INTERPRETING STUDENTS' EXPLANATIONS OF FRACTION TASKS, AND THEIR CONNECTIONS TO LENGTH AND AREA KNOWLEDGE

Submitted by

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Statement of Sources

This thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma.

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Anne Mitchell

Abstract

Much research on fractions has concentrated on the sub-constructs of measure, quotient, operator and ratio from Kieren's model of coordinated fraction knowledge (Kieren, 1980). In the primary school, partitioning, equivalence and unit-forming also can be used to describe children's approaches to fraction tasks (Kieren 1988, 1992, 1993, 1995). Given the approaches used to teaching fractions, other areas of the curriculum such as multiplicative thinking, measurement and spatial knowledge could affect students' understanding of fractions.

In one-to-one interviews, 88 Grade 6 students were asked 65 questions designed to ascertain their understanding of fraction, measurement, geometry and/or visualisation, and multiplication concepts. The students' answers and explanations were recorded on a record sheet at the time of interview and audio- and video-recording enabled later detailed analysis.

The associations between four categories based on Lehrer's key concepts (2003) for spatial measurement (attribute, additivity, units, and proportionality) and the measure sub-construct of fractions were analysed. The measure sub-construct was assessed using number lines, fraction comparison tasks, and length and area diagrams.

From detailed examination of students' explanations, insights into misconceptions were gained. Gap thinking in fraction pair size comparisons was discovered to be triggered at the same time as equivalence understanding began. The limitations of a part-whole double count approach to fractional area diagrams was noted.

Further, Kieren's four-three-four model (1988, 1992, 1993, 1995) describing coordinated fraction knowledge for analysing students' fraction understanding at the upper primary school level was evaluated. Use of the model enabled descriptions of students' responses to tasks to be placed in a framework of understanding which connected these three underlying concepts and the four sub-constructs.

Publications from the present study

Mitchell, A., & Horne, M. (2008). Fraction number line tasks and the additivity concept of length measurement. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Brisbane, pp. 353-360). Brisbane, Australia: MERGA.

Mitchell, A., & Horne, M. (2009). There are more than part-whole strategies at work in understanding non-equal-parts fraction area models. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides* (Proceedings of the 32nd annual conference of the Mathematics Research Group of Australasia, Wellington, pp. 371-378). Palmerston North, New Zealand: MERGA.

Mitchell, A., & Horne, M. (2010). Gap thinking in fraction pair comparisons is not whole number thinking: Is this what early equivalence thinking sounds like? In L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education* (Proceedings of the 33rd annual conference of the Mathematics Research Group of Australasia, Fremantle, pp. 414-421). Fremantle, Australia: MERGA.

Mitchell, A., & Horne, M. (2011). Measurement matters: Fraction number lines and length concepts are related. In J. Way & J. Bobis (Eds.), *Fractions: Teaching for understanding* (pp. 52-62). Adelaide, Australia: The Australian Association of Mathematics Teachers.

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Chapter 1: Introduction

This thesis is about a conceptual link between fraction and measurement understanding. It is a qualitative inquiry that seeks to investigate students' performance on fraction and measurement tasks as well as analyse the strategies that they offer in their explanations of their answers. It is an interpretive study that examines students' explanations and uses descriptive statistics to quantify associations between fraction and measurement constructs. In the present study, I conducted one-to-one task-based interviews with 88 Grade 6 students, offering 65 tasks that assessed their understanding of length and area measurement, dynamic imagery, multiplication, and fraction understanding. Each student was interviewed for up to three hours over several sessions, and my record sheets of the students' responses at the time, and subsequent transcripts of audio and video data, enabled me to classify their answers and explanations. Criteria for task selection focused on key concepts of measurement and key concepts of fraction understanding articulated by frameworks in the research literature. Correlations between these categories were calculated. The interpretation and implications of students' strategies (correct strategies and misconceptions), correlations between measurement and fraction understanding, and the explanatory power of Kieren's framework for fraction understanding (1995), are discussed in later chapters. In this chapter, I will outline the background to the study, present the research questions that developed out of my synthesis of the research literature, and provide a guide to the structure of the thesis by outlining the chapters to come.

1.1 Background to the Study

For students, fractions form the basis for other mathematical understandings, and underpin the development of proportional reasoning and later topics in mathematics, including algebra and probability. Researchers have suggested that

- to be able to think proportionally was a turning point in mental ability (Cramer, Post, & Currier, 1993),
- fractions led to proportional reasoning in ratios, rates, probability, percentages, and operators (Ohlsson, 1988), and
- the ability to see constants and variables developed from the ability to recognize proportional relationships (Lamon, 1999).

Hence fractions and decimals, as topics in themselves, were regarded as an enduring and important part of the primary and junior secondary mathematics curriculum (Lamon, 1999).

At the heart of this thesis is an interest in how children understand fractions. Before I began this thesis I was a teacher searching for

- a vocabulary to talk about fractions to students that would enable them to move towards an understanding of proportional reasoning,
- examples of tasks that would enable students to articulate the salient aspects of different representations of fractions,
- knowledge about how to have conversations with students so that they could generalise from one fraction task to another and recognise misconceptions,
- deeper personal knowledge of how different domains in mathematics intersect with fraction learning, and
- research-based discussion on the effect on children's understandings of fractions if less teaching time was given to the space and measurement domains in school mathematics term planners.

However, before I could answer the questions concerning classroom practice I needed to start with the more fundamental question: what are the understandings of fractions, space, and measurement of primary school children?

Curriculum documents provided a snapshot of the learning outcomes suggested for different grade levels in all domains of mathematics. These outcomes and the explicit and implicit strategies that they privilege changed over time. For example, the strategies for the comparison of the relative size of fractions have been revised three times in the last fifteen years in curriculum documents for Victorian schools. In the *Curriculum Standards Framework* (Board of Studies, 1995), children at the end of Level 4 (Grade 6) were expected to use common denominators to "compare and order fractions with different denominators (for example, using equivalent fractions)" (p. 46). A decade later, in a draft document of the (then) forthcoming *Victorian Essential Learning Standards* (Department of Education & Training, 2005), the common denominator procedure was described as one of several strategies that students should develop in order to work with fractions. Other strategies were described but not explicitly named. For example, students should "accurately estimate the size of fractions and decimals in the vicinity of 0 and 1 relative to 0, 1/2 and 1" (p. 23). In the research literature this strategy had been called the transitive strategy or reference point strategy in the United States (Behr, Wachsmuth, Post, & Lesh, 1984) and benchmarking in

Australia (Clarke & Roche, 2009). However, in the 2006 revised curriculum document, this benchmarking strategy was omitted (Department of Education & Training, 2006). Instead, neither common denominators nor benchmarking were mentioned as strategies to be used by Grade 6 students. They were to "use estimates for computation, and apply criteria to determine whether estimates are reasonable or not" (p. 23), but no elaboration of the strategies to be used (or taught) was provided. In the *Victorian Essential Learning Standards,* conceptual understandings were implicitly part of the curriculum because to make a reasonable estimate children would use strategies and number sense rather than a rote understanding of common denominator procedures. What was lost in the 2006 revision of the 2005 draft was the explicit inclusion of strategies that would guide the teaching and learning decisions made by practicing teachers. Without this explicit elaboration of useful strategies, a chance to provide examples of an understanding of fractions was lost.

There were several different misconceptions in children's fraction understanding that had been reported in the research literature. One misconception was confusing the number of pieces with the size of the fractional parts (Carrahar, 1996; Saxe, Taylor, McIntosh, & Gearhart, 2005). The double count – number of pieces shaded over number of pieces altogether- did not form a good basis for further knowledge (Kieren, 1993). The double count of shaded and unshaded parts became a misconception when students used this strategy for non-equal-parts diagrams, and this was described as the double count misconception.

In collecting some fraction assessment tasks from the research literature in order to pilot a one-to-one task-based fraction interview for my Master's project, I was interested in the types of factors that had impressed me as a classroom teacher when using the Early Numeracy Interview (Clarke et al., 2002; Department of Education & Training, 2001). These included the affective response of the children, the opportunity for students to self correct, and the finding that some children in my Grade 5/6 class had discarded strategies that used number sense in favour of algorithms (sometimes faulty). I interviewed Grade 5 students and asked them to identify Parts B and A in a circular model that had unequal sections (see Figure 1.1). This task had been designed to elicit the double count misconception if it were present (Cramer, Behr, Post, & Lesh, 1997). Some children named Part A as $^{1}/_{5}$ instead of $^{1}/_{6}$ of the whole circle because it was one out of five parts (Mitchell & Clarke, 2004), demonstrating the double count misconception. However, some children gave the answer of $^{1}/_{5}$ but had been comparing the smaller part (Part D) to the quarter (Part B), and had estimated $^{1}/_{5}$ as being smaller than $^{1}/_{4}$. They were not therefore demonstrating the double count misconception.

Children could give an incorrect answer with incorrect mathematical thinking or an incorrect answer but have a partially correct mathematical approach (M. Clements & Ellerton, 2005). The interview had revealed this distinction between a double count explanation or an estimation explanation, both with an answer of $^{1}/_{5}$.

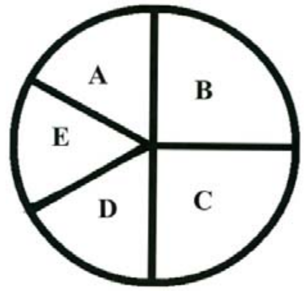


Figure 1.1. Fraction Pie task.

The use of length and area concepts in fraction tasks was of interest to me, and I investigated this using the Fold Me a Quarter task, in a Teacher Professional Leave study in 2005 (see Figure 1.2). I interviewed Grade 5 and Year 8 children using a one-to-one task-based interview and asked them to fold a square of paper into quarters, and then another identical square of paper into quarters another way (Mitchell, 2005). I then showed them two squares the same size as those they had just folded that were partitioned into square and triangle quarters (see Figure 1.2). I asked them which shaded piece, A or B, would give them more. Some children identified the triangle quarters as having "more" than the square quarters despite the large squares (or whole) being the same size. My experience as a teacher was that students have often been presented with length and area diagrams in fraction tasks or to support fraction activities. In short, children were often given fraction tasks in visual forms. To solve such tasks children had to interpret the diagram and work with the fraction content of the task. To do this they had to draw upon fraction knowledge and/or length and/or area knowledge, and/or properties of shape and/or the ability to mentally move parts of the diagram around. If students answered fraction tasks incorrectly, teachers might not know whether it was a lack of the fraction knowledge needed to complete the task, or whether the students did not have the measurement or spatial knowledge to complete the task successfully. This was reinforced by my analysis of the Fold Me a Quarter task and I suggested that some children did not have complete conservation of area (Mitchell, 2005).

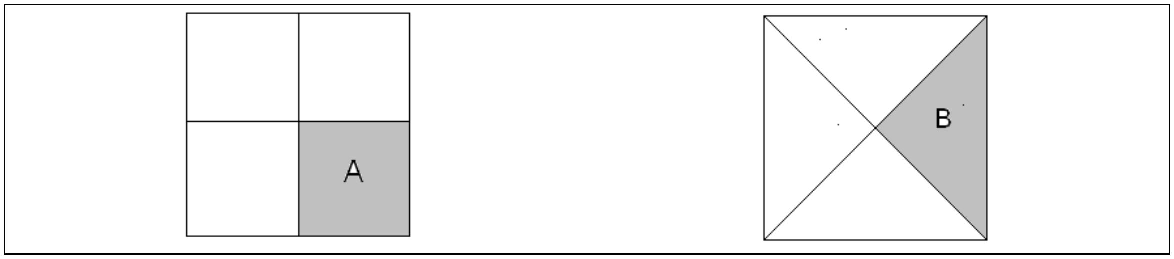


Figure 1.2. Representation of two papers folded into quarters and labelled.

The research literature had provided examples of fractions tasks that required conceptual understanding, such as the Fraction Pie (see Figure 1.1) and these could be used to elicit children's strategies. Kieren's five sub-constructs of rational number (1980) had framed my Master's project as well as my Teacher Professional Leave project. These five constructs were part-whole, measure, quotient, operator, and ratio. Kieren modified this model (1988, 1992, 1993, 1995) and

- incorporated part-whole into the other four sub-constructs,
- added three underpinning concepts: partitioning, equivalence, and unit-forming, and
- elaborated four levels of response to the concepts/sub-constructs: ethnomathematical, intuitive, technical-symbolic, and axiomatic deductive.

I have termed this later version the four-three-four model and discuss it in depth in section 2.1.5. Kieren also argued that leftover parts of units in measurement contexts were described using rational numbers (1976, 1980, 1988, 1992, 1993, 1995). In order to pursue my interest in children's strategies for solving fraction tasks, the link between fractions and measurement, and the importance of research-based frameworks for one-to-one interviews, I took leave from my teaching position and began my PhD in 2005.

1.2 Research Questions

Three questions emerged from the review of the literature:

- What strategies are evident in students' explanations of their thinking in a one-to-one task-based interview?
- Is there an association between performance on measurement tasks and performance on fractions tasks? Is there an association between the use of the use of dynamic imagery on visualisation tasks and performance on fractions tasks?
- Can we use Kieren's four-three-four model of fraction understanding (1988, 1992, 1993, 1995) to describe the fraction understandings of students in the present study?

These questions can be linked to very broad concerns in the mathematics education field. Firstly if we embrace constructivist learning then we are faced with the enormity of the role of the teacher in responding to the variety of correct and incorrect strategies that a class of children bring to every task. The elaboration of the variety of mathematical strategies is part of developing the ability of members of the teaching profession to embrace these learning contexts. Secondly, if we believe there to be a conceptual link between fractions and measurement, then uncovering an association in children's performance in these two domains raises issues for the design of the curriculum. Thirdly teachers in Victorian schools are using one-to-one task-based interviews as a normal classroom pedagogical tool (Department of Education and Early Childhood Development, 2009b; Department of Education & Training, 2001). If one-to-one interviews are supported by a sound connection to a theoretical framework then teachers are able to interpret and classify the detailed individual responses, use this as formative assessment and link this to classroom practice. Investigating the explanatory power of Kieren's four-three-four model (1988, 1992, 1993, 1995) contributes to this larger picture of making one-to-one task-based interviews a usable teaching strategy. None of these big picture questions can be answered in full in the present study, but they position the findings and their contributions to the field of mathematics education.

1.3 Structure of the Thesis

In this introductory chapter I have presented the background to the study and foreshadowed the questions that came out of the literature review and their significance.

In the Literature Review chapter, the research literature concerning length and area, multiplication, visualisation, and fractions provides examples of students' strategies for solving tasks in these domains. The literature on length and area measurement is framed using the constructs attribute, additivity, unit, and proportionality: an adaption of Lehrer's eight key concepts for measurement (2003). The terminology used in the field of visualisation has been synthesised using the research of Bishop (1983) and Presmeg (2006a). Kieren's four-three-four model for rational number knowing (1988, 1992, 1993, 1995) frames my interpretation of the fraction research literature. A critique of the methodologies used in the field of fractions research is offered. I conclude the chapter with the three research questions that emerged from the review of the literature.

The Methodology and Methods chapter has four sections:

- a discussion of the qualitative methodology of the present study,
- a brief description of the participants and local context, and a description of the instrument, research protocols, and validity,
- an elaboration of the methods of analysis, and
- a discussion of the limitations of the study.

The Results chapter is structured by constucts. Measurement tasks are reported first in relation to the key concepts attribute, additivity, unit, and proportionality. Visualisation and multiplication tasks are then reported. The fraction constructs are reported in the last section of the chapter, and have been categorised using Kieren's constructs (1988, 1992, 1993, 1995): the concept of equivalence, and the sub-constructs of measure, quotient, operator, and ratio. Quotations from transcripts of students' explanations and photos of students' inscriptions are presented as evidence for the classification of their strategies. The associations between performance in measurement categories and performance on fraction tasks are reported. Examples are provided that show that partitioning, equivalence and unit-forming concepts have been drawn upon in the students' responses to tasks assessing the different sub-constructs.

In the Discussion and Implications chapter, I consider the three research questions that emerged from the literature review. I interpret the results and discuss the ramifications of each of those questions.

The final chapter includes a summary of the discussion and interpretation of each research question, which I then connect back to the broad concerns that have been raised in this introduction, before elaborating my conclusions about the findings. The thesis concludes with suggestions for further research.

Chapter 2: Literature Review

I began the present study with an interest in the conceptual links between measurement and fraction tasks, and in the strategies that children have used to solve these tasks. In this review of the literature I examine the mathematical domains of measurement, visualisation and fractions to identify both the strategies used by children and the theoretical frameworks used by researchers. Boote and Beile's criteria for writing literature reviews (2005) provided the approach used in this chapter: the research is placed in its historical context, I resolve ambiguities in the terminology, the research questions are positioned within a theoretical framework, and the methodologies of research in the field are evaluated.

The constructs of the theoretical frameworks of the measurement, visualisation, and fraction domains provide the sections for these three research areas. The constructs of attribute, additivity, unit, and proportionality provide the theoretical framework for the literature on length and area and were adapted from Lehrer's eight key concepts of spatial measurement (2003). The ambiguities in the visualisation field caused by the use of different terms for the same concept or parts of concepts are resolved by developing a hierarchy list. In the fractions domain the constructs of Kieren's four-three-four model (1988, 1992, 1993, 1995) provided a framework for the examination of the research literature. The measure sub-construct of fractions is examined in more depth because the conceptual links to the measurement domain are more evident in these tasks. A discussion of a brief history of fractions research enables a critique of methodologies used in the field.

This chapter includes research up to 2007, as this was the literature that informed the present study and in particular the development of the data collection instrument. Research literature concerning children's explanations has been included in the Methodology and Methods chapter. The Discussion and Implications chapter includes two types of new literature: research on fractions and measurement published after 2007, and literature concerning classroom interactions that had been originally excluded from this literature review because the present study was not a classroom investigation. However, the terminology and concepts used in classroom interaction research has subsequently proved useful in framing the findings based on the children's explanations.

2.1 Critical Examination of the State of the Field

2.1.1 Procedural and conceptual knowledge.

Researchers in mathematics education have differentiated between procedural and conceptual understanding. Skemp (1976) described the kind of learning that was focused on procedures, for example, turning a fraction upside down and multiplying as a technique for dividing by a fraction, and called this instrumental understanding. Relational understanding on the other hand, described the making of a mental map that connected mathematical concepts. Analogously, Kieren (1976), distinguished between knowledge of fractions that was procedural in nature and an understanding of fractions that was intuitive and related to concept development. Hiebert and Carpenter (1992) used the terms procedural and conceptual knowledge to distinguish between instrumental and relational understanding (see also Hiebert & Lefevre, 1986).

2.1.2 Measurement.

The literature concerning length and area measurement is synthesised in the present study into four main constructs: attribute, additivity, unit, and proportionality. I include recognising increasingly complex formations such as straight paths, bent paths and perimeters in length measurement, and distinguishing between multiple attributes of a figure, as aspects of the *attribute* construct. For example, a single object can have a number of measurement attributes such as length, area, mass, temperature, or volume. The construct of *additivity* is defined in the present study as an understanding that the whole is the sum of the parts. This includes the concept of conservation, and the role of a zero-point on scales such as rulers. My synthesis of the construct of *units* includes describing a leftover part of a unit, using formal or informal units, and specifying identical or mixed units. An understanding of the inverse relationship between the size of the unit and the count forms the basis of the construct of *proportionality* in the measurement domain. These four constructs have been adapted from Lehrer's eight key concepts for spatial (length, area, area, volume) measures (2003), and concepts and strategies from the research on length and area measurement. Lehrer proposed eight key concepts for spatial measurement (2003):

- Unit-attribute relationship (units match the attribute being measured),
- iteration (a single unit can be moved to measure a spatial attribute, or the attribute can be subdivided into units),
- tiling (units fill lines, planes, volumes, and angles without cracks)

- identical units (if the units are identical a count represents the measure, and mixtures of units have to be specified),
- standardisation (formal units are used to facilitate communication),
- proportionality (the size of the unit is inversely proportional to the count of the units),
- additivity (the whole is the sum of the parts: conservation), and
- origin (zero-point) (any point can be used as a zero-point: e.g., the difference between 0 and 10 is the same as between 30 and 40).

The list did not represent a trajectory. Lehrer's eight key concepts represented a coordinated understanding of measurement.

Researchers in the measurement field have challenged the traditional teaching sequence based on Piaget's ideas about conservation: the use of gross comparison of length; the use of non-standard units and manipulative standard units; and the use of instruments such as rulers (see e.g., D. Clements, 1999). However, the concepts themselves have informed an understanding of the measurement domain and so research that used such trajectories is included in my re-categorisation of constructs. In the United States, length, area, volume and angle measurement have been classified as early geometry constructs (see e.g. Battista, 2007; Lehrer, Jenkins, & Osana, 1998). In Victorian curriculum documents, measurement has been a separate strand to space (geometry) (Board of Studies, 1995) and this is why in the present study I refer to measurement and geometry as separate domains.

2.1.2.1 Attribute.

The construct of attribute used in the present study encompasses both knowing that attributes are definable, and identifying them:

- measurable attributes
    - a continuous property of an object is an attribute that can be measured,
    - spatial attributes are length, area, volume, and angle; non-spatial attributes include mass, time, temperature,
- identifying attributes
    - attributes present with increasing complexity. For example, length: straight paths/bent paths/perimeter; or area: regular/non-regular/composite shapes,
    - different attributes of the same object can be distinguished and measured,

  - one attribute may be used to measure another in particular cases, for example length for area in rectangles of the same width, or area for volume in regular prisms, and
    - substitutions can be overgeneralised (e.g., the perimeter indicates area misconception).

A continuous attribute of an object can be measured. Wilson and Rowland observed, "We count discrete or separate objects and we measure continuous properties such as length, area, or volume" (1993, p. 176). Researchers have distinguished between the spatial attributes, length, area, volume, and angle (Lehrer 2003; Outhred, Mitchelmore, McPhail, & Gould 2003), and non-spatial measures, mass, temperature, time (Lehrer 2003), and gravity (Wilson & Osborne, 1992). Some frameworks were applicable for spatial measures only (Barrett & D. Clements, 2003; Lehrer 2003; Outhred et al., 2003). Other researchers have presented general principles for measurement (Wilson & Osborne, 1992; Wilson & Rowland, 1993).

Attributes can present with increasing complexity. For example length can involve straight paths, bent paths, or perimeter. Area can be of regular, non-regular or composite shapes. Children's performance on length tasks involving bent paths (Barrett, D. Clements, Klanderman, Pennisi, & Polaki, 2006; Battista, 2006) was lower than on length tasks involving straight paths, indicating greater complexity. Misconceptions about length measurement have been demonstrated in perimeter tasks (Barrett, Jones, Thornton & Dickson, 2003).

Different attributes of the same object can be distinguished and measured. Confusions about attributes could occur when using area models for fraction tasks: "We assume that students are thinking about area and that they notice that the figure has been divided into three congruent parts, but we know that many students confuse length and area" (Wilson & Rowland, 1993, p. 172). In a longitudinal study, primary school children incorrectly tried to use a unit of length to measure an area (Lehrer et al., 1998). A child reflecting on attributes said, "I used to think area was about the size of the edges" (Kidman, 2001). Traditional instructional sequences started with identifying the attribute (Outhred et al., 2003; Wilson & Rowland, 1993) but because instructional sequences often referred to early primary learning, teachers have not returned to the concept of identifying the attribute in later grades.

One attribute can substitute for another. Children have used length in area comparisons; attention is given to the length of fraction strips to differentiate between a half piece or third piece but the length indicates the area of the strip. Using the magnitude of the perimeter of two similar shapes indicates which is larger; a circle with a larger circumference has a greater area than one with a smaller perimeter. However, misconceptions have occurred when one attribute has been used to represent another inappropriately. When non-similar shapes do not follow this pattern it appears counter-intuitive. Piaget described this misconception: "by altering the form of an area we automatically extend or reduce its perimeter, and this in turn affects the topological intuitions which form the starting point for children's thinking in the realm of space" (Piaget, Inhelder, & Szeminska, 1960, p. 279). Activities in middle and upper primary school have been used to try to counter the power of this misconception. For example, the same rope was wrapped around a 5 by 8 units rectangle (area = 40 units, perimeter = 26 units) and a 6 by 7 rectangle (area = 42 units, perimeter = 26 units) and the areas compared (Barrett & D. Clements, 2003). The *perimeter indicates area* misconception was present in incorrect comparisons of square and triangle quarters in the Fold Me a Quarter task (Mitchell, 2005) (see Figure 1.2). Children incorrectly assumed that the piece with the larger perimeter had the larger area.

2.1.2.2 Additivity.

The construct of additivity used in the present study is categorised into conservation and comparison, and restructuring the whole:

- Conservation and comparison (position in space)
    - The whole may undergo a spatial transformation and the measure of the attribute will remain constant,
    - Direct comparison of two objects may be made using visual, numerical or non-numerical means,
    - The concept of transitivity may be used to compare two items to each other by means of an intermediary object,
    - Scales (e.g. rulers) have a zero point, either explicitly stated or implied (Origin: Lehrer (2003)).
- Restructuring the whole: space filling/subdividing
    - Conservation of the whole is maintained during subdivision of the whole into parts (Additivity: Lehrer (2003)),
    - Subdividing a whole into parts implies the use of iterated units (Iteration, and Tiling: Lehrer (2003)),

- The count of two spatial measures can be added if the objects are combined, for example two lengths, two areas, two volumes, two angles,
- Arrays are an important model for early area measurement,
- The area algorithm is multiplicative in nature and based on an array structure,
- Inference using geometric reasoning rather than direct iteration may be used to describe measurements, for example of non-straight paths, non-regular shapes, non-congruent units.

The concept of conservation describes the preservation of the whole when a) moved or rotated, b) partitioned, and c) subdivided and rearranged. The conceptual link between conservation and additivity was noted by Wilson and Osbourne (1992). Piaget's discussion of measurement used the ideas of end points and rotation to build up the concept of conservation in length and then area (1960). The whole could be moved in space (placed somewhere else) and the measure of the attribute would remain constant: the child "regards the new shape simply as an outcome of such a transformation rather than a new area to be compared with the original" (Piaget et al., 1960, p. 285). The direct comparison of two lengths could be through quantification or other means. In instructional models, making comparisons (unquantified) has been proposed as the next step after identifying the attribute (see e.g., Wilson & Rowland, 1992). The concept of transitivity could be used to compare two items by means of an intermediary object. Failure to attain any of the three foundational ideas (transitivity, conservation, and unit) was suggested as a barrier to further conceptual understanding (Wilson & Osbourne, 1992). Again, the comparisons could be numerical or non-numerical.

Non-quantified measurement could have sophisticated reasoning including the use of properties of shape (Battista, 2006). For example, to calculate the perimeter of the L shape (see Figure 2.1) a child would need to assume that lengths and widths added up to the same total even though they have been subdivided and rearranged (Battista, 2006). Using a not-to-scale diagram emphasised that geometric thinking was an easier strategy to use than measuring the lengths.

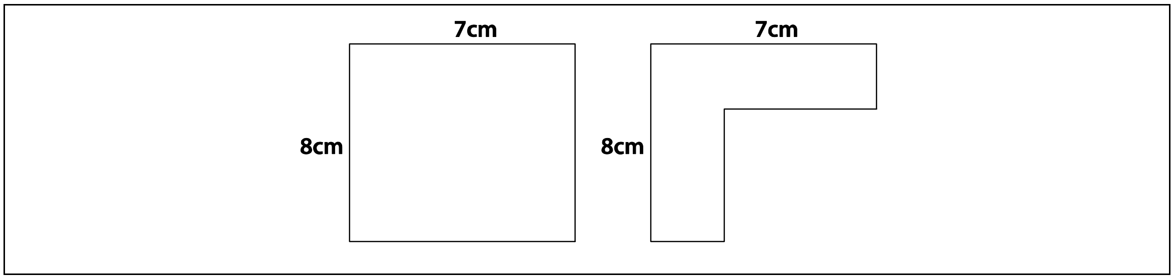


Figure 2.1. Perimeter of an L shape (adapted from Battista, 2006).

A ruler has the length scale on it, and has a zero-point explicitly marked. Any point could be chosen as a zero-point, Lehrer reminded us (2003), because the distance between 0 and 10, and 30 and 40 is the same. Broken ruler tasks have been used to investigate children's conceptual understanding of length (see e.g., Barrett & D. Clements, 2003; Bragg & Outhred, 2000, 2004; D. Clements, 1999; Hart, 1981; Lesh, Landau, & Hamilton, 1983). They also appear as items on national assessments (see e.g., Kamii & Clark, 1997). For example, children were asked to measure a line (9 cm long) with a ruler that had been cut at the 3.5 cm mark and labelled from 4 cm to 20 cm, necessitating the mental creation of a new zero-point (Bragg & Outhred, 2000).

The counting lines not spaces misconception was observed in some of these studies. Some children counted the mark at the edge of the object (the new zero-point) as *one*, thereby generating an incorrect count (one too many) for the length. Grade 6 students counted lines not spaces (Bragg & Outhred, 2004). In studies with younger children, Grade 1 students counted lines instead of gaps (McClain, Cobb, Gravemeijer, & Estes, 1999) and in a longitudinal study of Grade 1, 2 and 3 children (Lehrer et al., 1998) the incidence of the incorrect counting of the hash mark at the new zero-point as *one* decreased over the course of the teaching intervention. A variation of the misconception was to count the hash marks but not the edge of the object, thereby generating an incorrect count (one too few). For example, a Grade 4 child counted notches rather than lengths (Barrett, et al., 2003).

Proficient use of a ruler did not guarantee that children recognised conceptually that iteration of a length was taking place, and the "learner's goal may be to count rather than to measure, which may or may not match the teacher's goal for the learner" (Joran, Gabriele, Bertheau, Gelman, & Subrahmanyam, 2005, p. 6-7). Bragg and Outhred (2000) reported a Grade 5 student reflecting on his previous thinking about measuring, "No we just used to count stuff, you know,…like shoes and our hands, and them paddle pop sticks. I used to get things wrong 'cause I used to start at 1 like we counted" (p. 115). The researchers suggested that using informal units had focussed on counting and using a ruler was taught as a procedure.

The whole can be restructured by filling or subdividing. Conservation is required. The whole is the sum of the parts even if they are rearranged. Piaget suggested that the concept of conservation "which entails the complete coordination of operations of subdivision and order or change of position ... was found to have been reached by one subject in 10 of those aged 6-7, by half of those aged 7;0 – 7;6 [7 y to 7 $^{1}/_{2}$ y] and by three quarters of those aged 7;6 – 8;6. [7 $^{1}/_{2}$ y to 8 $^{1}/_{2}$ y]" (1960, p 114). A large British study found that 72% of 12 to 15 year olds could conserve both length and area (Hart, 1981), indicating that the frequency of children demonstrating conservation remained relatively unchanged from age eight and a half to age 15. Without the expectation of conservation, measurement would not convince children of the inaccuracy of their visual comparisons, argued Carpenter (1975), as they simply believed that the relation between quantities had changed or was being evaluated on some new unrelated dimension. The use of dynamic geometry computer programs enabled 14 year old students to explore conservation with polygons (Kordaki, 2003). Conservation presents with increasing complexity.

When tiling, children use units to fill lines, planes, volumes, and angles Lehrer argued (2003) and this could occur with physical objects or by subdividing a diagram. Wilson and Osbourne (1992) noted that gaps between units, or overlapping units, were tiling/iteration errors. Using wooden tiles could mask the principles of iteration because the children did not have to attend to tiling with no overlapping units; this was prevented by the raised edges of the tiles (Outhred & Mitchelmore, 2000). In a longitudinal study of children in Grade 1 to 3, Grade 2 to 4, and Grade 3 to 5, a preference was observed for using a unit that had a similar shape to the item being measured, for example using triangle units to measure a triangle (Lehrer, 2003). Lehrer (2003) explained that iteration could involve the re-using of units and that this drew upon an understanding of conservation. Providing fewer units than the total count was a strategy used by researchers to investigate children's iteration skills because for Grade 3 students picking up units and re-using them was conceptually more difficult than tiling (Lehrer Jaslow, & Curtis, 2003). Similarly, children in Grade 1 to 5 were forced to re-use units when given two paperclips to measure a line longer than that, demonstrating their understanding of unit iteration (Bragg & Outhred, 2000). Tiling and iteration have been included in instructional sequences as concepts and skills needed for informal measurement (Outhred & McPhail, 2000). Iterating a 30 cm ruler to measure a 93 cm streamer created a composite unit for length (Department of Education & Training, 2001), and 33% of Grade 4 students were successful at this and at calculating the difference to 1 metre (Clarke, Cheeseman, McDonough, & B. Clarke, 2003). In the same study, 88% of the students could measure a 20 cm straw with a ruler. Rows or columns were composite units in area measurement (Izsak, 2005).

Arrays can be a mental and physical restructuring of area diagrams. They have been described as meaningful subdivisions of space into a unit structure (Outhred & Mitchelmore 2000). The array structure was not always self-evident to Grade 2 children (Battista, 2003). A Grade 2 child did not visualise an array structure on a diagram with a partially marked array (see Figure 2.2. Note, the numbers shown represented the child's counting and did not appear on the task card shown to the child). Instead she counted in a "one dimensional path as if they were travelling along a road with no awareness of their surroundings as if in a tunnel" (Battista, D. Clements, Arnoff, K. Battista, & Borrow, 1998, p. 528).

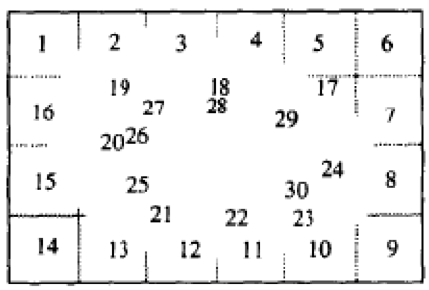


Figure 2.2. One dimensional counting strategy for the area of an array (Battista, et al., 1998).

Drawing an array rather than covering it with tiles was used as an assessment of Grade 1-4 children's conceptual understanding of the array structure (Outhred & Mitchelmore, 2000). The researchers deliberately printed a square unit on the paper so that it couldn't be picked up. Some children had difficulty subdividing an area physically into an array. Some children did not appear to recognise the convention that a single line represented both the end of one unit and the start of another for adjacent units and drew individual squares instead of an array. Outhred and Mitchelmore (2000) categorised the children's attempts at drawing arrays as incomplete covering, visual covering, concrete covering, and measurement. Only by Grade 4 were many children using an array structure to work out the area of the given rectangle. There were similar observations in other studies: Grade 2 children drew each unit in its entirety, rather than using a common edge (Battista, et al., 1998). Other children started by drawing the units on the edges of the shape but had difficulty co-ordinating a row by column structure

(Schifter & Szymaszek, 2003). Outhred and McPhail (2000) interviewed teachers and found that they taught area as a process of covering and counting not as subdividing a region.

The area algorithm is multiplicative in nature and based on an array structure. Covering rectangles with square tiles, a common classroom activity, was described as promoting additive, one-dimensional thinking, rather than the multiplicative thinking that the area formula represented (Outhred & Mitchelmore, 2000). Strategies observed for the numerical count of arrays in a Grade 2 class included count all, and skip counts of columns or rows (Lehrer et al., 2003). In a study of Grade 6 students, 50% of them did not have the conceptual understanding that area was multiplicative (Kidman, 2001). Many were still thinking additively about the count and were not able to use array structuring. Lehrer (2003) noted that the use of area models for fraction problems was potentially troublesome because it assumed knowledge of the array structure that may not be in place.

Conservation of the whole is maintained during multiplicative restructurings. The restructuring of area diagrams into arrays is possible because of the additive property of spatial measures: length, area, volume and angle. Researchers elaborated that while additivity did apply to some non-spatial measures such as mass it did not apply to others such as temperature (Lehrer 2003; Wilson & Osbourne, 1992; Wilson & Rowland, 1993).

Battista (2003, 2004, 2007) refined an earlier trajectory (Battista et al., 1998) for children's structuring of arrays as composite units in area contexts. The levels from his 2007 trajectory are presented below with the corresponding levels from 2004, 2003 and 1998 included in brackets, determined by tracking concepts and student examples in the articles:

- Level 1: absence of units-locating and organising-by-composites processes (Level 1, Level 1, Level 1),
- Level 2: beginning use of the units-locating and the organising-by-composites processes (Level 2, Level 2, Level 2 and 3A),
- Level 3: units-locating process sufficiently coordinated to eliminate double-counting errors (Level 3, Level 3, no corresponding level),
- Level 4: use of maximal composites, but insufficient coordination for iteration (Level 4, Level 4, Level 3B),
- Level 5: use of units-locating process sufficient to correctly locate all units, but less-than-maximal composites employed (Level 5, Level 5, no corresponding level),

- Level 6: complete development and co-ordination of both the units-locating and the organising-by-composites process (Level 6, Level 6, Level 3C), and
- Level 7: numerical procedures connected to spatial structuring, generalisation (Level 7, no corresponding level, no corresponding level).

Battista (2004) described a learning trajectory for the development of area measurement using arrays which provided a detailed framework for understanding the construct, in contrast to some instructional sequences which concentrated on the first few years of primary school and focussed on identifying attributes, comparing measures, using informal units and using a structure of repeated units (Outhred et al., 2003).

2.1.2.3 Unit.

The construct of units used in the present study contains two concepts: that attributes can be specified using appropriate units, and that spatial units and non-spatial units can be combined to make rates:

- Attributes can be specified using appropriate units (Unit-attribute relations: Lehrer (2003))
    - Iteration may result in "leftovers" and these leftovers can be described using rational numbers,
    - Numerical quantification and non-numerical qualitative comparisons can be made using iterated units (Identical units: Lehrer (2003)),
    - Informal, formal and standard units can be used as a unit of measure with identical or mixed units specified (Standardisation: Lehrer (2003)),
    - Iteration can involve composite units e.g. length: 30cm ruler repeated. Area: repeated addition of "rows", or a multiplicative understanding of row and column arrays,
- Spatial units can be combined with non-spatial units to make rates (e.g. speed: distance (length) over time), or to describe some non-spatial attributes (e.g. density: mass over volume)

Measuring with units can result in "leftovers" and those partial units have to be described. Wilson and Osbourne (1992) described measurement as a process whereby "First a suitable unit is chosen. Second, the unit is repeated, dividing the object into equal subdivisions with perhaps the fraction of a unit left over. Finally, the units are counted to produce a measurement of the object" (p. 91). Different strategies have been reported to describe such leftover parts of units:

- Qualitative statements could be made. For example, when describing a tower that was between 13 and 14 blocks high, children in Grade 2 and Grade 3 were happy to report the measure as 13 and a bit (Brown, Blondel, Simon, & Black, 1995). At these grade levels, the words *a half* could indicate *a bit* rather than a quantified fractional amount,
- Leftovers could be quantified using fractions. Grade 6 students in the study tied to describe the leftover part of the unit using fraction terminology (Brown et al., 1995),
- Leftover pieces could be combined using fractional reasoning. Children matched two half pieces to make units when measuring the area of a circle with a grid because the tiling did not match the boundary of the shape (Lehrer, 2003). On other non-regular shapes, a fourth of a unit plus a half of a unit plus a fourth of a unit made a whole (Lehrer et al., 2003),
- Smaller units could be used. Pre-service teachers used this strategy to confirm the area of Montana (Hodgson, Simonsen, Lubek, & Andreson, 2003),
- Mixed units could be specified. Mixtures of units needed to be clearly named, for example "5 yards and 3 inches" was not "8" (Lehrer, 2003). Mixed units were of different sizes and this was one strategy for managing partial units. In Australia, the use of the metric system for measurement has meant that many mixed units are related by powers of ten. For example, 1.7 m is 1 m and 70 cm.

Exposure to tasks in which measuring led to leftovers can draw attention to partial units. For example, measuring a straw that was four and a bit paperclips long was an assessment task for Grade Prep to 2 children in Victoria (Department of Education & Training, 2001). This measuring involved the iteration of a unit resulting in a leftover part.

