;=\; (\partial_m \;+\; \tfrac{1}{4}\;\not{\varpi}^{+}{}_m)\eta^i\,. \tag{4.42}$$

Using the projections Eq. (4.38) for each non-vanishing component of the pull-back of the $\mathfrak{su}(2)$ connection $\mathsf{A}^r{}_X\partial_m q^X$ we are left with

$$\nabla^{+}_m\eta^i = 0\,, \tag{4.43}$$

which is solved by constant spinors that satisfy the projection Eq. (4.36), *i.e.* if they are chiral in the 4-dimensional spaces of constant time.

It should be clear from the discussion of the gravitino variations, that, for some configurations, not all of the projections Π need be imposed, *e.g.* when turning on only an $\mathfrak{u}(1)$ in $\mathfrak{su}(2)_-$. The analysis of Eq. (4.37), however, indicates that still all 3 of the projections ought to be implemented. This is true if we disregard the possibility of a special coordinate dependency of the quaternionic map. As an extreme example we have the case with frozen hyperscalars which effectively is like not having them at all. A less-trivial example to this effect is fostered by the trivial uplift of the *c-mapped cosmic string* analyzed in [11, Sec. (4.4)], in which case the map is holomorphic.[9]

4.1.3 Supersymmetric solutions

In Section 3 we proved that timelike supersymmetric configurations solve all the equations of motions if they solve the Maxwell equations and Bianchi identities which we rewrite here in differential form language for convenience:

$$4^{\star}\mathcal{E}_I = -d^{\star}\left(a_{IJ}F^J\right) + \tfrac{1}{\sqrt{3}}C_{IJK}F^J \wedge F^K\,, \tag{4.44}$$

$$\mathcal{B}^I = dF^I\,. \tag{4.45}$$

We may evaluate these expressions for supersymmetric configurations using the formula (4.20). The result is

$$\mathcal{E}_I^0 = -\frac{\sqrt{3}}{2}f^2\left[\nabla^2_{(4)}\left(h_I/f\right) - \tfrac{1}{4}C_{IJK}\Theta^J\cdot\Theta^K\right]\,, \tag{4.46}$$

$$\mathcal{E}_I^m = -2\sqrt{3}f^{3/2}C_{IJK}h^J(\star_{(4)}d\Theta^K)^m\,, \tag{4.47}$$

$$\left(\star\mathcal{B}^I\right)^{0m} = -\sqrt{3}f^{3/2}(\star_{(4)}d\Theta^I)^m\,. \tag{4.48}$$

where, as usual, all the objects in the r.h.s. of the equations are written in terms of the 4-dimensional spatial metric h. The components $\left(\star_{(4)}\mathcal{B}^I\right)^{mn}$ vanish identically, and it is immediate to see that the KSI Eq. (3.15) is satisfied.

Then, the supersymmetric solutions have to satisfy only these two equations:

$$\nabla^2_{(4)}\left(h_I/f\right) - \tfrac{1}{4}C_{IJK}\Theta^J\cdot\Theta^K = 0\,, \tag{4.49}$$

$$d\Theta^I = 0\,, \tag{4.50}$$

which are identical to those found in Ref. [5] in absence of hypermultiplets, the difference being the quaternionic nature of the 4-dimensional space.

[9] In fact, part of Chen and Li's article [45] consists of showing that there are quaternionic maps between hyper-Kähler manifolds that are *not* holomorphic w.r.t. some complex structure.

4.2 Some explicit examples

In the recent paper Ref. [15] Jong, Kaya and Sezgin gave an explicit example with non-trivial and not-obviously-holomorphic hyperscalars taking values in the symmetric space $H_4 = SO(4,1)/SO(4)$. In this section we are going to use the same set-up to find 5-dimensional supersymmetric solutions and discuss the possible gravitational effects.

The four coordinates of the target are denoted by $q^{\underline{X}}$, $\underline{X} = 1, \ldots, 4$, and take the metric on the hypervariety to be

$$g_{\underline{X}\,\underline{Y}} = \Lambda^2 \delta_{\underline{XY}}\,, \qquad \Lambda(q^2) = \frac{1}{1\,-\,q^2}\,, \qquad q^2 \equiv q^{\underline{X}} q^{\underline{X}} \leq 1\,. \tag{4.51}$$

As one might have suspected this metric is Einstein, and since the space is conformally flat, it is also trivially selfdual, meaning that we are really dealing with an authentic 4-dimensional quaternionic-Kähler manifold.

A Vierbein for this metric is

$$E^X \;=\; \Lambda \delta^X{}_{\underline{Y}}\, dq^{\underline{Y}}\,, \qquad E_X \;=\; \Lambda^{-1} \delta_X{}^{\underline{Y}} \frac{\partial}{\partial q^{\underline{Y}}}\,. \tag{4.52}$$

In both the coordinate and the Vierbein basis the three complex structures are given by the 't Hooft symbols $\rho^r{}_{XY} (= \mathsf{J}^r_{XY})$, which are real, constant and antisymmetric matrices in the X, Y indices. Moreover they are anti-selfdual[10] and satisfy

$$\rho^r{}_{XY}\; \rho^s{}_{YZ} \;=\; -\delta^{rs}\; \delta_{XZ} \;+\; \epsilon^{rst}\; \rho^t{}_{XZ}\,, \tag{4.53}$$

$$\rho^r{}_{XY}\; \rho^r{}_{WZ} \;=\; \delta_{XW}\; \delta_{YZ} - \delta_{XZ}\; \delta_{YW} \;-\; \epsilon_{XYWZ}\,. \tag{4.54}$$

The anti-selfdual part of the spin connection is

$$\omega^{-XY} = 2\; \left(q^{[X} E^{Y]} - \tfrac{1}{2} \epsilon^{XYWZ} q^W E^Z \right)\,, \tag{4.55}$$

where $q^X \equiv \delta^X{}_{\underline{Y}}\; q^{\underline{Y}}$.

In order to construct the hyperscalars, we assume that also the base manifold is conformally flat, *i.e.*

$$h_{\underline{mn}} dx^{\underline{m}} dx^{\underline{n}} = \Omega^2 dx^{\underline{m}} dx^{\underline{m}}\,, \qquad \Omega = \Omega(x^2)\,, \qquad x^2 \equiv x^{\underline{m}} x^{\underline{m}}\,, \tag{4.56}$$

and thence take the Vierbein on the base manifold to be

$$V^m = \Omega \delta^m{}_{\underline{m}} dx^{\underline{m}}\,, \qquad V_m = \Omega^{-1} \delta_m{}^{\underline{m}} \partial_{\underline{m}}\,. \tag{4.57}$$

In this basis we can identify the complex structures of the base manifold with those of the hypervariety

[10] They can be seen as the three anti-selfdual combinations of generators of $\mathfrak{so}(4)$, *i.e.* the generators of the $\mathfrak{su}(2)_-$ subalgebra.

$$\mathsf{J}^r{}_m{}^n \;=\; \delta_m{}^X J^r{}_X{}^Y \delta_Y^n \;=\; \rho^r{}_{mn}\,. \tag{4.58}$$

The anti-selfdual part of the spin connection on the base manifold is

$$\xi^{-mn} = 2\frac{\Omega'}{\Omega^2}\left(x^{[m}V^{n]} - \tfrac{1}{2}\epsilon^{mnpq}x^pV^q\right) \tag{4.59}$$

where $x^m = \delta^m{}_{\underline{m}}\, x^{\underline{n}}$.

Now we analyze the conditions for supersymmetry on the hyperscalars $q^{\underline{X}}$. The first condition is that they must constitute a quaternionic map, *i.e.* Eq. (4.30), w.r.t. the chosen quaternionic structures. In our setting this equations takes the form

$$\partial_{\underline{m}}q^{\underline{X}} = \left(\delta_{\underline{mY}}\delta_{\underline{nX}} - \delta_{\underline{mX}}\delta_{\underline{nY}} - \epsilon_{\underline{mnYX}}\right)\partial_{\underline{n}}q^{\underline{Y}} \tag{4.60}$$

whose symmetric and antisymmetric parts give

$$\partial_{\underline{m}}q^{\underline{m}} \;=\; 0\,, \tag{4.61}$$

$$\partial_{[\underline{m}}q_{\underline{n}]} \;=\; -\tfrac{1}{2}\epsilon_{\underline{mnpq}}\partial_{\underline{p}}q_{\underline{q}}\,, \tag{4.62}$$

where $q_{\underline{m}} = q^{\underline{m}}$.

A solution to these equations is

$$q^{\underline{m}} \;=\; x^{\underline{m}}\, x^{-4}\,, \tag{4.63}$$

where we have chosen a possible multiplicative constant to be unity.

