Mathematics Cognition Reconsidered: On Ascribing Meaning

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In contrast to the common assumption that mathematics cognition involves the attempt to recognize a previously unnoticed meaning of a concept, here mathematics cognition is reconsidered as a process of ascribing meaning to the objects of one's thinking. In this paper, the attention is focused on three processes that are convoluted in the complex dynamics involved when individuals ascribe meaning to higher mathematical objects: contextualizing, complementizing, and complexifying. The aim is to discuss emerging perspectives of these three processes in more detail that speak to the complex dynamics in mathematics cognition.

Keywords: complexifying, complementizing, contextualizing, mathematics cognition, sense-making

Introduction

Mathematics cognition is a complex phenomenon that has been addressed and discussed in the literature in different ways and with various emphases. The work presented here arose from a primary cognitive tradition, focusing on critical processes in mathematical concept formation and their complex dynamics. In search for more dialogical possibilities in thinking about mathematics cognition, a new understanding of mathematics cognition emerged (see Scheiner & Pinto, 2017): mathematics cognition does not merely involve the attempt to recognize a previously unnoticed meaning of a concept but the attempt to ascribe meaning to the objects of one's thinking. The purpose of this paper is to provide deeper meaning to the complex processes involved when individuals ascribe meaning. In this paper, three processes are foregrounded: contextualizing, complementizing, and complexifying. Over the past few years, theoretical perspectives and insights emerged (in reanalyzing students' knowing and learning of the limit concept of a sequence) that advance our understanding of these processes. These new perspectives and insights inform research on mathematics cognition and enable one to see not only new phenomena in mathematical concept formation but to think about them differently. In this presentation, emerging interpretative possibilities in thinking about contextualizing, complementizing, and complexifying are discussed that speak to the complex dynamics in mathematics cognition.

Theoretical Orientations

The work presented here relies on and projects theoretical assertions made by Frege (1892a, 1892b). First, the meaning of a mathematical concept is not directly accessible through the concept itself but only through objects that fall under the concept (Frege, 1892a). Second, mathematical objects (different to objects of natural sciences) cannot be apprehended by human senses (we cannot, for instance, 'see' the objects), but only via some 'mode of presentation.' That we only have access to mathematical objects in using signs and representations, however, leads to what Duval (2006) called a 'cognitive paradox':

"how can they [individuals] distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representation?" (Duval, 2006, p. 107)

It seems to be an epistemological requirement to distinguish the 'mode of presentation' (or 'way of presentation') of an object from the object that is represented. Frege (1892b) revealed this critical insight, by proposing that an expression has a *sense*$_F$ ('Sinn') in addition to its *reference*$_F$ ('Bedeutung') (the subscript F indicates that these terms refer to Frege, 1892b). The reference$_F$ of an expression is the object it refers to, whereas the sense$_F$ describes a particular state of affairs in the world, the way that some object is presented. Thus, it seems to follow that we may understand Frege's notion of an idea$_F$ the manner in which we make sense of the world. Ideas$_F$ can interact with each other and form more compressed knowledge structures, called conceptions. A general outline of this view is provided in Figure 1.