Informal units have been elaborated (hands, paces, tiles), as have formal units (imperial or metric measures) and standard units (the meter, the kilogram, and the second). The use of informal units and formal units represented stages in instructional sequences (see e.g., Outhred et al., 2003; Wilson & Rowland, 1993) but not necessarily a learning trajectory.

Lehrer argued (2003) that choosing units involved both choosing units that matched the property being measured (e.g., area units for measuring area), and choosing the magnitude of those units (e.g., square foot or square inch). The construct of the unit has been a foundational idea in researchers' frameworks (see e.g., Wilson & Osbourne, 1992) and choosing an appropriate unit to measure an attribute was a step on instructional sequences (see e.g., Wilson & Rowland, 1993).

In regular arrays, rows or columns were described as composite units and used to skip count or to multiply (Outhred et al., 2003). The categorisations, no units, inexact units, exact units, and co-ordinated units (Barrett & D. Clements, 2003) corresponded to comparisons, informal, formal and composite units. The authors suggested that additive thinking was more common with inexact and exact units, while multiplicative thinking was used with co-ordinated units. For Grade 5 children, the number of units in a composite unit affected their choice to use composite units: four by four grids might prompt the use of composite units, but seven by nine grids did so less frequently (Izsak, 2005). In order to use composite units, children had to be able to conceptualise a row was four ones and one four (Izsak, 2005). Composite units require a coordinated understanding of units and additivity.

2.1.2.4 Proportionality.

The inverse relationship between the size of the unit and the magnitude of the count illustrates the construct of proportionality used in the present study:

- There is an inverse relationship between increasing size of the unit and the count (Proportionality: Lehrer (2003)),
    - Scales (e.g. rulers) use proportional markings,
    - If the size of the units and the count of the units remains the same, the spatial measure remains the same.

If the size of the unit is changed, the count changes. Young children in their first year of school could identify the direction of change by indicting who would get more or less when the size of the unit was changed (Sophian, Garyantes, & Chang, 1997). When shown a strip that measured four blocks long and asked how many smaller blocks would be needed to measure it (any answer greater than four was taken as correct), Grade 1 and Grade 2 children identified numerically, but not precisely, that more would be needed (Carpenter, 1975). In international testing 21% of Australian Grade 4 children identified who had the longest pace, given a chart with four children and the number of steps they had taken across a room (National Center for Educational Statistics, 2007). This indicated that comparing the size of the unit from the count was more difficult than predicting the direction of change in the magnitude of the count when comparing units. Hiebert (1979) demonstrated that non-success on basic Piagetian length conservation and transitivity tasks did not necessarily predict that Grade 1 children could not use units to measure. However, the children at this preoperational stage found a task coordinating different sized units difficult and fourteen out of sixteen of them showed no understanding of the inverse relationship between the unit size and the count.

Hiebert made a crooked "little roads for ants" with five black (7 cm) Cuisenaire rods and asked the child to make a straight road the same length using yellow (5 cm) rods (p. 244-255). This task required not just an understanding that more of the smaller unit would be needed but for the child to quantify this with concrete materials.

Pettito (1990) offered Grade 4 children rulers marked at 10, 20, 30 etc: one with equal spaces and three with non-equal spacings. The children chose to use the ruler with equal spaces more often than chance. Grade 1, 2, and 3 students in a longitudinal study were offered two seven inch rulers to measure a stapler and a nine inch book (Lehrer et al., 1998). One ruler had equal units and the other had unequal units marked and 67% of Grade 3 students iterated the equal-marked ruler to measure the book correctly.

Converting between related units draws on the inverse relationship between them. For example, in a Grade Prep to 2 assessment task, after measuring a 93 cm streamer, children were asked, "how far off one meter was I?" (Department of Education & Training, 2001). Calculating the answer relied on quantifying the inverse relationship between metres and centimetres. Thompson and Saldana (2003) elaborated that related units of measure were ratios. For example, 1 mile was 1760 yards.

2.1.3 Multiplication.

Multiplication has been seen as an important pre-requisite to fraction as ratio understandings. Proficiency with multiplication and division was described as a prerequisite skill for generating equivalent fractions or working proportionally with quantities (Stafylidou & Vosniadou, 2004).

Developmental taxonomies of multiplication have been proposed. The stages in Sherin and Fuson's (2005) trajectory were not based on proficiency at algorithms, but on strategy use: count all, additive calculation, count by, pattern based, learned products and hybrid strategies. Strategies for multiplication assessed in the Early Numeracy Interview included count all, skip count, recognising the array structure for multiplication, recalling multiplication and division facts, and using hybrid strategies (Department of Education & Training, 2001). Different types of multiplication problems elicited the use of different strategies (Kouba & Franklin, 1993). Some multiplication tasks were easier in automatic recall tests. Students were more successful doubling and squaring than calculating 9 x 8 in a study of Western Australian children from Grade 3 to Year 9 (Bana & Korbosky, 1995).

2.1.4 Visualisation.

Visualisation encompasses the use of physical inscriptions as input or output in a mathematical problem, or the use of mental visual imagery during mathematical thinking.

2.1.4.1 Visual and non-visual thinkers.

Children and adults can be classified as visual or non-visual thinkers. Krutetskii (1976) described this difference with the terms *geometric thinkers* (visual) and *analytic thinkers* (non-visual). Geometric thinkers thought holistically, often in pictures or diagrams, while analytic thinkers were characterised as thinking more sequentially and often numerically (see Figure 2.3, column I). The term *geometric thinkers* was distinct from *geometric reasoning*. Any domain could be approached visually or non-visually. For example, Bishop (1983) argued that both algebra and geometry could be approached visually or non-visually. He developed the term *visual processing* to describe rotating objects in the mind's eye and mentally viewing an object from a different point of view (1983). I use Presmeg's (1986) term *dynamic imagery* to describe this phenomenon. Bishop was careful to distinguish between figural input, visual thinking, and figural output (1977, 1986). For example, a child with a preference for analytic thinking (non-visual) might be presented with a diagram as input or be asked to produce a map as output but use analytic thinking to solve the problem. A child with a preference for visual thinking, a *visualizer* (Presmeg, 2006a), might use mental imagery despite the input and output of a task being symbolic.

Visual thinking could be prompted by tasks. For example, children were shown an image of 18 dots in a six by three array and asked to show thirds, sixths, and twelfths (Lamon, 2002). The prompt, do it without counting them all one by one, was designed to elicit a spatial restructuring strategy. To make thirds, a child might mentally restructure the array into three rows and choose one of those rows. The prompt was designed to dampen numerical thinking in which the child might draw on multiplication to remember that three times six was eighteen and look for a group of six.

Visual restructuring could work with any large friendly number that was divided into an array structure, whereas working numerically with such numbers would be onerous. On the other hand, diagrams could hinder understanding. An array structure may have helped with thinking about $^{3}/_{4}$ of 12 but not with visualising $^{3}/_{4}$ of 15 (M. Clements, 1983).

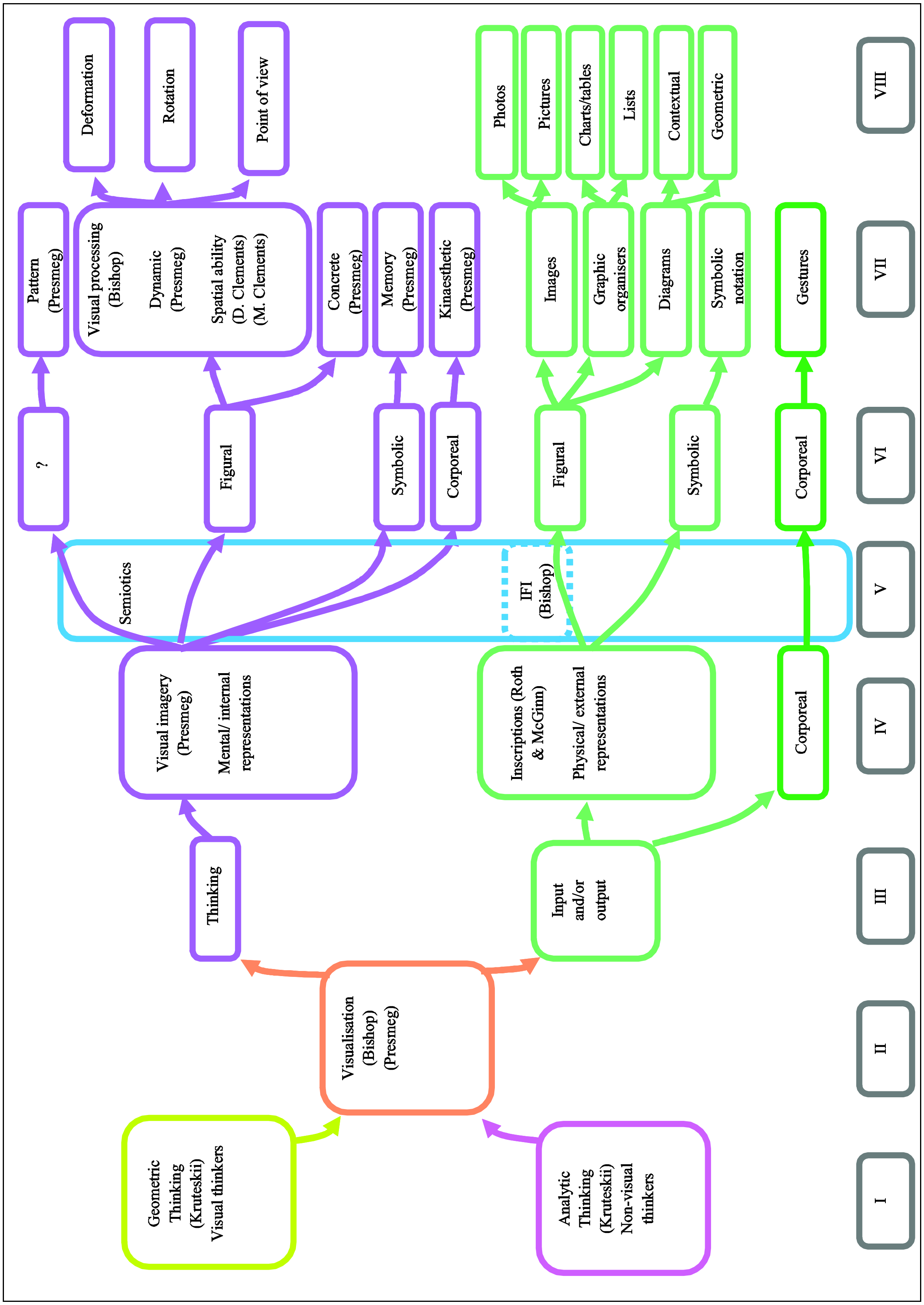


Figure 2.3. Synthesis of terminology used in visualisation research.

2.1.4.2 Visual imagery and inscriptions.

Visualisation described both mental/internal representations (thinking) and physical/external representations (input or output) (see Figure 2.3, see column II). Visual imagery (mental representations) included pattern, dynamic, concrete, memory, and kinaesthetic (Presmeg, 2006a). I have classified these under figural, symbolic, and corporeal representations (see column VI). A fourth category (unknown) classifies pattern imagery. Visual imagery concerned the thinking part of a task. Inscriptions were defined by Roth and McGinn (1998) as written representations including both diagrams and symbolic notation but excluded mental representations (see column IV), and convey information in a visual form: maps, charts, diagrams, tables and graphs, lists, photographs, spreadsheets, equations and histograms (see column VII and VIII). Inscriptions, concrete materials and corporeal actions concerned the input or output of a task (see column III).

2.1.4.3 Interpreting figural information and semiotic analysis.

Both visual imagery and inscriptions have been mediated by culturally specific understandings (see Figure 2.3, see column V). Bishop identified the role of *interpreting figural information* (semiotic considerations) of culturally specific mathematical diagrams by demonstrating that the dotted lines in the two dimensional images of a cube used in western countries was not a natural representation of a three dimensional object because it was not recognised by students in Papua New Guinea, despite their excellent visualisation skills in other tasks (Bishop, 1979). The interpretation of geometric diagrams required an understanding that was not cross cultural. Interpreting figural information "involves understanding the visual representations and spatial vocabulary used in geometric work, graphs, charts, and diagrams of all types" (Bishop, 1983, p.184). The description of mediated understandings for figural inscriptions (IFI in column V) signals the role that semiotic analysis had for all the interpretations of inscriptions, for interpretations of visual imagery, as well as for interpretations of corporeal actions (see column V). For example, in western cultures the holding up of one hand with fingers spread indicated five, but the holding up of one hand with fingers touching indicated stop.

Mathematical diagrams conveyed meaning through agreed semiotic conventions as to how they were to be decoded (Presmeg, 2006b). Diezman claimed that children should be "diagram literate" (2005, p. 286). Diagrams relied on conventions to depict both the components of the situation being represented and their organisation. Conventions had to be learned and understood before the diagrams could be understood and successfully used (Pantziara, Gagatsis, & Pitta-Pantazi, 2004). Some students were confused by symbolic notation: a fraction was "read" vertically, while written words in English were read left to right (Hunting, Pitkethy, & Pepper, 1990). Students in a multi-level Grade 3/4/5 did not necessarily interpret pre-partitioned physical models as connected to part-whole representations (Wenrick, 2003). These students interpreted concrete materials differently to inscriptions. Interpreting the dotted lines as representing the edges that cannot be seen in the two dimensional representation of the three dimensional Wattanawa Block (see Figure 2.4) was required before either spatial reasoning or geometrical reasoning were employed to solve the task (M. Clements, 1983). Diagrams had their own grammar and vocabulary that had to be interpreted.

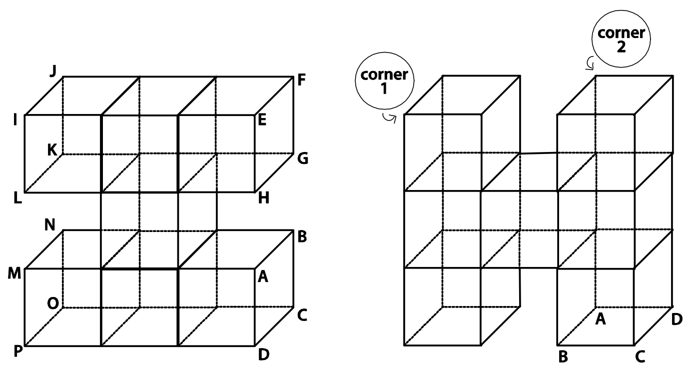


Figure 2.4. The Wattanawa Block task: dotted lines represent unseen edges.

2.1.4.4 Inscriptions.

I have classified the examples of inscriptions provided by Roth and McGinn (1998) into symbolic inscriptions and figural inscriptions (see column VI). Symbolic inscriptions include mathematical notation. Figural inscriptions are subdivided into images, graphic organisers, and diagrams (see column VII). Images include photos and pictures (see column VIII). Graphic organisers use spatial layout to indicate hierarchies or links, and include lists, charts, and tables. Diagrams are further categorised as representations of a context, and geometric figures. The role of a diagram in a task could be identification (e.g., labelling a circumference of a circle), comparison (e.g., analogue and digital clocks next to each other), showing a stage in a chain of events, or sequencing, (e.g., a sporting race showing ordinal finish), or a combination of the three types (Kidman, 2002). Geometric diagrams could be scale diagrams such as array tasks (see e.g., Figure 2.2) or not-to-scale diagrams such as the Perimeter of an L Shape task (see Figure 2.1). Contextual diagrams included children's detailed drawings of word problems. Younger children added detail to their mathematical drawings such as faces on human figures because they were still representing a concrete, not abstract image (Kamii & Housman, 2000). Later, stick figures or even marks could stand in for objects. I separate graphic organisers from diagrams, in contrast to Diezman and English (2001) whose definition of a diagram would include visual organisers: "A diagram is a visual representation that displays information in a spatial layout" (p. 77). I classify both as types of figural inscriptions and include images as well.

Diagrams of fractions were intended to help students develop visual imagery of fraction contexts. M. Clements (1981, 1983) noted the use of length and area diagrams in fraction instruction. In a study in which two classrooms were compared, one using traditional (symbolic) fraction instruction and the other using paper models of fractions, the results showed no significant difference between the overall achievement of the two groups (Marriott, 1978). However, students who had been instructed using the paper cut outs were closer to the real answer when they made errors than the students taught with symbolic inscriptions. In contrast, research on problem solving showed that "the efficient use of a diagram did not imply the successful solution of a problem and reversely the successful solution of a problem did not imply the efficient use of the accompanied diagram" (Pantziara et al., 2004). Diagrams as input in fraction tasks were helpful in one study. In contrast, in another study, diagrams as input or output in problem solving tasks were not always associated with success. Although it was unconventional to have unhelpful information in a diagram, some tasks from the Rational Number Project deliberately included perceptual distracters to test children's understanding of fractions (see e.g., Behr, Lesh, Post, & Silver, 1983; Behr & Post, 1981; Behr, Post, and Lesh, 1981).

Another use of diagrams has been in geometric proofs, but as the focus on explanations in the present study did not use geometric proofs, that research literature has not been included in this chapter.

2.1.4.5 Visual imagery.

Visual imagery is mental or internal representations (see Figure 2.3, column IV). I have classified visual imagery into figural, symbolic and corporeal representations (see column VI). Presmeg had elaborated five visualisation strategies: pattern, memory, kinaesthetic, dynamic, and concrete (1986) (see column VII). Pattern imagery was of pure relationships stripped of concrete details; memory imagery was of mathematical formulas and algorithms;

and kinaesthetic imagery was of physical movement such as walking with fingers on each end of a vector (mentally) (Presmeg, 1986; 2006a). I have classified memory imagery as symbolic, and kinaesthetic imagery as corporeal. Presmeg's concrete imagery was a picture in the mind and dynamic imagery involved an object moved or transformed in the mind's eye. These last two visual strategies have been categorised here by me as figural. The terms dynamic imagery (Presmeg, 1986; 2006a), visual processing (Bishop, 1983), and spatial ability (D. Clements & Battista, 1992; M. Clements, 1983) all referred to the same type of visual imagery that enabled the movement of objects in the mind's eye. There were three types of visual imagery (see column VIII) recognised in the mathematics education literature (D. Clements & Battista, 1992; M. Clements, 1983; Gorgorio, 1998) and in cognitive psychology literature (Hegarty & Waller, 2004):

- rotating the object while the mind looks on,
- changing the point of view, for example imagining the mind's eye moving around to the back of the object, and
- deformation of an object, analogous to the stretching or squashing of dynamic geometry objects using computer software (see e.g., Holzl, 1996).

The researchers in the Early Numeracy Research Project (Clarke, et al., 2002; Department of Education & Training, 2001) used the term dynamic visualisation to refer to the deformation of an object. In contrast, I have used the term dynamic imagery, as Presmeg did (2006a), to indicate all three types of movement in the mind's eye.

Longitudinal research into the use of visual imagery in fraction learning showed that pattern imagery was associated with conceptual understanding of fractions, and pictorial imagery was associated with a limited part-whole understanding (deWindt-King & Goldin, 2001).

Visualisation has become the focus for the investigation of both visual imagery and inscriptions in many domains, such as algebra, fractions, and geometry. Visualisation was first recognised as a separate topic at the Psychology of Mathematics Education Annual Conference in 1991. Bishop (1983) and Presmeg's (1985, 1986) work had preceded this. Spatial ability (dynamic imagery) had been a concern of cognitive psychology until the 1980s when mathematics educators began to study it (M. Clements, 1983). In geometry, spatial ability (dynamic imagery) had been categorised as a skill (D. Clements & Battista, 1992), but later spatial reasoning (visualisation) was described as underlying most geometric thought (Battista, 2007).

Researchers have made a distinction between dynamic imagery and geometrical thinking strategies. For example, in the Wattanawa Block task (see Figure 2.4) the student had to decide where corner 2 was on the second diagram. A spatial ability (dynamic imagery) strategy could be employed involving complicated rotations in the mind's eye. A non-visual geometric solution (it's always opposite corner A) was possible (M. Clements, 1983). If spatial tasks, similar to the Wattanawa Block task, could be correctly solved using geometric reasoning and not spatial ability (dynamic imagery), as indicated by M. Clements (1983), then cognitive psychologists using such tasks to assess spatial ability would have unidentified false positives: students who gave correct answers but had not used dynamic imagery. Bishop (1983) had identified the possibility of false negatives in his examples illustrating interpreting figural information (semiotics): students may give an incorrect answer because they cannot decode the conventions of the diagram, not because they have poor visual processing (dynamic imagery).

2.1.5 Fractions.

In Victorian primary schools children can encounter fractions when:

- the measurement of an object results in a count with a partial unit left over,
- the remainder from a division is renamed,
- an object or number is stretched or shrunk proportionally,
- ratio comparisons are made in probability contexts (e.g. you are more likely to pick a red marble if four out of five are red than if seven out of ten are red),
- decimals are used to extend the place value system to tenths and hundredths, and
- shaded parts of area diagrams are named as fractions.

I use the term fraction in the present study because I defer to the common usage in Victorian primary schools. "Fraction" signals early rational number understandings and the use of the $^{a}/_{b}$ notation. Thompson and Saldanha (2003) suggested that an understanding of the system of rational numbers was beyond primary students and that fractions were a better focus of research and curriculum design. Decimals are not the focus of the present study and so little research has been included in this literature review about misconceptions specific to decimal notation and diagrams.

Whole numbers are the first numbers children encounter and support discrete counting strategies. Baroody and Coslick (1998) pointed out that there was no "next" number when counting by fractions as there was when counting with whole numbers. Fractions are dense, as

Sophian and Wood (1997) elaborated "no matter how close one fraction is to another, there are always other fractions between them; therefore, they do not form a single fixed progression, as the counting numbers do" (p. 309). The continuous aspect of rational numbers meant that operating with fractions was not the same as operating with whole numbers.

In an effort to make children's introduction to rational numbers easier, some teachers have limited fraction instruction to the part of a whole model (Lamon, 2007). Ball (1993) commented that children "probably would have had limited experience, consisting primarily of shading pre-divided regions" in their early fraction learning (p. 169). Kieren (1988) noted that "part-whole models of fractions conveniently help produce fractional language" but that this tends "to orient the student to a static double count image" (p. 177). This fractional language produces a definition that limits the teaching that has accompanied the part-whole model of fractions: there are four parts and you take three of them (for three quarters). This definition makes improper fractions almost nonsensical: there are four parts and you take seven of them (seven quarters). The limited nature of definitions linked to procedures has been noted and alternatives proposed: "If we have $^{7}/_{3}$, the 3 tells the name or size of the parts (thirds) and the 7 tells us that we have 7 of those thirds (or $2^{1}/_{3}$)" (Clarke, Roche, & Mitchell, 2008, p. 375).

In some classrooms the use of the part-whole model was limited to a whole pre-divided into equal parts with students drilled in a procedure: count the shaded pieces (write this as the top number) and then count all the pieces (and write this as the bottom number) (Carrahar, 1996; Gould, 2005; Kieren, 1993; Ni & Zhou, 2005). Although counting without attending to the size of the pieces gives a correct answer when the diagram is divided into equal parts, if the same procedure is applied to diagrams with non-equal parts an incorrect answer results (the double count misconception). Some children articulated this confusion by asking "do you mean in size...or in amount?" (Post, Wachsmuth, Lesh, & Behr, 1985, p. 34). Using non-equal-parts diagrams has distinguished between students with the double count misconception and students who can mentally or physically repartition the diagram into equal parts.

Identifying the different contexts of rational numbers, Kieren (1976) proposed seven interpretations that included both conceptual understanding (decimal, equivalence, ratio, operator, quotient, measures) and procedural understanding (fraction algorithms). He refined this to a five part model (1980) which described only sub-constructs that formed a conceptual understanding of rational number: part-whole, measure, quotient, operator, and ratio. Kieren, a Canadian, worked closely with the American researchers from the Rational Number Project who used an adaption of the five part model to frame their first teaching experiment which took place in 1980 (Behr et al., 1983; Behr et al., 1981). This adaption was used in the design of the instructional materials and placed partitioning and part-whole before, and leading to, all the four other sub-constructs. Later articles from the Rational Number Project researchers named six interpretations of fractions: part-whole, measure, ratio, operator, quotient, and decimals (Behr & Post, 1992; Behr et al., 1984). Both Kieren and the Rational Number Project researchers used a concept of part-whole that was broad and not meant to be interpreted as the static double count practice of some instructional models. The depth of this part-whole understanding was visible in later work by Behr, Harel, Post and Lesh (1992). This conceptual understanding of part-whole was the first concept in the Rational Number Project's instructional sequence (Cramer et al., 1997) but they cautioned that careful assessment was required to ascertain whether the students' part-whole knowledge was robust, and to intervene if the double count misconception emerged (Cramer, et al., 1997; Lesh, Post, & Behr, 1988). This approach to countering the double count misconception was to include a broad range of part-whole contexts in instruction, and this was supported by other researchers (Baturo, 2004; Clarke, Roche, & Mitchell, 2007; Gould, 2005; Herscovics, 1996; Keijzer & Terwell, 2002; Moseley, 2005; Nabors, 2003).

Kieren's (1988) response to the double count misconception was to reframe the part-whole constructs within the other constructs of his 1980 model. Kieren's four-three-four model (1988) was not in opposition to his earlier five part model (1980), and the model used in the Rational Number Project research (Behr et al., 1983), but more of a refinement. In this adaptation (1988) the four sub-constructs of measure, quotient, operator and ratio remained, but were underpinned by three concepts, partitioning, equivalence and unit-forming (see Figure 2.5). These sub-constructs and constructs could be approached on four levels: ethnomathematic, intuitive, technical-symbolic and axiomatic deductive. I have called this model the four-three-four model to indicate these three different aspects of Kieren's model: *four* sub-constructs, *three* underpinning concepts, *four* levels of response. Kieren reiterated this model in later writing (1992, 1993, 1995). For Kieren, the measure and quotient sub-constructs provided conceptually richer ways of explaining the non-procedural part-whole examples used by the researchers in the Rational Number Project (1993, p. 57). Kieren (1995) encouraged the actions of partitioning as a way to develop the concept that underpinned this non-procedural part-whole understanding.

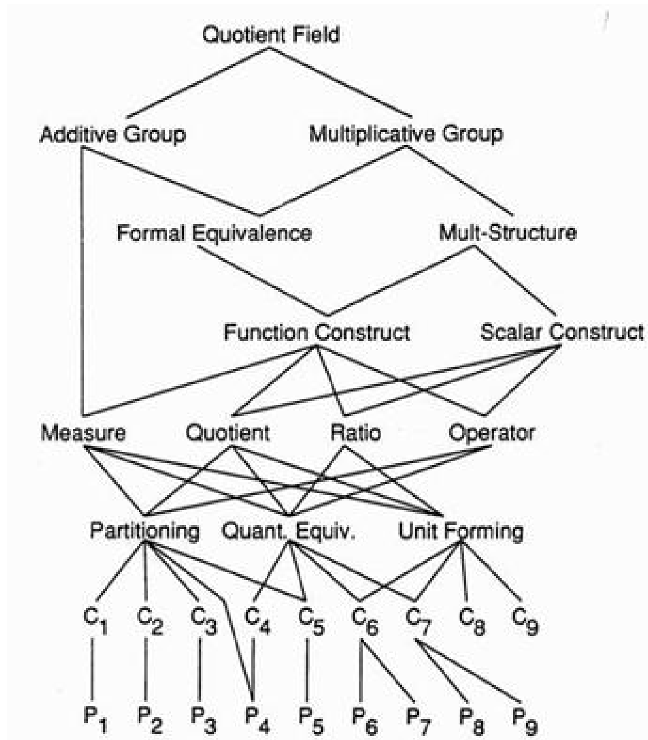


Figure 2.5. Kieren's model of rational number knowing (1993).

Kieren's four-three-four model (1988, 1992, 1993, 1995) was a framework of ideal rational number knowing. The four sub-constructs measure, quotient, operator, ratio, and three underpinning concepts, partitioning, equivalence, and unit-forming have been introduced above, and will be elaborated in later sections of this chapter. The model represented increasing conceptual complexity by vertical layers of constructs and a "mature rational number knower" would be able to engage with the whole range of constructs (Kieren, 1988). The constructs and concepts that combined to form a coordinated understanding of fractions had a hierarchical nature (see Figure 2.5): percepts (p-level), constructs (c-level), proto-mathematical knowledge (partitioning, quantitative equivalence, unit-forming), sub-constructs (measure, quotient, operator, ratio) and more formal multiplicative thinking and structural knowledge of rational numbers (Kieren, 1993). The lines connecting constructs represented the generative nature of understanding. Concepts at the lower level supported concepts at higher levels (1988). These lines represented where the knowledge might take you next. For example quantitative equivalence could be drawn upon by all four sub-constructs. Similarly, horizontal integration at every level was part of the idealised understanding. The model was not one pathway through the constructs but described different levels of constructs and different sub-constructs within each level.

2.1.5.1 Measure.

The measure sub-construct described the measurement context that generated rational numbers: "the notion of fractional numbers arises whenever one measures something. If the unit does not fit evenly within the object to be measured a whole number of times, how does one give a number which is the measure? Rational numbers answer this question" (Kieren, 1995, p. 37). A non-whole number count left partial units and these could be quantified in different ways (see section 2.1.2.3): fractions were one strategy.

The number line had been Kieren's example of the measure sub-construct in previous work (1976, 1980) but the number line was not the only example of the measure sub-construct. The measure sub-construct of fractions included linear and regional (area) measurement contexts because an area diagram could be thought of as a comparison to a unit, not just a part-whole model (Kieren, 1992). The measure sub-construct with its representations of continuous attributes, also provided "experiences with order" (Kieren, 1993, p. 59). Lamon (1999) had three key understandings for the measure sub-construct: being able to use a given unit interval to measure any distance from the origin; being able to find any number of fractions between two fractions; and being comfortable with partitions other than halving (1999). The examples of mathematical contexts used in the present study to illustrate the measure sub-construct are number lines, the relative size of fractions, and measurement contexts such as area diagrams.

2.1.5.1.1 Number lines.

Research on number lines in a fraction context concentrated on the distinction between making partitions and reading pre-marked partitions, and proper and improper fractions on number lines labelled 0 to 1, and 0 to greater than 1 (Bright, Behr, Post, & Wachsmuth, 1988; Ni, 2000; Novillis-Larson, 1980). Facility with number lines involved one of Lamon's (1999) key concepts of the measure sub-construct of fractions: being able to use a given unit interval to measure any distance from the origin.

Decimal number lines are common in Australia and New Zealand where the metric system is used for measuring. In the first year of a large numeracy project, 50.2% of Year 7 New Zealand students were successful at identifying 6.8 (or six and eight tenths) on a number line (see Figure 2.6) (Vince Wright, personal communication, January 23, 2008),

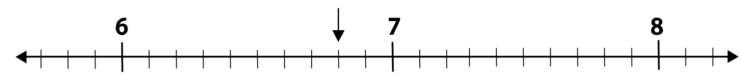


Figure 2.6. New Zealand assessment interview decimal number line.

Children have been confused between the representation of a whole and the length of the whole number line on number lines that were greater than 1. For example, they found *half of* a number line with some pre-marked partitions rather than where the *number half* would go on a number line marked with more than one whole (Kieren, 1993).

When reading pre-marked partitions on number lines successful students attended to the parts, rather than the vertical lines used to create the equal parts (Bright et al., 1988; Pearn & Stephens, 2007). Counting lines not spaces was a misconception in which the students counted the mark at 0 as one and hence read $^{3}/_{4}$ as $^{4}/_{5}$. However, strategy use on rational-number number lines and other measurement scales using similar fractions was not consistent Drake (2007).

A misconception became evident when children, asked to draw their own number lines, used a ratio representation. For example, some children drew a number line from 0 to 6 and labelled 4 as $^{2}/_{3}$ (Clarke et al., 2007). It was possible to use a number line to find $^{2}/_{3}$ of 6, but $^{2}/_{3}$ on a number line should be two thirds of the way between 0 and 1. The conceptual link between measurement and the measure sub-construct was explicitly made by Kieren (1995) and had informed his earlier work (1976, 1980, 1988, 1992, 1993). The similarity between the counting lines not spaces misconception in length broken ruler tasks and number line tasks suggested a conceptual link between measurement and the measure sub-construct of fractions. The double count misconception highlighted that children were not attending to the area attribute that represented the fractional meaning.

2.1.5.1.2 The relative size of fractions.

The relative size of fractions has included fraction pair comparisons and was called *order* in order and equivalence studies. The relative size of fractions was part of another of Lamon's key understanding for the measure sub-construct: being able to find any number of fractions between two fractions (1999). Strategies for comparing the relative size of fractions included:

- correct part-whole understandings,
- the use of common denominators and benchmarking, and
- whole number dominance misconceptions.

These strategies were described by the Rational Number Project researchers after interviews with Grade 4 and 5 children during a teaching experiment and were reported as strategies concerning either fractions with the same numerators, the same denominators or different numerators and denominators (see e.g., Behr, Wachsmuth, Post, & Lesh, 1984).

Fraction pair comparisons have been made by children using some simple correct strategies. Noting the same denominators and then comparing numerators worked for fractions with the same denominators. For example, $^7/_8$ was larger than $^3/_8$ because the denominators were the same and 7 was larger than 3 (Clarke et al., 2007). Attending to the numerators only, in this case, was termed whole number consistent by the Rational Number Project researchers. Although this strategy gave correct answers with same denominator pairs it could lead to misconceptions when other types of fractions were used (Behr et al., 1984).

Another correct strategy was to compare denominators if the numerators were the same. The most common examples of this were unit fractions such as $^1/_3$ and $^1/_4$. In these cases students could correctly claim that the bigger the denominator the smaller the fraction (Behr et al., 1984; Post & Cramer, 1987; Post et al., 1985). This strategy could also translate successfully to fractions with the same numerators such as $^2/_5$ and $^2/_3$ where "two fifths is less than two thirds because there are two pieces in each, but the pieces in two fifths are smaller, so a smaller amount of the unit is covered for two fifths" (Post et al., 1985, p. 20). Using this strategy for non-unit fractions was not always straightforward for Australian Grade 6 children, with 37.2% correctly identifying $^4/_5$ as larger than $^4/_7$ and offering a correct explanation. However, a further 21.4% who chose the correct answer ($^4/_5$) did so using the misconception of gap thinking (Clarke et al., 2007).

Residual thinking was a mathematically correct strategy useful for comparing fractions that were both one away from the whole: $^5/_6$ was one sixth away from the whole and $^7/_8$ was one eighth away from the whole; as one eighth was smaller, $^7/_8$ was closer to the whole (Clarke et al., 2007). This strategy had been previously identified by the Rational Number Project researchers (Cramer, Post, & DelMas, 2002; Post et al., 1986; Post & Cramer, 1987) and attributed to the use of fraction kit materials.

Two other correct mathematical strategies have been described in the research literature. The *transitive* strategy or *reference point* strategy (Behr et al., 1984; Post et al., 1986; Post & Cramer, 1987) was called *benchmarking* in Australia (Clarke & Roche, 2009). For example, $^5/_8$ was larger than $^3/_7$ because $^3/_7$ was less than a half and $^5/_8$ was more than a half. Using half as a benchmark was a strategy that children used when they had an understanding of the related value of different fractions without being dependent on an algorithm to find a common denominator. This could apply both to fractions represented in numerical notation (Sowder, 1988) and regional area representations (Keijzer & Terwell, 2001). The other correct strategy used was common denominators which required equivalence knowledge and was called the application of ratios (Behr et al., 1984).

The term *whole number dominance* was coined by the researchers in The Rational Number Project to describe strategies that they believed were inappropriate generalisations about whole numbers used in fraction comparisons by students. Four of these whole number dominance misconceptions are described below.

Some children incorrectly believed that a larger denominator indicated a larger fraction. For example, $^1/_3$ was less than $^1/_4$ because 3 is less than 4 (Behr et al., 1984). This whole number dominance misconception was also reported in other Rational Number Project articles (Behr, Post, & Wachsmuth, 1986; Post et al., 1986; Post & Cramer, 1987; Post et al., 1985).

Non-equivalent fractions were considered to be equal if the numerical difference between the numerator and denominator was one. For example, $^3/_4$ was equal to $^2/_3$ because the difference between 3 and 4 was 1 and the difference between 2 and 3 was 1 (Post & Cramer, 1987).

Some children chose the fraction with both numerator and denominator larger numbers than the other fraction. For example, $^3/_5$ is less than $^6/_{10}$ because "3 is less than 6, and 5 is less than 10" (Behr et al., 1984).

Students who used the incorrect *addition* strategy added the same number to the numerator and denominator to make an equivalent fraction, for example $^3/_4$ equals $^7/_8$ because you can add 4 to the numerator and 4 to the denominator of $^3/_4$ to get to $^7/_8$ (Behr et al., 1984). This misconception was reported as a whole number dominance misconception in other Rational number Project articles (Post et al., 1986; Post & Cramer, 1987).

The larger denominator indicating larger fraction misconception was observed in Australia (Clarke et al., 2007), and in Greece (Stafylidou & Vosniadou, 2004). Ni and Zhou (2005) also noted this misconception in their review of whole number bias in fraction understanding. In the present study I refer to this misconception as the *bigger denominator indicates bigger fraction* misconception.

The numerical difference strategy was known as the *gap thinking* misconception in Australia. Two fractions, both with a "gap" of one were the same. Gap thinking also encompassed choosing the fraction with the smaller gap between numerator and denominator. Pearn and Stephens (2004) used the term gap thinking to describe a Year 8 student's (incorrect) comparison of $^3/_5$ and $^5/_8$ where $^3/_5$ was larger because "there is less of a gap between the three and the five (in the first fraction)" than there is between the "five and the eight (in the second fraction)" (p. 434). Pearn and Stephens (2004) observed *comparing-to-a-whole* thinking in which the students claimed that $^2/_3$ was bigger than $^3/_5$, because $^3/_5$ "is two numbers away from being a whole" while $^2/_3$ "is one number away from being a whole" (p. 434). This was a correct answer but with a mathematically incorrect reason. Clarke, Roche, and Mitchell (2007, see also Clarke & Roche, 2009) combined Pearn and Stephens (2004) gap thinking and comparing-to-a-whole thinking in their definition of gap thinking. Gap thinking was evident in many of the responses to their fraction pair questions: 35.6% of the incorrect answers comparing $^3/_4$ and $^7/_9$ demonstrated gap thinking. Nearly 30% of all students said that $^5/_6$ and $^7/_8$ were equivalent (Clarke & Roche, 2009). In the present study, I use the term gap thinking to describe these related strategies.

The third whole number dominance strategy described above was called *higher or larger numbers* by Clarke et al. (2007). Pearn and Stephens (2004) observed this strategy: $^3/_5$ is larger than $^2/_3$ because the three is larger than the two (numerators) and the five is larger than the three (denominators). They viewed this as a variant of gap thinking. In the present study I use the term higher or larger numbers to describe this strategy.

2.1.5.1.3 Non-equal-parts area diagrams.

If area diagrams were thought of only as part-whole, then the measure construct in that representation was diminished. Kieren described how the part-whole concept (as used by the Rational Number Project researchers) integrated with the other sub-constructs:

> their part-whole sub-construct is subsumed under the quotient and measure sub-constructs as the dynamic comparison of a quantity to a dividable unit that allows for the generation of rational numbers as extensive quantities. The part-whole notion also relates to the operator sub-construct as the selected unit that forms the basis for operators as composite functions (see Dienes, 1971). It plays a similar role in the considerations of ratio numbers (e.g. mixtures) (Kieren, 1993, p. 57).

Lamon (2007) agreed with Kieren that "part-whole is not a separate construct, but really a case of the measure subconstruct (p. 659). By using the terminology of part-whole as a sub-construct, researchers, teachers and students have been ignoring the part-whole aspect of area and length diagrams in the measure sub-construct.