The second condition on the hyperscalars states that the anti-selfdual part of the spin connection of the base manifold must be related to the $\mathfrak{su}(2)$ connection induced from the target,

$$\xi^{-}_{mn}{}^{p} \;=\; -\vec{\mathsf{A}}_m\cdot\vec{\mathsf{J}}_n{}^p\,, \tag{4.64}$$

$$\vec{\mathsf{A}}_m \;\equiv\; \partial_m q^{\underline{X}}\,\vec{\omega}_{\underline{X}}\,, \tag{4.65}$$

where $\vec{\omega}_{\underline{X}}$ is the $\mathfrak{su}(2)$ connection of the target. We observe that the reasoning leading to the relation (4.64) can be applied on the target manifold as well,[11], where the involved connections are ω_{XY} and $\vec{\omega}_{\underline{X}}$ and therefore we may establish the following relation on the target

$$\omega^{-}_{XY}{}^{Z} \;=\; -\vec{\omega}_X \;\cdot\; \vec{J}_Y{}^Z\,. \tag{4.66}$$

By contrasting Eqs. (4.64)-(4.66) we conclude that in our settings the anti-selfdual part of the spin connection of the base manifold is induced from the one of the hypervariety,

[11] Indeed it can be applied in any four-dimensional Riemannian manifold.

$$\xi_m^{-np} = \partial_m q^{\underline{X}} \omega_{\underline{X}}^{-YZ} \delta_{YZ}{}^{np} \,. \tag{4.67}$$

This condition is satisfied if

$$\frac{\Omega'}{\Omega} = \frac{1}{x^2\,(x^6-1)} \,. \tag{4.68}$$

The solution to this equation is

$$\Omega = \left(1 \;-\; x^{-6}\right)^{1/3} \,, \tag{4.69}$$

where, as above, we chose a certain multiplicative integration constant. We would like to point out that in this case the whole spin connection on the base manifold, rather than only its anti-selfdual part, is induced by the connection on the hypervariety.

A small investigation of the curvature invariants for the metric on the base space, shows that the point $x^2 = 1$ corresponds to a naked curvature singularity.

We have, thus, found the following 1/8 BPS, static, asymptotically flat, spherically symmetric, solution with only unfrozen hyperscalars in the $SO(1,4)/SO(4)$ coset:

$$\begin{aligned} ds^2 &= dt^2 - \left(1 - \frac{1}{x^6}\right)^{2/3} dx^{\underline{m}} dx^{\underline{m}} \,, \\ q^{\underline{m}} &= \frac{x^{\underline{m}}}{x^4} \,, \end{aligned} \tag{4.70}$$

which, as was said above, presents a naked singularity at $x^2 = 1$. Since there are no conserved charges in this system, the *no hair* conjecture suggests that black-hole type (i.e. spherically symmetric) solutions of this and similar systems will always be singular, but a more detailed study is needed to reach a final conclusion since they may be excluded by a mechanism like the one discussed in Ref. [46, 47]. Furthermore, a higher-dimensional stringy interpretation of this, and similar solutions, is also needed as to interpret this singularity correctly.

As a further example let us now consider how solutions of minimal $N = 1, d = 5$ SUGRA[12] are deformed by the coupling to these hyperscalars. For the sake of simplicity we consider the simplest static ($\Theta = \omega = 0$) solution which is determined, according to Eq. (4.49), by a single function $f^{-1} = K$ which is harmonic w.r.t. the metric on the base manifold. The supersymmetric solution can be written as

[12] In our notation this means that $n_v = 0$, $C_{111} = 1$ and $h^1 = 1$.

$$
\begin{aligned}
ds^2 &= K^{-2}\,dt^2 \;-\; K\left(1+\frac{\lambda}{x^6}\right)^{2/3} dx^{\underline{m}}dx^{\underline{m}}\,,\\
A &= -\sqrt{3}\;K^{-1}\;dt\,,\\
q^{\underline{m}} &= \frac{x^{\underline{m}}}{x^4}\,.
\end{aligned}
\tag{4.71}
$$

If the harmonic function is chosen as to have an asymptotically flat, spherically symmetric solution with positive mass, the harmonic function is, with frozen hyperscalars,

$$
K = 1 + \frac{|Q|}{x^2}\,, \tag{4.72}
$$

and the solution is the 5-dimensional Reissner-Nordström black hole [48] which has an event horizon at $x = 0$ that covers all singularities.

When the hyperscalars are unfrozen and we have the above base manifold, K, determined again by imposing asymptotic flatness and spherical symmetry, is given by

$$
K \;=\; 1 \;+\; Q\;\frac{{}_2\mathrm{F}_1\left(\frac{1}{3},\frac{2}{3}\;;\;\frac{4}{3}\;;\;x^{-6}\right)}{x^2}\;, \tag{4.73}
$$

where ${}_2F_1$ is a Gauß hypergeometric function. It is easy to see that $\lim_{x^2\to\infty} K \;=\; 1$ and that ${}_2\mathrm{F}_1\left(\frac{1}{3},\frac{2}{3}\;;\;\frac{4}{3}\;;\;x^{-6}\right)/x^2$ is a real, strictly positive and monotonically decreasing function on the interval $x^2 \in (1,\infty)$. The real question then is: what happens at $x^2 = 1$? Eq. [49, 15.1.20] gives a straightforward answer

$$
{}_2F_1\left(\,\tfrac{1}{3},\tfrac{2}{3};\tfrac{4}{3};1\,\right) \;=\; \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \;\sim\; 1.76664\,, \tag{4.74}
$$

which implies that there is a naked singularity at $x^2 = 1$.

4.2.1 Solutions with an additional isometry

To make contact with the families of solutions with one additional isometry found in Refs. [1, 4] we make the following *Ansatz* for the 4-dimensional spacelike metric

$$
h_{\underline{mn}}dx^m dx^n = H^{-1}(dz+\chi)^2 + H\gamma_{\underline{rs}}dx^r dx^s\,, \quad r,s = 1,2,3\,, \tag{4.75}
$$

where the function H, the 3-dimensional metric $\gamma_{\underline{rs}}$, and the 1-form $\chi = \chi_{\underline{r}}dx^r$ are all independent of the coordinate z. This *Ansatz* covers all 4-dimensional metrics with one isometry. We also require all fields in the solution to be independent of z.

As we have seen, supersymmetry requires the anti-selfdual part of the spin connection of this metric to be identical to the pullback of the $\mathfrak{su}(2)$ connection of the hypervariety. With the orientation $\varepsilon_{z123} = +1$ and the Vierbein basis

$$V^z = H^{-1/2}(dz + \chi)\,, \qquad V^r = H^{1/2}v^r\,, \tag{4.76}$$

where the v^r is the Dreibein for the 3-dimensional metric $\gamma_{\underline{rs}}$, the anti-selfdual part of the spin connection 1-form is given by

$$\begin{aligned} \xi^{-\,zr} &= \tfrac{1}{2}H^{-3/2}[\partial_r H - (\hat{\star}\hat{d}\chi)_r]V^z \\ &\quad + \tfrac{1}{4}\varepsilon_{rst}H^{-3/2}\{[\partial_t H - (\hat{\star}\hat{d}\chi)_t]\delta_{su} - 2H\varpi_{ust}\}V^u\,, \end{aligned} \tag{4.77}$$

where hatted objects refer to the 3-dimensional metric.

Observe that the z-independence of all fields means that the pullback of the $\mathfrak{su}(2)$ connection has no z component. Then, the supersymmetry condition Eq. (4.27) leads to

$$\hat{d}H = \hat{\star}\hat{d}\chi\,, \quad \Rightarrow \quad \hat{\nabla}^2 H = 0\,, \tag{4.78}$$

which is a condition on the 4-dimensional metric, and

$$\xi_{\underline{r}}^{-\,zs} \;=\; -\tfrac{1}{2}\varepsilon^{stu}\,\varpi_{\underline{r}}{}^{tu} \;=\; -2\mathsf{A}^s{}_X\,\partial_{\underline{r}}q^X\,, \tag{4.79}$$

which is a condition on the hyperscalars and the 3-dimensional metric.

Observe that the above 4-dimensional metric is a generalization of the Gibbons-Hawking instanton metric [35]. The non-trivial 3-dimensional metric destroys the selfduality of the connection. However, the generalized metric admits a quaternionic structure which is the straightforward generalization of that of the Gibbons-Hawking metric [36] and is, therefore, associated to the three hyper-Kähler 2-forms

$$J^r \equiv V^z \wedge V^r - \tfrac{1}{2}\varepsilon^{rst}V^s \wedge V^t\,. \tag{4.80}$$

It is trivial to check that they satisfy the quaternionic algebra since the tangent space components of these 2-forms are identical to those of the Gibbons-Hawking metric and are proportional to the anti-selfdual generators of $SO(4)$. Unlike the Gibbons-Hawking case, however, the hyper-Kähler 2-forms are not closed. Instead, a simple calculation shows that they satisfy

$$dJ^r - \varpi^{rs} \wedge J^s = 0\,, \tag{4.81}$$

which, on account of Eq. (4.79), can be written in the form

$$dJ^r + 2\varepsilon^{rst}\mathsf{A}^s \wedge J^s = 0\,. \tag{4.82}$$

Thus, the 4-dimensional metric Eq. (4.75) and hyperscalars subject to Eqs. (4.78) and (4.79) (plus Eq. (4.30)) are the most general ones associated to supersymmetric solutions with one isometry. Using them it can be shown that the general solutions found in Ref. [4] are formally identical, the only difference being that the $2\bar{n}+2$ harmonic functions K^I, L_I, M, H on which these solutions depend, are harmonic functions w.r.t. the 3-dimensional metric $\gamma_{\underline{rs}}$.