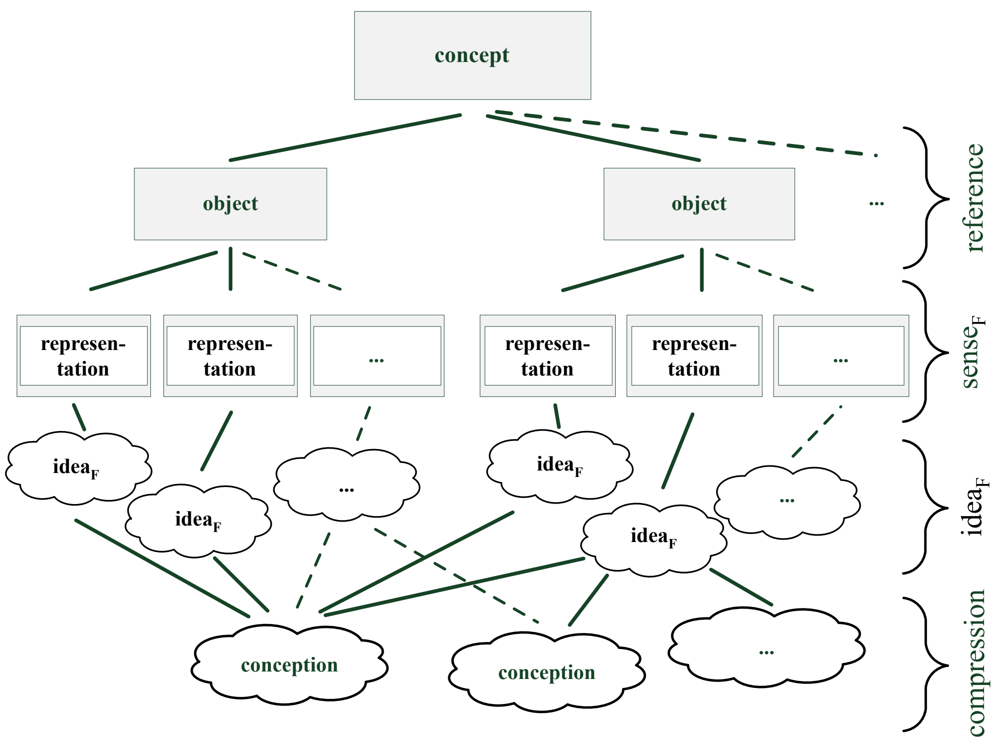

Figure 1: On reference$_F$, sense$_F$, idea$_F$, and compression (reproduced from Scheiner, 2016, p. 179)

Duval (2006) argued that via systematic variation of representation registers that is, "investigating representation variations in the source register and representation variations in a target register" (p. 125), one can detach a sense$_F$ from the represented object. This resonates a critical function of reflective abstraction that is, reflecting on the coordination of actions on mental objects (see Piaget, 1977/2001). The special function of reflective abstraction is extracting meaning of an individual's action coordination. Underlying these approaches is the assumption that meaning is inherent in objects and is to be extracted via manipulating objects (or representations of those objects).

Over the past few years, a new understanding of mathematics cognition emerged from reanalyzing students' knowing and learning the limit concept of a sequence (see Scheiner &

Pinto, 2014, 2017; Pinto & Scheiner, 2016): mathematics cognition does not so much involve the attempt to recognize a previously unnoticed meaning of a concept (or the structure common to various objects), but rather a process of ascribing meaning to the objects of an individual's thinking. That is, meaning is not so much an inherent quality of objects that is to be extracted, but something that is given to objects of one's thinking. Three processes are considered as critical in the complex dynamics involved when individuals ascribe meaning to higher mathematical objects: contextualizing, complementizing, and complexifying (see Scheiner & Pinto, 2017).

Contextualizing: Particularizing Senses$_F$

In Frege's view, a sense$_F$ can be construed as a certain state of affairs in the world and an idea$_F$ in which we make sense$_F$ of the world. In the work presented here, we started from an understanding of sense$_F$ as not primarily dependent on a mathematical object, but as emerging from the interaction of an individual with an object in the immediate context. That is, a sense$_F$ of an object at one moment in time can only be established in a more or less definite way when the process of sense$_F$-making is supported by what van Oers (1998) called *contextualizing*. Van Oers (1998) argued for a dynamic approach to context that provides for the "particularization of meaning" (p. 475), or more precisely, the particularization of a sense$_F$ that comes into being in a context in which an object actualizes.

Recent research suggests that individuals seem to reason and make sense from a specific perspective (see Scheiner & Pinto, 2017). It might be suggested that individuals take a specific perspective that orients their sense$_F$-making, or more accurately: in taking a certain perspective, individuals direct their attention to particular senses$_F$. Contextualizing, in this view, means taking a certain perspective that calls attention to particular senses$_F$. Attention in such cases, however, may not involve an attempt to 'sense' or 'see' anything, but it seems to be attentive thinking: attention as the direction of thinking (see Mole, 2011). As such, calling attention to particular senses$_F$, then, means directing mind to sense$_F$. In this respect, contextualizing is intentional: it directs one's thinking to particular senses$_F$.