Non-equal-parts fraction-area-diagrams have enabled researchers to observe whether children are counting or measuring by whether the child demonstrated the double count misconception or could name non-equal parts on an area diagram. The double count misconception became apparent in this context if the denominator in a child's response was the number of parts (Armstrong & Novillis-Larson, 1995; Saxe et al., 2005). In a study of 384 Grade 4, 5, and 6 students 25% identified the shaded eighth on a rectangle as one fifth (see Figure 2.7 left) (Saxe et al., 2005). In a doctoral study of 20 Grade 6 students, only two correctly identified part c of the circle as one sixth (see Figure 2.7 middle), while five wrote one fifth (Stewart, 2005). Using the pie task adapted from the same source (Cramer et al., 1997), Clarke et al. (2007) reported on 323 interviews with Grade 6 students in which 42.7% gave the correct answer of one sixth for the smaller piece, Part A (see Figure 2.7 right), with 13.6% answering one fifth. These three studies illustrated that correctly identifying parts in non-equal area models was difficult for some children, and the double count misconception (answers of one fifth) appeared evident in up to a quarter of the students' responses. Being comfortable with partitions other than halving, Lamon's (1999) first key concept of the measure sub-construct of fractions, was evident in students' successful answers of $^1/_6$ to the Fraction Pie task.

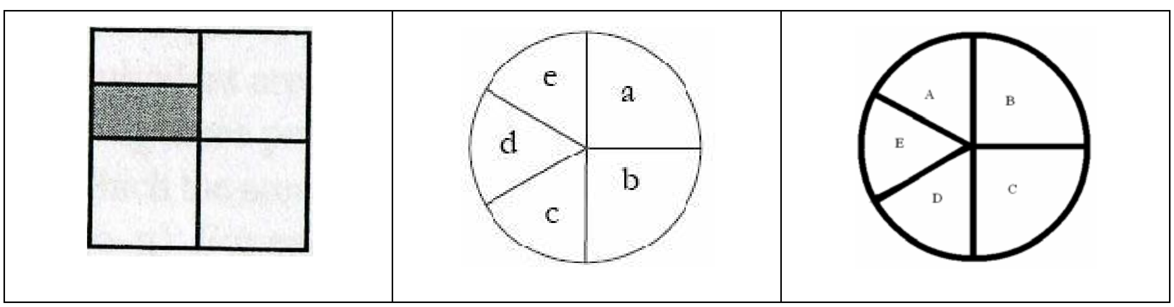


Figure 2.7 Non-equal-parts tasks.

Some answers of one fifth were not the double count misconception but rather an attempt to describe the smaller piece in relation to the quarter (Mitchell, 2005). The answer of one fifth as a part that was *nearly a quarter* used a mathematically correct approach, although the execution of the strategy was not sufficiently accurate.

2.1.5.1.4 Non-congruent parts area diagrams.

Research has indicated that some children find it difficult to think of conservation of area and fractions at the same time. When modelling equivalence using shapes partitioned in different ways children were asked to trust the evidence of their eyes. However, when a shape looked bigger despite having the same area as another, children were asked to override their visual perceptions (Fosnot & Dolk, 2002; Herscovics, 1996). For example, in a study on the use of area models in fraction learning, simple conservation tasks were offered to children (Walta, 1973). Two rectangle halves were cut from a square piece of paper, and two triangular halves cut from another, then when a triangular half and a rectangular half were put together 70% of the high achievers (in fraction performance) and 33% of the low achievers demonstrated an understanding of conservation by saying that the area was still the same as the original square. Kamii and Clark (1995) used Piagetian terminology to describe the confusion caused by non-congruent equal parts: "for example, *half* of a rectangle can be either rectangular or triangular. Although the triangular half may look bigger than the rectangular half from a *figurative* point of view, our *operative* knowledge enables us to deduce that two halves have the same area" (p. 369). Non-congruent parts of area models were difficult representations for children to use in the measure sub-construct.

2.1.5.2 Quotient.

The quotient sub-construct described a context in which sharing between two separate measure spaces took place. For example, three pizzas shared between five people generated shares of three fifths: $3 \div 5 = {}^{3}/_{5}$. Similar tasks appeared in the research literature (Behr, Post, Harel, & Lesh, 1993; Kieren, 1988; Lamon, 1999; Clarke et al., 2007).

The Dutch curriculum introduced fractions with a sharing context and elaborated several strategies that children use in solving such problems (see e.g., Keijzer & Terwell, 2001; Streefland, 1993). French division was the term used to describe cutting each pizza into enough pieces for everyone; each of the three pizzas would be divided into five parts and a piece from each pizza dealt out to each person, resulting in three one fifth shares. In this curriculum, the double count misconception became an incomplete understanding of the sharing context that every child encountered if they made unequal parts and was resolved because the contextual imperative to make fair shares was compelling.

Repeated halving was used as an intuitive strategy: cut every pizza in half and deal out all the pieces and see if that works, if not, divide every pizza into quarters and deal out all the pieces, repeat with eighths if quarters left a remainder (Pothier & Sawada, 1983). If children then subdivided the remainder their explanations sounded like "engineering reports" (Kieren, 1988). Children had trouble keeping track of the unit when doing non-equal partitioning (dividing each piece into half and then the leftover half into five) sometimes calling the result "a half and a fifth" instead of a half and a tenth or six tenths or three fifths (Lamon, 1999). Just less than a third (30.3%) of Grade 6 students could solve the pizzas shared between five people problem (Clarke et al., 2007). Very few of them used the fractions as division concept (Clarke, 2006) that three shared between five is three over five or three fifths.

Research on pre-school children's understandings of fractions has often focused on the ability to make fair shares (Hunting & Sharpley, 1988; Pearn, 1996; Pepper & Hunting, 1998; Pothier & Sawada, 1983; Sophian et al., 1997).

Other researchers found that the quotient construct produced greater conceptual understanding when used for early fraction learning than a traditional part-whole approach (Empson, 1999; Mamede, Nunes, & Bryant, 2005, Ni & Zhou, 2005). By introducing sharing contexts first and part-whole contexts second, the double count misconception was pre-empted.

2.1.5.3 Operator.

The operator sub-construct of fractions described, among other things, size transformations (Kieren, 1995). Two and a half times as large, or three quarters the size of, were size transformations that used fraction modifiers. This context had also been called "fraction composition" in the radical constructivist tradition (Izsak, 2008). Izsak (2008) analysed Behr, Harel, Post, and Lesh's *duplicator and partition reducer*, and *stretcher and shrinker operations* (1991), and recast them as a partitive model and a quotitative model respectively. For example, taking three quarters of something involved dividing by four and multiplying by three, or multiplying by three and dividing by four.

Simple operator tasks were relatively easy for Grade 6 students with 97.2% able to mentally calculate half of six (Clarke et al., 2007). Despite the use of pen and paper, only 17.6% were able to work out one third of a half (Clarke et al., 2007). Similar tasks were categorised as multiplicative thinking such as using a diagram to solve one third of a quarter (Kamii & Clark, 1995). Area models were often used as models for fraction multiplication.

2.1.5.4 Ratio.

The ratio sub-construct was often found in the context of mixtures or probabilities (Kieren, 1995). Lamon (1999) described ratios as used to convey a comparison of two quantities that may not be able to be represented as a single number. Kieren's refiguring of the part-whole concept meant that traditional discrete part-whole models (of arrays of dots for example) could be thought of as ratio understandings (Kieren, 1992). Instead of two thirds as two out of three dots, it could be thought of as two for every three dots.

Using re-unitising or spatial restructuring this definition linked ratio and equivalence understandings because $^{8}/_{12}$ could be seen as two for every three. Ratio understanding was linked closely to proportional reasoning (Lamon, 1993; Lesh et al., 1988). One method for assessing children's ratio understanding was a proportional thinking task in which three snakes of different lengths ate proportionally different amounts (Kamii & Clark, 1995). For example the second snake ate twice as much as the first, while the third snake ate three times as much as the first snake. By changing the amounts given to different snakes, the child's ratio and proportional conceptual understanding was elicited: if the first snake had one pellet, what would the others have; if the second snake had four pellets, what would the others have? This was originally a Piagetian task and it recurred in the research literature (Hart, 1981; Lamon, 1993; Resnick & Singer, 1993). Unsuccessful strategies included additive approaches: if the second snake ate four pellets, the first snake ate three because last time it ate one less. Although these tasks used whole numbers, they were assessing the ratio sub-construct of fractions.

2.1.5.5 Partitioning.

Kieren's description of partitioning was the folding and drawing actions required of the children when making equal parts (1995). Stages were elaborated for learning to partition including repeated halving and using the radius rather than the diameter of a circle to generate thirds and fifths (Pothier & Sawada, 1983). Repeated halving in paper folding activities were examples of splitting; an exponential rather than repeated addition/subtraction process (Confrey, 1994). The partitioning approach emphasised multiplicative understandings (Siemon, 2003) rather than just a static part-whole double count procedure. For example, to make forty-eighths, children in Kieren's study described folding a third and then a half, and half, and a half, and a half (1995). School experiences of part-whole double counting had not been as useful as partitioning in developing these multiplicative understandings (Kieren, 1995).

2.1.5.6 Equivalence.

The construct of equivalence had a specified place in Kieren's four-three-four model (1988, 1992, 1993, 1995). Kieren cautioned that drawing on equivalence as an internalised strategy in other tasks was not apparent in half the age cohort until age 12, and full common denominator reasoning occurred later (1992). Callingham and Watson, (2004) reported that equivalence understanding emerged slowly with students' recognition of:

- half of an even whole number,
- the use of equivalent fractions for a half, and
- tasks where "equivalence appears to become an important idea" such as $^{1}/_{2} + ^{4}/_{8}$

At Grade 6, 64.4% could identify $^{2}/_{4}$ and $^{4}/_{8}$ as the same in a fraction pair comparison task (Clarke et al., 2007). Equivalence understanding could be either procedural or conceptual (Wong & Evans, 2007. Equivalence knowledge could not be assumed by the end of primary school.

Ways of introducing equivalence included generating equivalent fractions and recognising the same fraction in different measure representations (Wong & Evans, 2007). Usually such diagrams could be spatially manipulated using dynamic imagery so that pieces could be rearranged to make the same area as the comparison fraction. The use of area diagrams to model equivalence depended on conservation. This was especially true if parts were compared that were non-congruent. For example children were asked to compare three fourths cut horizontally across a rectangle, with eighths cut vertically across the rectangle (Clark & Kamii, 1996). In this problem 32% of the Grade 5 children could equate three quarters with six eighths. In these cases quite sophisticated restructuring would be needed to superimpose the shapes on top of each other. This enabled the researchers to assess if operational thought was used because such thinking was based on relationships that weren't observable. Equivalent fractions were called commensurate fractions in the work of Steffe (Izsak, 2008).

Kieren's model for-grounded the intuitive knowledge of equivalence in which the understandings were not separate from the contexts offered in the classroom. Partitioning supported intuitive equivalence understanding because it could provide examples of absolute equality; one half could be subdivided into two quarters, so two quarters was equivalent to a half (Kieren, 1992). Understandings of equality could be observed in transitive reasoning, for example $^9/_6$ and $^{10}/_4$ were both one and a half. Traditional part-whole instruction had been used to model these two multiplicative aspects of equivalence. In these contexts, equivalence was "as many as" (Kieren, 1992, p. 350).

However, there was also an additive aspect to equivalence that traditional part-whole instruction ignored: $^a/_b + {}^c/_d = {}^e/_f + {}^g/_h$. In this context, equivalence was "as much as" (Kieren, 1992, p. 350). For example, $^3/_4 + {}^1/_2 = {}^4/_4 + {}^1/_4$

Equivalence was present in other models of fraction understanding, for example in the work of the Rational Number Project, but Kieren's four-three-four model made a place for both its additive and multiplicative characteristics. In some other frameworks, the numerical aspect of equivalence understanding had been positioned as early ratio understanding or bundled as order and equivalence (Behr et al., 1984). Equivalence was modelled with area diagrams within a part-whole introduction to fractions (Cramer et al. 1997).

2.1.5.7 Unit-forming.

The combining space described by Kieren (1995) included classroom activities with concrete materials that were used to show fractions as "additively combinable amounts" (p. 32). The concept of unit-forming had first been referred to as forming dividable units (Kieren, 1988). Unit-forming described "the kind of combining and reconfiguring mechanisms envisioned in the work of Behr et al" (Kieren, 1992, p. 116) and formed part of the broader part-whole conceptual understanding promoted by the Rational Number Project.

Unit-forming described the intuitive additive nature of fractions. Just as eight could be made of seven and one, or six and two, or five and three, so too fractions could be made from the sum of other fractional amounts (Kieren, 1995). The distinction between partitioning and unit-forming was that the addends were equal parts in partitioning but could be non-equal parts in unit-forming activities (1995). Unit-forming was not a new concept, it was a renaming and reframing of the additive nature of fractions. The addition of fractions did not use a counting on process but a put together process (Kieren, 1992). Algorithmic addition of fractions represented this additive process in symbolic form.

Kieren (1995) called the instructional aspect of unit-forming the combining space. An example of the unit-forming construct was making fractional amounts out of other fractional parts using a paper fraction kit. For example, primary (elementary) school children used the fraction kit to respond to the task: tell me five things you know about three fourths.

Explorations included adding one fourth, three eighths, and two sixteenths together to cover three fourths (Pirie & Kieren, 1994b); or recognising an additive equivalence in the "as much as" relation, $^{1}/_{2} + ^{1}/_{8} + ^{2}/_{16} = ^{3}/_{4}$ (Kieren, 1999). Students were challenged, in the context of the fraction kit, to find a missing fraction that was bigger than one fourth and smaller than three fourths (Pirie & Kieren, 1994a). Students were offered the open ended problem: "here are four rectangular pizzas cut in halves, quarters, sixths, and twelfths. Choose some pieces from at least three of these pizzas such that their "sum" is one pizza" (Kieren, 1993). Another unit forming classroom activity, Fraction Flags, emerged from work with fraction kits (Kieren, Davis, & Mason, 1996). The children made flags using fraction pieces on the coloured background of a half piece. They then added the parts to discover how much of the flag was covered by other colours (coloured fraction kit pieces). These activities explored addition by asking $^{3}/_{4}$ = ?, rather than $^{1}/_{4} + ^{1}/_{2}$ = ? In this unit forming context, the equals sign meant *as much as* rather than is equal to as a ratio.

The additive thinking used in unit forming activities was correct mathematical thinking. This correct additive thinking, thinking of fractions as made of other fractions, which was termed unit forming by Kieren (1988, 1992, 1993, 1995), had been categorised as part of part-whole in the Rational Number Project research (Behr, Harel, Post, & Lesh, 1992; Behr & Post, 1992). In other fraction tasks described in the research literature, the term "additive" had been used to describe incorrect mathematical thinking. In particular, this other incorrect additive thinking was associated with early attempts at proportion and with equivalence tasks in which children incorrectly used an additive relation rather than a multiplicative relation (see e. g. Behr, Lesh, Post, & Silver, 1983; Cramer et al., 1993; Post et al., 1986; Post & Cramer, 1987; Post, Cramer, Behr, Lesh, & Harel, 1993).

Partitioning, equivalence and unit forming "could build from and relate back to one's everyday experience" (Kieren, 1988, p. 170). The unit-forming concept was drawn upon in quotient contexts. For example, when sharing three pizzas between five people, students divided all three pizzas into halves and then divided the leftover half into five pieces. Each person received a half and a fifth of a half. This had been noted in earlier research and categorised as a subset of partitioning in which "the additive nature of partitioning is observed" (Kieren, Nelson, & Smith, 1985). I believe that this subset of partitioning (in 1985) became unit-forming (by 1988).

The review of the literature of length and area measurement, multiplication, visualisation, and fractions has revealed many correct strategies and incorrect misconceptions that children use when attempting mathematical tasks. This has led to the development of the first research question:

- What strategies are evident in students' explanations of their thinking in a one-to-one task-based interview?

2.1.5.8 Conceptual links between fractions and measurement.

In her literature review setting the research agenda for the next decade, Lamon (2007) noted the conceptual link between measurement and fraction understanding: "given the importance of measurement ideas to understanding not only rational numbers, but all of mathematics, a microanalysis of the development of measurement could provide another long-term research agenda with broad impact" (p. 661).

A conceptual link between measurement concepts and fraction concepts has been identified in the research literature on measurement. Researchers elaborated the conceptual links between the two domains: the "distinguishing feature of measuring is that it is concerned with quantities which are continuous; this requires an extension of the natural numbers into the rational numbers, and arguably eventually the real numbers" (Brown et al., 1995, p. 159). The use of area models in fraction instruction depended on an understanding of the measurement concept of area (Wilson & Osbourne, 1992). Measurement concepts were not just linked to the measure sub-construct of fractions, but also to ratios (Brown et al., 1995; Joram, Gabriele, Bertheau, Gelman, & Subrahmanyam, 2005). The (ratio) relationship between units remained the same: if seven toy cars were five wooden blocks long then 14 toy cars had to be ten wooden blocks long. Space, measurement and number were interrelated claimed Barrett, D. Clements, Klanderman, Pennisi, and Polake (2006). A lack of transfer across mathematical domains was of concern to some researchers (Brown et al., 1995; Lehrer, 2003).

Some researchers claimed that number developed out of measurement contexts (Davydov & Tsvetkovich, 1991; Dickson, Brown, & Gibson, 1984; Dougherty & Venenciano, 2007; Joram et al., 2005). Others claimed that fractions lay at an intersection between measurement and number: "Measurement with units is particularly interesting developmentally because it is at the interface between counting, on the one hand, and knowledge about rational numbers on the other" (Sophian, 2002). Further, units and iteration in fraction understandings were explored in a parallel way to measurement (Sophian & Wood, 1997). The measure sub-construct of fractions "shows the significant tie between the study of fractional numbers and geometry and space" (Kieren, 1993, p. 59). Piaget's explanations of the concept of fractions was explicitly linked to length and area diagrams and non-congruent parts (Piaget et al., 1960).

Performance on measurement tasks has shown that the achievement gap widens with age. Although Grade 1 children had differing success on length tasks, the gap appeared to widen by Grade 5, resulting in almost no change in performance between the lowest performing children in Grade 1 and Grade 5 (Bragg & Outhred, 2000). In a British study, the high performing Grade 2 students were outperforming the weaker Grade 8 at mass and length tasks (Brown et al., 1995). This achievement gap in measurement understandings might have implications for the ability of children to make conceptual links to other mathematical domains.

The importance of the conceptual link between fraction and measurement concepts resulted in the development of the second research question:

- Is there an association between performance on measurement tasks and performance on fractions tasks? Is there an association between the use of the use of dynamic imagery on visualisation tasks and performance on fractions tasks?

2.1.5.9 Four levels of response in Kieren's model.

Each of the four sub-constructs could elicit different levels of understanding: ethnomathematical, intuitive, technical-symbolic, and axiomatic-deductive. The four sub-constructs (measure, quotient, operator and ratio) were not sequential and independent (Kieren, 1993). These four levels can be mapped onto the eight levels of the Pirie and Kieren model of dynamical learning: primitive knowing, image making, image having, property noticing, formalising, observing, structuring, and inventising (Pirie & Kieren, 1994a, 1994b).

As sharing was one of the earliest ways of experiencing fractions, the quotient sub-construct had *ethnomathematic* understandings that the child could draw upon when responding to questions in this context. For example sharing situations using two measure spaces such as five pizzas shared between three people could provoke an ethnomathematic response in primary school children such as "each gets a bite and Mom puts the rest in the fridge" (Kieren, 1988, p. 172). The ethnomathematic response was also evident in non-exhaustive sharing in research on early fraction knowledge (see e.g., Pothier & Sawada, 1983). Ethnomathematic understandings were percepts, represented by the p-level on Kieren's diagram (see Figure 2.5). The measure, operator and ratio sub-constructs had fewer ethnomathematic contexts that were familiar to children (Kieren, 1993). I have mapped the ethnomathematic level onto the *primitive doing* level of the Pirie Kieren model (Kieren, 1993; Pirie & Kieren, 1994a, 1994b). In this first level of learning a child might partition or share in an appropriate context.

Intuitive approaches were planned mathematical activity, firmly located in a context developed from schooled or taught knowledge (Kieren, 1988). Explanations illustrating intuitive knowledge in sharing contexts included repeated halving and dealing out of pieces (Kieren, 1993). The Dutch curriculum introduced fractions with sharing activities, Kieren noted (1993), and the intuitive understandings that this developed were useful in developing children's overall understanding of fractions. Symbols could be used to record intuitive understandings, but in that context they were linked to a context in the child's mind. I have mapped Kieren's intuitive knowledge (1993) onto the *image making* and *image having* levels of the Pirie Kieren model (Kieren, 1993; Pirie & Kieren, 1994a, 1994b). In these two levels, a child might record sharing actions "in a way that is very closely tied to results" (Kieren, 1993, p. 73).

The Rational Number Project used the term intuitive to mean "qualitative knowledge about rational number and proportional situations" (Behr et al., 1992, p. 299). Other researchers used the word intuitive to describe the reasoning pre-school children had developed from concrete experiences before school activities (see e. g., Sophian et al., 1997). In this thesis, I use *intuitive* in Kieren's sense, meaning reasoning linked to school activities.

Technical-symbolic understanding manifested itself in standard language, notations and algorithms (Kieren, 1988). Two examples of this in the quotient sub-construct were using common denominators to compare the amount per person in two different sharing situations (Kieren, 1993), or using a fractions as division understanding, that three shared between five was $3 \div 5$ or $^3/_5$. Children using technical-symbolic understanding knew the result simply by working with symbols. Clarke (2006) described a technical-symbolic response in another quotient context, the Chocolate Game, where chocolate bars were shared between people. In the measure sub-construct, the number line was a technical-symbolic representation (Kieren, 1992). I have mapped the technical-symbolic level (Kieren, 1993) onto the three levels *property noticing, formalising*, and *observing* in the Pirie Kieren model (Kieren, 1993; Pirie & Kieren, 1994a, 1994b). Kieren (1993, p. 73). They distinguished the thinking between the levels with an example of using the denominator to reason about the size of a fraction:

- Image having: "as n gets bigger, the pieces 1/n get smaller."

- Property noticing: "I can make a smaller fraction by making the denominator bigger",
- Observing: "There is no least positive fractional number",

The *image having* thinking was tied to a context, while the property noticing and observing levels were characterised by not being tied to a particular context.

A combined-level response incorporated both intuitive and technical-symbolic understandings and was described by Kieren as an engineering report (1988). In this example, eight pizzas were shared between five people and the leftover was not ignored. A child reported that each person received $1 + {}^{1}/_{2} + {}^{1}/_{5}$ (meaning a fifth of the leftover half). This explanation combined intuitive and technical-symbolic understandings.

Axiomatic-deductive knowledge was "derived through logically situating a statement in an axiomatic structure" (Kieren, 1993). I have mapped the axiomatic-deductive level (Kieren, 1993) onto the two levels, *structuring* and *inventing* in the Pirie Kieren model (Kieren, 1993; Pirie & Kieren, 1994a, 1994b). Kieren (1993, p. 73) offered the example of the addition of fractions as the "logical consequence of field properties and the nature of formal equivalence." This level of knowledge would not be routinely observed in primary school classrooms.

2.1.5.10 The use of Kieren's four-three-four model in the research literature.

Kieren has been cited extensively in the research on fractions. However, it was the five-part model used by the Rational Number Project researchers that was used to frame much of the later research, not his revised four-three-four model. There were historical reasons for this. The 1980s was the Rational Number Project's decade. They used the five part model to frame their data collection in the early 80s and published extensively over that decade. These researchers wrote the review of the fraction domain in the 1992 *Handbook of research on mathematics teaching and learning* (Behr et al., 1992). Kieren's four-three-four model had only been proposed in 1988, while in contrast the Rational Number Project had generated much data from classrooms framed in the five-part model. The researchers kept the five part model because it had been important to the research of the previous decade and had "stood the test of time" (Behr et al., 1992, p. 298). Kieren's later elaborations of the four-three-four model were still to come (1992, 1993, 1995). Publications by other researchers cited the Rational Number Project papers and so the five-part model was entrenched.

It would appear that researchers as well as teachers were unwilling to reframe the concept of part-whole. Clark, Berenson, and Cavey (2003) referred to Kieren's four-three-four model but then added the part-whole sub-construct from the Rational Number Project research, ending up with Kieren's earlier five part model. Charalambouss and Pitta-Pantazi (2006) investigated quantitatively whether the five sub-constructs were separate or hierarchical concepts (the incorrect dating of Kieren's models in their literature review did not affect their results) and did not use Kieren's later four-three-four model (1988).

Some researchers in the Dutch tradition (Streefland, 1991, 1993) or from a multiplicative framework (Confrey, 1994) did not use Kieren's model. Steffe's model for early understanding of fraction composition was based on iteratation (Olive & Steffe, 2002; Steffe, 2003). In this model, a child made a conjecture about the size/name of the fraction and then iterated the part until the whole had been filled or subdivided exhaustively. A double count misconception would generate perturbation (cognitive conflict) because it would not iterate the right number of times into the whole. For example, if a fraction bar was divided into three pieces, a half and two quarters, the conjecture that each piece was a third would be challenged by the unsuccessful iteration; a fourth would iterate four times while the half would iterate twice but neither would iterate three times to make the whole (Norton & Wilkins, 2009).

With the four of the sub-constructs in Kieren's original five-part model (1980) in common with his four-three-four model (1999, 1992, 1993, 1995), research that was within these categories, such as investigations into children's understanding of number lines (see e.g., Pearn & Stephens, 2007), investigations into pre-service teachers understanding of the operator concept (Behr, Khoury, Harel, Post, & Lesh, 1997), or research comparing students' performance across tasks from different sub-constructs (Lamon, 2007; Moseley, 2005) remained unaffected by the distinctions between the old and the new model.

Other than Kieren's elaborations of his four-three-four model, there were few investigations of the model and its use as criteria for categorising tasks and analysing children's strategies. Millsaps (2005) used the four-three-four model for her investigation of teacher knowledge and classroom practice but she explained unit-forming as "the symbolic representation *a/b, b* not 0, is a number comprised if the unit *1/b* counted *a* times" (p. 24). She linked this to Steffe's unit iteration scheme. However, I believe that the concept that she described would not be unit-forming, but rather one aspect of Kieren's concept of partitioning. This lack of further research using Kieren's four-three-four model, despite its apparent suitability for describing fraction understanding, has led to my third research question:

- Can we use Kieren's four-three-four model of fraction understanding (1988, 1992, 1993, 1995) to describe the fraction understandings of students in the present study?

2.1.6 Research methods used in measurement and fraction research.

Mathematics education emerged as a field distinct from cognitive psychology and mathematics in the 1970s. This coincided with the influence of Piaget on instructional models, particularly a belief that children had always been constructing their own meanings of school mathematics. The Rational Number Project researchers, influenced by Piaget, used teaching experiments in the early 1980s to focus on two key areas: what would a constructivist curriculum look like, and what strategies did children use on such tasks? Their approach was top down; a new curriculum was needed for teaching in a constructivist environment. They implemented that curriculum, and so their research used a teaching experiment methodology. The early 1980s were a key moment when conceptual understanding rather than procedural understanding, supported by theoretical models (Kieren, 1976; Skemp, 1976), began to drive the classroom research agenda (Behr et al., 1981). The original Rational Number Project teaching experiment was an intervention and, using one-to-one interviews, enabled the researchers to "collect" the explanations children gave, which revealed the strategies they used for fraction tasks (see e.g. Behr, Wachsmuth, Post, & Lesh, 1984). Pen and paper testing was used to establish performance norms (Lesh et al., 1983). This was less effective at capturing thinking but could be conducted on a larger scale than interviewing. Only much later did the Rational Number Project team conduct a Treatment A/Treatment B experiment to demonstrate that their curriculum was superior to traditional teaching models (Cramer & Post, 1995; Cramer et al., 2002).

The 1990s and 2000s saw the flowering of interest in observing learning and the interaction of students and teachers in classrooms. For example, one-to-one interviewing gave insights into children's explanations and strategies (Clarke et al., 2007) but could not capture learning. Such research required research conducted in classrooms (Lesh & Kelly, 2000). The outcome of a curriculum intervention was not compared to a control group, instead classroom norms and sociomathematical norms were described (Cobb & Yackel, 1996).

Constructivist teaching experiments reported individual children's learning trajectories (see e.g., Steffe, 2003). Hypothetical Learning Trajectories were developed, mapping out the developmental scope of domains (see e.g. Battista, 2006, 2007). Curriculum trajectories were elaborated (see e.g. Outhred et al., 2003). Large scale pen and paper testing added a layer of evidence to such trajectories and Rasch analysis identified key tasks and concepts (see e.g., Callingham & Watson, 2004). Longitudinal studies provided evidence for progression through stages (see e.g., Steinle & Stacey, 2004).

Ideal models of understanding in a domain were proposed with global rather than linear links in conceptual understanding (see e.g. Kieren, 1995; Lehrer, 2003).

There were cognitive psychology investigations into children's thinking continuing alongside mathematics education research. Many used clinical interviews and collected data in an experimental format, investigating the effects of variables such as age or gender (see e.g., Sophian et al., 1997).

Large scale pen and paper testing of students in state, national, and international cohorts continued. The National Assessment of Educational Progress in the United States highlighted children's problems with fractions (see e.g., Stewart, 2005) and broken ruler tasks (see e.g., Kamii & Clark, 1997). The Third International Mathematics and Science Survey revealed, for example, Australian Grade 4 students' difficulty with the inverse relationship between the unit of measure and the count (National Center for Educational Statistics, 2007).

The measurement of teachers' pedagogical content knowledge was used to provide detail to others' observations of teacher effectiveness. The content of lessons was observed (see e.g., Outhred & McPhail, 2000). Teachers' questioning and instructional style was observed in classrooms (see e.g., Gearhart & Saxe, 2004). Interviews with teachers were used to ascertain their knowledge of concepts (see e.g., Outhred & McPhail, 2000), and Post, Harel, Behr, and Lesh (1988) gave a pen and paper ratio and rates test to Grade 4 to 6 teachers using items from the National Assessment of Educational Progress. Pen and paper tests were used to assess pre-service teachers' knowledge of fractions (see e.g., Cramer & Lesh, 1988).

There was a tension in these methods between generalisability and credibility. Large scale testing provided generalisable results but was conducted using pen and paper tests which were less effective at determining student thinking. Smaller studies could provide credibility if they were conducted meticulously and produced insightful analysis, but the results were not generalisable. Design experiments provided credibility by observing learning as it took place in classrooms but were not positioned as efficient at isolating variables unlike the approaches of cognitive psychology. Clinical interviews could provide insight into children's explanations but could not describe learning. Learning trajectories were often based on performance of students at different grade levels, but only longitudinal studies or constructivist teaching experiments could confirm that individual children moved through the levels of understanding. It was not that methodologies were right or wrong, but that they could be used strategically to build up a multifaceted understanding of mathematics education in the fields of length and area measurement, visualisation, and fractions.

2.2 Research Questions

The research questions which have developed out of this review of the literature reflect my interest in the theoretical underpinnings of the fraction domain, and an interest in children's explanations when thinking about mathematical constructs:

- What strategies are evident in students' explanations of their thinking in a one-to-one task-based interview?
- Is there an association between performance on measurement tasks and performance on fractions tasks? Is there an association between the use of the use of dynamic imagery on visualisation tasks and performance on fractions tasks?
- Can we use Kieren's four-three-four model of fraction understanding (1988, 1992, 1993, 1995) to describe the fraction understandings of students in the present study?

2.3 The practical significance of the present study

These three research questions can be linked to very broad scholarly concerns in the mathematics education field: if we embrace constructivist learning then we are faced with the complexity of the role of the teacher in responding to the variety of correct and incorrect strategies that a class of children bring to every task.

The elaboration of strategies (correct strategies and misconceptions) is part of developing the ability of members of the teaching profession to embrace constructivist learning contexts. This has practical significance. The focus for system improvement in Victoria is on the quality of individual teachers in classrooms. Many regional offices of the Department of Education and Early Childhood Development employ maths and literacy coaches to support school improvement in their areas. There are also regional professional development initiatives delivered in partnership with University researchers. The elaboration of strategies and misconceptions of Australian primary school children on measurement and fraction tasks could be immediately utilised in these departmental projects designed to support teachers in constructivist classrooms.

If we believe there to be a theoretical conceptual link between fractions and measurement, then uncovering this association in children's performance raises issues for the design of curriculum. The practical significance of the second research question regarding curriculum (measurement and fractions) will initially be at a school level because the Australian National Curriculum has recently been written and is about to be trialled in schools around the country. However, in Victoria, individual schools are responsible for the classroom level curriculum and individual teachers, who have a large degree of autonomy, are responsible for the links they help their students make between fractions and measurement. It is unusual to teach from one textbook in primary schools, and neither school in which I taught had textbooks on the book list for students to purchase (except a handwriting practice booklet). There are broad curriculum outcome statements to which teachers are accountable, but the activities, the order, and the pedagogical style are decided at an individual or school based level. Hence teachers could begin helping students make links between fraction and measurement contexts immediately.

Using one-to-one task-based interviews is a normal classroom pedagogical tool, as described in the background to the study, but they need to be supported by a sound connection to a theoretical framework. This enables teachers to use the detailed information that they generate to inform their practice. Investigating the explanatory power of Kieren's four-three-four model contributes to this larger picture of making one-to-one task-based interviews a usable pedagogical strategy and could have practical application to the development of teaching resources, the training of pre-service teachers, and the professional development of teachers. The Department of Education and Early Childhood Development recommends a research-based (Kieren's five-part model) one-to-one task-based interview as formative assessment for Grade 5 and 6 students (2009b). If the four-three-four model has explanatory power at the primary school level then its use as a clear domain level theory that makes sense of the variety of tasks and students' responses would have practical significance for classroom teachers and the development of further resources supporting this fraction interview.

Chapter 3: Methodology and Methods

This Methodology and Methods chapter describes what was done in this investigation and why it was done. The method of a one-to-one task-based interview was used in order to provide data to investigate the three research questions that had come out of the review of the literature:

- What strategies are evident in students' explanations of their thinking in a one-to-one task-based interview?
- Is there an association between performance on measurement tasks and performance on fractions tasks? Is there an association between the use of the use of dynamic imagery on visualisation tasks and performance on fractions tasks?
- Can we use Kieren's four-three-four model of fraction understanding (1988, 1992, 1993, 1995) to describe the fraction understandings of students in the present study?

3.1 Methodology

The present study had the three goals of research as identified by Neuman (2003): explanatory, descriptive, and exploratory. Descriptive research has been used to make a highly detailed and accurate picture of the object or subject or process chosen for observation. Investigating students' strategies is descriptive research. Exploratory research has been used to determine the feasibility of conducting further research. The investigation into associations between fraction understanding and measurement understanding is an exploratory study. Explanatory research has been used to confirm or refute the occurrence of phenomena predicted by a theoretical model. Investigating the explanatory power of Kieren's four-three-four model of rational number knowing (1988, 1992, 1993, 1995) for describing students' understanding of fractions in the upper primary school is explanatory research.

3.1.1. Interpretivism.

It is an assumption of the present study that it is possible to discover new phenomena or reclassify existing interpretations of mathematical behaviour. The research questions were framed by such an assumption: that a link between fractions and measurement might be found, or that a misconception might be discovered or reinterpreted. Mathematics education researchers had termed this *ontological innovation* (diSessa & Cobb, 2004).

Interpretive research has often been called qualitative research (Neuman, 2003). Observable behaviours cannot be understood without understanding the social rules that mediate that behaviour. For example, "The raising of my hand could be a signal for the revolution to take place, a gesture of welcome, or the seeking of attention. It all depends on what was intended" (Pring, 2005, p. 96). Theories that fell under the umbrella of interpretivism could be "a basis for considering how what is unknown might be organised" (Silverman, 2000, p. 78). In mathematics education research, Thompson (1982) has illustrated this legitimate field of investigation in one research question: "what is the problem that this student is solving, given that I have attempted to communicate to him the problem in my mind". This overarching interpretive theoretical perspective framed the investigations of the present study.

Three streams of interpretivism have been identified: hermeneutics, phenomenology, and symbolic interactionalism. Hermeneutics uses close reading of a text to "discover meaning embedded within text" (Neuman, 2003, p. 76). Phenomenology endeavours to present the subjects' own understanding of their actions in the social context in which such actions were made. For Patton (2002) the foundational question of symbolic interactionism was: "What common set of symbols and understandings has emerged to give meaning to people's interactions?" (p. 112).

Symbolic interactionsism enables me to report the students' explanations, interpret their strategies and overlay this with an interpretation determined by Kieren's four-three-four model. The students themselves did not frame their understanding of fractions in terms of Kieren's model. Transferability, credibility, confirmability and dependability (Denzin & Lincoln, 2008), authenticity (Neuman, 2003), usefulness (Silverman, 2000), and credibility, rigor, and integrity in analysis (Patton, 2002) have been identified as the truth claims made in qualitative research. Qualitative research could use descriptive statistics, for example, frequencies of success or frequencies of correlations. Conclusions are drawn from the qualitative research data in the present study but claims for causality are not made.

The conceptual framework that I, as a researcher, have brought to the present study has mediated my observations and analysis of the data. My undergraduate training was in ethnographic history. I later obtained a Diploma of Education and taught a combined Prep/1/2 class, Grade 1/2 classes and Grade 5/6 classes in State Government schools in Melbourne, Australia. I completed a Masters degree by coursework in Early Numeracy at the Australian Catholic University in my fourth and fifth years of teaching while working full time as a primary teacher. This move into the new discipline and "fieldwork" of Education built on my previous training as an historian. I hope that my ethnographic historian's ear will enable me to listen, through their explanations, to the students' understandings of the tasks.

3.1.1.1 Theories of learning.

A constructivist theory of learning underpinned Kieren's description of fraction understanding stratified into ethnomathematic, intuitive, technical-symbolic, and axiomatic-deductive engagement. It was "congruent with reflective abstraction" (Kieren, 1992, p. 349), dynamic and nonlinear (Kieren 1993; Pirie & Kieren, 1994b). When faced with more difficult concepts, students folded back to earlier understandings (Pirie & Kieren, 1994a; 1994b). The recursive nature of understanding maintained access to all these levels rather than discarding them as more sophisticated understandings were developed.

All constructivist theories of the learning of mathematics had a constructionist (non-positivist) epistemology, but they had different emphases on the processes of learning. The constructivism theories are theories of learning rather than of teaching. In their critique of constructivism, Lesh, Doerr, Camona, and Hjalmarson (2003) described the pedagogical implications of constructivism. Knowledge was actively constructed by the student and not simply passively received from the teacher. Constructivists looked at constructs and trajectories. Changes in concepts came about when the learner had to resolve conflicts, and students' reasoning was sought after the problem had been solved.

Social constructivists and radical constructivists positioned their investigations differently (Cobb, Stephen, McClain, & Gravemeijer, 2011). The focus of the present study is mathematical interpretations and reasoning (see Table 3.1). While not a study of classroom interaction, the present study has been influenced by the underlying beliefs of social constructivism. The term *classroom norms* described explicit and implicit expectations of teacher/student interaction. For example, Cobb and Yackel (1996) described Grade 2 students who were expected to explain their own strategies rather than guess what the teacher wanted them to say in classroom discussions. Sociomathematical norms codified the classroom practices of explanations. For example, "what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation" (Cobb & Yackel, 1996, p. 178). These normative standards for argumentation could apply to other mathematical activities such as reasoning with tools and inscriptions (Cobb, 2002).

Table 3.1

Corresponding Terms Between Social Constructivism and Radical Constructivism (Cobb et al., 2011)

Social Perspective (Social Constructivism)	Psychological Perspective (Radical Constructivism)
classroom social norms	beliefs about own role, others' roles, and the general nature of mathematical activity in school
sociomathematical norms	mathematical beliefs and values
classroom mathematical practices	mathematical interpretations and reasoning

For both social constructivists and radical constructivists, misconceptions were not nonsensical: they were understandings generalised inappropriately from one context to another. They were characterised in constructivist theories as "faulty extensions of productive prior knowledge" (Smith, diSessa, & Roschelle, 1993, p. 152). They had a basis in correct mathematical thinking but had been generalised inappropriately. Learning was concerned with "learning to use what you already know in either wider or more restricted contexts" (p. 136). The strength of misconceptions and their resistance to teaching lay in both this rootedness in being successful in a particular context and also because "Some misconceptions are powerful enough to influence what students actually perceive" (p. 162). For the descriptive investigation in the present study, this theorising of misconceptions was more useful than, for example, the cognitive task analysis approach from psychology (see e.g. Crandall, 2006).