To be explicit, in terms of these harmonic functions, the scalars, the closed selfdual 2-forms Θ^I, and the 1-form ω take the form

$$
\begin{aligned}
h_I/f &= C_{IJK}K^J K^K/H + L_I\,, \\
\Theta^I &= [(dz+\chi)\wedge d(K^I/H) + H\hat{\star}d(K^I/H)]\,, \\
\omega &\equiv \omega_5(d\psi+\chi) + \hat{\omega}\,, \\
\omega_5 &= M + \tfrac{3}{2}H^{-1}L_I K^I + H^{-2}C_{IJK}K^I K^J K^K\,, \\
\star_{(3)}d\hat{\omega} &= HdM - MdH + \tfrac{3}{2}(K^I dL_I - L_I dK^I)\,.
\end{aligned}
\tag{4.83}
$$

The function f has to be determined case by case using the constraint $C_{IJK}h^I h^J h^K = 1$, but an explicit expression for symmetric spaces is given in Ref. [4]. In the $n = 0$ case, *i.e.* only one function $K^0 \equiv K$ and one function $L_0 \equiv L$, it is given by

$$
f^{-1} = K^2/H + L\,. \tag{4.84}
$$

The metric of these solutions can be cast in the form

$$
\begin{aligned}
ds^2 &= -k^2[dz+B]^2 \\
&\quad + k^{-1}\left[\left(\tfrac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)(dt+\hat{\omega})^2 - \left(\tfrac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)^{-1}\gamma_{\underline{rs}}dx^r dx^s\right], \\
k^2 &= f^{-1}H^{-1} - f^2\omega_5^2\,, \\
B &= \chi + f^2\omega_5 k^{-2}(dt+\hat{\omega})\,.
\end{aligned}
\tag{4.85}
$$

In this form, comparing with the results of Refs. [10, 11] it is easy to see the form of the $N = 2, d = 4$ supersymmetric solution that will appear after dimensional reduction. The metric

$$
ds^2 = \left(\frac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)(dt+\hat{\omega})^2 - \left(\frac{fH^{-1}}{(f^{-1}H^{-1}-f^2\omega_5^2)^{1/2}}\right)^{-1}\gamma_{\underline{rs}}dx^r dx^s\,, \tag{4.86}
$$

is that of a solution in the timelike class, to which all $N = 2, d = 4$ supersymmetric black holes belong, and there is an additional scalar (k) and an additional vector field (B). If the 5-dimensional solution is static $\omega_5 = 0$ and the vector field $B = \chi$ is magnetic and corresponds to a KK monopole or a generalization thereof. This fact has been used in Refs. [21, 22, 23, 24, 25, 26, 27] to relate 4- and 5-dimensional black hole solutions.

4.3 The null case

Denote the null Killing vector by l^μ. Following the same considerations as in Refs. [1, 6], we find that we can choose null coordinates u and v such that

$$l_\mu dx^\mu = f du\,, \qquad l^\mu \partial_\mu = \partial_{\underline{v}}\,, \tag{4.87}$$

where f may depend on u but not on v, and the metric can be put in the form

$$ds^2 = 2f du(dv + H du + \omega) - f^{-2}\gamma_{\underline{rs}} dx^r dx^s\,, \tag{4.88}$$

where $r, s, t = 1, 2, 3$ and the 3-dimensional spatial metric $\gamma_{\underline{rs}}$ may also depend on u but not on v. Eqs. (4.9) and (4.11) state that the scalars are v-independent.

The above metric is completely equivalent to the one used in Refs. [1, 6], but we find this form more convenient; a Vielbein, and the corresponding spin connection and curvature for it are given in Appendix D.

In the null case the Fierz identities (A.23,A.24) and (A.25) imply that the 2-forms bilinears Φ^r are of the form

$$\Phi^r = du \wedge v^r\,, \tag{4.89}$$

where the 1-forms v^r are an orthogonal basis for the 3-dimensional spatial metric $\gamma_{\underline{rs}}$. Eq. (4.16) then implies the equation

$$du \wedge Dv^r = 0\,, \tag{4.90}$$

i.e. the spatial components of the pullback of the $\mathfrak{su}(2)$ connection are related to the spin connection coefficients of the basis v^r (computed for constant u) by

$$\varpi_{\underline{r}}{}^{st} \;=\; 2\varepsilon^{stp}\, \mathsf{A}^p{}_X\, \partial_{\underline{r}} q^X\,. \tag{4.91}$$

This equation is identical to the one found in Ref. [11] in the context of ungauged $N = 2, d = 4$ supergravity coupled to hypermultiplets. Actually, substituting the 2-forms we found into Eq. (4.14) we arrive at

$$\partial_r q^X f_X{}^{iA} \sigma^r{}_i{}^j = 0\,, \tag{4.92}$$

which is identical to the equation that the hyperscalars have to satisfy in a supersymmetric configuration of ungauged $N = 2, d = 4$ supergravity [11]. Observe that the last two equations together with Eq. (B.11) (for $\nu = -1$) imply that the Ricci scalar of the 3-dimensional metric γ satisfies

$$R_{rs}(\gamma) = -\tfrac{1}{2} g_{XY} \partial_r q^X \partial_s q^Y\,. \tag{4.93}$$

Let us now determine the vector field strengths: Eqs. (4.4,4.10) and (4.12) lead to

$$l^\mu F^I_{\mu\nu} = 0\,, \tag{4.94}$$

and, using the basis given in Appendix D, we can write

$$F^I = F^I{}_{+r} e^+ \wedge e^r + \tfrac{1}{2} F^I{}_{rs} e^r \wedge e^s = F^I{}_{+r} du \wedge v^r + \tfrac{1}{2} f^{-2} F^I{}_{rs} v^r \wedge v^s \,. \tag{4.95}$$

From Eq. (4.6) we get[13]

$$h_I F^I{}_{rs} = -\sqrt{3} \varepsilon_{rst} \partial_t f \,, \qquad \partial_t \equiv v_t{}^{\underline{s}} \partial_{\underline{s}} \,. \tag{4.96}$$

The same result can be obtained from $D \star \Phi^r$. From Eq. (4.13) we get

$$h^x_I F^I{}_{rs} \;=\; -\varepsilon_{rst} f \; \partial_t \phi^x \,, \tag{4.97}$$

which, together with the previous equation and the definition of h^x_I give

$$f^{-2} \; F^I{}_{rs} \;=\; \sqrt{3} [\hat{\star} \; \hat{d}(h^I/f) \;]_{rs} \,. \tag{4.98}$$

From the $++r$ components of Eq. (4.7) we get

$$h_I F^I{}_{+r} = -\tfrac{1}{\sqrt{3}} f^2 (\hat{\star} F)_r \,, \tag{4.99}$$

where

$$F = \hat{d}\omega \,. \tag{4.100}$$

The components $h^x_I F^I{}_{+r}$ are not determined by supersymmetry and we parametrize them by 1-forms ψ^I satisfying $h_I \psi^I = 0$. In conclusion, the vector field strengths are given by

$$F^I = [\tfrac{1}{\sqrt{3}} f^2 h^I \hat{\star} F - \psi^I] \wedge du + \sqrt{3} \hat{\star} \hat{d}(h^I/f) \,. \tag{4.101}$$

4.3.1 Solving the Killing spinor equations

Let us continue our analysis by plugging our configuration into Eq. (4.2): using the Vielbein, Eq. (4.97) and some Clifford algebra manipulations, we see that

$$0 \;=\; f^{-1} \left[\partial_u \phi^x \;+\; h^x_I \psi^I_r \; \gamma^r \;+\; \tfrac{f^2}{2} \partial_t \phi^x \; \varepsilon_{trs} \gamma^{rs} \gamma^- \right] \; \gamma^+ \epsilon^i \,, \tag{4.102}$$

so, if we want the scalars ϕ and the ψ^I to be non-trivial, we are forced to impose $\gamma^+ \epsilon^i \;=\; 0$.