Complementizing: Creating Conceptual Unity

Frege (1892b) underlined that a particular sense$_F$ "illuminates the reference$_F$ […] in a very one-sided fashion. A complete knowledge of the reference$_F$ would require that we could say immediately whether any given sense$_F$ belongs to the reference$_F$. To such knowledge we never attain" (p. 27). (Translated from Frege (1892b): "[mit dem Sinn] ist die Bedeutung aber […] immer nur einseitig beleuchtet. Zu einer allseitigen Erkenntniss der Bedeutung würde gehören, dass wir von jedem gegebenen Sinne sogleich angeben könnten, ob er zu ihr gehöre. Dahin gelangen wir nie"). This is to say, that just from sense$_F$-making of one representation that refers to an object, we are typically not in a position to know what the object is (see Duval, 2006). As contextualizing serves to particularize only single senses$_F$ of a represented object, the same object can be 're-contextualized' (see van Oers, 1998) in other ways that support the particularization of different senses$_F$ of the same object. Notice that senses$_F$ can differ despite sameness of reference$_F$, and it is this difference of senses$_F$ that accounts for the 'epistemological value' of different representations. It is the diversity of senses$_F$ that has 'epistemological significance' and forms conceptual unity (see structuralist approach, Scheiner, 2016), not the similarity (or sameness) of senses$_F$ (as might be advocated in an empiricist approach). This means, what matters is to coordinate diverse senses$_F$ to form a unity, a process called complementizing. However, the notion of 'complementizing' might be misunderstood as accumulating various senses$_F$ (until an individual has all of them); this is not the case. Complementizing means to coordinate different senses$_F$ to create conceptual unity.

As each idea$_F$ is partial in the sense of being restricted (in space and time) and biased (from a particular perspective), it needs to be put in dialogue with other ideas$_F$ that offers an epistemological extension. The function of complementizing, then, is extending the epistemological space of possible ideas$_F$. Complementizing as extending the epistemological space of possible ideas$_F$ brings a positive stance, indicating that seemingly conflicting ideas$_F$ can be productively coordinated in a way such that these ideas$_F$ are cooperative rather than conflicting. Hence complementizing is the ongoing expansion of one's epistemological space, the ever-unfolding process of becoming capable of new, perhaps as-yet unimaginable possibilities.

Complexifying: Creating a Complex Knowledge System

It is not only creating a unity of diverse senses$_F$, but creating an entity in its own right that forms a 'whole' from which emerges new qualities of the entity. That is, rather than treating the unity as a collection of different senses$_F$ that can be assigned to objects that actualize in the immediate context, it is the forming of the unity that emerges new senses$_F$ that might be assigned to potential objects. In forming a unity, senses$_F$ are not merely considered as the parts of the unity, but "they are viewed as forming a whole with distinct properties and relations" (Dörfler, 2002, p. 342). It is, therefore, not an unachievable totality of senses$_F$ (or ideas$_F$) that matters, but how senses$_F$ (or ideas$_F$) are coordinated that develop emergent structure. This brings to foreground a critical function of complexifying that has not been attested yet: blending previously unrelated ideas$_F$ that emerge new dynamics and structure (for a detailed account of conceptual blending, see Fauconnier & Turner, 2002). The essence of conceptual blending is to construct a partial match, called a cross-space mapping, between frames from established domains (known as inputs), in order to project selectively from those inputs into a novel hybrid frame (a blend), comprised of a structure from each of its inputs, as well as a unique structure of its own (emergent structure). This strengthens Tall's (2013) assertion that the "whole development of mathematical thinking is presented as a combination of compression and blending of knowledge structures to produce crystalline concepts that can lead to imaginative new ways of thinking mathematically in new contexts" (p. 28).

Discussion

Mathematics cognition, as asserted here, evolves in the dialogue of contextualizing, complementizing, and complexifying. As such, mathematics cognition is ongoing and cannot be pre-stated. That is, mathematics cognition does not follow a determinable developmental trajectory, but the evolution of mathematics cognition is directional: it seems to move toward higher levels of internal diversity, interactions, and decentralization of ideas$_F$.