Ernest described an example of the conventions in mathematics that young children encounter: "3 divided by 4 ($^3/_4$) is at first an impossible task. Later it is not only a possible task, but $^3/_4$ names the answer to it, i.e., becomes a new kind of semiotic object, a fractional numeral" (2006, p. 76). Fraction notation has its own semiotic conventions that link symbols to rational number sub-constructs. For social constructivists, mathematics is a culturally transmitted (or emerging), internally consistent, highly resilient discourse that describes pattern, order, magnitude, space and relationships.

For Piaget, knowledge was actively built: assimilation was not bringing material from the environment into the organism but treating *new* material as an instance of something *known* (Von Glasersfeld, 1995). For example, Piaget, Inhelder and Szeminska (1960) described children's responses to a conservation of area task where two identical rectangles were rotated. Child A's response was not generalisable: he said that they were the same because he compared them. Child B, on the other hand said that they were the same because he "regards the new shape as simply the outcome of a transformation rather than a new area to be compared with the original" (p. 285). The new situation was recognised as an example of something already known. Children can assimilate ideas but not realise that these ideas contradict their schema. Von Glasersfeld offered the analogy of a card punch machine that sorts cards with three specific holes punched from the rest. However, cards with extra holes punched are included too, as long as the three key holes are punched; just as the organism "remains unaware or disregards whatever does not fit into the conceptual structure it possesses" (1995). The most common cause of accommodation was linguistic iteration, particularly teaching: children needed to hear something that contradicted their schema several time before it was noticed that it could not be assimilated.

If the child could not assimilate the new material there would be perturbation and the formation of a new scheme as accommodation occurred (Von Glasersfeld, 1995). Reflective abstraction is a branch of constructivism, grounded in Piaget's later work and describes not perturbation, but reflection, as the key mechanism in resolving cognitive conflict. Reflective abstraction had been used to describe fraction learning (Simon, Tzur, Heinz, & Kinzel, 2004). For adherents to a reflective abstraction interpretation of learning, other constructivist theory generated a learning paradox because "a child who has no conception of multiplication will not perceive multiplicative relationships in any situation, including those considered by the teacher to transparently display multiplication (e.g. an array)" (Simon et al., 2004, p. 310). Children had "no access to a mathematics that is independent of their ways of knowing" (p. 306). Being shown something to contradict a misconception assumed that the child could see the mathematical metaphor in the new materials. Perturbation might produce accommodation, but how could perturbation be generated if children could not see what they did not know?

In Simon, Tzur, Heinz, and Kinzel's research a child was shown two identical square pieces of paper and told they were cookies (2004). One cookie was then cut in half vertically and the other diagonally. The child was then offered the choice of either of the two identical vertical halves, followed by the choice of either of the two diagonal halves, lastly followed by the choice between a vertical half or a diagonal half. One child opted for the half cut on the diagonal over the half cut vertically because "it is bigger" (p. 315). In order for the child to move on from the conception that the cutting made two distinct subsections, not two equal halves, the researchers argued, a pedagogical intervention needed to take place in which the child reflected on his or her understanding. Reflective abstraction occurred when the child reviewed the sequences of actions and results and looked for patterns. Activities involving partitioning and iteration were proposed by the researchers as suitable activities to prompt reflective abstraction, in order for the child to develop a stronger understanding of the cookie problem posed above (Simon et al., 2004).

One outcome of radical constructivist research was descriptions of the learning process of moving from one stage to the next (the how) and also the sequence of constructs that defined those stages (the what). Hypothetical learning trajectories were one of the descriptions of construct development. They described, not a child's individual constructions of mathematical ideas, but key conceptual ideas that children understood about a mathematical domain (Simon & Tzur, 2004). They were hypothetical because they could not predict actual children's learning pathways (Simon, 1995). Hypothetical learning trajectories elaborated the fine detail of concept development in a mathematical domain, but in contrast to Piaget, they were not linked to age (Steffe & Weigel, 1996). The teaching component of hypothetical learning trajectories was also important: the learning goal, the learning activities, and the thinking and learning in which the students might engage were all parts of a hypothetical learning trajectory (D. Clements & Sarama, 2004). Steffe (2004) reported a case study of an actual learning trajectory of two children and their teacher, engaged in learning experiences related to equivalent fractions, because he believed that real learning trajectories needed to be documented as part of the research into hypothetical learning trajectories.

Kieren's model for fraction understanding (1995) and Lehrer's key concepts of measurement (2003) were not hypothetical learning trajectories. In the present study I was not seeking to test or devise a hypothetical learning trajectory. I was interested in children's explanations and how they co-ordinated their understandings. To support this interpretive approach, I chose to use one-to-one task based interviews as the method of investigation.

3.1.2 Advantages and disadvantages of task-based interviews.

There were three main questions to answer in order to justify the use of a one-to-one task-based interview in the present study:

- why interviews and not classroom observations?
- why task-based?
- why one-to-one?

There were several research methods that could have been used to collect data to answer some of the research questions, but the one-to-one task-based interview could be used to investigate all three questions and was a practical method for a single researcher.

Why were interviews used and not classroom observations?

The main criticism of the use of interviews in educational research was the "white room effect" (diSessa, 2007). The white room effect caused people to behave in ways that they normally would not, because they were in an unfamiliar environment. As an interpretive device, interviews had the disadvantage of generating knowledge about individuals in a limited context (diSessa, 2007). On the other hand, Ginsberg argued (1997) this non-classroom environment could prompt children to attempt to think differently about tasks because, unlike classroom questioning, they were asked questions that they were not always expected to be able to answer immediately. In other research, the interview format itself has been normalised as part of classroom practice, for example in cognitively guided instruction (see e.g., Fennema, Franke, Carpenter, & Carey 1993) or in the teaching experiments of the Rational Number Project (see e.g., Behr, Wachsmuth, Post, & Lesh, 1984). The use of the Early Numeracy Interview was suggested but not mandated in 2001 (Department of Education & Training, 2001). One-to-one interviews in Victorian schools have been a useful pedagogical practice (McDonough, B. Clarke, & Clarke, 2002; Clarke, Mitchell, & Roche, 2005), not just a research tool. For students, interviews have been a familiar, but not frequent, way of interacting with a teacher. Although students' behaviour and even thinking might be different in an interview, because one-to-one task-based interviews were part of *school* learning they did not have the same *white room* effect as clinical interviews in other settings.

Interviews have been a useful method for assessing the viability of a theoretical model (Clement, 2000). They could provide empirical support for predictions based on a theory. In investigating the plausibility of Kieren's four-three-four model, the framework had to have explanatory power when used with real students' explanations. Interviews usually generated data that could sustain interpretive analysis (Clement, 2000). The present study might be considered a pilot study for a later instructional design research project.

Why were the interviews task-based?

Task-based interviews gave researchers an opportunity to gather rich data on children's descriptions of their mathematical strategies (diSessa, 2007). For example, there has been greater emphasis in using mental computation to "uncover children's thinking rather than covering it up or generally ignoring it", not merely to test speed and accuracy (Sparrow & McIntosh, 2004, p. 155). In contrast, Lesh and Kelly argued (2000), teaching experiments provided better access to individual students' *learning.*

The students engaged with an interviewer and with the task thus enabling a topic to be explored in depth (Goldin, 2000). A prompt for an explanation could be given by the interviewer and this could generate further data. Ginsberg (1997) suggested four interviewer strategies to elicit further information to investigate whether the child was giving the right answer for the wrong reason or giving incorrect answers but had greater understanding than was indicated by the answer: rephrase the question, modify the task, probe, or offer a counter suggestion. These interviewer techniques could be considered a "repair" in conversational analysis terms (Wooffitt, 2005).

For appropriate task selection it is important to have criteria (Ginsberg, 1997). In the present study several key aspects of Kieren's four-three-four model (1988, 1992, 1993, 1995) provided the criteria. A research based categorisation of measurement tasks based on Lehrer's key concepts for measurement (2003) was also used. The uniformity of data collection in interviews was a strength of the method. The children's responses to mathematical tasks could be coded (Clement, 2000). The data can then be used to calculate the frequencies of success or the frequencies of particular strategies The use of mathematical tasks in the interview protocol enabled investigations into children's explanations and strategies.

Why were the interviews one-to-one?

The one-to-one task-based interview was able to detect whether a student gave the right answer for the wrong reason (M. Clements, 1980; M. Clements & Ellerton, 2005) or the wrong answer despite full or partial understanding (M. Clements & Ellerton, 1995). Both students' explanations and their answers were evaluated. Self-correction was possible in one-to-one interviews for two reasons. Firstly, in explaining the answer, a student could correct a careless mistake (M. Clements, 1982). Secondly, the child had time to think without being influenced by other students' answers. Interviews guaranteed wait time (Ginsberg, 1997), or "free problem solving" (Goldin, 2000), if the researcher desired it. In a one-to-one interview the student did not need to adjust their explanation for the understanding of another student. There were certain assumptions that the student could make about the extent of the teacher's understanding of the task: the student had to explain their own thinking, but did not have to check whether the teacher "knew the maths".

The explanations that the child offered were always in the context of a student talking to a teacher because the interview format was similar to other assessment interviews conducted in Victorian primary schools. I was introduced to the students as a teacher, and so in order to keep the idea that the students positioned me as a teacher in their responses, I have referred to them as students rather than children in my reporting of results.

In an interview, the interviewee transferred some authority to the interviewer, particularly in regard to task choice (diSessa, 2007). Therefore, it was not possible to gain insights into aspects of children's thinking that the researcher had not chosen to investigate because the student was unlikely to volunteer this information: it was tacitly agreed that the interviewer was defining the field of inquiry.

The interviewing of individuals "sensitised" researchers to new observations, although it could take several individual's responses before researchers recognised something they had not been seeking in the explanations (Clement, 2000). Similarly to Thompson (1982), Ginsberg (1997) suggested that "a useful approach to interpreting a "wrong" answer is to discover the question to which the child's answer is correct" (p. 12). The individual nature of the interview had advantages for identifying undocumented mathematical strategies.

Thus one-to-one task-based interviews were chosen as the method for the present study for several reasons. This method enabled tasks, chosen using a research based criteria, to be offered to many students. The data could be analysed for the specific strategies that students used, for correlations between fractions and measurement understandings, and for the explanatory power of Kieren's model of fraction knowledge. Interviews enabled the gathering of data on children's explanations in a context that was familiar, albeit not necessarily common, for the students.

3.2 Method: one-to-one task-based interview

A one-to one task-based interview was developed assessing length and area measurement, multiplication, and fraction understanding. There were also tasks to assess students' use of dynamic imagery. The interview had 65 tasks and took up to three hours (over several sessions) to complete. 88 Grade 6 students were interviewed and their answers and explanations recorded on a record sheet, audio taped, and more than half were videotaped.

In this section I describe

- the participants and their schools,
- the interview used in the present study, and
- how the interview was conducted, and the protocols for the collection and coding of data.

3.2.1 The Present study.

This section describes the students, their schools and the local context in specific detail. However, details have been withheld if they would be identifying. Four government Prep to Grade 6 primary schools in Melbourne, Victoria, Australia were chosen for the study. They were given the pseudonyms Casuarina Primary School, Wallaby Flat Primary School, Lone Pine Primary School, and Four Hills Primary School. Fourteen Grade 6 students from Casuarina Primary School and Wallaby Flat Primary School were interviewed for the pilot study in Term 4, 2007 and 88 Grade 6 students from Wallaby Flat Primary School, Lone Pine Primary School, and Four Hills Primary School were interviewed for the main study (the present study) in Term 1 and 2, 2008 (see Table 3.2). Two Grade 5 students from Casuarina Primary School were also interviewed using the main study interview protocol in 2008 in order to assess whether the tasks would be appropriate for a range of students from the beginning of Grade 5 to the end of Grade 6.

Table 3.2

Profile of Participants and Schools for the Main Data Collection 2008

	Schools		
	Wallaby Flat PS	Lone Pine PS	Four Hills PS
Number of participants	17	22	49
Gender breakdown	7 girls 10 boys	9 girls 13 boys	27 girls 22 boys
School size	200-300	200-300	Over 600
Data collection (2008)	Feb 11 to March 7	March 4 to April 21	April 22 to June 24
Grade structure	5/6	5/6	6
Class groups sampled	2	3	5
Socio economic profile 2009	mid-high	high	low-mid
Proportion of students with English as a second language	mid	mid	high
% return rate of consent forms	60-70%	60-70%	35-45%

The four schools (including the pilot schools) were in the Northern Metropolitan Region of Melbourne, Australia. All students who returned consent forms for the pilot study and the present study were interviewed. In 2008 there were five different socio-economic levels used to categorise government primary schools in Victoria: low, low-mid, mid, mid-high, high (Department of Education and Early Childhood Development, 2009c) and three schools with different socio-economic levels were chosen for the present study. Between 40% and 65% of the students in Grade 6 in each school were interviewed. The students came from 10 different class groupings, each with their own classroom teacher. Wallaby Flat Primary School and Lone Pine Primary School had composite Grade 5/6 classes while Four Hills Primary School had straight Grade 6 classes.

The school year in Australia runs from late January to December, and in Victoria is broken up into four terms. In Victoria, the first year of school is called Prep and students begin in January if they have turned five or will do so by April 30 in this first year of schooling. Grade 6 students in Victoria turn 11 by April 30, so are aged 11-12 years old. Students spend Prep and Grade 1 to 6 in the primary school and 7-12 in the secondary school. In the present study the Grade 6 students were in classes of around 25 students which would be considered a normal size. In February 2008 there were 44,134 Grade 6 students in Government schools in

Victoria (Department of Education and Early Childhood Development, 2008). Further summary statistics are available in Appendix C.

Government schools in Victoria are funded by the State Government and use the curriculum documents developed by the Department of Education and Early Childhood Development. Australia uses the metric system and students' length measurement activities using formal units would be with millimetres, centimetres and metres. Many scales referred to in everyday life use decimal notation. For example, temperature is measured in degrees Celsius and mass in kilograms.

The state government department responsible for government schools in Victoria has had several name changes in the past fifteen years:

- Department of Education & Training (1995-2006),
- Department of Education (2007), and
- Department of Education and Early Childhood Development (2008-2011).

The Victorian Curriculum and Assessment Authority is a separate but related State Government body.

3.2.2 The instrument.

A pilot interview was conducted with 14 Grade 6 students (Term 4). It was used to

- identify any confusing wording in the tasks
- confirm the order of tasks in some length and area categories
- check timing
- confirm that the interview was suitable for Grade 5 and 6 students.

No child was correct on every task and no child was incorrect on every task.

The interview used in the present study was then further developed and refined from the pilot interview protocol. This was then used with 88 Grade 6 students in Terms 1 and 2 of the school year. This section of the chapter describes how the interview was conducted and identifies the nature of the data and how they were recorded. The interview tasks (questions, task cards, referencing) are included in full in Appendix A.

Tasks were chosen to assess the students' understandings of length and area measurement, dynamic imagery, multiplication, the equivalence concept of fractions, and the measure, quotient, operator, and ratio sub-constructs of fractions. As the measure sub-construct and equivalence concepts were the particular focus of the present study, more tasks were used to assess these aspects of fraction understanding than the other sub-constructs. The concepts of partitioning and unit-forming were not directly assessed in the interview, but tasks were chosen that might reveal whether students drew on partitioning, equivalence and/or unit-forming in the measure and quotient sub-construct contexts. Kieren's four-three-four model of fraction knowledge (1988, 1992, 1993, 1995) and Lehrer's key concepts for measurement (2003) were the domain level theories that informed the criteria for task selection.

3.2.2.1 Multiplication.

The multiplication and division section of the interview was offered to the students first. The tasks were taken from the number section of the Early Numeracy Interview multiplication and division section (Department of Education & Training, 2001). The Tennis Balls task was the first task in the interview used in the present study. This task was not difficult and was used to settle the student into the interview and establish the classroom norms of the interviewer listening to a response, asking how the student worked out their answer and not directing the student's thinking through prompts or teaching. I did not follow the same protocol as the Early Numeracy Interview which was to continue as long as the student answered correctly. Instead, I offered all of the tasks after the Tennis Balls task, to each student in the present study.

The multiplication and division tasks had been used extensively in another research project with Grade Prep to 2 students (Clarke et al. 2002), and later with the some of the same cohort in Grade 3 and 572 of them in Grade 4 (Clarke, 2005). The interview questions were also used successfully with the 323 of the cohort in Grade 6 in a follow up project. The tasks had been suggested as a formative assessment tool for use in Victorian Primary Schools (Department of Education & Training, 2001).

3.2.2.2 Fractions.

Kieren's four-three-four model for an ideal fraction knowledge described constructs and also levels of engagement with those constructs (1988, 1992, 1993, 1995). It was chosen as the model of fraction knowledge that positioned the data collection and analysis of data in the present study because it

- elaborated the measure sub-construct as more than proficiency with number lines,
- could theorise an association between measurement understanding and fraction understanding,
- pedagogically made a space for equivalence, partitioning and unit-forming,

- further delineated the ethnomathematic, intuitive, and technical-symbolic levels of understanding of the sub-constructs,
- attempted to counteract the double count misconception by a reframing solution rather than a word of caution, and
- was a model that assumed many pathways to fraction understanding rather than one hypothetical learning trajectory.

Tasks were selected to assess the concept of equivalence, and the sub-constructs measure, quotient, operator and ratio. Continuous (length and area), discrete and symbolic contexts were represented.

3.2.2.2.1 Equivalence.

The concept of equivalence was assessed using area and length diagrams, concrete materials, and symbolic representations in tasks similar to others in the research literature (see e.g. Baturo, 2004; National Center for Educational Statistics, 2007). Equivalence tasks were specifically designed to assess the students' recognition of $^1/_4$, $^1/_6$ and $^2/_3$ in length and area diagrams (see Figure 3.1). Equivalence was raised as a possibility in the Fraction Pair task by the wording of the question asked by the interviewer, "please point to the larger fraction or tell me if they're the same". The addition of the words *or tell me if they're the same* was an adaption made to this task for the present study in order to cue the students into considering equivalence.

Equivalence:

- Continuous
    - Length
        - Fraction Sort (Q. 19t)
    - Area
        - Fraction Sort (Q. 19c, n, r, s, v, w)
        - Crossroads (Q. 28)
- Discrete
    - Fraction Sort (Q. 19i, j, k)
    - Golden Beans (Q. 21b, d)
- Symbolic
    - Fraction Pairs (Q. 22b), and after data collection: Q. 22f
    - After data collection Q. 26c

Figure 3.1. Classification of tasks for the equivalence concept of fractions.

3.2.2.2.2 Measure.

In the research literature the measure sub-construct of fractions was assessed using:

- number lines (Kieren, 1992, Lamon, 1999, Ni, 2000),
- area diagrams (Kieren, 1992),
- length contexts of measuring (Kieren, 1992, Lamon, 1999), and
- the comparison of the relative size of fractions (Kieren, 1993; Lamon, 1999, Ni, 2000).

Kieren had suggested that some aspects of the part-whole character of fractions could be reframed in the measure sub-construct (1993). Non-equal-parts area diagrams had been categorised as part-whole in the research literature (see e.g., (Clarke et al., 2007; Heinz, Kinzel, Simon, & Tzur, 2000) but were used in the present study to assess one aspect of the measure sub-construct. Number lines, and length and area diagrams, and symbolic inscriptions of fraction size comparisons were used to assess the measure sub-construct of fractions, with tasks adapted from the research literature (see Figure 3.2). Discrete inscriptions were not used for the measure sub-construct tasks because they have been categorised by Kieren (1992) as ratio tasks drawing on the partitioning concept. Both length and area diagrams were used.

Measure

- Continuous
  - Length
    - Number lines (Q. 16)
    - Tightrope Walker (Q. 15)
    - Density (Q. 25)
    - Puff Machine. (Q. 23)
  - Area
    - Fraction Sort (Q. 19a, b, e, g, h, l, m, o, p, q, u,)
    - Fraction Pie (Q. 14)
    - Fold Me a Quarter (Q. 13)
- Discrete
  - (not used in measure construct, discrete "part-whole" can be thought of as early ratio)
- Symbolic
  - Fraction Pairs (Q. 22)

Figure 3.2. Classification of tasks for the measure sub-construct of fractions.

The number line questions had been designed to assess whether students could both make partitions and read partitions. Two contexts were also used; number lines labelled 0 to 1, and number lines that were labelled greater than 1 (see Table 3.3).

Table 3.3

The Selection of Number Line Tasks to Represent Research-based Criteria

	Number lines 0 to 1	Number lines 0 to >1
Make partitions	16a	Proper fractions: 16c, Improper fractions:16b
Read partitions	16e, 16f (non-equal-parts)	Improper Fractions: 16d, 16g, 16h

One example of a deliberate choice about diagram construction in the measure sub-construct was the Fraction Pie Task (Q. 14). In the literature, the sixth to be identified was on the left hand side of the image (Cramer et al., 1997; Clarke et al., 2007; Mitchell, 2005) (see Figure 3.3, left). In the present study I reversed the image so that the more difficult fraction part to identify was at the intuitive zero-point; rotating clockwise from 12 o'clock (see Figure 3.3, right). This was to give students the best chance of identifying the fraction part. In addition, I altered the labelling of Parts A and B on the diagram so that they corresponded to Parts A and B of the task.

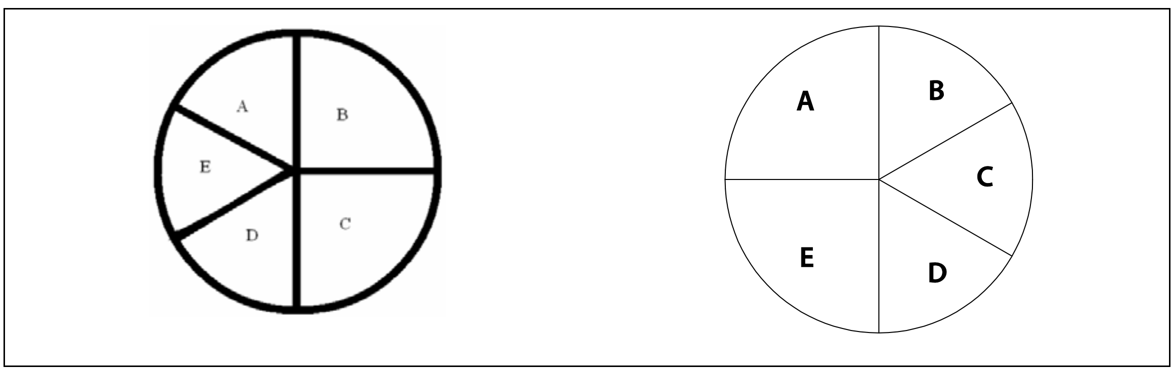


Figure 3.3. Fraction Pie task diagram used in the research literature (left) and the diagram used in the present study (right).

3.2.2.2.3 Quotient.

The quotient sub-construct was assessed using a sharing situation similar to sharing tasks in the research literature using pizzas and people (Clarke et al., 2007; Kieren, 1988, 1993; Lamon,1999) or people and tables (Streefland, 1991). Length and area representations were used, but pizzas were not chosen as the context because of the concern that children had preconceived ideas about the number of pieces into which they could be divided (see Figure 3.4).

Quotient sub-quotient

- Continuous
  - Length
    - Sharing Custard Tarts and Liquorice (Q. 20a, d; liquorice tasks)
  - Area
    - Sharing Custard Tarts and Liquorice (Q. 20b, c; custard tart tasks)
- Discrete
  - (not used, except for whole number division with no remainder. For example in multiplication and division section)

Figure 3.4. Classification of tasks for the quotient sub-construct of fractions.

3.2.2.2.4 Operator.

Three types of tasks assessing the operator sub-construct had been noted in the research literature:

- an area context using concrete materials (pattern blocks)
- the nominal context, "of" questions, such as two thirds of nine (see e.g., Clarke et al., 2007), and
- fraction multiplication (see e.g., Behr et al., 1997).

these three aspects of the sub-construct were assessed (see Figure 3.5).

Operator sub-construct

- Continuous
  - Pattern Blocks (Q. 17)
  - If area model used in Simple Operators (Q. 18d and e) when pen and paper permitted)
  - after data analysis, Fraction Pie (Q. 14b)
- Nominal
  - Simple Operators (Q. 18a, b, c)
- Symbolic
  - Fraction Algorithms (Q. 26e)

Figure 3.5. Classification of tasks for the operator sub-construct of fractions.

3.2.2.2.5 Ratio.

The ratio sub-construct was assessed using a classic Piagetian task, calculating the food needed for fish of different lengths, (Piaget, cited in Resnick & Singer, 1993). This task had been replicated in the literature with slightly different representations (Hart, 1981; Clark & Kamii, 1996). In the present study, the context was four bookworms who ate different numbers of books according to their length (see Figure 3.6). In Kieren's reframing of the part-whole concept into the four sub-constructs, discrete "part-whole" diagrams and contexts could be thought of as early ratio understandings, and the discrete category of ratio tasks included some questions that could be thought of as traditional discrete part-whole diagrams.

Ratio

- Continuous
  - Length
    - Bookworms (Q. 12)
  - Area
    - No task offered
- Discrete
  - Fraction Sort (Q. 19f, x)
  - Golden Beans (Q. 21a, c)
  - Show Me Thirds (Q. 27)

Figure 3.6. Classification of tasks for the ratio sub-construct of fractions.

3.2.2.3 Measurement.

There were three criteria for measurement tasks:

- the four concepts, attribute, additivity, units and proportionality;
- a differentiation between conceptual tasks and tools and procedures tasks; and
- the two contexts of length and area.

The four concepts attribute, additivity, units, and proportionality were based on Lehrer's (2003) key concepts but my synthesis also included conceptual elaborations by Outhred and Mitchelmore (2000), Barrett and D. Clements, (2003), D. Clements (1999), Battista, D. Clements, Arnoff, Battista, and Borrow (1998), Bragg and Outhred (2000; 2004), and Pettito (1990) (see sections 2.1.2.1 (the concept of attribute), 2.1.2.2 (the concept of additivity), 2.1.2.3 (the concept of units), and 2.1.2.4 (the concept of proportionality). Length and area were the most common contexts used in fraction tasks. Volume and angle, the other spatial measures, were not included in the assessment of measurement concepts, nor were non-spatial measures.

The abbreviations used to identify these categories begin with a distinction between conceptual (C) and tools and procedures (TP) tasks. The next section of the abbreviation indicates the relevant key concept, for example attribute (AT), additivity (AD), unit (UN), or proportionality (PR). The last section of the abbreviation indicates whether the task or tasks used length (L) or area (A) diagrams. For example, the tools and procedures (TP) task for the key concept of units (UN) using an area (A) context was TPUNA.

The terminology "conceptual tasks" and 'tools and procedures" that is used in the present study is based on the distinctions made by Skemp (1976) and Kieren (1976) and is comparable to the definitions of conceptual and procedural understanding (Heibert & Carpenter, 1992). The same interview was offered to all students in the present study, but because of the use of entry-level tasks in some of the measurement categories, there were seven instances in the interview protocol where different follow up tasks could be offered; a harder or easier task. There were conceptual tasks assessing the four concepts of measurement in both length and area contexts (see Figure 3.7).

Attribute

- Length (CATL)
  - Similar shapes (Q. 36g)
- Area (CATA)
  - Similar shapes (Q. 36h)

Additivity (CADL)

- Length
  - Straightening wires (Q. 43)
  - Freddo (Q. 41) *entry-level task*
  - Footy Card (Q. 42)
- Area (CADA)
  - If Fold Me a Quarter Incorrect, Square To Triangle Sequence (Q. 34) offered
  - Missing oval (Q. 35, the count only) if not already correct count on Staircase Array (Q. 62, pen and paper after Q. 12)
  - Area Calculation, Half Rectangle (Q. 33) *entry-level task*
  - Area Calculation, Triangle (Q. 53)

Unit

- Length (CUNL)
  - Using Paper Clips to Measure (Q. 40a)
  - Keyboard (Q. 39) *entry-level task*
  - Swimming pool (Q. 65)
- Area (CUNA)
  - Cuisenaire Array (Q. 48)
  - Array With Leftovers (Q. 46) *entry-level task*
  - Packing boxes (Q. 47)

Proportionality

- Length (CPRL)
  - Using Paper Clips to Measure (Q. 40b)
  - Steps (Q. 44) *entry-level task*
  - Choosing Rulers(Q. 45)
- Area (CPRA)
  - Draw Your Own Array (Q. 38b)
  - Four Triangles (Q. 37)

Figure 3.7. Classification of the conceptual tasks of the concepts of measurement.

The pilot study had been used to ascertain whether the entry-level task protocol would give valid results. The students were asked all three CADL tasks in the pilot study; the Straightening Wires task, the Freddo task; and the Footy Card task. There was a range of performance by the students on the three tasks: two students scored 0, four students scored 1, four students scored 2, and four students scored 3. All of the students' results followed the entry-level sequence: no student was incorrect on a task lower than the highest one at which they were successful. As the entry-level sequence was validated by the pilot study in which students were asked all three questions, it was decided in the present study that students would be asked two questions (entry-level task and more difficult task or entry-level task and easier task) with the assumption, validated in the pilot study, that a student unsuccessful at the Freddo task would also be unsuccessful at the Footy Card task.

The pilot study had been used to verify the entry-level protocol of the CADA sequence: successful count on the Staircase task (Q. 62) or Missing Oval task (Q. 35); Area Calculation, Half Rectangle (Q. 33); and Area Calculation, Triangle (Q. 53). Thirteen Grade 6 students were asked all four of these questions, and there was a range of correct and incorrect responses. All of the students' correct and incorrect responses followed the entry-level sequence: no student was incorrect on a task lower than the highest one at which they were successful. Note that students could be correct on the count for either the Staircase task or the Missing Oval task as these tasks counted together as the easier task. It was decided in the present study that students would be asked an entry-level task with the assumption, validated in the pilot study, that a student unsuccessful at the Area Calculation, Rectangle (Q. 33) would also be unsuccessful at the Area Calculation, Triangle task (Q. 53).

There were tools and procedures tasks assessing the four concepts of measurement in both length and area contexts (see Figure 3.8).

Attribute

- Length (TPATL)
    - Blocks of Ice (Q. 54a)
- Area (TPATA)
    - Blocks of Ice (Q. 54b)

Additivity

- Length (TPADL)
    - Measure DVD with Ruler (Q. 32)
    - Streamer (Q. 31a)
- Area (TPADA)
    - Area Calculation, Rectangle (Q. 63)

Units

- Length (TPUNL)
    - Dragonfly (Q. 64)
- Area (TPUNA)
    - Staircase Array (Q. 62), Area Calculation, Rectangle, (Q. 63), and if needed, Missing Oval (Q. 35).

Proportionality

- Length (TPPRL)
    - Streamer (Q. 31b)
- Area (TPPRA)
    - Draw Your Own Array task (Q. 38a)

Figure 3.8. Classification of the tools and procedures tasks of the concepts of measurement.

Both the Streamer task and the Measure a DVD task were offered to all students but one in the pilot study, but a larger margin for error (0.5 cm) was allowed when measuring 19 cm. With this larger margin for error, the pilot data supported the interview protocol of offering the Measure a DVD task only to students who were unsuccessful at measuring the streamer. However, because a student could measure from the edge of the ruler and not 0 and obtain a measure of 18.5 cm, a more accurate answer was needed to distinguish this inaccurate measuring from a correct measurement and so a smaller margin for error was used in the main data collection interviews.

3.2.2.4 Visualisation.

Some tasks could be attempted using dynamic imagery. The research literature had indicated that a student's ability to use dynamic imagery, in particular rotating objects in the mind's eye was difficult to assess (M. Clements, 1983). To distinguish between dynamic visualisation and geometric reasoning on spatial tasks, the children were asked for their explanations of how they worked out their answer. Five tasks were adapted from the research literature on dynamic imagery/spatial ability/visual processing (see e.g., M. Clements, 1983) and from respected assessment protocols (see e.g., Australian Council for Educational Research, 1978; Department of Education & Training, 2001; National Center for Educational Statistics, 2007). The dynamic imagery involved mentally rotating diagrams or illustrations in the student's mind's eye, but not mentally reorienting the student's mental point of view.

Other geometrical tasks were also included to explore two other areas; the relationship between the number of cuts and the number of pieces, and location using array coordinates. The geometry taught in the primary curriculum was not sufficiently developed to afford a comparison study between fractions and geometric reasoning. Thus the geometric tasks were not a major component of the present study.

3.2.3. Interview protocols.

The length of the interview could be up to three hours. Tasks were completed over several sessions, and no child was interviewed in one sitting for more than an hour. The breaks in the interview differed for each child, determined by external forces (classroom timetables) and how quickly they progressed through the questions. Most interviews took two and a half hours and were spread over three sessions; this meant that most interviews carried over into the next day.

The interview began with the gaining of informal consent that the students were still happy to participate in the interview (see section 3.3.4 for formal consent protocols).

> Are you happy to do some maths with me today?
>
> I am interested in how you think when you are doing maths. I have a whole lot of tasks to do with you here. I won't tell you whether you get an answer right or wrong. But I will probably always say, and how did you work that out? You can tell me what you were thinking while you were working out the problem. Or, sometimes you just know an answer, so then you can explain how you know that you are right. If you change your mind about an answer while you are explaining it, that's fine, you just tell me your new answer.
>
> Some of the questions might be easy. Some might be hard. Some of the things you might not have been taught yet, so just do your best. (Interview Protocol, see Appendix A)

It did not take long to establish the relationship with the students that I required for the one-to-one task-based interview because most students in the study had some familiarity with the Early Numeracy Interview (Department of Education & Training, 2001). Some had done this either in their first year of school, in later years of primary school, or as part of a mathematics support program. Thus the context of a teacher listening to students' understandings, rather than guiding them to a teacher determined strategy and answer, was established. The multiplication and division tasks were used to begin the interview so that this relationship was established by the time fraction and measurement tasks were offered. The Tennis Balls task was relatively easy and so the students could become familiar with thinking about verbalising an explanation without being overly concerned about the calculations.

I conducted all the interviews at the students' schools and I was introduced to them as a teacher. In the students' minds, I might at any time stop listening without comment on their strategies and start teaching, and they were alert to the possibility that these classroom norms might change. For this reason I deliberately did no teaching during the interview and provided no feedback on whether they were correct or incorrect. Most answers were followed up with "and how did you work that out" in the same tone whether their answer had been correct or incorrect. On some occasions, I did not record the strike (correct) or dash (incorrect) on the answer square on the record sheet if I thought that it would indicate success or otherwise to the students. There would be enough information from the strategy recorded and, if need be, the video or audio files, to determine the correctness of the answer during data entry if it had not been recorded on the record sheet. To the students, I was a teacher and so, to remind myself and the reader that the explanations that the students gave were always *to a teacher*, I have referred to them in the present study as *students*.

Prompts had to be strictly monitored in order to maintain the classroom norm of the student offering their own explanation for tasks, not what they thought the teacher wanted to hear. However, this was balanced by a need to gather as much information as possible because having record sheets or video or audio footage was only as good as the information that was sought and recorded. A confirmatory question could be used in interviews (diSessa, 2007), but this was used rarely in the present study because it tended to undermine the relationship between the interviewer and the student. However, neutral probing was sometimes necessary to distinguish between different strategies.

3.2.3.1 Recording of data.

The data collected for interpretation in the present study included students' answers, explanations, and inscriptions in response to mathematical tasks and questions. All interviews were audio-taped and some were video-taped, with parental consent. The interviews were audio-taped on a digital recording device and the files downloaded to an external hard drive using Olympus software. More than half of the interviews were also videotaped using a digital video recorder with an internal hard disk and files were downloaded to an external hard drive using Windows Media Player software.

Researchers recommended recording as much as possible of the interviews for later analysis (Clement, 2000; Ginsberg, 1997; Goldin, 2000, Patton, 2002). During the interview, notes were made on a record sheet. As I am right handed, students were seated to my left. A box was included for each part of each task to record success (correct answer and correct explanation) or failure (correct answer with incorrect explanation, incorrect answer with correct mathematical thinking, incorrect answer with incorrect mathematical thinking) using the criteria of M. Clements and Ellerton (2005).

In order to minimise the time needed for written recording, common strategies were included as dot points on the record sheet so that they could be circled quickly. Four methods were employed to choose strategies to include as dot points:

- piloting the interview,
- a literature review of children's mathematical approaches to tasks,
- adaptation of record sheets from previous fraction and multiplication and division interviews (Department of Education & Training, 2001; Clarke et al., 2007), and
- revision of the record sheet after the first five to ten interviews to include strategies that were emerging from the present study.

For example, dot points were used on the record sheet for the Fraction Pairs task (Q. 22) (see Figure 3.9) to enable the faster recording of common correct strategies (benchmarking, common denominators, and residual thinking) or incorrect strategies (higher or larger numbers, and gap thinking). Every task also had space to include detail of explanations that did not fit the dot points.

□	g 5/6 and 7/8 or same benchmarks to ½ converts to common denominator other (satisfactory) residual thinking higher or larger numbers gap thinking other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D

Figure 3.9. Example of record sheet, Fraction Pair task (Q. 22g).

Some tasks required extensive note taking, such as the Fraction Sort task (Q. 19) (see Figure 3.10). There was space on the record sheet to record different kinds of visual imagery. Presmeg's categorisations of visualisation were used: concrete pictorial imagery, pattern imagery, memory images of formulae, kinaesthetic imagery, and dynamic imagery (1986). All the data on the record sheets was supported by audio, and sometimes video, recordings.

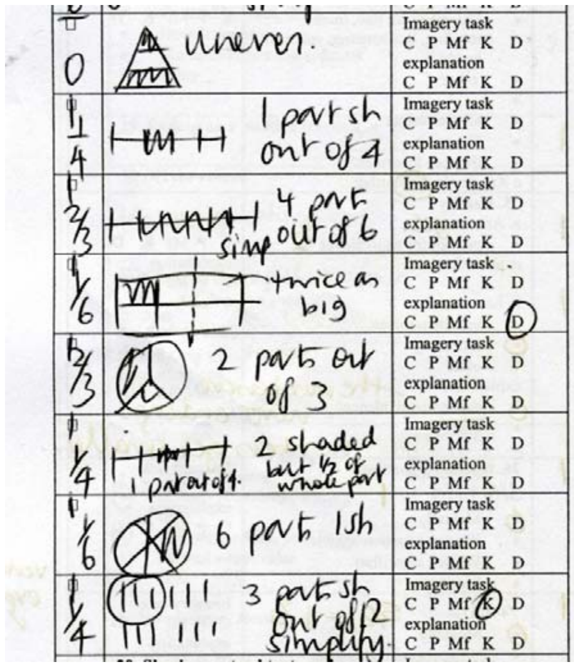


Figure 3.10. Scan of recording for part of Fraction Sort task (Q. 19).

3.3 Analysis of Data

Qualitative research distinguished "what is *observed* from what is *inferred*, and this is especially important when what is observed is already complex and qualitative" (Goldin, 2000). I have presented this thesis in a traditional format: Introduction, Literature Review, Methodology and methods, Results, Discussion and implications, and Conclusion. This format clearly delineated between the selection and presentation of results and the further analysis of the discussion, implications and conclusions based on those results. The theoretical perspective outlined in the first part of this chapter, together with a research-based criteria for task selection stemming from the literature review, were realised in the interview protocol. Initial interpretive analysis follows in the Results chapter and further interpretive analysis continues in the Discussion and Implications chapter: the misconceptions and correct strategies offered by the students are described and investigated and linked back to the research literature; correlations between performance on measurement tasks and performance on fraction tasks are investigated; and the explanatory power of Kieren's four-three-four model is evaluated. The conclusions of the present study are then related back to the significance of the study as described in the Introduction.