As is usual in wave-like supersymmetric solutions, the $-$ component of the susy variation (4.1) is identically satisfied by an v-independent spinor, and the remainder of the components simplify greatly due to the lightlike constraint: The ones in the r-directions reduce, after using Eqs. (4.96,4.99), to

[13] Unless stated otherwise (as is the case of $F^I{}_{rs}$) all quantities with flat spatial indices refer to the 3-dimensional metric and Dreibein basis.

$$
\begin{aligned}
0 \;&=\; f\,\mathsf{D}_r\epsilon \;=\; f\left[\partial_r \;-\; \tfrac{1}{4}\varpi_{rst}\gamma^{st} \;+\; i\vec{\mathsf{A}}\cdot\vec{\sigma}^T\right]\,\epsilon \\
&=\; f\left[\partial_r \;+\; \mathsf{A}_r^p\gamma^p\left(1 \;-\; i\gamma^p(\sigma^{(p)})^T\right)\,\right]\,\epsilon\;,
\end{aligned}
\tag{4.103}
$$

where in the last step we made use of Eq. (4.91). If we then introduce the projection operators (no sum over p!)

$$
\Pi_p \;=\; \tfrac{1}{2}\left(1 \;-\; i\gamma^p(\sigma^{(p)})^T\right) \quad;\quad \Pi_p^2 \;=\; \Pi_p \quad;\quad [\,\Pi_p\;,\;\Pi_q\,] \;=\; 0\;,
\tag{4.104}
$$

the above equation is solved by imposing the condition $\Pi_p\epsilon = 0$, for every p for which A^p does not vanish, leading to a Killing spinor that can only depend on u.

The penultimate equation that needs to be checked is the gravitino variation in the u-direction.

$$
0 = \partial_u\epsilon \;+\; \tfrac{1}{4}v_r{}^{\underline{t}}\partial_u v_{s\underline{t}}\;\gamma^{rs}\epsilon \;+\; i\vec{\mathsf{A}}_u\cdot\vec{\sigma}^T\epsilon \;=\; \partial_u\epsilon \;-\; \left[\mathsf{A}^p{}_u \;+\; \tfrac{1}{4}\varepsilon_{prs}v_r{}^{\underline{t}}\partial_u v_{s\underline{t}}\right]\;\gamma^p\epsilon\;.
\tag{4.105}
$$

Generically the factor $v_r{}^{\underline{t}}\partial_u v_{s\underline{t}}$ is spacetime dependent, which, in order to avoid an inconsistency with the x-independency of the Killing spinor, means that we must have

$$
\mathsf{A}^p{}_u \;=\; -\tfrac{1}{4}\varepsilon_{prs}\;v_r{}^{\underline{t}}\;\partial_u v_{s\underline{t}}\;.
\tag{4.106}
$$

A consequence of this analysis is that the Killing spinor is constant.

Eq. (4.3) is the only one left to be analyzed. In fact it is straightforward to see that, given the constraints obtained thus far, Eq. (4.3) is tantamount to (4.92) contracted with ϵ_j. In order to get this far, however, one has to make use of all the constraints, meaning that if we do not want even more constraints, Eq. (4.92) must hold.

4.3.2 Equations of motion

In the null case, the KSIs contain far less restrictive information than in the timelike case, and as one can see from Eqs. (3.18)-(3.22), there are more equations of motion to be checked.

In order to get on with the show, let us analyze the gauge sector: the non-vanishing components of the Bianchi identities are immediately found to be

$$
{}^\star\mathcal{B}^{I\,+-} \;=\; \sqrt{3}f^3\hat{\nabla}^2(h^I/f)\,,
\tag{4.107}
$$

$$
f^{-1\star}\mathcal{B}^{I\,-r} \;=\; [\hat{\star}\hat{d}(\tfrac{1}{\sqrt{3}}f^2h^I\hat{\star}F - \psi^I)]_r + \sqrt{3}\left[\hat{\star}\partial_{\underline{u}}\hat{\star}\hat{d}(h^I/f)\right]_r\,,
\tag{4.108}
$$

and the Maxwell equations take the form

$$4^\star\mathcal{E}_I = -\sqrt{3}du \wedge \left\{ f\hat{d}h_I \wedge F + \tfrac{1}{\sqrt{3}}\left[\hat{d}(\hat{\star}\psi_I/f) - 2C_{IJK}\psi^J \wedge \hat{\star}\hat{d}(h^K/f)\right]\right\} , \tag{4.109}$$

and satisfy the KSIs Eqs. (3.19) and (3.20).

Eq. (4.107) is solved by $\bar{n} \equiv n_v + 1$ harmonic[14] functions K^I:

$$h^I/f \;=\; K^I \quad , \quad \hat{\nabla}^2\, K^I \;=\; 0\,, \tag{4.110}$$

$K^I \neq 0$, which, as in the timelike case, determines f to be

$$f^{-3} \;=\; K_I K^I\,, \quad K_I \;\equiv\; C_{IJK}K^J\, K^K\,. \tag{4.111}$$

Since the K^I are harmonic, we may introduce $\bar{n}$ local, 3-dimensional 1-forms $\alpha^I = \alpha^I_{\underline{r}}(u,\vec{x})dx^r$ which satisfy

$$\hat{d}\alpha^I = \hat{\star}\hat{d}K^I\,, \tag{4.112}$$

such that each α^I is determined, up to a 3-dimensional gradient, in terms of K^I and γ. This gauge freedom will be relevant soon.

Eqs. (4.108) become

$$\hat{d}\psi^I = \tfrac{1}{\sqrt{3}}\hat{d}\left(f^2h^I\hat{\star}F\right) + \sqrt{3}\hat{d}\dot{\alpha}^I\,, \tag{4.113}$$

where $\dot{\alpha} \equiv \dot{\alpha}^I_{\underline{r}}\, dx^r$. The general, local solution to this equation is

$$\psi^I = \tfrac{1}{\sqrt{3}}f^2h^I\hat{\star}F + \hat{d}M^I + \sqrt{3}\dot{\alpha}^I\,, \tag{4.114}$$

where the M^Is are some functions. The constraint $h\cdot\psi = 0$ implies

$$\tfrac{1}{\sqrt{3}}f^2\hat{\star}F + h_I\hat{d}M^I + \sqrt{3}h_I\dot{\alpha}^I \;=\; 0\,. \tag{4.115}$$

Due to the relation $F = \hat{d}\omega$, the above is the equation that, if we manage to fix the Ms, will determine ω.

Plugging Eq. (4.114) into the Maxwell equations we see that

$$\hat{\nabla}^2 L_I \;+\; \sqrt{3}C_{IJK}\left[\hat{\nabla}_r\left(K^J\dot{\alpha}^K\right)_r + \partial_r K^J\left(\dot{\alpha}^K\right)_r\right] \;=\; 0\,, \tag{4.116}$$

where we have defined the combinations

$$L_I \;\equiv\; C_{IJK}\, K^J\, M^K\,. \tag{4.117}$$

At this point we take advantage of the gauge freedom of (4.112) in order to simplify the Maxwell equations: fix the gauge by imposing

[14] In this section, harmonic means harmonic on the 3-dimensional Euclidean space with metric γ.

$$C_{IJK}\left[\hat{\nabla}_r\left(K^J\dot{\alpha}^K\right)_r+\partial_r K^J\left(\dot{\alpha}^K\right)_r\right] \;=\; 0\,, \qquad (4.118)$$

thus determining α^I completely in terms of the K^I and γ. In this gauge the functions L_I are harmonic,

$$\hat{\nabla}^2\, L_I \;=\; 0\,, \qquad (4.119)$$

and we determine the functions M^I in terms of the harmonic functions K^I and L_I by Eq. (4.117).

Another advantage of the above gauge is that the equation for ω, Eq. (4.115), takes on the rather nice form:

$$\hat{\star}\hat{d}\omega \;=\; \sqrt{3}\left(L_I dK^I \;-\; K^I dL_I\right) \;-\; 3K_I\dot{\alpha}^I\,. \qquad (4.120)$$

In the analysis of the Einstein equations it is useful to perform the following change of variables

$$H \;=\; -\tfrac{1}{2}L_I M^I \;+\; N\,. \qquad (4.121)$$

With this redefinition $\mathcal{E}_{++}$ becomes

$$\begin{aligned}\mathcal{E}_{++} \;=\;& -f\nabla^2 N+f\Big[\nabla_r(\dot{\omega})_r+3(\dot{\omega})_r\partial_r\log f+\tfrac{1}{2}f^{-3}(\ddot{\gamma})_{rr}+\tfrac{1}{4}f^{-3}(\dot{\gamma})^2-\tfrac{3}{2}f^{-4}\dot{f}(\dot{\gamma})_{rr}\\ &-3C_{IJK}K^I\left(\dot{K}^J\dot{K}^K+(\dot{\alpha}^J)_r(\dot{\alpha}^K)_r+\tfrac{2}{\sqrt{3}}(\dot{\alpha}^J)_r\partial_r M^K\right)+12f^3\left(K_I\dot{K}^I\right)^2\\ &+\tfrac{1}{2}f^{-3}g_{XY}\dot{q}^X\dot{q}^Y\Big]\,. \end{aligned}\qquad (4.122)$$