Scheiner and Pinto (2017) suggested that individuals take a specific perspective in ascribing meaning to the limit concept of a sequence. For instance, individuals, who take the perspective of a limit sequence as approaching, may activate dynamic ideas$_F$ (such as the idea$_F$ of a sequence of points 'getting closer' to a limit point). Consider Figure 2: one might activate the idea$_F$ of a limit that can be approached monotonically (see idea$_F$ A) or the idea$_F$ of a limit that can be approached from above and below (see idea$_F$ B) in making sense$_F$ of the respective representation (see Cornu, 1991; Davis & Vinner, 1986; Tall & Vinner, 1981). On the other hand, individuals, who take the perspective of closeness in thinking about the limit concept of a sequence, might activate rather static ideas$_F$ (such as the idea$_F$ of points of a sequence 'gathering around' the limit point). One might activate the idea$_F$ that infinite many points of a sequence can lie within a given epsilon strip (see idea$_F$ U or V) in making sense$_F$ of the representations (see Przenioslo, 2004; Roh, 2008; Williams, 1991).

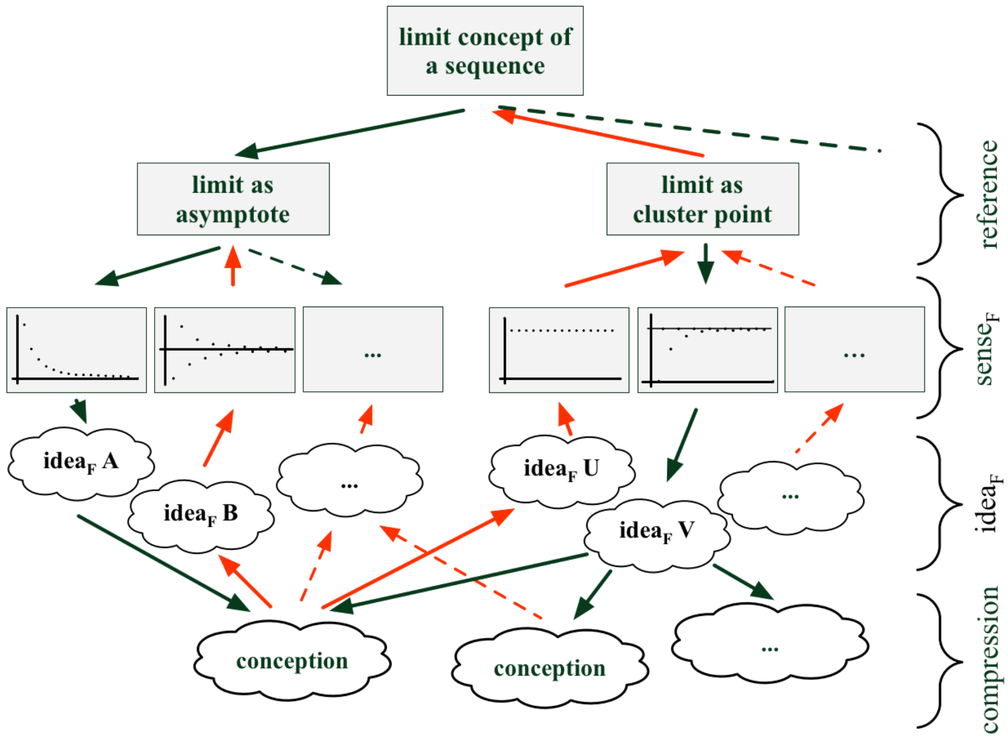

Figure 2: On the complex dynamics in mathematics cognition

The critical point here is that it is not one single idea$_F$ around one's thinking is to be organized (such as the idea$_F$ that only finite many points lie outside a given epsilon strip), but a variety of diverse ideas$_F$ that provides a resource of activating productive ideas$_F$ and of making sense$_F$ in the immediate context. Decentralization and internal diversity of ideas$_F$, however, are not only critical for making sense$_F$ in the immediate context, but also for creating novel ideas$_F$. Whereas analogy theory typically focuses on compatibilities between ideas$_F$ simultaneously connected, blending is equally driven by incompatibilities (see Fauconnier & Turner, 2002).

Creating novel ideas$_F$, however, only occurs if there is a certain level of interaction between existing ideas$_F$. That is, only if ideas$_F$ can compensate for each other's restrictions and limitations, one is able to extend the space of possibilities in thinking about a mathematical concept. In this view, novel ideas$_F$ can ascribe *new* meaning to the objects of one's thinking (see Figure 2). This substantiates the assertion that mathematics cognition is as much concerned with creating a meaning of a concept as it is with comprehending it (see Scheiner, 2017).

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