3.3.1 Coding protocols.

The coding of data was a three stage process. This reflected the interpretive nature of the examination of the data. Neuman (2003) recommended three passes at coding. Firstly, the use of open coding enabled themes to emerge. Secondly, axial coding divided existing codes into subcategories, or combined specific codes to make broader categories. Finally, selective coding was used to re-examine tasks of interest. In the present study, the first pass at coding included piloting the interview, developing a record sheet, recording the interviews, and assigning a code to every response and collating that in a Microsoft Word Excel spreadsheet. The second pass at coding included choosing categories to record in an SPSS database, and refining categories for use in double coding of the data. The third pass at coding included the transcription of the students' explanations, from audio or video tapes, in key tasks, and the refinement of some interpretive categories during analysis. In broad terms these three passes at coding were chronological. However, there were checks for accuracy implemented throughout the process.

Each afternoon after interviewing in a school, I returned to the University and downloaded the video files and, using the record sheet, entered the basic coding of correct/incorrect for each question into an Excel spreadsheet. To be coded as correct, a student had to have a correct answer and have given a mathematically correct explanation. This process enabled me to quickly ascertain if I had inadvertently missed offering a task to a student and I could then do so the next day if needed. Audio files were downloaded regularly but not every day.

After data collection was complete, I used the information on individual record sheets to assign a code to every type of explanation, even if only one child had used that strategy, and recorded the codes and a description of the strategies in the Excel spreadsheet. This was an open coding process which used strategies identified in the literature but was open to the identification of new strategies. There was no category of *other*. Voice recognition software (Microsoft Word 2003) was used for much of the data entry and errors were corrected immediately. The detailed coding was completed task by task not student by student, enabling me to be open to new strategies as suggested by Goldin (2000). The database included the actual answer, any self correcting, any prompts, the reasoning given for the answer, and a code for the type of inscription they created if appropriate. Some questions required specific information such as whether the child marked the streamer with their finger, a pen or another object. Missing data of strategies on the record sheet was obtained from the video footage or audio files.

This detailed coding also enabled a check of my original data entry of correct/incorrect coding. After I had entered the codes for a question, I would print out the spreadsheet for that task and manually check that the code, 1 correct or 2 incorrect, matched the answer given by the student and the code for reasoning (which started with a 1 if it was mathematically correct and a 2 if it was mathematically not correct).

I made notes on the task in a separate document, noting the frequency of success, any interesting responses, the specifics of the interview protocol such as prompting, and suggested comparisons with other tasks.

After the completion of this first pass at coding, I double checked the data entry. This entailed examining each individual record sheet again, assigning a code from the list I had prepared for each task and then checking that this was what had been entered in the Excel spreadsheet.

Schools then received aggregate data on their students' performance. Only one parent did not give permission for feedback to be given to their child's school. A summary report was compiled of some of the tasks with frequencies of success and frequencies of selected strategies.

The second pass at coding included assigning scores to some of the:

- constructs (e.g., multiplication, equivalence, measurement categories),
- tasks (e.g., Number Line, Fraction Pairs), and
- strategies (e.g., gap thinking).

In order to use factor analysis on the data (the validity of this is discussed in section 3.3.2 below, and the results of this in section 5.4.2.1), the responses had to be ranked with more than two levels (correct/incorrect). Secondly, to facilitate the calculation of correlations, measurement categories were given a score using a rubric: where three tasks were used to investigate a concept, a score of 0 indicated unsuccessful attempts at both the entry-level and easier task, a score of 1 indicated success at the easier task after an incorrect response to the entry-level task, a score of 2 indicated success at the entry-level task but not the harder task, and a score of 3 indicated success at both the entry-level and harder task. These ordinal scales were transferred into an SPSS database.

Coding descriptors had been included in the Excel spreadsheet but coding decisions were recorded in a hard copy "coding book" because some questions had margins for error, or coding rubrics. Changes to rubrics were noted in the coding book and then any required changes were made to the database of tasks and to any "scores" if appropriate in both the Excel spreadsheet and the SPSS database. In the final write up stage, the databases were checked for synchronisation by hand (by an engineer with a PhD in robotics) to ensure that frequencies and correlations were accurate.

The third pass at coding included refinement of some interpretive categories during analysis, choosing specific categories for double coding, and the transcription of the students' explanations in key tasks. Parts of this phase occurred during data collection and throughout the analysis phase. For example, preliminary results of 29 students' responses to the multiplication, division, number lines and CADL (conceptual understanding of the measurement concept of additivity in a length context) tasks were coded more exhaustively during the data collection phase while preparing a conference paper (Mitchell & Horne, 2008).

3.3.1.1 Double coding of data.

Six tasks, or parts of tasks, were double coded by six different second coders. The results of the double coding are reviewed in the Results chapter as the results of the six tasks are reported. Double coding could be related to whether the students were correct or incorrect, and/or of their strategy use. The texts examined could include their inscriptions, video footage of their response to tasks, and/or transcripts of their explanations. A summary of the interview questions that were double coded in the present study is presented in Table 3.4.

Table 3.4

Double Coding of Students' Responses in the Present Study

Task and Coder	Aspect of Task	Sample	Section
Draw Your Own Array R	TPPRA Size of units	all inscriptions (88)	5.2.4.4
Number Lines S	Q. 16a:	all inscriptions (88)	5.4.1
Fraction Sort M	Equivalence: correct/incorrect	10 students (video)	5.4.2.1
Algorithm J	$^1/_3 + ^1/_2$ Did correct answers use equivalence?	all inscriptions (88)	5.4.2.1
Fraction Sort M	Equivalence strategy	10 students (video)	5.4.2.2
Fraction Pairs A	Correct/incorrect	56 students (video)	5.4.3.1
Fraction Pairs A	Gap Thinking Strategy	All instances identified by either coder (video or audio transcripts)	5.4.3.1.2
Fraction Pairs A	Whole Number Strategies	All instances identified by either coder (video plus audio transcripts)	5.4.3.1.3
Fraction Pie L	Correct/incorrect	58 students (video plus transcripts)	5.4.3.2.1
Fraction Pie L	Answers of $^1/_3$, $^1/_5$, $^2/_5$, $^1/_7$, $^2/_7$	All answers of $^1/_3$, $^1/_5$, $^2/_5$, $^1/_7$, $^2/_7$ (video or audio transcripts – 1 missing)	5.4.3.2.2
Fraction Sort M	Partitioning: correct/incorrect	10 students (video)	5.4.3.3
Fraction Sort M	Partitioning: strategies	10 students (video)	5.4.3.3

3.3.2 Descriptive statistics and correlations.

This section describes the descriptive statistics that were used to make inferences about the data: frequencies of success, Kendall's Tau b correlation coefficient and factor analysis. Quantification should not be confused with quantitative research. The present study made use of frequencies and correlations but was a qualitative study. Quantification was not ruled out in non-positivist research (Bouma & Ling, 2004; Crotty, 1998). The use of statistical tests for correlation did not make this a quantitative study, because their use was to indicate possible patterns in the data, rather than provide proof of a causative relationship. The results of tests for correlations were not the end of the analysis, they were the beginning. They indicated where deep qualitative analysis was required.

Dimensional sampling (Cohen, Manion, & Morrison, 2007) created samples with at least one respondent for each identified factor of interest: Government schools were chosen from three different socio-economic levels and Grade 6 boys and girls were interviewed from each school. The sample in the present study was not representative. Neither the frequency of success, the frequency of particular strategies, nor a correlation could be generalised to the wider population. However, some tasks were chosen from the literature that had been used on large samples in order to ascertain whether the results of the present study were outliers or within the general range of representative testing.

One descriptive statistic calculated in the present study was frequency of success. As there were 88 students in the study, the frequency of success was presented as a percentage. Percentages enabled comparison between samples of different sizes, and also made it easier for the reader to judge the magnitude of differences across tasks in the same study. Frequency of a particular strategy out of several correct answers and explanations, or conversely frequency of a particular misconception, were also reported using percentages.

The variables in the study were coded using ranked or ordinal data. Responses in the present study were coded in three main ways:

- as correct or incorrect,
- as a score where all students were offered all the tasks in a category, or
- as a score when a entry-level task determined whether an easier or more difficult task was offered.

Coding students in the SPSS database used the numbers 1 for success and 0 for non-success and this produced a rank (this classification was not categorical (nominal) because a score of 1 was clearly better than a score of 0). In the case of only two ranks, there were many students with tied ranks. For some categories of tasks, a score was generated based on frequency of success on a group of tasks. For example, several multiplication tasks were offered and four of them were used to create a score from 0 to 4 (each question correct added one to the student's score). This score ranked the students; a score of 4 being higher than a score of 3. Even if many tasks generated a category, such as equivalence understanding with possible scores of 0 to 13, there were still many tied ranks in the sample of 88 students. The entry-level structure of task sequences in some measurement categories produced scores of 0 to 3 and generated many tied ranks. All three of these scoring systems produced ordinal data.

The second research question was concerned with examining the correlation of students' performance on measurement tasks and their performance on fraction tasks. Calculating a correlation provided one way of quantifying whether a linear association existed between two variables (tasks or concepts). The appropriateness of a test of association depended on whether the data required parametric statistics (interval or ratio data) or non-parametric statistics (ordinal or nominal data). The data in the present study was ordinal, had many tied ranks, and did not meet the assumptions of normal distribution. The correlations were a bivariate analysis (comparing two variables). Kendall's Tau was the correlation coefficient appropriate for analysing the significance and effect size of associations in the data because it was designed for non-parametric data with many tied ranks and did not assume normality. It also needed relatively small sample sizes.

Chi-squared tests were suitable for non-parametric data but were more appropriate for nominal data. Odds ratios would be useful for controlling for further variables, such as gender or school, but the data were not robust enough for this test to be used because the sample was not large enough. Pearson's correlation was the test for association most commonly used in the research literature but was not used in the present study because the assumptions for parametric testing were not met: the sample could not be assumed to have a normal distribution and the variables used were not interval or ratio measures. Spearman's Rho was the test for association analogous to Pearson's r but used on non-parametric ranked data (Tilley, 1993). However Spearman's Rho was not used because of the presence of many tied ranks.

Correlations were used to help answer questions about data. Vaske, (2002) identified three questions that researchers should ask of associations:

- did this pattern of results happen by chance?
- if the effect was real, how large was it? and
- did this have practical importance?

The significance of the result answered the first questions. A significance level of $p < .05$ was used suggesting that the observed pattern of results could come about by chance only five percent of the time. If the observed pattern in the data *did in fact* occur by chance then a Type I error had occurred and the researcher had claimed that a relationship existed when in fact it did not (Neuman, 2003). This would be a false positive. A lower p value indicated that there was less chance, for example a one percent chance ($p < .01$), that the pattern of results occurred by chance. A lower p value had no bearing on the effect size (Vaske, 2002).

If a statistical test suggested that there was no relationship when in fact there was, then a Type II error had occurred. This was a false negative. In general researchers in the social sciences accepted an 80% chance of finding an effect if indeed there was one, and the 20% risk of a Type II error occurring (Field, 2009). Intuitively, the more participants there are, the more likely an effect can be seen. The statistical power was a measure of a Type II error occurring and a statistical power of .8 (20% probability of a false negative) was the common standard in social sciences research, but was calculated differently for each statistical test type (Bonett & Wright 2000). Field (2009) calculated minimum sample sizes using Pearson's correlation coefficient (r) to detect a

- small effect ($r = .1$), 783 participants,
- medium effect ($r = .3$), 85 participants, and
- large effect ($r = .5$), 28 participants.

This assumes a statistical power of .8 using a p value of < .05 in a one-tailed test. Two-tailed tests would require a larger sample size for the same statistical power. However, a false negative, reporting a non-correlation between fractions and measurement when in fact there was one, would be a conservative result. Interviews were conducted with 88 students in the present study, hence the present study had the statistical power to detect large effect sizes.

If there were a significant linear association ($p < .05$), the effect size was of interest and was represented by the correlation coefficient which lay between -1 and +1. In the present study, Kendall's Tau was calculated using SPSS version 17 software. The commonly agreed standards for describing effect sizes were based on Pearson's r. Effect sizes were often categorised as small (.1), medium (.3), and large (.5) (Field, 2009). Vaske's terminology (2002) of minimal, typical and substantial effect, which correspond to small, medium and large effect have been used in the present study. *Typical* highlighted that such associations were common in the behavioural sciences and *substantial* reflected the fact that educated readers would agree that there was an association just by looking at the data without doing inferential statistics (Vaske, 2002). Looking at data graphically before running a statistical analysis was suggested by Field (2009). Contingency tables were used to present data visually from two variables. Kendall's Tau used a different metric to Pearson's r (Strahan 1982) so a .3 effect size in one was not the same as a .3 effect size in the other. It was possible to use tables

(Gilpin 1993; Strahan 1982) to relate a value of Kendall's Tau to Pearson's r to compare effect sizes. Kendall's Tau values of minimal, (.07), typical (.20) and substantial (.34) effect sizes had the same common variance as the Pearson's r categories of small (.1), medium (.3), and large (.5) (see Table 3.5). This terminology of minimal, typical and substantial effect size was used when reporting results in the present study.

Table 3.5

Magnitude of Kendall's Tau Correlation Coefficient and Sample Sizes to Detect Differing Effect Sizes of Associations Between Variables Using Gilpin's Tables (1993)

	Effect Size			
	Minimal relationship	Typical relationship	Substantial relationship	Very Substantial
Pearson's r	.10	.30	.50	.9
Covariance: r^2	1%	9%	25%	81%
Kendall's tau	.07	.20	.34	.72
Sample size for a power of 0.8	783	85	28	

Sample size: power of .8, p < .05, one-tailed test, from Field (2009).

A correlation co-efficient could reveal if an association was positive or negative. A positive correlation indicated that as one variable increased so did the other, for example as the scores on one category increased, so did scores in another. A negative correlation indicated that as one value increased, the other decreased, for example as one score increased, one misconception decreased. Using two-tailed tests effectively split the 5% margin for error (Type I error) into a 2.5% margin for error at the positive end and a 2.5% margin for error at the negative end of the distribution. This enabled the detection of positive and negative correlations and I used two tailed tests because I could not assume that all performance would be positively linked.

The co-variance between two variables was a common indicator of the magnitude of the effect size (Strahan, 1982). When using Spearman's Rho this was done by squaring the co-efficient, for example a correlation coefficient of .50 accounted for .25 or 25% of the variance between the two variables. It was noted that squaring a coefficient lost its negative direction, if it were present (Walker, 2003). However, the square of Kendall's Tau was even more different from the square of Spearman's Rho because the two unsquared coefficients used a different metric (Strahan, 1982). By converting between Kendall's tau and Pearson's r (see

Table 3.5), the covariance of the two variables could be matched to values of Kendall's tau (Gilpin, 1993).

An association between two variables does not imply causality and that was taken into account when interpreting any association. A third variable or lurking variable may have been responsible for the association, or there may be no causal link. However, while correlation does not imply causality it remains a possibility. Positivist research, as opposed to interpretive research, sought to reveal and explain causality. Correlation on its own was not enough evidence to prove causality. As Miller (2004) elaborated, consistency of association, strength of association, temporal relationships, and a mechanism were necessary to posit a causal relationship. Quantification of these factors in a representative sample was often part of supporting the truth claims of this type of research. In the present study, descriptive statistics (correlations) were used to aid an interpretive analysis.

It was a mistake, Vaske cautioned (2002), to think that *significant* meant *of practical importance*. The practical significance of a relationship required a value judgement by the researcher and the readers. For example, if it were shown that there was a stronger correlation (larger effect size) between performance on fractions tasks and performance on conceptual measurement tasks as opposed to tools and procedures measurement tasks, that result would have practical importance because using broken ruler tasks as well as assessing whether a student can use a ruler accurately was not an onerous change to instruction. On the other hand, if there were a significant association between a specific numeracy program and a minor improvement in test scores, then the result would be significant but not of practical importance to the wider teaching profession would be

Factor analysis was used to explore the data. This suggested further qualitative investigation of the correlation between several variables. Factor analysis was used to discover which variables appeared to show similar performance, possibly suggesting similar underlying constructs. In order to use factor analysis, the students' responses were coded with greater detail than just correct or incorrect. Responses were rated, in layperson's terms, as:

- correct with a correct explanation,
- just a slip up,
- right strategy but not fully executed,
- some relevant mathematical thinking but an incorrect answer, and
- incorrect mathematical thinking.

The factor analysis itself was not used to justify the inclusion of tasks in specific categories, for example the construct of equivalence. Rather, further interpretive analysis justified the inclusion of a task within a category or score.

The test for association using Kendall's Tau only reveals linear correlations, so some data was graphed to see if there were any non-linear correlations substantial enough to be observed without statistical tests.

3.3.3 Validity.

The section on validity contains three parts:

- Construct validity, face validity, content validity
- Reliability
- Authenticity

3.3.3.1 Construct validity, face validity, content validity.

Internal validity is an evaluation of the effectiveness of the research design for detecting cause and effect (Neuman, 2003) and so is not applicable in the present study.

Construct validity is a measure of whether the instrument actually measures what was intended. The mathematical constructs investigated in the present study were fractions, measurement, multiplication and division, and dynamic imagery. I used a theoretical model for task selection, as recommended by Goldin (2000) so that my preconceptions were explicit and not implicit. A distinction was made between relational and instrumental thinking (Skemp, 1976), or conceptual and procedural understanding (Hiebert & Carpenter, 1992). In the present study, this is referred to as conceptual understanding and a knowledge of tools and procedures. I chose, where possible, tasks from the research literature that had been subjected to peer review. The acknowledgments of sources in the description of the instrument (section 3.2.2) and in the interview protocol in Appendix A make clear this aspect of construct validity.

Only the presented task was subject to experimental control not the interpreted task Goldin cautioned (2000). The literature had revealed that "spatial" tasks could be solved correctly by students using either dynamic visualisation or geometric reasoning: the task itself was not inherently a dynamic visualisation task or a geometric task (M. Clements, 1983). Although tasks were chosen to assess different categories of knowledge of fractions or measurement, there would be no guarantee that students would use the relevant constructs in their solutions. An examination of the students' explanations would help to determine if the task had in fact assessed the desired construct. A student's explanation only revealed their preferred strategy (Presmeg, 1985). If a child used geometric thinking on a spatial task (successfully or unsuccessfully), that did not mean that he or she could not use dynamic thinking: all it indicated was that that geometric thinking was the first strategy tried on this particular task. These constraints were noted in the analysis of data.

Face validity is less precise than construct validity: at face value, do the tasks appear to assess the constructs that they were chosen to assess, and what practical measures were taken to determine this. One measure of face validity was peer acceptance that the tasks tested the construct as categorised by the researcher (Neuman, 2003). During the development of the instrument, tasks were shown to other mathematics education researchers and their suggestions incorporated into the final interview protocol. For example, a conservation of area task was changed from being diagram-based to being hands-on with the children cutting and moving the paper themselves (Jill Cheeseman, personal communication). The tasks were also shown to a practising primary teacher to confirm that students would have some entry point into the concepts. The interview was piloted so that I, as the researcher, could assess its face validity. After data collection, peer-reviewed conference papers and book chapters (Mitchell & Horne, 2008, 2009, 2010, 2011) also provided an opportunity for other researchers to comment on the validity of the tasks and my interpretations of them.

Language use was another factor that was taken into account in developing the tasks. Previous research had identified that everyday language and mathematical language could be confused by children, such as the word "bigger" (Mitchell, 2005). The imprecision of everyday language and the importance of context may contribute to this. For example, "size" did not automatically map to "area". "Length" could be distance travelled or distance from the starting point. "Longer" could be a measurement of length or a comparison of endpoints. The phrasing of the questions was carefully considered, and many of the tasks in the final data collection were used in a pilot study to ensure that the language was not interfering with the understanding of the task.

Content validity concerns whether the task criteria covers all of the various aspects of a construct (Neuman, 2003). Two aspects of content validity are discussed here: coverage of a mathematical domain and difficulty of the questions.

In the domain of fractions, the focus of the present study was on the measure sub-construct and the concept of equivalence and so several questions were developed to cover different aspects of these concepts. For example, the research literature had identified that for number line questions, there was a difference in presenting number lines from 0 to 1 and from 0 to greater than 1. There was also a difference, proposed in the literature, between reading partitions and making partitions. Thus, eight number line tasks were developed (or adapted or replicated) to cover these different aspects of number line knowledge (see Table 3.3).

I had ethics approval for an interview of not more than three hours in total, so not every aspect of fractions and measurement could be assessed. The fraction sub-constructs of operator and ratio were not the main focus of the present study, and only one task was used to assess them. This was not enough to give a full picture of students' performance on these sub-constructs but time constraints prevented further questions being offered. As a consequence, correlations between these sub-constructs and measurement concepts have limited content validity.

The research literature had identified that the students who lagged behind grade level performance could be significantly behind (Brown et al., 1995). Hence questions were included at different levels of difficulty to assess students' understandings of a concept. In order to save time, some measurement concepts had an entry-level task that was offered to every child: a harder task was only offered to those correct at the entry-level task and an easier task to those who were unsuccessful on the entry-level task. This protocol was confirmed in piloting of the interview where all three tasks were offered and no student was successful on the harder task if he or she had not been successful at the entry-level task.

A range of performance was anticipated for equivalence concepts and tasks were included to assess this spectrum. Questions included equivalences to one half, other unit fractions (one quarter and one sixth), and non-unit fractions (two thirds). Later data analysis also identified another more difficult category of equivalence questions in which equivalence was used to benchmark or make common denominators.

3.3.3.2 Credibility and reliability.

Patton (2002) synthesised "rigorous methods for doing fieldwork", "integrity in analysis", "the credibility of the researcher", and "philosophical belief in the value of qualitative inquiry" (p. 552-553) as key concepts that underpinned the credibility of qualitative research. Good data can be interpreted badly, diSessa had cautioned (2007).

Interviewers make inferences based on what they *can* observe (Goldin, 2000). It was important to maximise the students' engagement through external representations such as materials, their own inscriptions, and explanations (Goldin, 2000) because that was what would be observable. I included the instructions to retell the thinking or to justify an answer because children do one of three things when they are given a mental calculation – know the answer immediately, decide immediately that they cannot do it, or adopt a strategy, successful or not, for arriving at an answer (McIntosh, De Nardi, & Swan, 1994). An explanation could be a justification for an answer (Lesh et al., 1983), so I did not assume that an explanation was automatically a window into a student's thinking.

The preamble to the interview in the present study also made clear that I was interested in the student's thinking, as suggested by Ginsberg (1997), and would not be trying to direct that thinking. Goldin (2000) cautioned that interviews do not give the interviewer access to the participants' thinking, reasoning, cognitive processes, internal representations, meanings, knowledge structures, schemata, affective or emotional states, and the like. Goldin (2000) suggested two types of prompting in interviews: neutral questions, such as "why do you think so?", and heuristic suggestions, such as "do you see a pattern in the cards?" (p. 125), but only the first type of neutral prompt was used in the present study.

Trustworthiness was based on triangulation of the data. In the present study, the students completed an extensive interview but their responses were not triangulated using different instruments. However, quotations of explanations are presented so that readers can make their own judgments about the validity of the interpretations, as suggested by Clement (2000). The double coding and transcripts were used in reporting the data to support the present study's claim for authenticity (Neuman, 2003). By choosing more than one school it was possible to demonstrate that if a misconception were present in children from different schools, it was not due to the idiosyncratic teaching of individual teachers, but possibly caused by developmental factors or the state curriculum.

The use of a theoretical model to guide task selection and analysis was used to support the present study's claims for trustworthiness, credibility, transferability, and confirmability as described by Denzin and Lincoln (2008).

The use of an interview script as suggested by (Goldin, 2000), improved reliability of the instrument by framing the questions in the same way and offering questions in the same order. The advantage of having the same interviewer (me) was that it was easier to deliver the interview in the same way. The disadvantage was that the data collection took five months, so a Grade 6 student in the study might be a Grade 6 student at the beginning of Term 1 (February) or the end of Term 2 (June). However, the topic of fractions had not been taught to the students before interviewing took place.

The interview protocol remained the same as suggested by Clement (2000). The study involved children, so it was not possible to eliminate the variables of illness, a "bad day", or nervousness. A one-to-one task-based interview was one assessment on one day and so may not be able to produce identical results for the same student on a different occasion. The re-test reliability (Bouma & Ling, 2004) of a one-to-one task-based interview could not be 100% because previous exposure to a task might affect a child's subsequent performance, unlike measuring another variable such as height which is unaffected by multiple recordings.

Patton (2002) emphasised practice and experience with interviewing as important contributors to well conducted interviews. I had used the number domains of the Early Numeracy Interview (Department of Education & Training, 2001) with Grade 1 and 2 students and also with Grade 5 and 6 students in my capacity as a practising classroom teacher. I had developed a fractions interview for a Masters project and interviewed primary school children (Mitchell & Clarke, 2004). I had also interviewed Grade 6 students using the Early Numeracy Interview as a research assistant for a larger project (see e.g.,Clarke et al., 2007). Familiarity with conducting, recording and coding one-to-one task-based interviews contributed to the reliability of the data collection in the present study.

One measure of reliability was whether another researcher would code the data in the same way as the investigator (Bouma & Ling, 2004). Verification was important (Pring, 2005) and concerned that which could be observed. For example in the present study whether a student gave an answer of three and three quarters to the Keyboard task (Q. 39) can be verified by the use of audio or video data. The coding of explanations into categories could not be verified as it involved a layer of interpretation. Instead the credibility, rigour and integrity (Patton, 2002) could be increased by the process of double coding of the data by another researcher. Clement noted (2000) reliability measures were of observations, not of theories. The interpretation of the explanatory power of Kieren's model could not be double coded, but it could be subject to academic critique.

3.3.4 Ethics.

Three principles guided the scholarly research and writing in the present study: protecting the rights and welfare of the participants, ensuring the accuracy of scientific knowledge, and protecting intellectual property rights (American Psychological Association, 2010). The present study was approved by the Human Research Ethics Committee of the Australian Catholic University (V20050679) and by the Department of Education & Training (now known as the Department of Education and Early Childhood Development) (SOS003302). The letters granting approval to undertake the research and the letters to participants and consent forms are included in Appendix B.

3.3.4.1 Protecting the rights and welfare of the participants.

The present study was conducted within the guidelines of the Human Research Ethics Committee of the Australian Catholic University and the guidelines of the Department of Education and Early Childhood Development for conducting research in schools. The rights and welfare of the participants were broken down in three categories: consent, confidentiality and respect.

Informed consent was gained from the participants in the present study. The participants were under the age of 18 and were students in state government primary schools. Permission to approach principals of state government primary schools was obtained from the Research Branch of the Education Policy and Research Division in the Office for Policy, Research and Innovation of the Department of Education and Early Childhood Development, Victoria. As requested, a courtesy letter was sent to the Regional Director of the schools chosen to be in the study. Four principals were approached and permission requested to conduct the research in their schools and to send information letters home to parents. Written consent was obtained from all four principals. Information letters were sent home with the students in the senior grades and these included consent forms for parents or guardians to sign and consent forms for the children to sign. In addition, I asked every student at the beginning of the interview itself, "Are you happy to do some maths with me today?" The participants had the right to discontinue their participation at any time and this simple verbal check combined with their signed consent form confirmed that they themselves were happy to participate. No child refused to do the interview, but if they had, I would have discontinued the interview despite their parents' written permission to conduct it. The parents and students consented to

- one-to-one task-based interviews, including a pen and paper test (total 3 hours),
- the interview being audio-taped,
- the interview being video-taped if written permission was included (a box ticked),
- the video footage being used in professional development and conferences, including on the internet,
- the data obtained being written up in my PhD and in conference papers and articles
- feedback on the child's performance being given to the school if written permission was included (a box ticked).

More than half of the participants gave permission for the interview to be video-taped. Only one parent refused permission for feedback to be given to the school. This option for feedback to the school was added after the initial piloting of the interview when it was discovered that confidentiality prevented me from commenting to the teachers about the students' performance, and this necessitated an amendment to the ethics application. Consent was given by the students and their parents or guardians to take part in the research and for this to be written up and shared in research publications, teacher professional development activities and feedback to the school.

Confidentiality of the students' identities was maintained. It was made clear in the information letter that teachers and other students would be aware that the participants were taking part in the research because it involved being withdrawn from the classroom. However, in the use of the data, confidentiality was maintained. The children's names were written on the record sheets of the interview and could be heard on the audio or video recordings but these data were kept in a locked cabinet at the University and were not accessible to others. Written reports on the data (this PhD, conference papers etc.) used pseudonyms if the students were quoted and only anglo-celtic names were chosen because ethnically appropriate names may have identified the students or the school. The gender of the student was identified in the name or pronoun used. In the present study, information about the schools was broad rather than specific so that they were not identified. A code rather than a name was used to identify each student in electronic databases (Excel and SPSS). Some of the tasks were double coded by other researchers and they agreed to maintain confidentiality if they heard a child's name on the video. A report was prepared for each school on some of the tasks and aggregated data of only that school's students were made available to them (excluding the one child whose parents had refused permission for feedback to the school). In these ways the confidentiality of the students' responses to the interview tasks was maintained.

Respect for the participants was an underlying consideration of the interview protocol of the present study. I was the only interviewer and the parents were informed that I was registered as a teacher with the Victorian Institute of Teaching and had a current criminal records check. My experience as a teacher of children at this level enabled me to develop a rapport with the students. I was appreciative of the consideration shown to me by the teachers and principals as I withdrew students from classes. In designing the interview, I tried to have some interesting materials to work with throughout the assessment, as suggested by Ginsberg (1997), and the students enjoyed using a doll, golden beans and figurines. In a preamble to the interview I informed the students that they would not be told if their answers were wrong or right but would probably always be asked "and how did you work that out?" This set up "classroom norms" in which I listened to their answers but they did not have to guess what I might prefer them to think. This established a powerful pedagogical relationship based on my respect for their explanations, and prevented a "why" question triggering doubt in the students as to the correctness of their answers, as cautioned by Patton (2002). Respect for the students was demonstrated through both the design and conduct of the interview.

One task (Q. 41) used a graphic of a Freddo Frog (an Australian chocolate confectionary made by Cadbury Pty. Ltd.). Data collection was completed in 2008. In 2010, Kraft Foods then acquired Cadbury. However in 2009, Kraft Foods Limited became signatories to the Australian Food and Grocery Council's Responsible Marketing to Children Initiative. Freddo Frogs did not meet the "sensible solutions" criteria because they did not contain ingredients at a nutritionally meaningful level. Kraft therefore, stopped marketing them to children under 12 years old. Thus, it would now be inappropriate for me to use that graphic with primary school students.

3.3.4.2 Ensuring the accuracy of scientific knowledge.

The present study was written with a commitment to the accuracy of the research findings presented within it. This was demonstrated in the analysis of data, the presentation of data, and the administration of the data. In publications stemming from this project, previous papers on the same data were cited (e.g. conference papers) so that the reader was clear about which data were being reanalysed as recommended in the APA 6th style guide (American Psychological Association, 2010). Some tasks were excluded from the results because the children were confused by the way the question was asked or were unfamiliar with the materials, not because the results were unexpected. A detailed reporting of all results was not possible within the word limits of the thesis, but it is hoped that some tasks not reported in depth here will be reported in subsequent journal articles. Unexpected results were included – a finding is a finding whether it confirms a prediction or contradicts it. The interview protocol is included in Appendix A so that other researchers could

- examine the detail of the tasks to ascertain whether they would categorise the tasks in the same way as I did in my task selection criteria
- examine the detail of the tasks to assess my interpretations of the students' explanations
- compare the results of the present study to other studies
- replicate the study

The Results chapter is detailed, including quotations of students' explanations, so that other researchers could evaluate my interpretations. Data will be retained for at least five years after publication.

3.3.4.2.1 Conflict of interest.

I was an employee of the Department of Education and Early Childhood Development and a student at the Australian Catholic University. The Early Numeracy Interview was developed by a team of researchers including some of my supervisors at the Australian Catholic University and is available on the Department of Education and Early Childhood Development website (Clarke et al., 2002; Department of Education & Training, 2001). The Fractions online interview and associated classroom activities book was developed by researchers, including one of my supervisors, at the Australian Catholic University and is available on the Department of Education and Early Childhood Development website (Department of Education and Early Childhood Development, 2009a, 2009b). I received payment for some work on the online Fractions Interview. I do not believe that my participation in (nor the participation of my supervisors in) the research listed above constitutes a conflict of interest in the present study.

3.3.4.3 Protecting intellectual property rights.

Research findings and analysis by other researchers is referenced in the text. Some tasks have been used or adapted from other sources and acknowledgement appears in the text. I have obtained permission for the use of unpublished one-to-one interview tasks developed by fellow researchers, and acknowledged their authorship in the text. The graphics used in the tasks were commissioned and paid for by me.

Permission to reproduce the image of a Freddo Frog in this thesis was given by Kraft Foods (see Appendix B).

Permission to report frequencies of state and national item responses to specific questions on the Achievement Improvement Monitor (AIM) and National Assessment Program: Literacy and Numeracy (NAPLAN) tests from the NAPLAN Data Service was given by The Victorian Curriculum and Assessment Authority (external request ID 472)

3.4 Limitations

The present study interpreted students' understanding. It did not investigate learning, nor did it investigate teaching. Students' explanations have been interpreted without commenting on the specifics of the teaching that they had received, nor the learning they had engaged in, in their particular classroom and schools.

The research design of the present study was interpretive research. I have tried to describe faithfully the strategies that students' were using in their explanations but I have interpreted them within a framework and terminology that the students might not use. The present study did not seek to present the students' explanations from a phenomenological standpoint. I categorised responses and grouped variations, but the students did not participate in this process.

An interview could not capture change over time. The variety of students' explanations have sometimes been interpreted as indicating pathways. Some strategies of students who are less successful might be compared to the performance of more successful students, as if these represented stages that all children will pass/have passed through. Only a longitudinal study could provide evidence of a pattern of learning.

Although the present study set out to produce original analysis, it did not set out to develop new domain level theory. Theory-generating research was used to generate domain specific theories: for example design experiments in education (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) or grounded theory (Strauss & Corbin, 1998). The associations between measurement categories and fraction performance rely on the construct validity of the tasks chosen to assess them. If the tasks do not assess the construct that they have been chosen to represent, then associations are invalid.

It was never going to be possible in this thesis to examine in depth the responses to every task. All tasks were coded for frequency of success and every strategy evident from the students' explanations was also coded. There was a substantial time investment in preparing task results for presentation: transcribing explanations, reviewing original coding criteria, analysing related performance in other tasks, refining categories for double coding, explaining the coding criteria, locating start times on video and audio data, and discussing strategies post double coding. There were other results that could have been analysed in depth but because of the word limit of this thesis not all of them are presented here.

Summary of Methodology

In this chapter, I began with the research questions generated by an examination of the research literature in the previous chapter, and described a methodology (interpretivism) and a research method (one-to-one task-based interview) that could investigate these questions. A one-to-one task-based interview was developed for data collection with research-based criteria for task selection. The interview protocols were described, and the coding protocol was elaborated and the proposed analysis outlined. The sample of students from whom data was collected was described. Issues of validity were addressed and the ethical considerations of the present study were described. The limitations of the present study were outlined to conclude the discussion of methodology and methods.

Chapter 4: Results

The structure of the Results chapter is based on concepts. This chapter is divided into five sections:

- describing the baseline and upper limits of the students' performance as a group,
- measurement tasks results,
- visualisation tasks results,
- multiplication tasks results, and
- fraction tasks results.

I examine the frequency of success of particular tasks, the strategies offered by students, and correlations between some fraction tasks and the measurement concepts.

In the Discussion and Implications chapter which follows after this Results chapter the results are discussed in a thematic order: strategies (misconceptions and correct strategies) evident in children's explanations, correlations between fraction and measurement concepts, and the fractions concepts of Kieren's four-three-four model evident in children's explanations.

Draft findings of some of the results of tasks in the present study have been reported in other places (see e.g., Mitchell & Horne, 2008, 2009, 2010, 2011). Some of these preliminary findings have been cited in further research (see e.g., Cunningham, 2009; Chinnappan & Pandian, 2009; Petit, Laird, & Marsden, 2010)

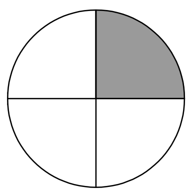
The interview protocol is presented in Appendix A. The diagrams (task cards) presented in this Results chapter are often much smaller than the actual task card used during the interview.

4.1 Students' Baseline and Ceiling Performance

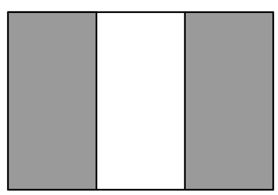
The baseline performance of the students' in the present study is specified by describing the tasks which had 100% frequency of success. The upper limit of the students' performance is specified by describing the tasks that had 0% frequency of success.

There were five questions to which every student in the present study gave a correct answer with a mathematically correct explanation (see Figure 4.1). These five questions provided a description of the baseline performance of the specific sample of students in the present study.

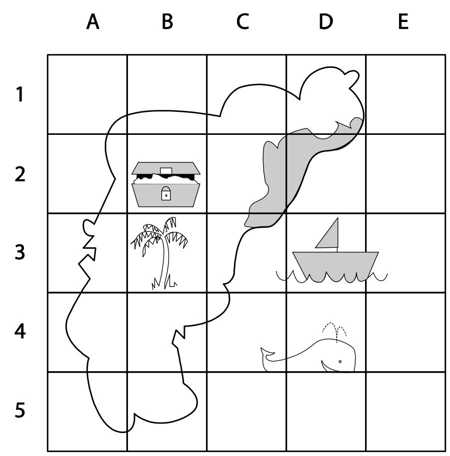
All students correctly identified as one quarter the one shaded part in a circle divided into four equal parts (Q. 19g); all correctly identified as two thirds the two shaded parts in a rectangle divided into three equal parts (Q. 19p); all correctly used a letter and number code to identify a square on a grid overlaying a treasure map (Q. 50); all could describe a length in a three dimensional context of a picture of a block of ice (Q. 54a), either formally as "the length" or informally, for example, "how high" or "how wide"; and all successfully mentally calculated the total number of balls in four packets of three tennis balls when shown one packet (Q. 1). Success on these tasks demonstrated that all students had a basic understanding of unit and non-unit fractions; of an array structure; of the attribute of length as a straight line; and of repeated addition (or possibly, multiplication).



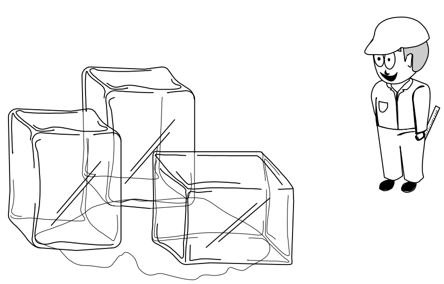
Q. 19g



Q. 19p



Q. 50



Q. 54

Q. 1

Figure 4.1. Task cards or materials used in tasks with 100% frequency of success.

There were several tasks that showed where the first brittleness in the students' knowledge began to appear (see Figure 4.2). These were the tasks where all but one or two students answered correctly with mathematically correct explanations. For example, all but two students could correctly identify as one sixth the one shaded part in a circle divided into six equal parts (Q. 19a); all but two students could identify as one quarter the one shaded piece of a square divided into four equal triangle pieces (Q. 19h); all but one student could identify as two thirds the two shaded pieces of a circle divided into three equal sized pieces (Q. 19q); all but one student could state that half of six was three (Q. 18a). While all the students had some basic understanding of unit and non-unit fractions, this knowledge did not extend to all the standard inscriptions for every student.

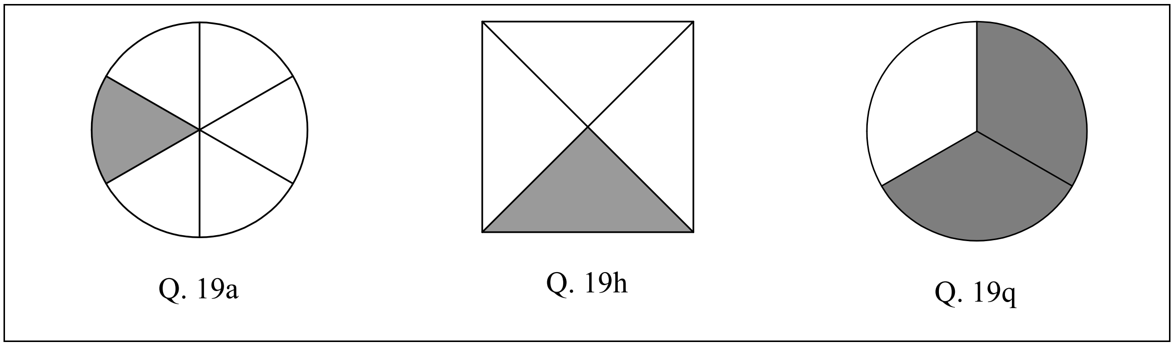


Figure 4.2 Diagrams used in tasks with a 97-99% frequency of success (Q. 18a was verbal).