In general there is a gauge freedom in setting the one-form ω given in (4.120), corresponding to shifts in the coordinate v. If we choose to fix this gauge freedom by demanding

$$\begin{aligned}\nabla_r(\dot{\omega})_r+3(\dot{\omega})_r\partial_r\log f \;=\;& -\tfrac{1}{2}f^{-3}(\ddot{\gamma})_{rr}-\tfrac{1}{4}f^{-3}(\dot{\gamma})^2+\tfrac{3}{2}f^{-4}\dot{f}(\dot{\gamma})_{rr}-\tfrac{1}{2}f^{-3}g_{XY}\dot{q}^X\dot{q}^Y\\ &+3C_{IJK}K^I\left(\dot{K}^J\dot{K}^K+(\dot{\alpha}^J)_r(\dot{\alpha}^K)_r+\tfrac{2}{\sqrt{3}}(\dot{\alpha}^J)_r\partial_r M^K\right)\\ &-12f^3\left(K_I\dot{K}^I\right)^2\,, \end{aligned}\qquad (4.123)$$

then $\mathcal{E}_{++}$ vanishes identically if N is a real, harmonic function. $\mathcal{E}_{+r}$ becomes

$$\mathcal{E}_{+r}=-\tfrac{1}{2}\nabla_s(\dot{\gamma})_{rs}+\tfrac{1}{2}\partial_r(\dot{\gamma})_{ss}+\tfrac{3}{2}f^3\dot{K}_I\partial_r K^I+\tfrac{1}{2}g_{XY}\dot{q}^X\partial_r q^Y\,, \qquad (4.124)$$

whereas $\mathcal{E}_{rs}$ is identically satisfied by the configuration as we have it.

4.3.3 u-independent solutions

The equations that need to be solved, simplify greatly if we consider the case that the solutions do not depend on the coordinate u: in that case the gauge-fixings Eqs. (4.118,4.123) and the remaining equation of motion, Eq. (4.124), vanish identically, meaning that now the solutions are completely determined by the hyperscalars, the 3-dimensional metric and the $2\bar{n}+1$ real, harmonic functions L_I, K^I and N. Given these ingredients, in order to fully specify the solution we need calculate f, H, ω and ψ^I through the following, simplified equations.

$$
\begin{array}{rclcrcl}
f^{-3} & = & K_I\, K^I & , & L_I & = & C_{IJK}K^J\, M^K\,, \\
H & = & -\frac{1}{2}L_I\, M^I\, +\, N & , & \hat{\star}\hat{d}\omega & = & \sqrt{3}\left[L_I\hat{d}K^I\, -\, K^I\hat{d}L_I\right]\,, \\
h^I(\phi) & = & f\, K^I & , & \psi^I & = & f^3K^I(L_J\hat{d}K^J - K^J\hat{d}L_J) + \hat{d}M^I\,.
\end{array}
\tag{4.125}
$$

Solutions that belong to this family, but depending on a smaller number of harmonic functions have been given *e.g.* in Refs. [7, 50, 51].

Apart from being one of the nicest subclasses of solutions, the u-independent one becomes doubleplus interesting when we observe that if we reduce a solution in the null class over the spacelike direction $\sqrt{2}y = u - v$, which implies u-independence, we end up with a solution in the timelike class of $N = 2$ $d = 4$ SUGRA. In fact, comparing the constraints in this section with the ones in [11, Sec. (5)], one finds the same constraints on the hyperscalars and the 3-dimensional metric.

The metric Eq. (4.88) can be put in an y-adapted system, and one finds

$$
\begin{array}{rcl}
ds^2 & = & -k^2[dy+A]^2 + k^{-1}\left[\left(\frac{f^3}{1-H}\right)^{1/2}(dt+\frac{1}{\sqrt{2}}\omega)^2 - \left(\frac{f^3}{1-H}\right)^{-1/2}\gamma_{\underline{rs}}dx^r dx^s\right]\,, \\
k^2 & = & (1-H)f\,, \\
A & = & -(1-H)^{-1}(Hdt + \frac{1}{\sqrt{2}}\omega)\,.
\end{array}
\tag{4.126}
$$

The 4-dimensional solutions can be easily read from these. Apart from the scalar k and the vector field A, which is purely electric if the 5-dimensional solution is static ($\omega = 0$), the metric takes the form

$$
ds^2 = \left(\frac{f^3}{1-H}\right)^{1/2}(dt+\tfrac{1}{\sqrt{2}}\omega)^2 - \left(\frac{f^3}{1-H}\right)^{-1/2}\gamma_{\underline{rs}}dx^r dx^s\,, \tag{4.127}
$$

and belongs to the $N = 2, d = 4$ timelike class to which all black-hole-type solutions belong in $d = 4$.

This 4-dimensional solution should be compared to the one in Eq. (4.85), which is the one one obtains when imposing an extra isometry on the four dimensional spacelike manifold in the timelike case. the main difference between them is the electric or magnetic nature of the KK vector field. In the simplest case this solutions would give a 4-dimensional electric KK black hole and the other one a 4-dimensional magnetic KK black hole, related by 4-dimensional electric-magnetic duality, as we discussed in the introduction. In the more general case, the relation between these solutions is more complicated and we hope to say more about it in the near future.

5 Conclusions

In this paper we have found new families of supersymmetric solutions with unfrozen hypermultiplets. These families are very general and the form and physical properties of each solution depend on the details of the choices of hypervarieties, harmonic mappings and harmonic functions made. This opens a new wide range of possibilites that needs to be explored. More work is need to find out what happens with black hole attractors[15] and black hole entropy when hyperscalars are unfrozen [47], to find and explain the generic features of these solutions (are they always singular?), to find out to which stringy configurations these solutions correspond to etc.

One of the families of solutions describes generically strings with pp-waves propagating along it and can be dimensionally reduced to supersymmetric $N = 2, d = 4$ black holes. This raises new question about how the 4-dimensional attractor mechanism is implemented in the 5-dimensional setting, taking into account that these 5-dimensional solutions belong to the null class and the standard attractor mechanism is proven only for solutions in the timelike class. The 5-dimensional origin of the 4-dimensional entropy can (and must) be investigated.

We hope to report on some of these issues in the near future.

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[15]For a recent, pedagogical, review, see Ref. [52].

A Conventions

Our conventions can be obtained from those of Ref. [37] by changing the sign of the metric (to have mostly minus signature), multiplying all γ^as by $+i$ and all γ_as by $-i$ and setting $\kappa = 1/\sqrt{2}$, but we collect here the main features of our conventions to use them as a reference. In particular, Section A.3 contains the relevant Real Special Geometry identities for $\kappa = 1/\sqrt{2}$ (those in Appendix C of Ref. [37] are only valid for $\kappa = 1$).

A.1 Gamma matrices and spinors

We use mostly minus signature.

The first four of our 5-dimensional gamma matrices are taken to be identical to 4-dimensional purely imaginary gamma matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ satisfying

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\,, \tag{A.1}$$

and the fifth is $\gamma^4 = -\gamma^{0123}$, so it is purely real, the above anticommutator is valid for $a = 0, \cdots, 4$ and, furthermore, $\gamma^{a_1\cdots a_5} = +\varepsilon^{a_1\cdots a_5}$ and, in general

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]}}{(5-n)!}\varepsilon^{a_1\cdots a_n b_1\cdots b_{n-5}}\gamma_{b_1\cdots b_{n-5}}\,. \tag{A.2}$$

On the other hand, γ^0 is Hermitean and the other gammas are anti-Hermitean.

To explain our convention for symplectic-Majorana spinors, let us start by defining the Dirac, complex and charge conjugation matrices $\mathcal{D}_\pm, \mathcal{B}_\pm, \mathcal{C}_\pm$. By definition, they satisfy

$$\mathcal{D}_\pm\,\gamma^a\,\mathcal{D}_\pm^{-1} = \pm\gamma^{a\,\dagger}\,, \qquad \mathcal{B}_\pm\,\gamma^a\,\mathcal{B}_\pm^{-1} = \pm\gamma^{a\,*}\,. \qquad \mathcal{C}_\pm\,\gamma^a\,\mathcal{C}_\pm^{-1} = \pm\gamma^{a\,T}\,. \tag{A.3}$$

The natural choice for Dirac conjugation matrix is

$$\mathcal{D} = i\gamma^0\,, \tag{A.4}$$

which corresponds to $\mathcal{D} = \mathcal{D}_+$. The other conjugation matrices are related to it by

$$\mathcal{C}_\pm = \mathcal{B}_\pm^T\mathcal{D}\,, \tag{A.5}$$

but it can be shown that in this case only $\mathcal{C} = \mathcal{C}_+$ and $\mathcal{B} = \mathcal{B}_+$ exist and are both antisymmetric. We take them to be

$$\mathcal{C} = i\gamma^{04}\,, \qquad \mathcal{B} = \gamma^4 \quad \Rightarrow \mathcal{B}^*\mathcal{B} = -1\,. \tag{A.6}$$

The Dirac conjugate is defined by

$$\psi^\dagger\mathcal{D} = i\psi^\dagger\gamma^0\,, \tag{A.7}$$

and the Majorana conjugate by

$$\psi^T \mathcal{C} = i\psi^T \gamma^{04} \,. \tag{A.8}$$

The Majorana condition (Dirac conjugate = Majorana conjugate) cannot be consistently imposed because it requires $\mathcal{B}^*\mathcal{B} = +1$. Therefore, we introduce the symplectic-Majorana conjugate in pairs of spinors by using the corresponding symplectic matrix, *e.g.*

$$\psi^{i\,c} \equiv \varepsilon_{ij} \psi^{j\,T} \mathcal{C} \,, \tag{A.9}$$

then the symplectic-Majorana condition is

$$\psi^{i\,*} = \varepsilon_{ij} \gamma^4 \psi^j \,. \tag{A.10}$$

To impose the symplectic-Majorana condition on hyperinos ζ^A the only thing we have to do is to replace the matrix ε_{ij} by $\mathbb{C}_{AB}$, which is the invariant metric of $Sp(n_h)$.