There was also a ceiling on the students' performance. One task proved too difficult for all of the students and none answered successfully. This question was an equivalence card in the Fraction Sort task (Q. 19r) (see Figure 4.3) where the students had to mentally re-partition the shape into six ninths before being able to give a mathematically correct explanation for why the card should be placed in the two thirds pile. Two students successfully mentally re-partitioned the triangle into nine smaller triangles and saw that two horizontal rows of these would be six out of nine parts, but neither of them realised that two thirds was another name for that. The other students, who placed this card in the two thirds pile and gave the reason that there were two parts shaded, were not coded as correct on this question as they neither demonstrated an understanding of the geometric complexity of the task, nor had used equivalent fraction reasoning.

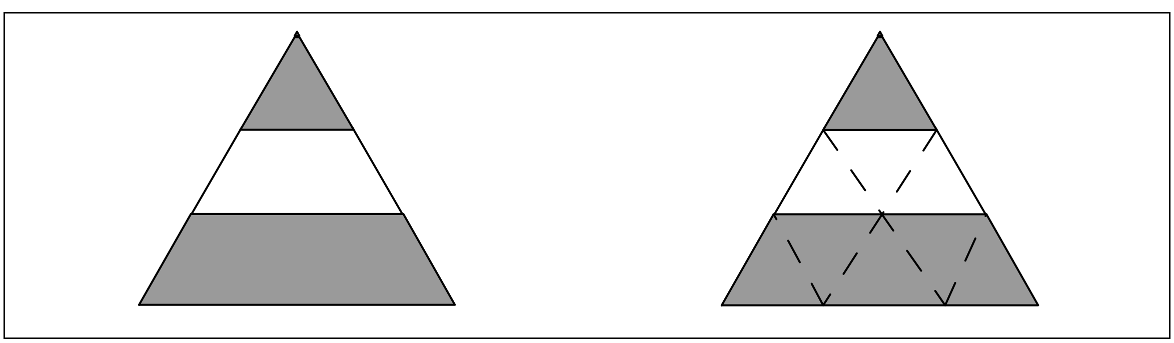


Figure 4.3 Triangular $^{2}/_{3}$, Q. 19r (left) and mental re-partitioning required (right).

There were three schools used in this study, all of them performed above the base line measures described above. The performance of the three schools differed with respect to each other, and this is not unexpected given that they represented three different socio-economic groups of students. However, differences between the schools were not the focus of this study and results are presented as aggregate percentages.

4.2 Length and Area Measurement Results

The measurement tasks are reported by key concept category: attribute, additivity, unit, and proportionality. Conceptual tasks using length and area diagrams, and tools and procedures tasks using length and area diagrams were investigated (see section 3.2.2.3).

4.2.1 Attribute.

The frequency of success on the conceptual and tools and procedures tasks assessing the identification of the attribute of length and area is presented in Table 4.1. A score of 0 indicates an incorrect response or a right answer for the wrong reason, while a score of 1 indicates a correct answer with a mathematically correct explanation.

Table 4.1

Attribute: Frequency of Success

Concept	Task type	Context	Score	Frequency of success	Tasks used to rank success at the concept
Attribute	Conceptual	Length	0	62.5%	
			1	37.5%	Q. 36g Similar Shapes
Attribute	Conceptual	Area	0	33%	
			1	67%	Q. 36h Similar Shapes
Attribute	Tools and Procedures	Length	0	0%	
			1	100%	Q. 54 Blocks of Ice (length)
Attribute	Tools and Procedures	Area	0	67%	
			1	33%	Q. 54 Blocks of Ice (area)

4.2.1.1 CATL: conceptual tasks, attribute concept, length context.

Four pairs of shapes (see Figure 4.4*)* were used for the Similar Shapes task (Q. 36). Students were asked to compare the shapes' perimeters and the shapes' areas. The first three pairs, squares, circles and non-similar rectangles, were not included in the score for the attribute concept. The perimeter comparisons were made successfully by 96.6%, 92%, and 45%

respectively. The area comparisons were made successfully by 96.6%, 97.7%, and 71.6%, respectively. The perimeter and area comparisons of the shaded shapes were 37.5% and 67% respectively. However, only 4.5% of the students were successful on all eight questions.

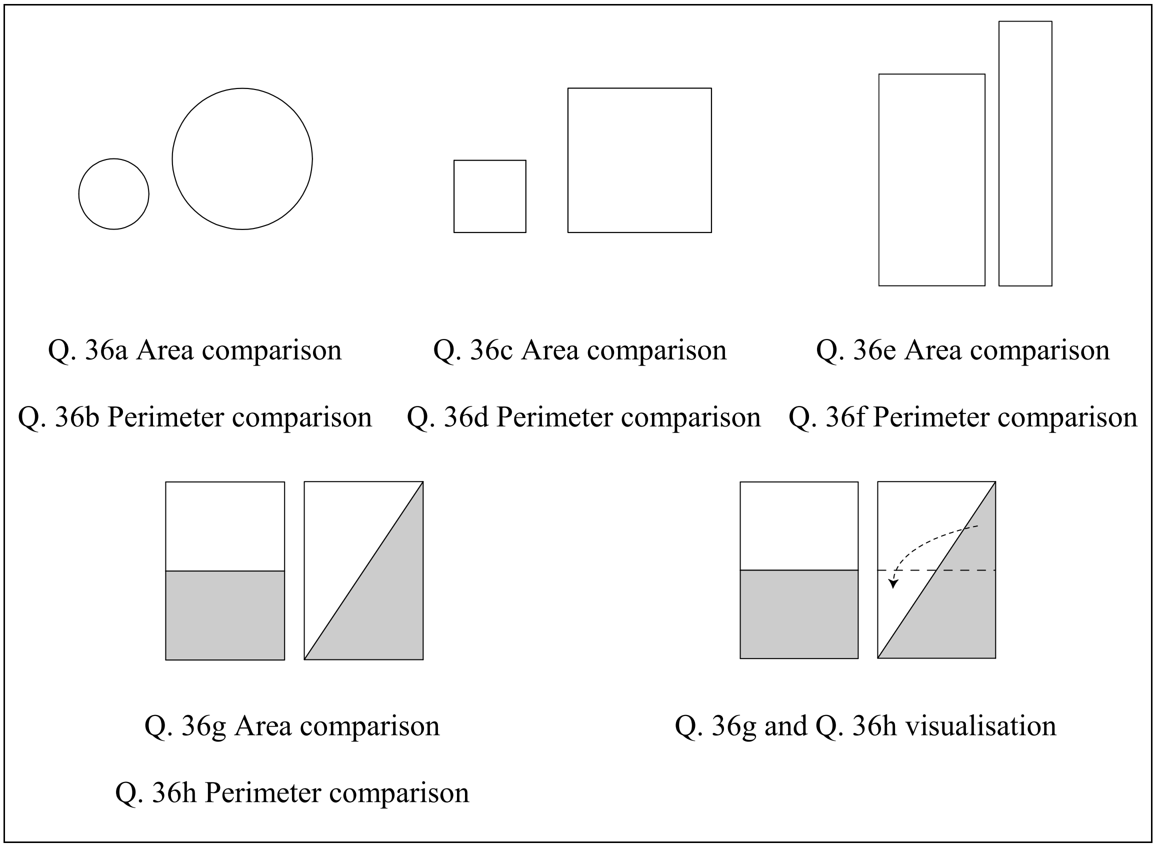


Figure 4.4. Diagrams used for Similar Shapes task (Q. 36), and the geometric visualisation explained by some students to show that both shaded parts were half.

The length context of this attribute category (CATL) was assessed using only the perimeter comparison of the shaded pair of similar shapes (Q. 36g) and 37.5% of the students correctly explained that the triangle half of a rectangle had a bigger perimeter than a rectangle half (see Table 4.1).

The most common incorrect response was that the perimeters were the same because the shaded parts were both half and 45.5% of the students offered an explanation with the word half or halves in it. For example, Sylvie explained that the perimeters were the same because the shapes were "halved in different ways" (see Table 4.2). A further 8% of the students similarly concluded that the perimeters were the same by geometrically breaking and rearranging the two areas using visualisation to show that they were the "same" (see Figure 4.4). And one other student (1%) offered a similar geometric reason but with less sophisticated language. In the perimeter question (Q. 36g) over half of the students linked perimeter to area in the non-congruent halves in the Similar Shapes task.

Table 4.2

Explanations for the Perimeter Comparison in Q. 36g

Strategy	Explanation from transcript
The perimeter of the triangle is longer	Cameron: Probably this one [points to triangle] Interviewer: Why's that? Cameron: Because this line here [hypotenuse] is probably about like two of these [traces vertical line of rectangle] of these sides; the height. And these two [two sides of triangle] should be longer than them two together as well [points to two horizontal lines of rectangle]. Claire: This one's longer [points to triangle] than that one [points to rectangle] Interviewer: And how do you know? Claire: Because it goes up most of the shape, and this one only goes across half of it.
The perimeters are the same (fractional area reasoning)	Sylvie: They're the same. Interviewer: How do you know? Sylvie: Because they're both the same like shape and equal like shape, but they're halved in different ways

4.2.1.2 CATA: conceptual tasks, attribute concept, area context.

The task assessing the area context of the conceptual aspect of the concept of attribute was the comparison of the areas of non-congruent halves. One mathematically correct explanation of why the areas of the shaded parts had the same area used the fraction reasoning that the shaded parts were both halves. Cameron (see Table 4.3) used fraction reasoning to conclude that the shaded shapes were "both halves" but also noted that the triangle looked bigger. Cameron's explanation also revealed that dynamic reasoning could be used to justify fraction reasoning. Claire and Sylvie attended to the actions of halving and explained that both shapes had been "coloured in half" or "cut in half" (see Table 4.3). The words half or halves were used in a fractional reasoning strategy by two thirds of the 67% of students who were correct on Q. 36h. The other correct students used dynamic imagery (see Figure 4.4) or global size comparisons.

Table 4.3

Explanations for the Area Comparison in Q. 36h

Strategy	Explanation from transcript
the areas are the same (both halves)	Cameron: Yeah they're the same. Because they're both halves but at the same time this might look bigger [points to triangle] but it's actually, they're both halves, and if you got some pieces of the square [points to rectangle] and put them all on to that [points to triangle], on this triangle, then yeah it's the same. And this would be the same.
	Sylvie: They're the same. Interviewer: How do you know? Sylvie: Because um they are like the same size and the same of it is cut in half
	Claire: They're the same Interviewer: And why's that? Claire: Because the pieces of paper are the same Interviewer: Hmmm, and? Claire: And they're both coloured in half.

Only 22.7% of students were correct on both Q. 36g and Q. 36h. A quarter of the students who correctly used the fraction explanation (halves) in the area comparison had successfully explained why the perimeter of the shaded triangle half was bigger in the previous question (see for example, Cameron's explanation in Table 4.2).

Of the 77.3% of students who had offered the correct *answer* to the area comparison (Q. 36h)

- 20 of them explained that the areas were the same and had explained that the perimeter of the triangle was longer in the previous question,
- 39 of them reasoned that the areas were the same but had stated that the perimeters were the same in the previous question, but were coded correct on the area question,
- 8 of them specified that the magnitude of the area (the same) was due to the perimeter and were coded incorrect, and
- 1 was coded incorrect on the area comparison despite offering the correct answer because her explanation was not mathematically correct.

More than two thirds (39 out of 59) of the 67% of students who were coded as correct at the area comparison used correct fraction or geometric visualisation reasoning but had unsuccessfully employed this reasoning for the previous perimeter comparison question (see for example, Sylvie's explanations in Table 4.2 and 4.3).

Some students argued in their explanation of the area comparison that the areas of the two shaded parts were the same *because* the perimeters were the same (which they were not). They were coded as incorrect because they offered a correct answer but with a mathematically incorrect reason. For example, Bella explained "Because they're just the same, the same length and width. And like if the perimeter is the same, the area would be the same too". The 9.1% of students who specified that the areas were the same because the perimeters were the same were included in the 33% who were coded incorrect on this part of the task. One further student gave the correct answer with a mathematically incorrect explanation and was also included in the 33% who were coded incorrect on this part of the task.

In comparing the areas in Q. 36h, 11.4% chose the triangle as the larger area (incorrectly) because the perimeter was larger. Some of the 37.5% of students who had identified that the perimeter of the shaded triangle was larger than the shaded square then used this intuitively to conclude, incorrectly, that one area was larger than the other.

The misconception that perimeters indicate area was also evident in the non-similar rectangle pair in the Similar Shapes task (Q. 36e, see Figure 4.4, and not included in the CATA category). After carefully using geometric reasoning to establish that the perimeters in the non-similar rectangles were the same, Bella's response to the area comparison was to explain that "they would probably be the same", ignoring the geometric reasoning that would have suggested that she could put the tall rectangle inside the fat rectangle with minimal restructuring and relying instead on the premise that perimeter and area were always related, adding that "Because if the perimeter would be the same, the area would be the same too" .

4.2.1.3 TPATL: tools and procedures tasks, attribute concept, length context.

The Blocks of Ice diagram (see Figure 4.5) was used for both the length and area tools and procedures questions assessing the key concept of attribute. In a three dimensional context 100% of the students correctly identified an example of the attribute of length on the diagram of the Blocks of Ice (see Table 4.1). The actual word *length* was volunteered by 38.6% of the students. As this was an open question, some also volunteered the more informal, but correct words; the *width* or *how wide*, the *depth* or *how deep*, the *height* or *how high*, or *how long* as well. A further 31.8% volunteered one or more of these less formal terms but not the formal word length. And 29.5% did not verbally volunteer any of these terms describing the attribute of length, but when prompted to show a length could successfully indicate on the diagram the dimensions of a length measurement.

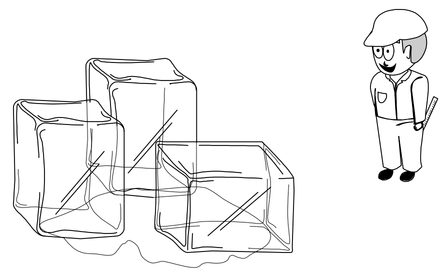


Figure 4.5. Diagram used in Blocks of Ice task, Q. 54.

4.2.1.4 TPATA: tools and procedures task, attribute concept, area context.

A two dimensional image of a three dimensional context (the Blocks of Ice) was used to assess students' knowledge of the attribute of area (see Figure 4.5). Area in a three dimensional context proved more confusing than length for the students with 33% of the students either volunteering or being prompted for the area and successfully describing what that would be in the three dimensional context (see Table 4.1). The most common error was to describe *what's inside* and indicate the volume of a block; 52.3% of the students did this. This error occurred in all three schools. This error had a higher frequency than the correct answer and explanation. On the other hand, every student who used the word *face* successfully described an area on the block.

In responding to the Blocks of Ice task (Q. 54) many children struggled to volunteer any attributes other than length. The open nature of this question enabled students to suggest any attributes that could be measured about the blocks and 14.8% suggested mass, 2.3% suggested temperature, 1.1% suggested hardness, 1.1% suggested opacity, and 1.1% suggested angle. However they did not always use these formal attribute terms. There were no suggestions to measure the attribute of time, but the blocks of ice context may have been more suggestive of mass and temperature than time.

4.2.2 Additivity.

The frequency of the students with each score on conceptual and tools and procedures questions, in both length and area contexts, assessing the key measurement concept of additivity are shown in Table 4.4. The entry-level protocol was used for three of the categories. All students were offered the entry-level task and the percentages in brackets beside some of the tasks used to rank success at the concept show the frequency of success of the entry-level task. If unsuccessful the students were offered the easier task, or if correct they were offered the harder task. In the tools and procedures length category, only two tasks were used: an entry-level task and an easier task.

Table 4.4

Additivity: Frequency of Success

Concept	Task type	Context	Score	Score frequency	Tasks used to rank success at the concept
Additivity	Conceptual	Length	0	11.4%	None of three below correct
			1	31.8%	Q. 43 Straightening wires
			2	19.3%	Q. 41 Freddo (56.8%)
			3	37.5%	Q. 42 Footy Card
Additivity	Conceptual	Area	0	5.7%	None of the three below correct
			1	51.1%	Correct count on Q. 62 Staircase Array or Q. 35 Missing Oval, if needed (94.3%)
			2	22.7%	Q. 33 Area Calculation, Half Rectangle (43.2%)
			3	20.5%	Q. 53 Area Calculation, Triangle
Additivity	Tools and Procedures	Length	0	22.7%	Neither of the two below correct
			1	19.3%	Q. 32 Measure DVD with a ruler
			2	58%	Q. 31a Streamer (58%)
Additivity	Tools and Procedures	Area	0	42%	
			1	58%	Q. 63 Area Calculation, Rectangle

4.2.2.1 CADL: conceptual tasks, additivity concept, length context.

To assess a conceptual understanding of additivity in a length context, (CADL), the Freddo task (Q. 41) was offered to all students (see Figure 4.6). If they could successfully say how long the Freddo Frog was, students were offered the Footy Card task (Q. 42), or if unsuccessful the Straightening Wires task (Q. 43) was offered instead (see Figure 4.6). This entry-level task protocol that enabled students' performance to be ranked for the CADL category had been validated in the pilot study. If unsuccessful at the entry-level Freddo task and the Straightening Wires task where students had to explain which bent wire was longer, a student was assigned a score of 0. If unsuccessful at the entry-level Freddo task but successful at the Straightening Wires task, a student was assigned a score of 1. If successful at the entry-level Freddo task but not the Footy Card task, a student was assigned a score of 2. Students who were successful at both the entry-level Freddo task and the Footy Card task were assigned a score of 3. The response frequencies in Table 4.4 show how many students achieved each score. So while 19.3% of students achieved a score of 2 in the CADL category and all of them correctly answered the Freddo task, the frequency of success on the Freddo task itself was 56.8% because the other 37.5% who were correct on the Footy Card task had also correctly answered the Freddo task.

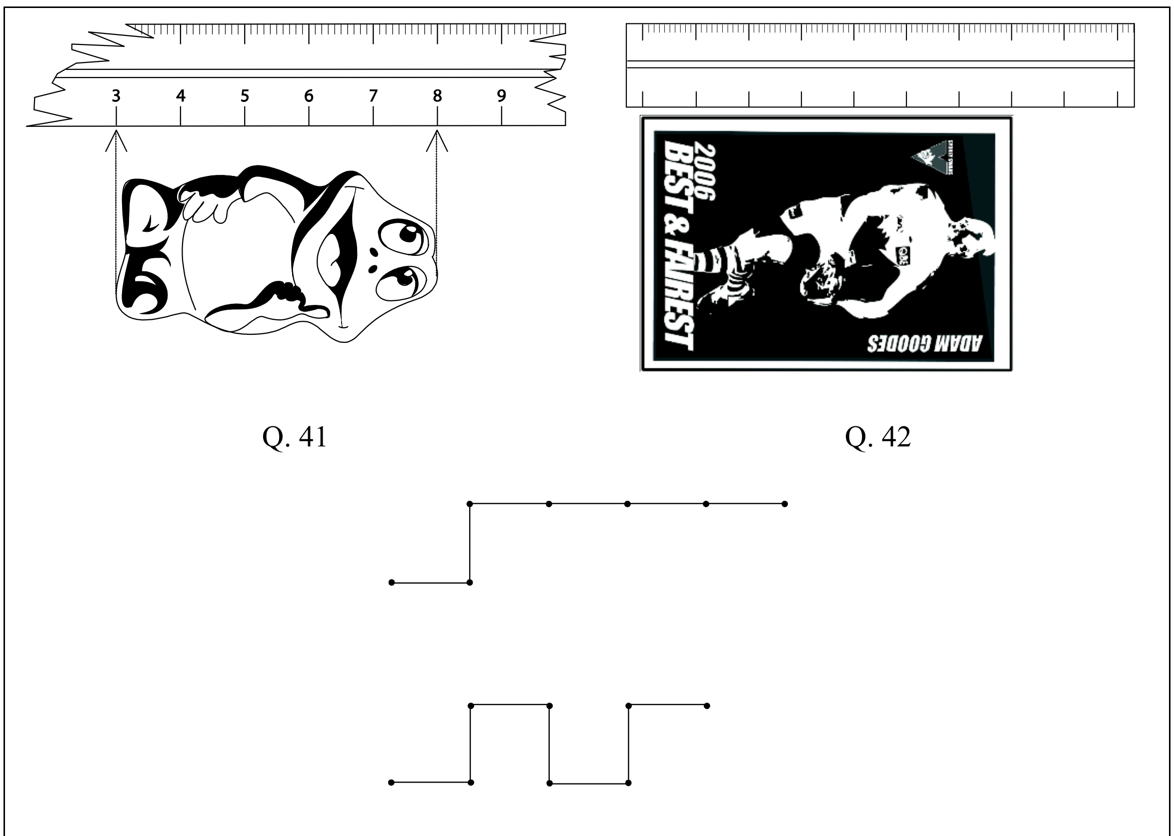


Figure 4.6. Diagrams used in tasks assessing a conceptual understanding of additivity in a length context.

In the present study, there were several different strategies employed by students who were successful at the Freddo task which all attempted because it was the entry-level task.

Numerical reasoning with the numbers three and eight was used by 20% of the students. The next most common approach in the present study was to imagine the three as a zero or a one and then to count from there and 14.8% of the students did that successfully. A similar number of students, 11.4%, counted the hash marks successfully, and a further group of students, 10.2%, counted spaces successfully.

The misconception of counting lines not spaces, incorrectly starting the counting sequence at the zero point and calling it *one*, was evident in the data. In the response to the Freddo task, 29.5% of the students demonstrated this strategy. Only students who had answered the Freddo task successfully were offered the Footy Card task and a third of them then demonstrated the counting lines not spaces misconception in the more difficult task. So overall, 48.9% of the students demonstrated this misconception, when the task was difficult enough.

4.2.2.2 CADA: conceptual tasks, additivity concept, area context.

To assess a conceptual understanding of additivity in an area context, CADA, the Area Calculation, Half Rectangle task (Q. 33) (see Figure 4.7) was offered to all students as a entry-level task. The students had to work out the area of the non-shaded part of the rectangle and explain their reasoning. The entry-level sequence had been validated in the pilot study. Students who were successful at this task were offered the Area Calculation, Triangle task (Q. 53) in which they had to identify the area of the shape and explain their reasoning. Students who were successful on both these tasks were assigned a score of 3. Students who were successful on the entry-level task, Area Calculation, Half Rectangle, but not the Area Calculation, Triangle were assigned a score of 2. There were two tasks that were used to define a baseline level of performance at the CADA category; the Staircase Array (Q. 62) and the Missing Oval task (Q. 35). In both tasks, students had to determine the area of the rectangles. They were not required to offer correct units to be correct in this category because that aspect of the task was used to assess the units category. Students who were unsuccessful at the entry-level task but could offer a correct count for one (or both) of the array tasks (Q. 62 or Q. 35) were assigned a score of 1. Students who were unsuccessful at the entry-level task, Area Calculation, Half Rectangle and were unable to offer a correct count of the area on both the Staircase Array task and the Missing Oval task were assigned a score of 0.

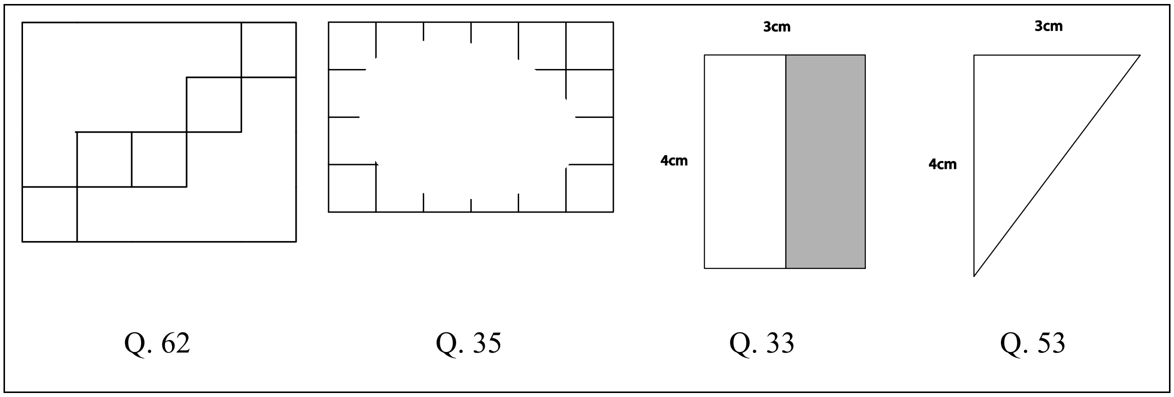


Figure 4.7. Diagrams used in tasks assessing a conceptual understanding of additivity in an area context.

The 20.5% of students with a score of 3 (see Table 4.4) who were successful at calculating the area of the triangle (Q. 53) all offered a fraction-based explanation, such as it's half of twelve, while none of them used the half base by height formula. A further 22.7% of students were successful at the entry-level task, Area Calculation, Half Rectangle (Q.33) but not the harder task. Hence 43.2% of the students had been able to calculate the area of half a four by three cm rectangle. Some successful students used fraction reasoning. Some students attempted to calculate the algorithm 4 x $1^{1}/_{2}$ but were unsuccessful. Just over half the students were successful on only the array tasks, and were unable to calculate the area of a half rectangle. A further 5.7% of students did not offer a correct count to the Staircase Array task (Q. 62) or if needed, the Missing Oval task (Q. 35). Errors with the count of the area on the Staircase array task came from all three schools.

4.2.2.3 TPADL: tools and procedures tasks, additivity concept, length context.

The entry-level task assessing the students' tools and procedures understanding of the concept of additivity in a length context was the Streamer task (Q. 31). Students iterated a 30cm ruler to measure a 93cm streamer and were assigned a score of 2 if they gave an answer between 92 and 94cm. There was no harder task and so a score of 3 was not possible. The less difficult task was the Measure a DVD task (Q. 32) and this was offered to students who were unsuccessful at the Streamer task. If the students were successful at measuring a 19cm DVD case with a 30 cm ruler (18.8 – 19.2 cm) they were assigned a score of 1, and if unsuccessful assigned a score of 0. A high degree of accuracy was required with an error of 1.1% allowed for both tasks.

The students demonstrated some difficulty with measuring the streamer and 58% of them were able to measure with the required degree of accuracy. Assuming that the students who measured the streamer successfully could also measure a 19cm DVD case, 77.3% of the students could use a ruler to measure.

4.2.2.4 TPADA: tools and procedures tasks, additivity concept, area context.

One task was used to assess students' tools and procedures knowledge of additivity in an area context. This was the pen and paper Area Calculation, Rectangle task (see Figure 4.8, or see Q. 63, before Q. 12 in Appendix A) in which students were asked, what is the area of this shape? Similarly to the CADA category, the students only had to give the correct count for the area of the 4 cm by 3 cm rectangle in this additivity category, but give the correct units, cm^2, to be successful in the units category aspect of the task. The correct count of units, 12, was given by 55.7% of the students (see Table 4.4). The most common incorrect answer was 14, possibly indicating the addition of the four lengths.

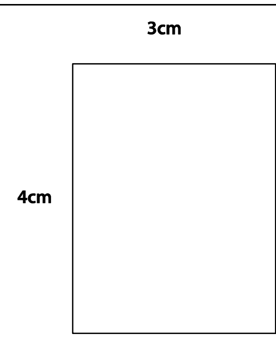


Figure 4.8. Area Calculation, Rectangle task.

The calculation of the area of a rectangle may be a prerequisite task for the calculation of the area of a half rectangle. The contingency table (see Table 4.5) shows that 33 of the students were correct on both of these tasks (area of a rectangle and area of a half rectangle). Of those who were only correct on one of the tasks 15.8% (3 out of 19) of them correctly calculated the area of the half rectangle and 84.2% (16 out of 19) of them could calculate the area of the rectangle (see Table 4.5). This suggests that the students were less likely to be able to calculate the area of a half rectangle before being able to calculate the area of a rectangle.

Table 4.5

Two Way Table of Association Between the Calculation of the Area of a Rectangle (Q. 63) and *Area of a Half Rectangle (Q. 33)*

	Area ½ rectangle correct	Area ½ rectangle incorrect
Area calculation rectangle correct	33	16
Area calculation rectangle incorrect	3	36

4.2.3 Units.

The percentage of the students with each score on conceptual and tools and procedures questions, in both length and area contexts, assessing the key measurement concept of units is presented in Table 4.6. The percentages in brackets beside some of the tasks used to rank success at the concept show the frequency of success of the entry-level tasks as these were the only tasks offered to all students. Three of the categories in the units tasks had questions that dealt with units and leftovers: CUNL, CUNA, and TPUNL. And the fourth category assessed students' knowledge of the correct standard units used with area measures: TPUNA.

Table 4.6

Units: Frequency of Success

Concept	Task type	Context	Score	Score Frequency	Tasks used to rank success at the concept
Units	Conceptual	Length	0	12.5%	Neither of the two tasks below correct
			1	33%	Q. 40a Using paperclips to measure
			2	54.5%	Q. 39 Keyboard (54.5%)
Units	Conceptual	Area	0	87.5%	
			1	12.5%	Q. 46 Array with leftovers (12.5%)
Units	Tools and Procedures	Length	0	15.9%	
			1	84.1%	Q. 64 Dragonfly (84.1%)
Units	Tools and Procedures	Area	0	52.3%%	Offers incorrect units (one or more times)
			1	10.2%	Offers informal units and/or no units only
			2	37.5%%	Offers cm^2 and no incorrect units for Staircase task (Q. 62) and Area of Rectangle (Q. 63) and, if needed Missing Oval (Q. 35).

4.2.3.1 CUNL: conceptual tasks, units concept, length context.

All students were offered both the Keyboard task (Q. 39) (see Figure 4.9) and the Swimming Pool task (Q. 65) (see Appendix A, before Q. 12). However, the Swimming Pool task was a pen and paper task and the students had difficulty understanding the question and therefore it has not been included in the results. Students who were successful on the Keyboard task were assigned a score of 2 (a score of 3 was not possible for this category). If unsuccessful on the entry-level Keyboard task, the students were offered the Using Paperclips to Measure task (Q 40a). The Keyboard task required a quantified description of the leftover part (and three quarters), while the Using Paperclips to Measure task only required a qualitative description of the leftover part (and a bit). Students successful on the Using Paperclips to Measure task were assigned a score of 1 and if unsuccessful they were assigned a score of 0.

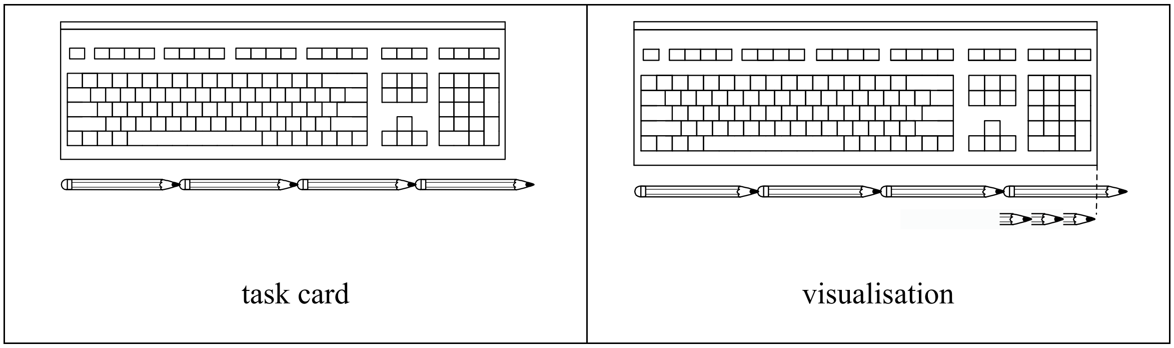


Figure 4.9. Task card (left) used for the Keyboard task (Q. 39), and mental visualisation (right).

The keyboard was exactly three and three quarter pencils long, but the students were allowed a margin for error in their estimation, as they could not draw on the task card. On this task 54.5% of students were successful (see Table 4.6). Correct answers (as long as they were accompanied by mathematically correct explanations) included:

- three and two thirds,
- three point seven,
- three and three quarters (or three and six eighths or three point seven five),
- three and four fifths (or three point eight),
- three and five sixths, and
- three and seven eighths.

The margin of error allowed was 3.75 +/- 0.125. Three and five eighths would have been an acceptable answer but was not offered by any student. Answers that fell outside of this margin on the lower estimate included: three and a bit, three and a half (or three and five tenths or three and three sixths) and three and just over a half. Answers that fell outside the margin on the upper estimate included: three and nine tenths (or three point nine) and three and four quarters (or three and three thirds).

Some students successfully used the excess part of the pencil as a sub-unit and iterated it back along the partial unit of the pencil that was measuring the keyboard (see Figure 4.9 right), counting three of these parts in the leftover part. Four ways of iterating the excess part were demonstrated:

- using a pincer grip to iterate the lengths,
- using the width of a finger as an informal unit
- making imaginary hash marks the width of the excess part, and
- comparing the excess part by eye to the whole pencil.

All these processes happened from right to left. The iterated unit was the excess part beyond the edge of the keyboard so its size was fixed and it was mapped back onto the rest of the pencil from right to left. This was demonstrated mainly with gestures and is evident on the video recordings of some interviews. Freya articulated this strategy, "Um, three pencils and. Three pencils and three quarters I think. Because this pencil [far right] doesn't exactly go against the keyboard, and I'm guessing if that much [touches part beyond keyboard] was all over this, it would be four [iterates three more times from right to left of last pencil] so three pencils and three quarters." Claire also used the left over part, explaining her answer of 3.75, "Because it's basically a quarter of a pencil too long".

Other students worked from left to right, proposing a sub-unit and estimating with that. This may explain some of the variety of fractional answers. Noah explained how he got his answer of 3.8, "Well, well obviously there's three pencils. There's four pencils there but the fourth one is a little too long. So I thought that I'd just divide it into tenths, I'd use tenths and eight, point eight would be about there."

Other students used splitting (Confrey, 1994) and worked from the middle of the last pencil to halve and halve again.

Some students elected to use a ruler to help them with this task. Some students successfully compared the length of the pencil and the length of the leftover part. For example, Jack explained, "well I measured the end pencil and it was three and a half, and the end of the keyboard was two and a half. So three pencils [traces them with his finger] and it was going to be around three quarters of a pencil." Chris used repeated addition for his ratio of two "point fives" less than seven "point fives" (2.5 cm and 3.5 cm), and renamed this five sevenths in his answer of three and five sevenths. These students were essentially working left to right. However, for some students, using the ruler led them to giving incorrect answers in combined units, pencils and centimetres such as George's response, "Is it three pencils and two centimetres and five millimetres?"

Some students were more holistic in their visualising. For example Lachlan explained how he got his answer of three and three quarter pencils, "Just by eye. I reckon it could be different, but."

Students who were unsuccessful at the Keyboard task were offered a hands-on measuring context with leftovers and different sized units, measuring a DVD case with paperclips, and 12.5% of the students were unsuccessful at both tasks (see Table 4.6).

4.2.3.2 CUNA: conceptual tasks, units concept, area context.

The area context of the units conceptual tasks was assessed by the Array with Leftovers task (Q. 46). There were sixteen whole units and four partial units in the array (see Figure 4.10). Students were given some wooden blocks which were 2 cm by 2 cm exactly the same size as the squares on the array, but there were not enough to cover the whole of array, and asked to find the area of the shape. The task had been used in the pilot study and had a frequency of success of 41.7%. This had indicated that it would be a good entry-level task. However the frequency of success was much lower in the present study, 12.5% (see Table 4.6) and this had the domino effect that very few students were offered the harder task, Packing Boxes (Q. 47). For this reason the Packing Boxes task has been excluded from the results reported here. Unfortunately, the materials in the (expected to be) easier task, the Cuisenaire Array, (Q. 48), proved unfamiliar for some students and this made the task harder than it was designed to be. For this reason the results of the Cuisenaire Array task have also been excluded from the results reported here. Students who were correct on the Array with Leftovers task were assigned a score of 1 and those unsuccessful were assigned a score of 0.

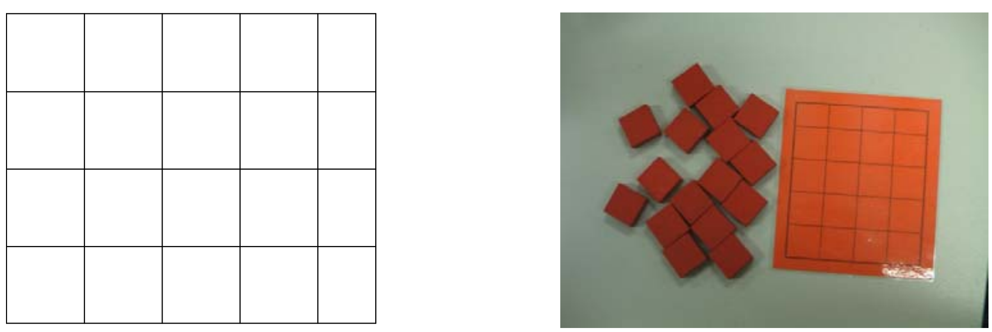


Figure 4.10. Array with Leftovers task (squares and wooden blocks were 2 cm square).

The Array with Leftovers (Q. 46) and the Keyboard task (Q. 39) used the same leftover: the length task had three quarters of a pencil leftover and the area array task had four three quarter squares left over. The same margin for error for describing the leftover piece was allowed in the area task; that is correct answers could be between $^5/_8$ and $^7/_8$ (0.75 +/- 0.125). 19 and $19^1/_5$ were the correct answers offered by the students. They had identified the partial squares as three quarters or four fifths. There was less variation in the description of the leftover part than in the Keyboard task. The students who were correct came from all three schools.

The two stage difficulty of the area task was illustrated by the further 11.4% of students who correctly identified the leftover parts as three quarters of a square, but made errors in combining the four leftover amounts. This response was present in all three schools. In addition, one student described the partial square unit as two thirds but miscalculated the addition of the four leftover parts. The students explained two different ways of adding the leftover parts. Some children successfully numerically added the leftovers, while others successfully used dynamic imagery to move parts of parts in an effort to make wholes with the leftover pieces. Two children successfully used dynamic rearrangement without quantifying the leftover part itself; they made wholes and counted those.

4.2.3.3 TPUNL: tools and procedures tasks, units concepts, length context.

The Dragonfly task (Q. 64) was the tools and procedures length task assessing the key concept of units (see Figure 4.11). The students had to use mixed decimal units, centimetres and millimetres, to describe the leftover, and 84% of them were successful at this task (see Table 4.6). As this was a pen and paper task, the strategies that the students used to arrive at their answers were not elucidated.

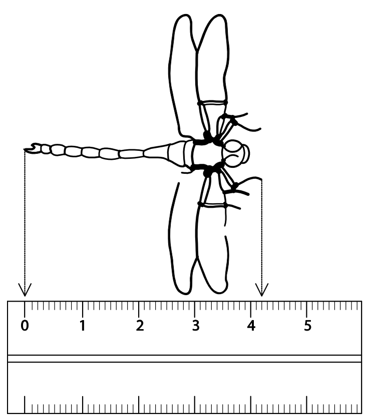


Figure 4.11. The Dragonfly task.