Our conventions on $SU(2)$ indices are intended to keep manifest the $SU(2)$ covariance. In $SU(2)$, besides the preserved metric, there is the preserved tensor ε_{ij}. We also introduce ε^{ij}, $\varepsilon_{12} = \varepsilon^{12} = +1$. Therefore we may construct new covariant objects by using ε_{ij} and ε^{ij}, for instance $\psi_i \equiv \varepsilon_{ij}\psi^j$ (whence $\psi^j = \psi_i \varepsilon^{ij}$). With this notation the symplectic-Majorana condition can be simply stated as

$$\psi^{i\,*} = \gamma^4 \psi_i \,. \tag{A.11}$$

We use the bar on spinors to denote the (single) Majorana conjugate:

$$\bar{\psi}^i \equiv \psi^{i T} \mathcal{C} \,, \tag{A.12}$$

which transforms under $SU(2)$ in the same representation as ψ^i does. We also lower its $SU(2)$ index: $\bar{\psi}_i \equiv \varepsilon_{ij} \bar{\psi}^j$. In terms of single Majorana conjugates the symplectic Majorana condition reads

$$\left(\bar{\psi}^i\right)^* = \bar{\psi}_i \gamma^4 \,. \tag{A.13}$$

Finally, observe that after imposing the symplectic Majorana condition the following simple relation between the single Dirac and Majorana conjugates holds:

$$\psi^{i\dagger} \mathcal{D} = \bar{\psi}_i \,, \tag{A.14}$$

which is very useful if one prefers to use the Dirac conjugate instead of the Majorana one.

The bilinears that can be constructed from Killing spinors will in general be 2×2 matrices that can be written as linear combinations of the Pauli matrices $\sigma^{\hat{r}}$ ($\hat{r} = 0, \ldots, 3$) where $\sigma^0 = \mathbb{I}_{2\times 2}$. Therefore, we are bound to need the Fierz identities

$$\begin{aligned} \left(\bar{\lambda} M \varphi\right)\left(\bar{\psi} N \chi\right) &= \tfrac{p}{8} \left\{ \left(\bar{\lambda} M \sigma^{\hat{r}} N \chi\right)\left(\bar{\psi}\sigma^{\hat{r}}\varphi\right) + \left(\bar{\lambda} M \gamma^a \sigma^{\hat{r}} N \chi\right)\left(\bar{\psi}\gamma_a \sigma^{\hat{r}}\varphi\right) \right. \\ &\quad \left. - \tfrac{1}{2}\left(\bar{\lambda} M \gamma^{ab}\sigma^{\hat{r}} N\chi\right)\left(\bar{\psi}\gamma_{ab}\sigma^{\hat{r}}\varphi\right) \right\} \,, \end{aligned} \tag{A.15}$$

where the $SU(2)$ indices are implicit and $p = (-)1$ for (anti-)commuting spinors.

A.2 Spinor bilinears

With one commuting symplectic-Majorana spinor ϵ^i we can construct the following independent, $SU(2)$-covariant bilinears:

$\bar{\epsilon}_i\, \epsilon^j$: It is easy to see that

$$\begin{aligned} \bar{\epsilon}_i\epsilon^j &= -\varepsilon^{jk}(\bar{\epsilon}_k\epsilon^l)\varepsilon_{li}\,, \\ (\bar{\epsilon}_i\epsilon^j)^* &= -\bar{\epsilon}_j\epsilon^i\,, \end{aligned} \tag{A.16}$$

The first equation implies that this matrix is proportional to $\delta_i{}^j$ and the second equation implies that the constant is purely imaginary. Thus, we define the $SU(2)$-invariant scalar

$$f \equiv i\bar{\epsilon}_i\epsilon^i = i\bar{\epsilon}\sigma^0\epsilon \quad , \quad \bar{\epsilon}_i\epsilon^j = -\tfrac{i}{2}\, f\, \delta_i{}^j\,. \tag{A.17}$$

All the other scalar bilinears $i\bar{\epsilon}\sigma^r\epsilon$ $(r = 1, 2, 3)$ vanish identically.

$\bar{\epsilon}_i\gamma^a\epsilon^j$: This matrix satisfies the same properties as $\bar{\epsilon}_i\epsilon^j$, and so we define the vector bilinear

$$V^a \equiv i\bar{\epsilon}_i\gamma^a\epsilon^i = i\bar{\epsilon}\gamma^a\sigma^0\epsilon \quad , \quad \bar{\epsilon}_i\gamma^a\epsilon^j = -\tfrac{i}{2}\,\delta_i{}^j\, V^a\,. \tag{A.18}$$

which is also $SU(2)$-invariant, the other vector bilinears being automatically zero.

$\bar{\epsilon}_i\gamma^{ab}\epsilon^j$: In this case

$$\begin{aligned} \bar{\epsilon}_i\gamma^{ab}\epsilon^j &= +\varepsilon^{jk}(\bar{\epsilon}_k\gamma^{ab}\epsilon^l)\varepsilon_{li}\,, \\ (\bar{\epsilon}_i\gamma^{ab}\epsilon^j)^* &= \bar{\epsilon}_j\gamma^{ab}\epsilon^i\,, \end{aligned} \tag{A.19}$$

which means that these 2-form matrices are traceless and Hermitean and we have three non-vanishing real 2-forms

$$\Phi^{r\,ab} \equiv \sigma^r{}_i{}^j\, \bar{\epsilon}_j\gamma^{ab}\epsilon^i \quad , \quad \bar{\epsilon}_i\gamma^{ab}\epsilon^j = \tfrac{1}{2}\sigma^r{}_i{}^j\, \Phi^{r\,ab}\,. \tag{A.20}$$

$r = 1, 2, 3$, which transform as a vector in the adjoint representation of $SU(2)$, and the fourth $\bar{\epsilon}\gamma^{ab}\sigma^0\epsilon = 0$.

Using the Fierz identities Eq. (A.15) for commuting spinors we get, among other identities,

$$V^a V_a = f^2\,, \tag{A.21}$$

$$V_a V_b = \eta_{ab} f^2 + \tfrac{1}{3} \Phi^r{}_a{}^c \Phi^r{}_{cb}\,, \tag{A.22}$$

$$V^a \Phi^r{}_{ab} = 0\,, \tag{A.23}$$

$$V^a ({}^\star\Phi^r)_{abc} = -f \Phi^r{}_{bc}\,, \tag{A.24}$$

$$\Phi^r{}_a{}^c \Phi^s{}_{cb} = -\delta^{rs}(\eta_{ab} f^2 - V_a V_b) - \varepsilon^{rst} f \Phi^t{}_{ab}\,, \tag{A.25}$$

$$\Phi^r{}_{[ab}\Phi^s{}_{cd]} = -\tfrac{1}{4} f \delta^{rs} \varepsilon_{abcde} V^e\,, \tag{A.26}$$

$$V_a \gamma^a \epsilon^i = f \epsilon^i\,, \tag{A.27}$$

$$\Phi^r_{ab}\gamma^{ab}\epsilon^i = 4if\epsilon^j \sigma^r{}_j{}^i\,. \tag{A.28}$$

A.3 Real Special Geometry

The geometry of the n physical scalars ϕ^x $(x = 1, \ldots, n)$ of the vector multiplets is fully determined by a constant real symmetric tensor C_{IJK} $(I, J, K = 0, 1, \ldots, \bar{n} \equiv n + 1)$. The scalars appear through $\bar{n}$ functions $h^I(\phi)$ constrained to satisfy

$$C_{IJK} h^I h^J h^K = 1\,. \tag{A.29}$$

One defines

$$h_I \equiv C_{IJK} h^J h^K\,, \quad \Rightarrow h_I h^I = 1\,, \tag{A.30}$$

and a metric a_{IJ} that can be use to raise and lower the $SO(\bar{n})$ index

$$h_I \equiv a_{IJ} h^J\,, \qquad h^I \equiv a^{IJ} h_J\,. \tag{A.31}$$

The definition of h_I allows us to find

$$a_{IJ} = -2C_{IJK} h^K + 3 h_I h_J\,. \tag{A.32}$$

Next, one defines

$$h^I_x \equiv -\sqrt{3} h^I{}_{,x} \equiv -\sqrt{3}\frac{\partial h^I}{\partial \phi^x}\,, \tag{A.33}$$

and

$$h_{Ix} \equiv a_{IJ} h^J_x = +\sqrt{3} h_{I,x} \,, \tag{A.34}$$

which satisfy

$$h_I h^I_x = 0 \,, \qquad h^I h_{Ix} = 0 \,, \tag{A.35}$$

due to Eq. (A.29). The h^I enjoy the following properties of closure and orthogonality

$$\begin{pmatrix} h^I \\ h^I_x \end{pmatrix} \begin{pmatrix} h_I & h^y_I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta^y_x \end{pmatrix} \,, \quad \begin{pmatrix} h_I & h^x_I \end{pmatrix} \begin{pmatrix} h^J \\ h^J_x \end{pmatrix} = \delta^J_I \,. \tag{A.36}$$

Therefore any object with $SO(\bar{n})$ index can be decomposed as

$$A^I = \left(h_J A^J\right) h^I + \left(h^x_J A^J\right) h^I_x \,. \tag{A.37}$$

The metric of the scalars $g_{xy}(\phi)$ is the pullback of a_{IJ}:

$$g_{xy} = a_{IJ} h^I_x h^J_y = -2 C_{IJK} h^I_x h^J_y h^K \,, \tag{A.38}$$

and can be used to raise and lower x, y indices. Other useful expressions are

$$a_{IJ} = h_I h_J + h^x_I h_{Jx} \,, \tag{A.39}$$

$$C_{IJK} h^K = h_I h_J - \tfrac{1}{2} h^x_I h_{Jx} \,, \tag{A.40}$$

and

$$h_I h_J = \tfrac{1}{3} a_{IJ} + \tfrac{2}{3} C_{IJK} h^K \,, \tag{A.41}$$

$$h^x_I h_{Jx} = \tfrac{2}{3} a_{IJ} - \tfrac{2}{3} C_{IJK} h^K \,. \tag{A.42}$$

We now introduce the Levi-Cività covariant derivative associated to the scalar metric g_{xy}

$$h_{Ix;y} \equiv h_{Ix,y} - \Gamma_{xy}{}^z h_{Iz} \,. \tag{A.43}$$

It can be shown that

$$h_{Ix;y} = \tfrac{1}{\sqrt{3}} (h_I g_{xy} + T_{xyz} h^z_I) \,, \tag{A.44}$$

$$h^I_{x;y} = -\tfrac{1}{\sqrt{3}} (h^I g_{xy} + T_{xyz} h^{Iz}) \,, \tag{A.45}$$

$$T_{xyz} \;=\; \sqrt{3}h_{Ix;y}h^I_z = -\sqrt{3}h_{Ix}h^I_{y;z}\,, \tag{A.46}$$

$$\Gamma_{xy}{}^z \;=\; h^{Iz}h_{Ix,y} - \tfrac{1}{\sqrt{3}}T_{xy}{}^w = 8h^z_I h^I_{x,y} + \tfrac{1}{\sqrt{3}}T_{xy}{}^w\,. \tag{A.47}$$

B Quaternionic-Kähler manifolds

In this appendix we review the definition and basics of quaternionic-Kähler manifolds. We refer the reader to Ref. [53] for a more comprehensive introduction to quaternionic manifolds with original references.

A *quaternionic-Kähler manifold* is a real $4n$-dimensional manifold ($n > 1$) such that[16]

1. There exists on it a triplet of complex structures $J^r{}_X{}^Y$, $r = 1,2,3,\ X,Y = 1,\ldots 4n$ which satisfy the algebra of imaginary unit quaternions,

$$J^r J^s \;=\; -\delta^{rs} \;+\; \varepsilon^{rst}\; J^t\,, \tag{B.1}$$

   which is known as *hypercomplex or quaternionic structure*. A manifold with this property is an *almost hypercomplex* of almost *quaternionic manifold*.

2. The hypercomplex structure is integrable, *i.e.* it is covariantly constant with respect to the standard Levi-Cività connection and a non-trivial $\mathfrak{su}(2)$ connection (*i.e.* with non-vanishing curvature):

$$\partial_X J^r{}_Y{}^Z - \Gamma_{XY}{}^U J^r{}_U{}^Z + \Gamma_{XU}{}^Z J^r{}_Y{}^U + 2\varepsilon^{rst}\omega_X{}^s J^t{}_Y{}^Z = 0\,, \tag{B.2}$$

   where $\omega_X{}^r$ is the $\mathfrak{su}(2)$ connection. In this case the manifold is a *quaternionic manifold*. (If this equation is satisfied with a trivial $\mathfrak{su}(2)$ connection the manifold is a *hypercomplex manifold*.)

3. There is a metric which is invariant under the action of the three complex structures

$$g_{XY} = J^{(r)}{}_X{}^Z J^{(r)}{}_Y{}^U g_{ZU}\,, \quad \text{(no sum over } r!)\,. \tag{B.3}$$

   This property makes it a (quaternionic) Kähler manifold.

The combination of the complex structures with the metric gives us the three hyper-Kähler 2-forms

[16] Clearly, the definitions given below are just too weak to be useful when $n = 1$, and one defines a 4-dimensional manifold to be quaternionic-Kähler, iff it is Einstein and selfdual. For a supergravity justification of this definition see *e.g.* [53].

$$J^r{}_{XY} = g_{XZ} J^r{}_Y{}^Z \,. \tag{B.4}$$

They are covariantly closed respect to the $\mathfrak{su}(2)$ connection,

$$dJ^r + 2\varepsilon^{rst}\omega^s \wedge J^t = 0 \,. \tag{B.5}$$

The holonomy of a quaternionic-Kähler manifold is contained in $SU(2)\cdot Sp(2)$ and the tangent space indices are split accordingly into pairs of $SU(2)$ and $Sp(n)$ indices $i,j,k = 1,2$ and $A,B,C = 1,\ldots,2n$ respectively. The Vielbein is defined to be $f_{iA}{}^X$ and is related to the metric by

$$g_{XY} \;=\; f_X{}^{iA}\, f_Y{}^{jB}\, \mathbb{C}_{AB}\, \varepsilon_{ij} \,, \tag{B.6}$$

where

$$f_X{}^{iA}\, f_{iA}{}^Y \;=\; \delta_X{}^Y \,, \qquad f_{iA}{}^X\, f_X{}^{jB} \;=\; \delta_i^j\, \delta_A{}^B \,, \tag{B.7}$$

and $\mathbb{C}_{AB}$ is the $Sp(n)$-invariant metric. The Vielbein also satisfies the reality condition

$$\left(f_X{}^{iA}\right)^* \;=\; \varepsilon_{ij}\, \mathbb{C}_{AB}\, f_X{}^{jB} \,, \tag{B.8}$$

and they are covariantly constant under the combination of the Levi-Cività, $\mathfrak{su}(2)$- and $\mathfrak{sp}(n)$ connections. The Vielbein also gives us the tangent version of the complex structures. The constant matrices $-i\sigma^r$ satisfy the algebra Eq. (B.1), and we have

$$J^r{}_X{}^Y \;=\; f_X{}^{iA}\, \mathsf{J}^r{}_{iA}{}^{jB}\, f_{jB}{}^Y \,, \qquad \mathsf{J}^r{}_{iA}{}^{jB} \;\equiv\; -i\sigma^r{}_i{}^j\, \delta_A{}^B \,. \tag{B.9}$$

The spin connection can be split into its $\mathfrak{su}(2)$ and $\mathfrak{sp}(n)$ components as follows:

$$\omega_{X\,iA}{}^{jB} \;=\; \tfrac{i}{2}\, \omega_X{}^r\, \mathsf{J}^r{}_{iA}{}^{jB} \;+\; \omega_{X\,A}{}^B\, \delta_i^j \,. \tag{B.10}$$