4.2.3.4 TPUNA: tools and procedures tasks, units concept, area context.

The tools and procedures area tasks assessed whether or not students volunteered the formal units cm^2 when calculating an area of a rectangle. Two initial pen and paper opportunities were provided: the Staircase Array (Q. 62) and the Area Calculation, Rectangle (Q. 63) (see

Figure 4.7 and Figure 4.8). For the TPUNA category, a successful count was not needed, just the choice of unit. The Staircase Array task and the Area Calculation, Rectangle task were both pen and paper tasks, hence there was no opportunity to prompt for units if none were offered. There were two reasons that a student may have needed a third opportunity to volunteer units: either no units had been offered in the first two tasks or only informal units had been offered. The third task, the Missing Oval task (Q. 35) (see Figure 4.7) was offered in the interview, where a prompt for units could be made. If a student offered an informal unit, for example, twenty four squares, they were not prompted for a formal unit.

Students were assigned a score based on their volunteering of formal units and whether they offered incorrect formal units (see Table 4.6):

- score of 2 (37.5%): student volunteered correct formal units, cm^2, without volunteering incorrect formal units (cm) in either the other one or two tasks,
- score of 1 (10.2%): student volunteered correct informal units and/or no units, but no incorrect formal units (cm), on the three tasks, or
- score of 0 (52.3%): student volunteered incorrect formal units (cm) on any of the two (or three) tasks

Most of the correct responses were written as cm^2 but one example of "2cm" after the number for the count was also accepted because it indicated centimetres square even if it was not written conventionally. Verbal descriptions accepted were *centimetres squared* and *square centimetres*.

Two of the tasks were on a pen and paper test so students' explanations were not elucidated. In the third task I prompted if the student did not offer any units, but I did not ask them to explain why they decided to offer the chosen unit. There is no data of students' explanations about their use of formal, informal or incorrect units.

The two way table for TPADA and TPUNA shows that many of children could do the calculation for area, (three times four) before they could also attribute correct units (see Table 4.7). More students calculated the area of the rectangle but also offered incorrect units (cm) on one or more of the tasks (19 students) than offered only correct formal units (cm^2) but could not calculate the area of the rectangle (6 students).

Table 4.7

Two Way Table of Association Between TPADA and TPUNA

	TPUNA Score		
	2 (correct cm^2)	1 (informal units)	0 (incorrect cm)
Area calculation correct	27	5	19
Area calculation incorrect	6	4	27

4.2.4 Proportionality.

The frequencies of success on the variations of the proportionality concept are reported in Table 4.8. Only one task each was used to assess the other three categories.

Table 4.8

Proportionality: Frequency of Success

Concept	Task type	Context	Rank	Response Frequency	Tasks used to rank success at the concept
Proportionality	Conceptual	Length	0	1.1%	None of the below correct
			1	18.2%	Q. 40b Paper clips
			2	33.0%	Q. 44 Steps (80.7%)
			3	47.7%	Q. 45 Choosing Rulers
Proportionality	Conceptual	Area	0	0%	
			1	100%	Q. 38b Array units
Proportionality	Tools and Procedures	Length	0	20.5%	
			1	79.5%	Q. 31b Streamer (diff to 1m)
Proportionality	Tools and Procedures	Area	0	10.2%	
			1	89.8%	Q. 38a Draw your own array

4.2.4.1 CPRL: conceptual tasks, proportionality concept, length context.

An entry-level task was offered for CPRL category: the Steps task (Q. 44). If students were successful at the Steps task they were offered the more difficult Choosing Rulers task (Q. 45). Students successful on both these tasks were assigned a score of 3. If successful on the entry-level Steps task but not the Choosing Rulers task they were assigned a score of 2. If unsuccessful on the entry-level task, they were asked Part B of the Paperclips task (Q. 40b). If successful on Part B of the paperclips task they were assigned a score of 1. If unsuccessful on both the entry-level task and Part B the Paperclips task, the assigned score was 0.

In the Steps task, students had to identify who would take a longer pace across a room (see Figure 4.12). By comparing the number of paces taken by four children presented in a table 80.7% of students successfully chose Tim because he took less steps (see Table 4.8).

Name	Number of steps
Jack	10
Emily	8
Max	9
Tim	7

Figure 4.12. Steps task (Q. 44).

In the Choosing Rulers task, the students had to choose a ruler to measure a pie but some of the rulers had uneven markings (see Figure 4.13). Just under half of the students (see Table 4.8) chose the equal interval ruler and used it consistently, or chose a non-equal-interval ruler but explained that they were imagining moving the spacings to make them equal. This second explanation was not common, but because the length of the pie being measured was close to 60 units, some students used a ruler that had sixty in the correct place between 0 and 100, explaining that the sixty was in the correct place even though the other numbers weren't.

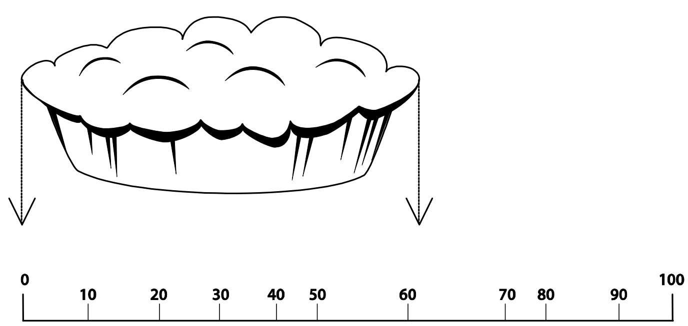


Figure 4.13. Choosing Rulers (Q. 45): pie image with one of the rulers.

In Part B of the Paperclips task (Q. 40b), students were shown two sizes of paperclips and asked, would you need more, or less, or the same number of (large) paperclips to measure the DVD than with small paperclips. The answer did not have to be quantified and one student was unsuccessful (see Table 4.8).

4.2.4.2 CPRA: conceptual tasks, proportionality concept, area context.

The task used to assess the area context of the concept of proportionality was Part B of the Draw Your Own Array task (Q. 38b). The students were asked to consider another rectangle the same size as the one they had restructured in Part A of the Draw Your Own Array task (Q. 38a, see section 4.2.4..4) and decide if they would need more or less or the same number of the new units than those used in Part A (see Figure 4.14). Only students who had been incorrect on the area comparison of the shaded shapes (Q. 36h), assessing the attribute concept, were offered Part B of the Draw Your Own Array task. It was assumed that if they could identify that non-congruent halves were the same, that they would be able to identify that fewer units of a larger size would be needed to measure an area. All students offered Part B of the Draw Your Own Array task answered correctly (see Table 4.8). This task was analogous to the easier task in the length context, Part B of the Paperclips task (Q. 40b) and the direction of change but not a quantified response was required. I had included the Four Triangles task (Q. 47) in the data collection interview as the harder CPRA task, but listening to the students' explanations of their answers to this task, I realised that it was an additivity task, not a proportionality task and so have not reported the results here. As the frequency of success on the TPPRA category was 100% and no other tasks were found to be suitable (or developed) for this category, it is not possible to calculate correlations using the category CPRA.

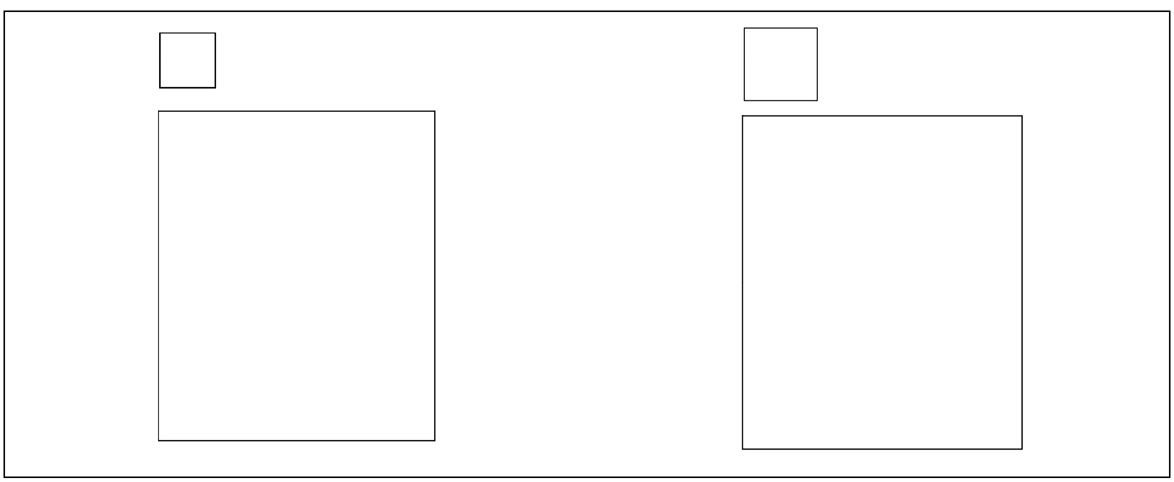


Figure 4.14. Draw Your own Array task Q. 38b.

4.2.4.3 TPPRL: tools and procedures tasks, proportionality concept, length context.

Part B of the Streamer task (Q. 31b) was used to assess students' understanding of converting between centimetres and metres. The accuracy of their answer was based on the difference to one metre from their measure (whether this was within the range for success at TPADL or not), and 79.5% were successful (see Table 4.8). If a student had offered a length with a fraction part for Part A of the Streamer task (e.g. $92^1/_2$ cm) the second part of the part of the question was rephrased so that he or she did not have to calculate the difference between a fractional answer and 100 cm (e.g., what is the difference between 92 cm and 1 m).

4.2.4.4 TPPRA: tools and procedures tasks, proportionality concepts, area context.

Students had to restructure a rectangle into an array in Part A of the Draw Your Own Array task (Q. 38a) (see Figure 4.14 left) to calculate the area. The unit (a 2 cm by 2 cm square) was printed on the paper and could not be moved onto the rectangle. Students had access to a ruler but did not have to use it. In order to be coded as correct, students had to meet three criteria: a) an answer of 30 squares or 120 cm^2 had to stated, b) the rectangle had to be subdivided into 30 units, or have the units marked on a length and width, and c) the units had to have dimensions between 1.5 and 2.5 cm. If only one unit was too small or too big, the student was still coded as correct. However, if a row or column was outside the 1.5 cm to 2.5 cm dimensions, the student was coded as incorrect even if there were 30 units. These criteria were double coded by a mathematics research assistant using the students' drawings from the interview and 89.8% of the students created an acceptable array or indicated the array with unit marks on a length and width (see Table 4.8).

Every student drew an array or indicated an array with marks on a length and a width but nine students did not draw an array with regularly sized units. Of these nine students, seven had a TPUNA score of 0 and had volunteered cm instead of cm^2 as units for an area calculation, one had a TPUNA score of 1 and had offered informal units only, while one student had a TPUNA score of 2 and used formal units correctly.

Some students had used the counting lines not spaces misconception in the Freddo task (Q. 41) or the Footy task (Q. 42) indicating some confusion over the first mark on the scale indicating the beginning of one unit but not the end of another. All students used one line to indicate the edge of adjacent units in their restructuring of the array in Part A of the Draw Your Own Array task. However, of the nine students who were unsuccessful drawing the array, most demonstrated misconceptions about the zero-point. Two demonstrated the counting lines not spaces misconception and a further two used dynamic imagery to realign the edge of the Freddo to the mark of 1 on the ruler on the Freddo task. All four of these students demonstrated a misconception about the zero-point and gave the answer of six (one too many). Another student could not use the broken ruler to measure and gave an answer of zero. Three of the nine students used the counting lines not spaces misconception on the Footy Card task (having been successful on the Freddo task). One of the nine students who could not draw the array accurately enough was successful on both the Freddo and Footy Card tasks.

4.3 Visualisation

Dynamic imagery and geometric reasoning were two strategies used to solve the visualisation tasks. The Flag task (Q. 56) used a real flag to model blowing in different directions, and the task card had four options for a flag blowing in the opposite direction to the target flag (see Figure 4.15). Alex gestured with her hand, flipping from the back to the palm, to indicate that she had used dynamic imagery, while adding "I imagined it blowing the other way". Students' geometric reasoning attended to the position of the shapes, for example that the square should be closest to the pole. Using either strategy, 90.9% of the students successfully chose the first flag.

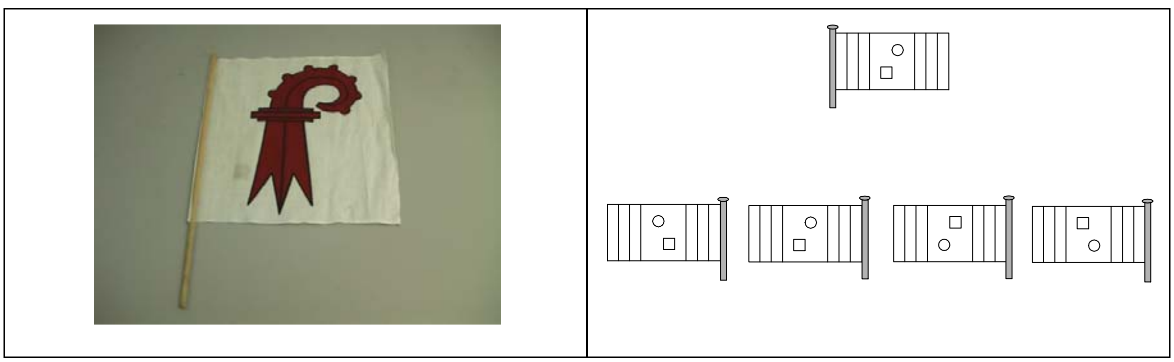


Figure 4.15. Flags task.

The Design task (Q. 59) required students to locate which of four smaller sections did not appear in the whole (see Figure 4.16). Students' geometric reasoning used a single identifying feature of the four fragments. For example, Nicky explained why he had chosen the correct piece (bottom right) saying, "Because, um, for this, there's no square that's by itself". The use of dynamic imagery involved mentally picking up each fragment as a whole and rotating it over the design to find where it superimposed over an identical part. It is possible that Alex was using dynamic imagery: "Because I looked at all of those [points to other 3] and they were all in, in it. And this one wasn't [points to bottom right]". She answered enthusiastically, "Yeah, yeah" when a further confirmatory question was asked, "were you sort of picking this shape up in your head and doing this [rotates fingers]". Using either strategy, 72.7% of the students successful chose the bottom right fragment.

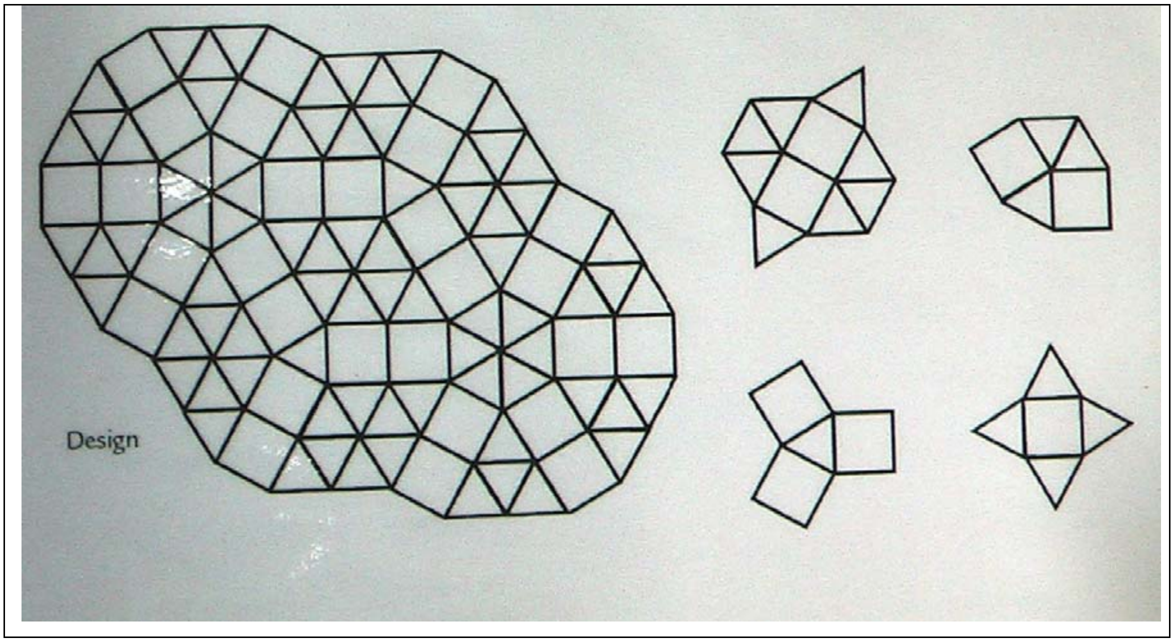


Figure 4.16. Design task.

The Cubes task (Q. 58) was modeled using a diagram and concrete materials (see Figure 4.17). The students were asked to make the target arrangement out of 2 cm wooden cubes. A premade model of the task card using more cubes was then shown to the student, who could then look at either the task card or the model to decide which one of the four options was the same arrangement of cubes. Nicky appeared to use dynamic imagery, explaining "Because it's just been turned upside down". He agreed, "yeah" when asked a confirmatory question, "did you move that around in your head." It was not possible to determine whether Alex used dynamic imagery or geometric reasoning in her explanation: "Ah, 'cause, 'cause I was looking around seeing, because I was like, because this one has two going that way and that way, and there's one like this one [compares her choice to top model] and that one's only got one on top like this one, and that's only got one there, and it's exactly the same". Alex agreed to a confirmatory question: "Ok so did you, were you just looking for, you know, so it's got one on top and then I need to look for the other bits?". But she also agreed to a confirmatory question proposing dynamic imagery: "Or were you imagining this in your head and imaging it picked up and moved around?". It is possible that Alex was using a mixture of the two strategies, but it is also possible that she was agreeing with the interviewer, and her final response, "Kind of both, yeah both" could be read either way. Using either strategy, 71.6% of the students successfully chose the first option.

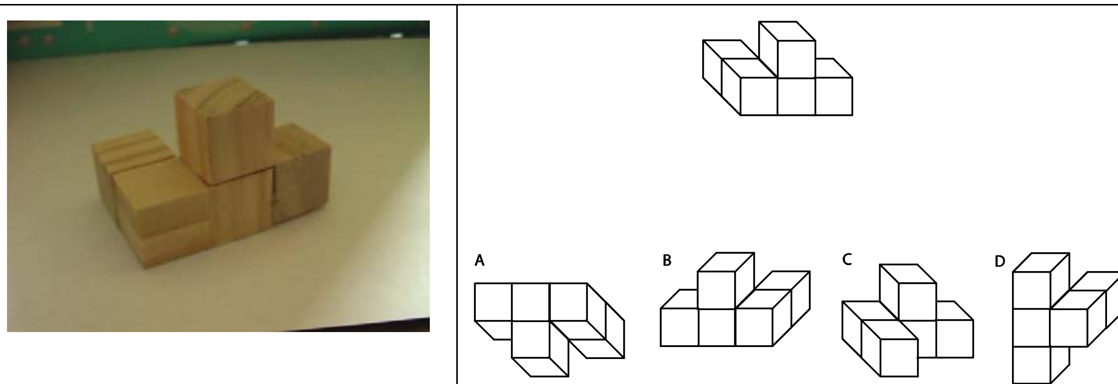


Figure 4.17. Cubes task.

The Puzzle task (Q. 57) required the students to choose shapes to fit together to make a square (see Figure 4.18) and then physically arrange them (see Appendix A, Q. 57) and 40.9% of students were successful.

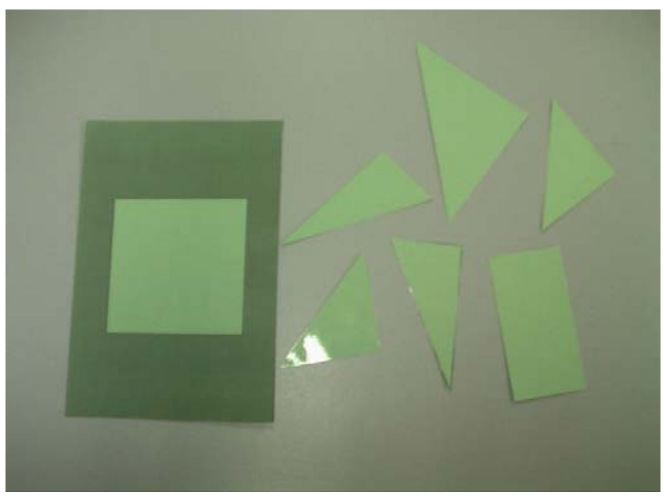


Figure 4.18. Puzzle task (Q. 57) pieces.

The Wattanawa Block task (Q. 60) could be solved using sophisticated dynamic imagery, or simple geometry if the student attended to that feature of the task (see Figure 4.19 left. Note: there is one dashed line that should be solid on the diagram with the corners labelled, but this was the diagram shown to the students). A model of the diagram made of wooden blocks was used in the question (see Fig. 4.19 right). Students were shown the first block model. This was then placed on the left and the other waved in the air indicating a rotation before it was placed on the right in the configuration that matched the rotation in the diagram. I then pointed to "corner 2" and asked what colour it would be. This adaptation was easier than the diagram as only eight corners were coloured.

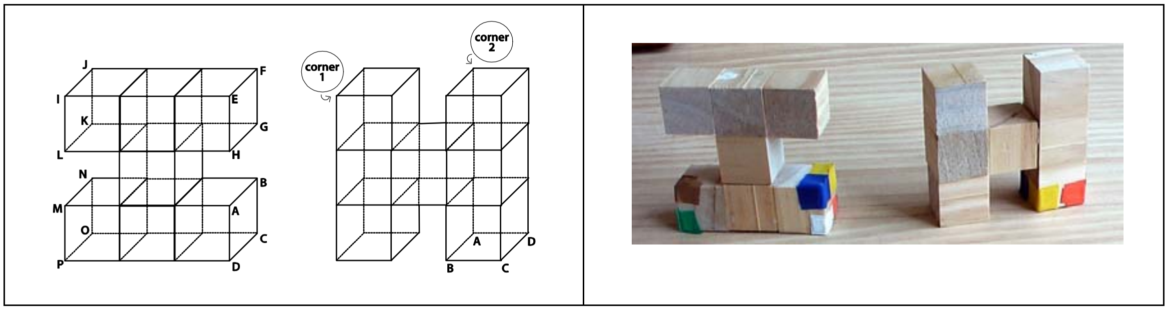


Figure 4.19. Wattanawa Block task.

Jack, who was successful on all the visualisation tasks appeared to use dynamic imagery, explaining "Well I pretended to flip the shape around til I got the orange and the yellow facing me and the blue down under there, and I looked to see what was in that corner." Alex appeared to use dynamic imagery but was unable to offer the correct answer, perhaps because she could not coordinate the two stage rotation, explaining "'Cause I, because in my head I flipped it, so that like it was the exact same as that. And then I thought that was there, and then I saw the green and the brown and the white one there."

Geometric reasoning concentrated on the edge between the blue and the unknown corner, and Nicky explained "'cause the blue is in line with the, is in line with the brown. Those two are in line [touches blue and brown corners on model 1] and those two are in line [touches blue and corner 2 model 2]".

However, it was difficult to determine whether some students' explanations indicated dynamic imagery or geometric reasoning. For example, Noah may have self corrected part way though his attempt, explaining "Well I turned this one around in my head [indicates model 1] and I noticed that one was there [points to brown corner on model 1] and I just realised if that's there, if that block was that block, so this would be there [brown corner to corner 2]". Lily's explanation was satisfactory and, while it suggested geometric reasoning, it may have been a justification of dynamic imagery: "On this side [model 2] on the same corner as this on the bottom there's blue." After a non-specific probing question, "Mmm", Lily elaborated, "So I looked, I checked for blue and it had this one". Using either strategy, 39.8% of the students were successful on this task.

The frequency of success on each of the five visualisation tasks is reported in Table 4.9. The tasks fell into three broad groupings:

- Group A - the Flags task (90.9% correct)
- Group B - the Design and the Cubes tasks (71.6-72.7% correct), and
- Group C – the Puzzle and the Wattanawa Block tasks (39.8-40.9% correct.

Table 4.9

Visualisation: Frequency of Success

	Flags (Q. 56)	Design (Q. 59)	Cubes (Q. 58)	Puzzle (Q. 57)	Wattanawa (Q. 60)
Success	90.9%	72.7%	71.6%	40.9%	39.8%

Two students did not get any of the visualisation tasks correct and 17% were successful on all five tasks. Students were given a score based on number of tasks correct (0 to 5). To be classified as following the trajectory Group A, Group B and Group C tasks

- students with a score of 1 was successful on the Group A task (Flags),
- students with a score of 2 were successful on the Group A task and one of the Group B tasks (Design or Cubes),
- students with a score of 3 were successful on the Group A task and both of the Group B tasks, and
- students with a score of 4 were successful on the Group A task, both Group B tasks, and one of the Group C tasks (Puzzle or Wattanawa Block).

Just under three quarters of the students with a score of 1 to 4 (52 out of 71) were successful in the order Group A, Group B, Group C (students with a score of 0 or 5 by definition followed this trajectory). However, due to difficulties distinguishing dynamic imagery from geometric reasoning, the visualisation score is not a dynamic imagery score.

The Puzzle task had a similar frequency of success to the Wattanawa Block task at around 40%. However, as the two way table shows (see Table 4.10) while 70.5% of the students were either successful on both tasks or unsuccessful on both tasks, the remaining 29.5% were split evenly in the order of difficulty of the task. So although the two tasks were superficially equally hard, performance on one did not necessarily predict performance on another.

Table 4.10

Two Way Table of Association Between The Puzzle task and the Wattanawa Blocks task

	Wattanawa Blocks task correct	Wattanawa Blocks task incorrect
Puzzle task correct	23	13
Puzzle task incorrect	13	39

4.4 Multiplication Results

Three tasks were used to generate a Multiplication score from 0 to 4 reflecting the students' success on four possible questions:

- Multiplication Tables (Q. 6, of which all eight had to be correct),
- 23x4 (Q. 10),
- Missing Number (Q. 11a, in which students were asked to think of a number that when multiplied by 54 would end in a 2, and
- Missing Number (Q. 11b), in which children were asked could any other number be multiplied by 54 and end in a 2.

If a correct answer was not offered in Part A of the Missing Number task (Q. 11a) then Part B (Q. 11b) was not offered (and assumed incorrect). These two parts were coded separately and were treated as two items in respect of the Multiplication score because this enabled the top 60% of the students to be split again into a top 20% of the group (see Table 4.11). Some children were assigned a Multiplication score of 0 because the Tennis Balls task (Q. 1), a task with 100% frequency of success, was not included in the score. The range of scores in all three schools was 0 to 4, with 4 indicating success on the four selected tasks. The division tasks were not reported here as there was not a division task difficult enough to split the top half of the students into smaller segments.

Table 4.11

Percentage of Students With Each Multiplication Score

	Multiplication Score				
	0	1	2	3	4
Percentage of students at each score	8%	13.6%	18.2%	39.8%	20.5%

4.5 Fraction Results

This section presents the frequency of success on some fraction tasks; an elaboration of some of the students' explanations to six fraction tasks; and correlations between fraction and measurement categories. The results of the fraction tasks are reported under the categories equivalence, measure, quotient, operator and ratio. The main focus of the present study was the measure sub-construct and equivalence understandings. The tasks classified as the measure sub-construct context reported in this chapter are:

- number lines (the Number Lines task (Q. 16),
- the relative size of fractions (the Fraction Pairs task (Q. 22), and
- area diagrams (the Fraction Pie task (Q. 14), the Fraction Sort task (Q19), and Fold Me a Quarter task (Q. 13)).

Kieren's four-three-four model of fraction understanding (1988, 1992, 1993, 1995) proposed that the concepts of partitioning, equivalence, and unit-forming could be drawn upon in measure sub-construct and quotient sub-construct contexts (see Figure 2.5). My interpretation of the tasks selected to represent the measure and quotient sub-construct includes a description of the concepts of partitioning, equivalence and unit-forming evident in the students' explanations.

4.5.1 Equivalence.

In the criteria for task selection, three tasks had questions that were used to assess equivalence knowledge in the interview criteria. These three tasks were the Fraction Sort task (Q. 19), the Golden Beans task (Q. 21), and the Fraction Pairs task (Q. 22). Two equivalence questions were included in the Golden Beans task (Q. 21). Equivalence was also assessed by the fraction pair comparison $^{2}/_{4}$ and $^{4}/_{8}$, (Q. 22b). Analysis of other fraction tasks determined that two further questions also assessed equivalence understanding: the fraction pair $^{3}/_{7}$ and $^{5}/_{8}$ (Q. 22f), and the fraction addition algorithm $^{1}/_{3} + ^{1}/_{2}$ (Q. 26c).

4.5.1.1 Fraction Sort task.

Eight cards from the 24 in the Fraction Sort Task (see Figure 4.20) were designed to include:

- area and discrete models for $^{2}/_{12} = ^{1}/_{6}$ (Q. 19c and d),
- different orientations of discrete models for $^{3}/_{12} = ^{1}/_{4}$ (Q. 19i and j)
- different discrete models for $^{1}/_{4}$; $^{2}/_{8}$ (Q. 19k) and $^{3}/_{12}$ (Q. 19i and j)
- an area and a length model for $^{2}/_{3}$ (Q. 19s and t), and
- an area model that required complex geometric re-structuring $^{6}/_{9} = ^{2}/_{3}$ (Q. 19r).

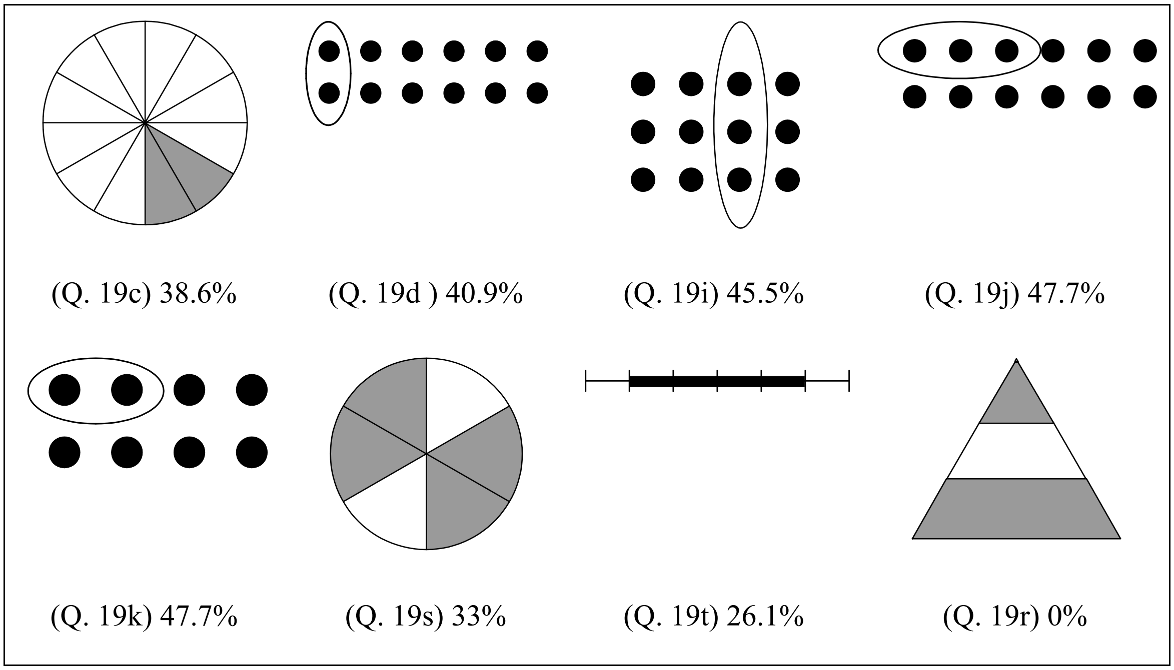


Figure 4.20. Task cards used for assessing equivalence in the Fraction Sort task (Q. 19) and frequencies of success.

Ten students' explanations for their sorting of the above eight Fraction Sort cards were double coded by a mathematics education lecturer, using video footage of the students completing the task. The double coder agreed with all correct/incorrect coding of the eight cards.

The Fraction Sort task (Q. 19) consisted of 24 individual cards, each with one fraction diagram, which the students had to sort into four piles: one quarter, one sixth, two thirds, and other. There were inscriptions of area, length and discrete contexts. The students held the cards and could orient them any way they chose. They could also place one to the back of the pile if they wanted to come back to it later. The cards were shuffled, but in general, one of the two cards depicting one quarter as one of four equal areas was placed on top as the first card for the student to consider. The cards were not presented in the same order for each student because the order could be changed by the student as they worked through the pile, and indeed some students did do this. The sorting piles were indicated by cards with a symbolic inscription ${}^{1}/_{4}$, ${}^{1}/_{6}$, ${}^{2}/_{3}$ and *other*, but the interviewer also read this to the student as one quarter, one sixth, two thirds, and "if it's not any of those, it's other".

The double coding of the Fraction Sort task (Q. 19) by a mathematics education lecturer also included coding students' explanations. The categories used were fewer than the exhaustive coding I had used to describe every variation of response in my initial coding. After discussion, the following seven categories were used successfully to clarify whether students had an understanding of the concept of equivalence and whether renaming a fraction was even considered. On the eight equivalence cards (see Figure 4.20) the students either:

- successfully identified the equivalence;
- stated the answer, for example, *it's other* (successfully or unsuccessfully);
- named the two quantities in the diagram, for example, three twelfths, but did not recognise that as one quarter;
- named one of the quantities, usually the denominator and rejected all the fraction possibilities because of this, for example, sorting the diagram of $^{2}/_{8}$ into the *other* pile because $^{1}/_{4}$, $^{1}/_{6}$, and $^{2}/_{3}$ didn't have an eight;
- miscounted one of the quantities and therefore was unable to simplify to an equivalent fraction from the incorrect starting fraction. This miscounting affected up to 6.8% of students and was more common on the diagrams with 12 parts than on diagrams with six or eight parts;
- demonstrated semiotic confusion about the conventions of the diagram and therefore did not state the correct initial fraction value, and was unable to simplify it to one of the three fraction possibilities. For example, some students, counted the lines not spaces in the length diagram and called four sixths, five sevenths, which made the recognition of two thirds as equivalent to four sixths impossible in that particular question;
- gave an explanation that was not one of the above.

In the present study, the first two categories were coded as correct if the correct pile was chosen. Explanations three and four were categorised as not showing awareness of the concept of equivalence.

Correct explanations of the placement of the equivalent fractions in the $^{1}/_{4}$, $^{1}/_{6}$ and $^{2}/_{3}$ piles of the Fraction Sort task included numerical explanations and spatial re-structuring explanations. The equivalences to a quarter had the highest frequencies of success (see Figure 4.20):

- 47.7% of the students correctly explained why the discrete $^{2}/_{8}$ (Q. 19k) was a quarter,
- 47.7% of the students correctly explained why the discrete $^{3}/_{12}$ (Q. 19j) was a quarter, and
- 45.5% of the students correctly explained why the discrete $^{3}/_{12}$ (Q. 19i) was a quarter.

For example, Rohan explained that "This one, with eight dots and it's got circled two, would be one quarter" (Q. 19k); that "This is one quarter 'cause it is circled three and there's twelve dots" (Q. 19j); and that "That one's one quarter as well, because it's got the dots, three out of twelve circled" (Q. 19i).

The equivalences to a sixth had a lower frequency of success (see Figure 4.20):

- 40.9% of the students correctly explained why the discrete $^{2}/_{12}$ (Q. 19d) was a sixth, and
- 38.6% of the students correctly explained why the circle $^{2}/_{12}$ (Q. 19c) was a sixth.

For example, Daniel explained that "two four six eight ten twelve, two out of twelve, would be one sixth" (Q. 19d) and Rohan explained that "And this one will go under one sixth 'cause it's cut up into twelve and it's got two shaded" (Q. 19c).

The equivalences to two thirds had a lower frequency of success than the unit fractions (see Figure 4.20):

- 33% of the students correctly explained why the circle $^{4}/_{6}$ (Q. 19s) was two thirds, and
- 26.1% of the students correctly explained why the length $^{4}/_{6}$ (Q. 19t) was two thirds.

For example, Rohan explained that "this one will go under two thirds because it's four sixths and then it's equivalence to two thirds" (Q. 19s) and that "And this is two thirds 'cause it's four sixths and if you halve it it will be two thirds" (Q. 19t).

There was evidence in Jade's explanation, of spatial re-structuring of the 12 dots as four columns (see Figure 4.21) leading to the circled three dots being described as "One quarter, because there's one part circled and three parts that aren't circled".

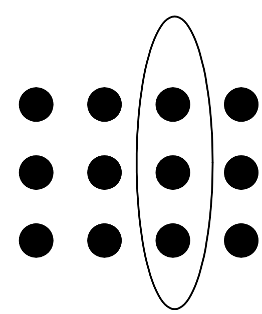


Figure 4.21. Three of twelve dots or one of four columns: numerical and spatial equivalence.

The Fraction Sort task required the students to *volunteer* equivalence because the word equivalence was not used in the task explanation, nor specifically hinted at; they had to reject an equivalent fraction to be incorrect. An example of a student's explanation illustrating this was Jess who placed the circle $^{2}/_{12}$, (Q. 19c) in *other* and explained, "Ah, I put that there [*other*] because um there was twelve there and there was two shaded; and I didn't think that was one sixth or two thirds or one quarter." Although not asked to consider equivalent fractions, Jess rejected one sixth as a possible other name for two twelfths.

Some students had the information needed to be able to see an equivalent relationship but did not appear to be looking for it. These students named both parts (numerator and denominator) but placed the card in other, like Jess above. It was as though students using this strategy did not think about fractions as having many names; equivalence was not "on their radar". Children from all three schools demonstrated this behaviour. The percentage of students who named both parts on each equivalence question in the Fraction Sort task is reported in Table 4.12. The frequency of correct explanations on the Fraction Sort cards equivalent to $^1/_4$ and $^1/_6$ cards was about 40-50%, and a further 35-40% stating the two quantities correctly but failed to connect that to the appropriate equivalent sorting pile.

Table 4.12

Counting behaviour in explanations of equivalence Fraction Sort cards

	Fraction Sort cards							
	Q.19i $^3/_{12}$	Q.19j $^3/_{12}$	Q.19k $^2/_8$	Q.19c $^2/_{12}$	Q.19d $^2/_{12}$	Q.19s $^4/_6$	Q.19t $^4/_6$	Q.19r $^4/_6$
Success (%)	45.5	47.7	47.7	38.6	40.9	33	26.1	0
Both N and D identified (equivalence not recognised)	39.8	36.4	36.4	37.5	40.9	52.3	50	2.3
Only one of N or D verbalised	2.3	4.5	5.7	12.5	9.1	9.1	4.5	-
Miscount of N or D	3.4	4.5	1.1	6.8	2.3	1.1	0	-
Semiotic confusion	0	0	0	0	0	0	9.1	-
other type of error	10.2	6.8	8	4.5	6.8	3.4	8	97.7

4.5.1.2 Golden Beans task.

The Golden Beans task used real lima beans, white on one side and spray painted gold on the other, as a discrete context for sixths. The students rolled the beans and named what they had rolled, and then were asked for an equivalent fraction in Part B of the task. In the second part of the task (Q. 21c and d) after the students' roll of their six beans, I added three more beans so that there were six of one colour (white or gold) and three of the other colour showing. The students were then asked to name the fraction of the gold or white beans (whichever was six out of nine). They were then asked for another name for that, or equivalent fraction if a further prompt was needed.

Exactly half the students were successful on Part B, renaming the fraction they had rolled with the six beans. In Part D, 47.7% were successful at offering an equivalent fraction for the six out of nine beans "rolled".

There were two students who misnamed their original roll of three gold and three white beans as three thirds and six thirds respectively, but then were able to rename the fraction as a half. These students were not coded as successful on the equivalence criteria because they could not offer two names for the same fraction, despite appearing to simplify in Part B. Simplifying or doubling were the only strategies used to generate an equivalent fraction by the successful students.

Two out of the eleven students who rolled $^5/_6$ incorrectly offered $^2/_3$ as an equivalent fraction. This may have been generated using the faulty mathematical reasoning that $^2/_3$ is equivalent to $^5/_6$ because the numerators were both one number away from the denominators.

The incorrect responses to renaming a fraction in the Golden Beans task (Q. 21) included:

- only being able to restate the quantities, for example three *out of* six for three sixths,
- gap thinking like behaviour as described above, for example $^2/_3$ for $^5/_6$,
- using the same numbers but restating an equivalence as a possible decimal, for example three and six *hundredths* for three sixths,
- flipping the numerals, for example six thirds for three sixths,
- Decimal-like renaming, for example three point nine for three ninths, and
- Other incorrect responses.