Some useful identities are

$$R_{XY}{}^r \;=\; \tfrac{1}{4}\nu\, J^r{}_{XY} \,, \tag{B.11}$$

$$2 f_{[X}{}^{iA} f_{Y]jA} \;=\; i J^r{}_{XY}\, \sigma^r{}_j{}^i \,, \tag{B.12}$$

$$2 f_{(X}{}^{iA} f_{Y)jA} \;=\; g_{XY}\, \delta_j{}^i \,. \tag{B.13}$$

The constant ν is given in terms of the dimensionality of the manifold $4n$ and its Ricci scalar R by

$$\nu = \frac{R}{4n(n+2)} \,. \tag{B.14}$$

C The $d=5$ conformastationary metric

In the timelike case we find the conformastationary metric Eq. (4.17) which we rewrite here for convenience:

$$ds^2 = f^2\,(dt+\omega)^2 - f^{-1}h_{\underline{mn}}dx^m dx^n\,, \qquad \omega = \omega_{\underline{m}}dx^m\,, \qquad m,n=1,\cdots,4\,. \tag{C.1}$$

We choose the Vielbein basis

$$(e^a{}_\mu) = \begin{pmatrix} f & f\omega_{\underline{m}} \\ 0 & f^{-1/2}V^n{}_{\underline{m}} \end{pmatrix}\,, \qquad (e^\mu{}_a) = \begin{pmatrix} f^{-1} & -f^{1/2}\omega_m \\ 0 & f^{1/2}V^{\underline{n}}{}_m \end{pmatrix}\,, \tag{C.2}$$

where

$$h_{\underline{mn}} = V_{\underline{m}}{}^p V_{\underline{n}}{}^q \delta_{pq}\,, \qquad V_m{}^{\underline{p}} V_n{}^{\underline{q}} h_{\underline{pq}} = \delta_{mn}\,, \qquad \omega_m = V_m{}^{\underline{n}}\omega_{\underline{n}}\,. \tag{C.3}$$

The non-vanishing components of the spin connection in this basis are

$$\omega_{00m} = -2\partial_m f^{1/2}\,, \quad \omega_{0mn} = \omega_{m0n} = \tfrac{1}{2}f^2\,(d\omega)_{mn}\,, \quad \omega_{mnp} = -f^{1/2}\xi_{mnp} - 2\delta_{m[n}\partial_{p]}f^{1/2}\,, \tag{C.4}$$

where, from now on, all the objects in the r.h.s. of these equations refer to the 4-dimensional metric $h_{\underline{mn}}$ and, in particular

$$(d\omega)_{mn} = V_m{}^{\underline{p}}V_n{}^{\underline{q}}\,(d\omega)_{\underline{pq}} = 2V_m{}^{\underline{p}}V_n{}^{\underline{q}}\partial_{[\underline{p}}\omega_{\underline{q}]}\,. \tag{C.5}$$

Thee non-vanishing components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -\nabla^2 f + f^{-1}(\partial f)^2 - \tfrac{1}{4}f^4(d\omega)^2\,, \\ R_{0m} &= -\tfrac{1}{2}f^{-1/2}\nabla_n[f^3(d\omega)_{nm}]\,, \\ R_{mn} &= fR_{mn} - \tfrac{1}{2}(d\omega)_{mp}(d\omega)_{np} + \tfrac{3}{2}f^{-1}\partial_m f\partial_n f - \tfrac{1}{2}\delta_{mn}[\nabla^2 f - f^{-1}(\partial f)^2]\,, \end{aligned} \tag{C.6}$$

and the Ricci scalar is given by

$$R = -fR + \tfrac{1}{4}(d\omega)^2 + \nabla^2 f - \tfrac{5}{2}f^{-1}(\partial f)^2\,. \tag{C.7}$$

We define, following Ref. [1] we define the decomposition

$$fd\omega = G^+ + G^-\,, \tag{C.8}$$

so

$$de^0 = f^{-1}df\wedge e^0 + G^+ + G^-\,. \tag{C.9}$$

Further, since in this basis $\hat{V} = fe^0$, we have

$$\begin{aligned} d\hat{V} &= 2df \wedge e^0 + f(G^+ + G^-)\,, \\ \star d\hat{V} &= 2\star(df \wedge e^0) + (G^+ - G^-) \wedge \hat{V}\,. \end{aligned} \tag{C.10}$$

D The null-case metric

$$ds^2 = 2f du(dv + H du + \omega) - f^{-2}\gamma_{\underline{rs}} dx^r dx^s\,, \qquad r, s = 1, 2, 3\,. \tag{D.1}$$

Orthonormal 1-form and vector basis for this metric are given by

$$\begin{aligned} e^+ &= f du\,, & e_+ &= f^{-1}(\partial_{\underline{u}} - H\partial_{\underline{v}})\,, \\ e^- &= dv + H du + \omega\,, & e_- &= \partial_{\underline{v}}\,, \\ e^r &= f^{-1}v^r\,, & e_r &= f(v_r - \omega_r \partial_{\underline{v}})\,, \end{aligned} \tag{D.2}$$

where $v^r = v^r{}_{\underline{s}} dx^s$ and $v_r = v_r{}^{\underline{s}}\partial_{\underline{s}}$ are orthonormal basis 1-forms and vectors for the 3-dimensional spatial positive-definite metric $\gamma_{\underline{rs}}$

$$\delta_{rs} v^r{}_{\underline{t}} v^s{}_{\underline{q}} = \gamma_{\underline{tq}}\,, \qquad v_t{}^{\underline{r}} v_q{}^{\underline{s}} \gamma_{\underline{rs}} = \delta_{tq}\,. \tag{D.3}$$

The non-vanishing components of the spin connection are

$$\begin{aligned} \omega_{+r+} &= \partial_r H - \partial_{\underline{u}}\omega_{\underline{s}} v_r{}^{\underline{s}}\,, & \omega_{rs+} &= -\tfrac{1}{2} f^2 F_{rs} - f^{-2}\partial_{\underline{u}} f \delta_{rs} - f^{-1} v_{(r|}{}^{\underline{t}} \partial_{\underline{u}} v_{|s)\underline{t}}\,, \\ \omega_{+r-} &= \tfrac{1}{2}\partial_r f = \omega_{-r+} = -\omega_{r+-}\,, & \omega_{+rs} &= \tfrac{1}{2} f^2 F_{rs} - f^{-1} v_{[r|}{}^{\underline{t}} \partial_{\underline{u}} v_{|s]\underline{t}}\,, \\ \omega_{rst} &= f\varpi_{rst} - 2\delta_{r[s}\partial_{t]} f\,, \end{aligned} \tag{D.4}$$

where all the quantities in the r.h.s. of all these equations refer to the 3-dimensional metric and Dreibein and

$$F_{rs} = v_r{}^{\underline{t}} v_s{}^{\underline{p}} F_{\underline{tp}}\,, \qquad F_{\underline{rs}} \equiv 2\partial_{[\underline{r}}\omega_{\underline{s}]}\,. \tag{D.5}$$

The non-vanishing components of the Ricci tensor are

$$
\begin{aligned}
R_{++} &= -f\nabla^2 H - \tfrac{1}{4}f^4F^2 + f\nabla^{\underline{r}}\dot{\omega}_{\underline{r}} + 3\dot{\omega}_{\underline{r}}\partial^{\underline{r}}f + \tfrac{1}{2}f^{-2}\gamma^{\underline{rs}}\ddot{\gamma}_{\underline{rs}} + \tfrac{1}{4}f^{-2}\dot{\gamma}^{\underline{rs}}\dot{\gamma}_{\underline{rs}} \\
&\quad -\tfrac{3}{2}f^{-3}\dot{f}\gamma^{\underline{rs}}\dot{\gamma}_{\underline{rs}} - 3f^{-2}\left[\partial^2_{\underline{u}}\log f - 2\left(\partial_{\underline{u}}\log f\right)^2\right] , \\
R_{+-} &= -\tfrac{1}{2}f^2\nabla^2\log f , \\
R_{+r} &= -\tfrac{1}{2}\nabla_s\left(f^3F_{sr}\right) - \tfrac{1}{2}v_r{}^{\underline{r}}\gamma^{\underline{st}}\nabla_{\underline{s}}\dot{\gamma}_{\underline{rt}} + \tfrac{1}{2}v_r{}^{\underline{r}}\partial_{\underline{u}}\left(\gamma^{\underline{st}}\partial_{\underline{r}}\gamma_{\underline{st}}\right) + \tfrac{3}{2}v_r{}^{\underline{r}}\dot{\gamma}_{\underline{rt}}\partial^{\underline{t}}\log f \\
&\quad -\tfrac{3}{2}\partial_r\partial_{\underline{u}}\log f - \tfrac{3}{4}\gamma^{\underline{st}}\dot{\gamma}_{\underline{st}}\partial_r\log f + \tfrac{3}{2}\partial_{\underline{u}}\log f\,\partial_r\log f , \\
R_{rs} &= f^2R_{rs}(\gamma) - \delta_{rs}f^2\nabla^2\log f + \tfrac{3}{2}\partial_r f\partial_s f ,
\end{aligned}
\tag{D.6}
$$

and the Ricci scalar is

$$
R = -f^2R(\gamma) + 2f^2\nabla^2\log f - \tfrac{3}{2}\left(\partial f\right)^2 . \tag{D.7}
$$

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