4.5.1.3 Fraction Pairs task.

The fraction pair comparison $^2/_4$ and $^4/_8$, (Q. 22b) drew on equivalence understanding. Most of the students who offered the correct answer that the fractions were the same explained using the words double or half, but few specified whether this was (correct) additive or multiplicative thinking. Jack described the process of looking at the difference between the numerator and the denominator, explaining "they're both half. Of the bottom number". Julia's use of the word half in her explanation could have been additive (correctly) or multiplicative: "'cause um four is half of eight and two is half of four, so they're both half of what the whole is." The word "half" was used as both an object and an operator in Penny's explanation, "Well two quarters is a half and then four eighths is half of eight so that would also be a half" and may have indicated a common denominators approach. Five children explicitly used correct additive thinking in their explanations of why $^2/_4$ and $^4/_8$ were the same. This was demonstrated in their use of the word "plus" in their explanations of the doubling of the numerator or halving of the denominator (see Table 4.13).

Table 4.13

Explanations using the word "plus" in correct responses to the fraction pair $^2/_4$ and $^4/_8$

Strategy	Explanation from transcript
Correct additive thinking	Emma: Well 'cause, you just, they're all, they're halved, so they would be the same. So there's four plus four is eight and two plus two is four.
	Patrick: they're both the same, because two plus two will equal four, and four – they're both half numbers
	Jordan: 'cause two plus two is four and four plus four is eight, which means they're like half.
	Hannah: 'cause if you do two plus two is four and four plus four is eight. So they are pretty much times two.
	Maxine: Because two plus two is four and four plus four is eight

Some explanations using the words "simplify", "twice" or "times" showed evidence of multiplicative thinking. Three students used the word "simplify" in their correct explanations. For example, Nicky said "'cause two quarters and four eighths; if you simplify four eighths you make it go down to two quarters, you simplify that again it would be one half". Jai explained that the fractions were the same "because they're both equal because this is times by two to get that, and this is times by two, so it's both equal." Aiden's explanation was "because two goes into four twice and four goes into eight twice." Along with Hannah (see Table 4.13) these were the only three students to use the words "times" or "twice". However, Hannah's explanation combined correct additive thinking with the use of the word "plus" and possible multiplicative thinking with the use of the words "times two".

The most frequent unsuccessful explanation was larger or higher numbers and this strategy accounted for just over a third of the incorrect responses.

No student was successful in comparing the fraction pair $^3/_7$ and $^5/_8$ (Q. 22f) without either using common denominators or benchmarking to a half. Lily used benchmarking to explain that $^{5}/_{8}$ was larger: "Because half of eight is four and means that's gone more, that's more than a half. And that one's three point five. And to go over a half, that has to be four." Two students used common denominators to calculate that $^{35}/_{56}$ was larger than $^{24}/_{56.}$ For example, Tom explained that $^{5}/_{8}$ was larger "Because I made these denominators the same by times-ing this by eight and this by seven, and that makes these fifty-six. And then I times-ed this by eight as well and this is twenty-four over fifty-six. And I times-ed this by seven and this is thirty-five over fifty-six".

In the initial design of the interview, there were 11 questions assessing equivalence understanding. However, after the interviews were completed, factor analysis on all the fraction tasks, revealed that a further two questions appeared to be associated with other tasks assessing fraction equivalence understanding. Although the interview data did not meet the assumptions for regressive analyses, a factor analysis was applied to the coded data as a rough tool to suggest where further interpretive analysis might prove useful. Tasks that had a frequency of success below 10% and above 90% were excluded because they would not provide useful correlations. The questions used in the factor analysis included:

- one multiplication question (Q. 11b),
- one division task (Q. 7),
- nearly all individual fraction questions,
- the measurement categories (e.g. CUNA), and
- all visualisation tasks.

As a factor analysis required more than a yes/no ranking, the explanations to the tasks were generally ranked into five categories:

- correct answer with a correct explanation,
- just a small slip up,
- right strategy but not executed properly,
- some relevant mathematical thinking, and
- incorrect mathematical approach.

These rankings did not provide the interval measure needed for factor analysis. After generating a 24 factor solution, I looked at the tasks that had clustered together and tried to suggest a common understanding, or context that might explain their association.

The factor analysis suggested two extra tasks that might be considered equivalence tasks. Detailed examination of the explanations and strategies of the students confirmed that the fraction pair $^3/_7$ and $^5/_8$ (Q. 22f) and the algorithm $^1/_3$ + $^1/_2$ (Q. 26c) were only successfully completed using equivalence knowledge.

By reviewing the notes taken at the time on the record sheets, and analysing the strategies used by successful students using video footage and transcripts of some student's explanations, it was determined that no student was successful comparing the relative sizes of the fractions $^3/_7$ and $^5/_8$ (Q. 22f) who did not use equivalence understanding to do so. Benchmarking to a half required equivalence comparisons, as did the use of common denominators. This fraction pair was therefore added to the list of tasks used to assess equivalence understanding. No student was successful on the fraction pair size comparison $^3/_4$ and $^7/_9$ who had not also been successful at the pair $^3/_7$ and $^5/_8$ and the inclusion of the former question may have further stratified the performance of the more successful students. However, I decided that this level of graduation of performance was not necessary and so did not include the fraction pair $^3/_4$ and $^7/_9$ in the questions that made up the Equivalence score.

4.5.1.4 Fraction Algorithms: Addition.

The addition algorithm $^1/_3$ + $^1/_2$ (Q. 26c) proved difficult for the students with 14.8% successful. The students' inscriptions which had been filed with their interview record sheets were double coded by a mathematics education lecturer. Students were not asked to explain their answers, but seven of the successful students used some form of common denominators, making equivalent fractions and left evidence of this in their written working out. One further student had notes on her record sheet indicating a verbal commentary of a common denominator approach. Five students left no evidence of their strategy as an inscription. However, both coders agreed these students would have used some form of equivalence knowledge in mentally working out their answer, as non-equivalence methods (for example, using wooden fraction models, or drawings) could be excluded. This interpretive analysis was prompted by the appearance of the addition algorithm question clustering with other equivalence tasks in a factor analysis of the data (see section 3..5.1.3 above for more detail).

4.5.1.5 Frequency of success on equivalence tasks.

The 13 questions from four fraction tasks categorised as equivalence questions spanned a range of difficulty (see Table 4.14). The easiest equivalence task in the present study was recognising halves in symbolic inscriptions (Q. 22b) and 72.7% of the students were successful. Part B of the Golden Beans task required an equivalent fraction to something out of six depending on the students' roll. However, some students rolled three sixths which may have been easier than other rolls. Nine questions, many of Fraction Sort cards, had frequencies of success ranging from 47.7% to 26.1% The next most difficult tasks were the questions that required equivalence as one step in multi-step reasoning, such as benchmarking or common denominators (Q. 22f and Q. 26c). The most difficult task (Q. 19r) required both geometric re-structuring and numerical equivalence and proved too difficult for all the students

Table 4.14

Frequency of Success on Thirteen Equivalence Tasks

Equivalence task	Frequency of success	Fraction
22b fraction pair $^2/_4$ $^4/_8$	72.7%	$^2/_4 = {}^4/_8 = ({}^1/_2)$
21b golden beans to $^x/_6$	50%	Sixths equivalent
19k discrete $^2/_8$ as $^1/_4$	47.7%	$^2/_8 = {}^1/_4$
19j discrete $^3/_{12}$ as $^1/_4$	47.7%	$^3/_{12} = {}^1/_4$
21d golden beans $^3/_9 = {}^1/_3$	47.7%	$^3/_9 = {}^1/_3$
19i discrete $^3/_{12}$ as $^1/_4$	45.5%	$^3/_{12} = {}^1/_4$
19d discrete $^2/_{12}$ as $^1/_6$	40.9%	$^2/_{12} = {}^1/_6$
19c circle $^2/_{12}$ equiv to $^1/_6$	38.6%	$^2/_{12} = {}^1/_6$
19s circle $^4/_6$ as $^2/_3$	33%	$^4/_6 = {}^2/_3$
19t length $^2/_3$ (no zero point)	26.1%	$^4/_6 = {}^2/_3$
26c algorithm $^1/_3 + {}^1/_2$	15.9%	Common denominators: sixths
22f fraction pair $^3/_7$ $^5/_8$	13.6%	Benchmark to $^1/_2$
19r triangle $^6/_9$ as $^2/_3$	0%	$^6/_9 = {}^2/_3$

The students' performance on the 13 equivalence questions was spread between a score of 0 (poorest performance) to 12 (highest actual performance). A score of 0, 1 or 2, was achieved by 37.4% of the students and 13.7% scored 10, 11 or 12 (see Table 4.15). As no students correctly re-partitioned the Fraction Sort triangle into $^6/_9$ and renamed this as $^2/_3$, no students achieved an Equivalence score of 13. There were students with scores of 0 and scores of 12 in all three schools.

Table 4.15

Spread of Equivalence Questions Correct

	Equivalence Score													
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
score frequency (%)	17	15.9	4.5	9.1	2.3	3.4	10.2	11.3	5.7	6.8	2.3	5.7	5.7	0

4.5.1.6 Equivalence pathways.

Frequencies of success (see Table 4.14) suggest a trajectory, but they do not prove that the same students were correct on the difficult tasks, or that the same students were incorrect on the easier tasks. To describe a general pathway of development requires looking at individual students' performances. This showed that there was a broad pathway of increasing levels of achievement:

- Level 1, equivalences to $^1/_2$: the fraction pair $^2/_4$ and $^4/_8$ (Q. 22b), and the initial roll of the Golden Beans task (Q. 21b);
- Level 2, the equivalence $^1/_3 = ^3/_9$ in the Golden Beans task (Q. 21d); and equivalences to $^1/_4$, $^1/_6$ and $^2/_3$ in the Fraction Sort task (Q. 19c, Q. 19d, Q. 19i, Q. 19j, Q. 19k, Q. 19s, Q. 19t, (not Q. 19r));
- Level 3, both of the benchmarking fraction pair $^3/_7$ and $^5/_8$ (Q. 22f) and the addition algorithm $^1/_3 + ^1/_2$ (Q. 26c), and
- Level 4, the $^2/_3$ equivalence to $^6/_9$ in the triangle fraction sort card (Q. 19r).

Level 2 had the most questions and the students' performance can be further classified by the number correct (one to four, and five to eight), but not by success on particular tasks. There was also a group of students whose performance spanned the overlap between Level 2 and Level 3: one or both of the questions for Level 3 correct but not all questions in Level 1 and 2 questions. There were also students who had an Equivalence score of 0 (see Table 4.15). Seven Equivalence Bands of performance are described below.

The students in Band A had an Equivalence score of 0 and were not successful on any of the 13 equivalence questions (see Figure 4.22). None of these students by definition and in actual fact were successful at any questions in higher bands.

		Equivalence Band A														
		Ricky	Harry	Finn	Zannah	Jonno	Milly	Caitlin	Ash	Clara	Josh	Olivia	Tamika	Cassie	Frank	Cath
	Score	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Level 1	Q. 22b	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 21b	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Level 2	Q. 21d	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19k	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19j	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19i	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19d	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19c	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19s	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19t	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Level 3	Q. 26c	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 22f	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
4	Q. 19r	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

x denotes incorrect response, c denotes correct response

Figure 4.22. Students' performance on Equivalence tasks: Band A

Students in Band B had an Equivalence score of 1. Except for Jess, they were successful either at renaming the fraction rolled in the Golden Beans task or at explaining that ${}^{2}/_{4}$ and ${}^{4}/_{8}$ were the same (see Figure 4.23). The four students who successfully offered an equivalent fraction in Part B of the Golden Beans task had rolled a half (three gold and three white beans). All the students in Band B (except for Jess) were not successful on any of the tasks in Level 3 and above, which included Part D of the Golden Beans and the Fraction Sort cards; the fraction algorithm and the other fraction pair ${}^{3}/_{7}$ and ${}^{5}/_{8}$. Jess had rolled four out of six with the Golden Beans and so was not given the opportunity to name an equivalent fraction to a half in this question which would have classified her as Band C.

		Equivalence Band B													
		Josie	Tyler	Simon	Courtney	Jasmine	Ebony	Isabel	Maxine	Will	Sylvie	Sam	Rose	Angus	Jess
	Score	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Level 1	Q. 22b	c	c	c	c	c	c	c	c	c	x	x	x	x	x
	Q. 21b	x	x	x	x	x	x	x	x	x	c	c	c	c	x
Level 2	Q. 21d	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19k	x	x	x	x	x	x	x	x	x	x	x	x	x	c
	Q. 19j	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19i	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19d	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19c	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19s	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 19t	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Level 3	Q. 26c	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 22f	x	x	x	x	x	x	x	x	x	x	x	x	x	x
4	Q. 19r	x	x	x	x	x	x	x	x	x	x	x	x	x	x

x denotes incorrect response, c denotes correct response

Figure 4.23. Students' performance on Equivalence tasks: Band B

Students in Band C were all of those with an Equivalence score of 2 to 5 and half of those with an Equivalence score of 6 (see Figure 4.24). They were successful on four or less of the tasks in Level 2 (Part D of the Golden Beans task and the Fraction Sort cards excluding Q. 19r) and at either/both of the tasks in Level 1. None of them were successful on either of the tasks at Level 3.

	Equivalence Band C																						
		Cadel	Hannah	Kit	Annabelle	Patrick	Kelly	Mia	Brad	Ruby	Lara	Chloe	David	Meg	Aiden	Kate	Matthew	Julia	Rebecca	Claire	Bella	Declan	Adele
	Score	3	3	3	2	2	2	3	4	5	5	2	3	3	3	3	4	5	6	6	6	6	6
Level 1	Q. 22b	x	c	c	c	c	c	c	c	c	c	x	c	c	c	c	c	c	c	c	c	c	c
	Q. 21b	x	x	x	x	x	x	x	x	x	x	c	c	c	c	c	c	c	c	c	c	c	c
Level 2	Q. 21d	x	x	x	c	c	x	x	x	x	c	x	c	c	c	c	x	x	c	c	x	c	x
	Q. 19k	x	x	x	x	x	x	x	c	x	c	x	x	x	x	x	x	c	x	c	x	c	c
	Q. 19j	c	x	c	x	x	x	x	c	c	c	c	x	x	x	x	x	c	x	c	c	c	x
	Q. 19i	c	x	c	x	x	x	c	c	c	c	x	x	x	x	x	x	c	x	x	x	c	x
	Q. 19d	x	c	x	x	x	x	c	x	c	x	x	x	x	x	x	x	x	c	x	c	x	c
	Q. 19c	x	c	x	x	x	x	x	x	x	x	x	x	x	x	x	c	x	c	c	c	x	c
	Q. 19s	x	x	x	x	x	c	x	x	c	x	x	x	x	x	x	c	x	c	x	c	x	c
	Q. 19t	c	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Level 3	Q. 26c	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 22f	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
4	Q. 19r	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

x denotes incorrect response, c denotes correct response

Figure 4.24. Students' performance on Equivalence tasks: Band C

Students in Band D had an Equivalence score of 6 to 10, but were not all the students with these scores. Students in Band D were successful at five to eight of the questions at Level 2 and either/both of the questions at Level 1 (see Figure 4.25). These students were consolidating their understanding of equivalence tasks presented in length and area diagrams, but were not successful on either of the tasks in Level 3. Some of the students who were unsuccessful at Part B of the Golden Beans task (Q. 21b) may have rolled $^5/_6$ which made the task harder than for those who rolled a half.

		Equivalence Band D																				
		Elsie	James	Cameron	Zak	Freya	Amelia	Alex	Shannon	Daniel	Ben	Penny	Emma	Brooke	Michael	Fiora	Emily	Nicky	Tony	Jack	Yani	Jade
	Score	6	6	6	6	7	7	7	8	7	7	7	7	7	7	8	9	9	9	9	9	10
Level 1	Q. 22b	x	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
	Q. 21b	x	x	x	x	x	x	x	x	c	c	c	c	c	c	c	c	c	c	c	c	c
Level 2	Q. 21d	x	x	c	x	c	c	c	c	c	c	c	c	c	x	c	c	c	c	c	c	c
	Q. 19k	c	c	c	c	c	c	c	c	c	c	x	c	c	c	c	c	c	c	c	c	c
	Q. 19j	c	c	c	c	c	c	c	c	x	c	c	x	c	c	c	c	c	x	c	c	c
	Q. 19i	c	c	c	c	c	c	c	c	x	c	x	x	c	c	c	c	c	c	c	c	c
	Q. 19d	c	c	x	c	c	c	x	c	c	c	x	c	x	c	c	c	c	c	c	c	c
	Q. 19c	c	c	x	x	c	c	x	c	c	x	c	c	x	x	c	c	c	c	c	x	c
	Q. 19s	x	x	c	x	x	x	c	c	x	x	c	x	x	c	x	x	c	c	x	c	c
	Q. 19t	c	x	x	c	x	x	c	x	c	x	c	c	c	x	x	c	x	c	c	c	c
Level 3	Q. 26c	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	Q. 22f	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
4	Q. 19r	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

x denotes incorrect response, c denotes correct response

Figure 4.25. Students' performance on Equivalence tasks: Band D

Students in Band E included all the rest of those with Equivalence scores of 7 to 10 and all of them with an Equivalence score of 11. They successfully calculated the fraction algorithm (Q. 26c) and/or successfully compared the fraction pair $^3/_7$ and $^5/_8$ (Q. 22f), but were not successful on every question at Level 1 and Level 2 (see Figure 4.26). Both the questions at Level 3 required the use of equivalence knowledge, such as benchmarking or using common denominators, to complete another strategy.

	Equivalence Band E											
		Jordan	Felix	Jai	George	Adam	Seth	Noah	Sarah	Dylan	Alec	Tom
	Score	7	9	8	8	8	10	11	11	11	11	11
Level 1	Q. 22b	c	x	c	c	c	c	c	c	c	c	c
	Q. 21b	x	c	c	c	c	c	c	c	c	c	c
Level 2	Q. 21d	c	c	c	x	c	c	c	c	c	c	c
	Q. 19k	c	c	x	c	c	c	c	c	c	c	c
	Q. 19j	c	c	c	c	x	c	c	c	c	c	x
	Q. 19i	x	c	c	c	x	c	c	c	c	c	c
	Q. 19d	x	c	x	c	c	x	c	c	c	c	c
	Q. 19c	x	c	c	c	c	x	c	c	c	c	c
	Q. 19s	c	c	x	x	c	c	c	c	c	c	c
	Q. 19t	c	x	x	x	x	c	x	c	c	x	c
Level 3	Q. 26c	c	x	c	x	c	c	c	c	c	c	c
	Q. 22f	x	c	c	c	x	c	c	x	x	c	c
4	Q. 19r	x	x	x	x	x	x	x	x	x	x	x

x denotes incorrect response, c denotes correct response

Figure 4.26. Students' performance on Equivalence tasks: Band E.

All the students in Band F had an Equivalence score of 12. They were correct on all of the equivalence questions in Levels 1, 2 and 3 (see Figure 4.27). No students could be categorised as being in Band G as no students were successful at the Level 4 task of the triangle equivalence (Q. 19r) in the Fraction Sort task.

		Equivalence Band F				
		Rohan	Lachlan	Chris	Lily	Steven
	Score	12	12	12	12	12
Level 1	Q. 22b	c	c	c	c	c
	Q. 21b	c	c	c	c	c
Level 2	Q. 21d	c	c	c	c	c
	Q. 19k	c	c	c	c	c
	Q. 19j	c	c	c	c	c
	Q. 19i	c	c	c	c	c
	Q. 19d	c	c	c	c	c
	Q. 19c	c	c	c	c	c
	Q. 19s	c	c	c	c	c
	Q. 19t	c	c	c	c	c
Level 3	Q. 26c	c	c	c	c	c
	Q. 22f	c	c	c	c	c
4	Q. 19r	x	x	x	x	x

x denotes incorrect response, c denotes correct response

Figure 4.27. Students' performance on Equivalence tasks: Band F.

The spread of performance across these descriptive Bands demonstrated (see Table 4.16) that the students in the present study could be classified into five roughly equal sized groups. These were Band A, Band B, Band C, Band D, and Band E and F combined. Students with an Equivalence score of 0 to 5 appeared to follow a similar trajectory: not successful at equivalence tasks, can recognise or make equivalences to a half, and begins to recognise equivalences to $^{1}/_{4}$, $^{1}/_{6}$, or $^{2}/_{3}$. Students with an Equivalence score of 6 could be found in either

Band C or D. The was no specific Fraction Sort question in Level 2 at which every student in Band D was successful. These students were consolidating equivalences to $^1/_4$, $^1/_6$, or $^2/_3$.

A conceptual leap in equivalence understanding was represented by entry into Band E. Students in this Band began using equivalence knowledge in other strategies such as benchmarking or the use of common denominators. Jordan in Band E with an Equivalence score of 7 represented the earliest transition into this stage of the trajectory. Students in Band E were not necessarily correct on every task in Level 2. Students in Band F had consolidated the equivalence knowledge up to this point. However, they did not demonstrate sophisticated spatial restructuring and equivalence on the triangle Fraction Sort card (Q. 19r).

Table 4.16

Equivalence Bands: Seven Broad Groupings of Performance on Equivalence Questions

Band	Question type	Frequency	Equivalence score
A	None correct	17%	0
B	Equivalences to 1/2	15.9%	1
C	Length and area diagrams, discrete objects (1-4 correct)	15.9%	2-6
D	Length and area diagrams, discrete objects (5-8 correct)	23.8%	6-10
E	Benchmarking and/or common denominators	12.5%	7-11
F	Consistency in Levels 1-3	5.7%	12
G	Sophisticated spatial restructuring and equivalence	0%	13

4.5.1.7 Correlations between Equivalence score and measurement categories.

Equivalence scores rather than Equivalence Bands were chosen for the calculation of correlations with other categories because in similar calculations for other categories, scores were used. There were correlations between students' Equivalence scores and their performance in measurement categories (see Table 4.17). There was a substantial association between the students' Equivalence score and the conceptual tasks in the additivity concept (both length and area contexts). There was a substantial association between the students' Equivalence score and tasks assessing the units concept (the conceptual tasks in a length context and the tools and procedures tasks in an area context). These substantial associations were to the broken ruler tasks, calculating the area of half rectangles, the Keyboard task, and offering cm^2 as the units for area calculations. The only categories that did not have a relationship with the students' Equivalence scores were the area and perimeter comparisons of the shaded halves in the Similar Shapes task (CATL (Q. 36g) and CATA (Q. 36h)). TPATL and CPRA were not calculated. The students' Multiplication score had a substantial association with their Equivalence score.

Table 4.17

Correlations Between Equivalence Score and Measurement Concepts

	Correlation		
	Minimal $\tau > .07$	Typical $\tau > .20$	Substantial $\tau > .34$
Attribute	TPATA $\tau = .195$, $p = .034$		
Additivity	TPADL $\tau = .172$, $p = .050$	TPADA $\tau = .273$, $p = .003$	CADL $\tau = .377$, $p < .000$ CADA $\tau = .370$, $p < .000$
Unit		CUNA $\tau = .315$, $p = .001$ TPUNL $\tau = .215$, $p = .019$	CUNL $\tau = .486$, $p < .000$ TPUNA $\tau = .414$, $p < .000$
Proportionality		CPRL $\tau = .240$, $p = .006$ TPRL $\tau = .245$, $p = .008$ TPPRA $\tau = .245$, $p = .008$	
Multiplication			MULT $\tau = .428$, $< .000$

All of the unit concept categories showed a correlation with Equivalence score. The CUNL category (CUNL) had the strongest effect size of all the reported associations, equating to a common variance of 47%. One task used to assess the CUNL category was the Keyboard task (Q. 39). Of the fifteen students who had an Equivalence score of 0, one gave the correct answer of three and three quarters to the Keyboard task (Q. 39). The incorrect descriptions of the length of the keyboard included many answers with four as the whole number instead of three: four and an inch, four and a half (two students), four and a quarter, four and two quarters, three and four quarters, three and two halves, three and a half, three and just over a half, three and a sixth, twelve centimetres, three and two and a half centimetres, and three point nine (imprecise estimation, two students). Of the fourteen students who had an Equivalence score of 1, four gave the correct answer of three and three quarters, three and two thirds (two students), and three point seven to the Keyboard task (Q. 39). The incorrect descriptions of the length of the keyboard also included answers with four as the whole number: four and three quarters, four and a half, four and a bit, three and three sixths, three and five tenths, three and a third, three and a quarter, ten pencils, three quarters (missing whole number), and three and nine tenths (imprecise estimation). There were 12 answers with four as the whole number in the Keyboard task and eight of them were offered by students with Equivalence scores of 0 or 1.

The students' Equivalence score had a substantial association with offering standard units for area (TPUNA). The Equivalence Bands offer another way of examining the association between equivalence understanding and TPUNA. Around 70% of students in Equivalence Bands A, B and C had a TPUNA score of 0 and had offered cm incorrectly as a unit for an area calculation (see Table 4.18). The students in Bands D and E were more likely to offer the correct units of cm^2 in area calculations. However, the percentage who had a TPUNA score of 0 was very similar in both Bands (30% and 36%). The successful use of formal units was demonstrated by 100% of the students in Equivalence Band F.

Table 4.18

Volunteering cm^2 and Equivalence Band

	TPUNA score		
	0	1	2
Band A: Equivalence Score 0	11	2	2
Band B: Equivalence Score 1	10	2	2
Band C: Equivalence Score 2-6	15	2	5
Band D: Equivalence Score 6-10	6	4	10
Band E: Equivalence Score 7-11	4	0	7
Band F: Equivalence Score 12	0	0	5

4.5.2 The measure sub-construct.

4.5.2.1 Number lines.

The number line task consisted of eight questions that were devised and selected in order to assess both students' ability to read and partition number lines, from 0 to 1 and from 0 to greater than 1. The results of students' explanations of three questions in particular are reported in more depth:

- Q. 16a, draw a number line and mark $^2/_3$ on it,
- Q16d reading 3 $^3/_4$, and
- Q. 16e reading $^5/_6$.

Frequencies of success on all eight questions and Number Line scores are reported.

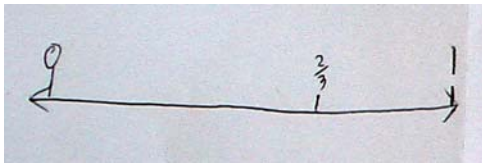
4.5.2.1.1 Draw a number line and mark two thirds on it.

For the first number line question students were asked to draw a number line and put two thirds on it. As the number lines were drawn by the students, they were not of uniform length. Drawing two thirds by eye was acceptable, although some students elected to use a ruler to help them draw the number line and mark two thirds. I decided it reasonable that a mark up to but not including $^3/_4$ would be an acceptable representation of two thirds, given that the task could be completed without aids such as a ruler, the upper and lower limits were determined by what would be a reasonable margin for error by eye – when it should look wrong. Using similar reasoning for the lower band resulted in the successful range being 58% and 74% of distance between 0 and 1. The categorisations of the number line inscriptions were

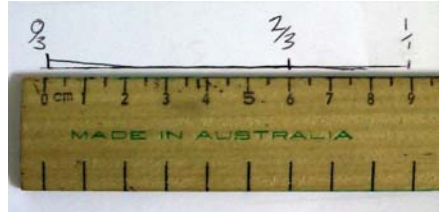
- Successful positioning of $^2/_3$ (between 58% and 74% of the distance between 0 and 1)
- Ratio misconception: $^2/_3$ marked at a correct ratio of $^2/_3$, e.g. 6 out of 9
- Unsuccessful positioning of $^2/_3$ because of the placement of 1; but $^2/_3$ positioned between 58% and 74% of the distance along *the line*
- other

All the inscriptions made by the students in drawing their number line (Q. 16a) were double coded by an independent coder with a science PhD. The coder measured the marks on the number lines to check four categorisations and this double coding did not require teacher judgement. The students' inscriptions were then discussed and agreement reached about categorisation of the students' drawn responses. Some of the students' explanations for their drawings were transcribed from video and audio recordings and used for more detailed strategy descriptions.

The frequency of success on this question was 33%. There were two main correct strategies demonstrated by students in drawing a number line and marking $^2/_3$ on it. Some students drew the number line by hand and iterated an imagined third (see Figure 4.28). Daniel iterated with a pincer grip several times until he was sure that his thirds were equal. Other students used a ruler and coordinated the ratio understanding and the conventions of a number line, for example, Alec explained "well I did nine centimetres and you can divide that by three and each third is three centimetres, so I just went up to the six which is the second third and I put the mark there".



Daniel



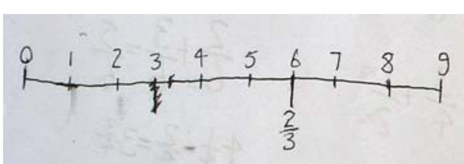
Alec

Figure 4.28. Number lines with $^2/_3$ correctly placed.

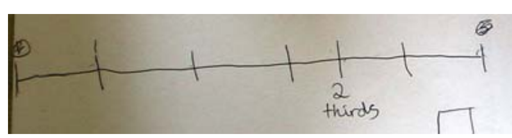
One misconception was to use a ratio understanding but not coordinate this with the conventions of the number line. For example, a number line nine units long might be used to calculate $^2/_3$ of 9 but the point $^2/_3$ cannot be at the same point on the number line as 6. Rohan demonstrated successful ratio knowledge, but not an understanding of the conventions of a number line in his explanation, "I made it so that I could have nine numbers and so it would equally divide up into three parts in the end. Or we could have had three numbers and that would have made it a bit easier" (see Figure 4.29). Seth used the ratio 8 out of 12 but included the measurement units of centimetres on his drawing. His explanation indicated that he understood ratios and had included a measurement context, "Um I did a twelve centimetre long line and two thirds of that is eight centimetres", but his mark was not $^2/_3$ of the distance between 0 and 1.

Cameron

Seth



Rohan



Ruby

Figure 4.29 Examples of ratio misconceptions for placing $^2/_3$ on a number line.

Cameron also called on a ratio understanding for $^2/_3$ but circled two parts on his number line, (see Figure 4.29) indicating correct partitioning, explaining "To begin with twenty four is an even number and eight, sixteen, twenty four; which is like eight times three and there's three parts to it so from zero you jump from eight and then sixteen and that's two, thirds". There were several different ratios used by the students: 2 of 3, 6 of 9, 6.75 out of 10, 8 of 12, and 16 of 24.

One category of inscriptions in which $^2/_3$ was two thirds of the way along the line but 1 was placed incorrectly included the ratio misconception. Ruby placed her $^2/_3$ at the fourth mark out of six (see Figure 4.29) but it was not apparent that this was a ratio understanding until after she had been prompted to mark 1, because she had not labelled two scales as Rohan had done. Ruby's explanation confirmed this ratio interpretation, "Well, I kind of made it the same as the tight rope. And I put two there and two there and two there cause if you have six it would make thirds." If we include Ruby in the frequency of ratio understandings the frequency of ratio strategies was 14.8%. Of this group, five of the thirteen students also used a ratio approach to Q. 16c, placing their mark for $^1/_4$ on a number line labelled 0, 1 and 2, at $^1/_2$, which was a quarter of two. But the majority of ratio users in Q. 16a did not use the same strategy in Q. 16c.

There were two other types of inscriptions in which $^2/_3$ was two thirds of the way along the line but 1 was placed incorrectly. For some students, 1 seemed to clearly indicate $^1/_3$ (see Figure 4.30). The misplaced 1 meant that these students were coded as incorrect. If they had written *one third* rather than *one,* they would have then been prompted to mark 1. When four students had done this, I suspected a misunderstanding of the question and for the other students asked, where would *one whole* go, rather than where would *one* go. For this reason, I have not reported the frequency of this misconception.

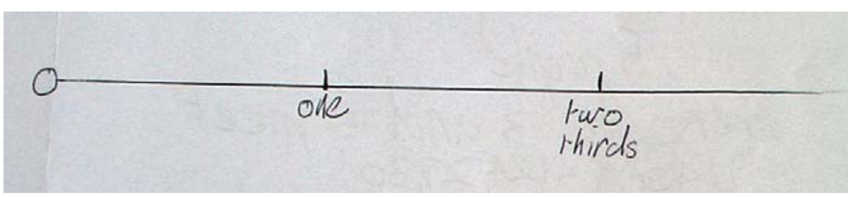
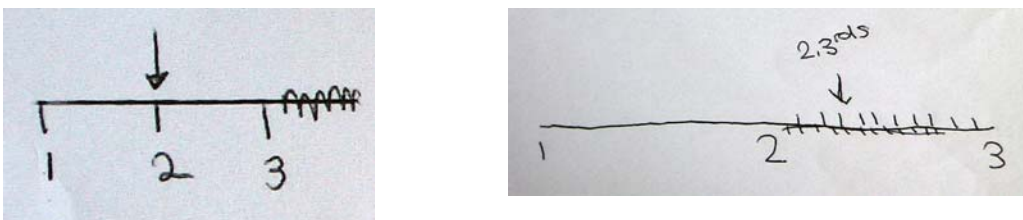


Figure 4.30. Two thirds on a number line with 1 at one third.

The other category of misplaced 1s was "other", in which the 1 was too close to 0 to make a ratio with the $^2/_3$ as marked. However, as the 1 was added by eye later, some of these inscriptions may have been using ratio understandings but further probing was required (and wasn't carried out) to confirm this strategy. In this last category, as in the first two of misplaced 1s, the $^{2}/_{3}$ was two thirds of the way along the line but the 1 was not in the correct place for a number line interpretation.

Some of the students' inscriptions looked like incorrect ratios but the accompanying explanation could be interpreted as a double counting misconception. Adele explained her number line below (see Figure 4.31), "Um, two thirds is like um, like um, three parts, yeah and um, so like, there are two parts shaded in, so I went to two". This type of inscription was not included in the frequency of the ratio misconception above, and 6.8% of students drew number lines like this without a 0. These students' responses to the Fraction Pie task and two non-equal parts Fraction Sort cards, showed a range of responses from double counting to success on three of the questions with an answer of *other* for the fourth non-equal-parts diagram. A double counting misconception could not be tracked across these number line inscriptions and area diagrams.



Adele Julia

Figure 4.31. Possible double count inscription, and three-past-two inscription.

Another misconception was $^{2}/_{3}$ as three-past-two or 2.3. Julia explained her drawing (see Figure 4.31) "Because there's two, there's three and there would be something between, and I did ten little ones, and that would be the third." Other similar drawings did not break the space between two and three into ten parts, but kept the semantic pattern, two and then three. The frequency of this semantic approach was 9.1%, and a further 4.5% indicated the whole segment between 2 and 3. I classify these responses as whole number misconceptions.

The use of a ratio misconception demonstrated that the students had some fractional understanding. For example, they knew that two thirds of nine was six, or that two thirds of twenty four was sixteen. Of the 29 students who correctly draw a number line and marked two thirds on it, 79.3% (17 out of 29) of them had a CADL score of 2 and had correctly used the broken ruler to measure the length of the Freddo (See Table 4.19). Of the students who were unsuccessful on the number line task, those who had used the ratio misconception and those who had used partitioning inappropriately had similar (to the successful number line students) success on the Freddo task: 69.2% (9 out of 13) and 70% (7 out of 10) respectively. Students who were unsuccessful on the number line task for other reasons were markedly less successful on the broken ruler task: with 30.5% (11 out of 36) successful on the Freddo task. The frequency of success on the harder broken ruler task, the Footy Card task, was similar for the students who were successful on the number line task or who had used the ratio misconception: 58% (17 out of 29) and 53% (7 out of 13) respectively. However, this performance was not duplicated by students who had used partitioning inappropriately. Only 10% (1 out of 10) of the students who used partitioning (incorrectly) on the number line task were successful on the Footy Card task (CADL score of 3), which was lower than the performance by the students who had other unsuccessful strategies on the number line task (22% (8 out of 36) were successful on the Footy Card task).

Table 4.19

Number Line Strategies (Q. 16a) and CADL score

	Correct	Incorrect		
	Correct (33%)	Ratio (14.8%)	Partitioning (11.4%)	Other incorrect (40.9%)
CADL score 0 or 1	6	4	3	25
CADL score 2	6	2	6	3
CADL score 3	17	7	1	8

4.5.2.1.2 Reading $3^3/_4$ on a number line.

The number line Q. 16d required students to read $3^3/_4$ on a number line partitioned into quarters (see Figure 4.32). Successful students were able to coordinate whole numbers and fraction numbers. For example, Emily explained "I started at the three, 'cause that's the closest number behind the arrow. I counted how many lines were along, so after the three, the three was zero, so one two three four, which means it was quarters. And then I counted how many along from the three again, how many of the four quarters there was. And ended up three and three quarters."

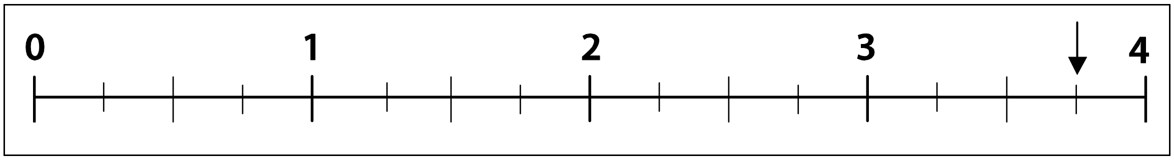


Figure 4.32. Number line diagram Q. 16d, reading 3 $^{3}/_{4}$.

Some students demonstrated the misconception that fraction number lines are decimal number lines and read the marks as tenths. One illustration of this was the counting of the marks after 3 as point one, point two, point three as demonstrated in Sylvie's explanation "'Cause there's three [points to 3] and then if you get one two three, three [points to lines]. So it's three of these, so point three". Eight students (from two of the schools) used this incorrect strategy. There were two other variations of this *assuming decimal number lines* misconception. One student correctly identified 3.5 on the number line and then counted the next mark (with the arrow) as 3.6. Some students gave the answer of 3.9 but it was difficult to ascertain whether they were estimating, such as Jai's explanation, "Um, because four was there so three point nine" or whether they were assuming that the hash mark before a whole number was point nine. Tamika appeared to be assuming that a number line could be counted in tenths but was not convinced, explaining, "I said three point nine but that doesn't make sense."

Some students did attempt to estimate the length using decimals. This was not an example of the assuming decimal number lines misconception but was a mathematically correct strategy. However, they were coded incorrect if they did not answer three point seven five. For example, Amelia answered "Um three point, [pause] um three point eight or something." When prompted by the interviewer "And how did you work that out?" she replied, "Well if that's three, it's not quite four yet, because it's before. Take that as a half and just then a bit more than half." This just imprecise estimate was not an example of the decimal misconception, as Sylvie's was because Amelia did not assume the marks on the number line were automatically tenths.

The keyboard in the Keyboard task (Q. 39) was three and three quarter pencils long. In both the number line question (16d) and the Keyboard task, the students had to identify 3 $^{3}/_{4}$ and explain their answers appropriately. Of the eight students who used an inappropriate decimal strategy for the number line showing 3 $^{3}/_{4}$ (Q. 11d), five of them answered correctly on the Keyboard task: four said three and three quarters and one said three and two thirds. None of the remaining three offered decimal answers for the length of the Keyboard. Decimal answers for the Keyboard task were decimal estimations, but on the 3 $^{3}/_{4}$ number line some decimal responses indicated a misconception that all hash marks were tenths.

Some students omitted the whole number in their answer to the 3 $^{3}/_{4}$ number line giving the answer three quarters. Of these eight students, six of them answered three and three quarters successfully on the Keyboard task. One student offered three quarters as an answer for both tasks.

4.5.2.1.3 Reading $^{5}/_{6}$ on a number line.

The number line Q. 16e required students to read $^{5}/_{6}$ on a number line partitioned into six equal parts (see Figure 4.33). Some successful students counted spaces, for example, in explaining his answer of $^{5}/_{6}$, Nicky said "I counted up how many spaces it's divided into, and it's pointing to." Other successful students counted lines, as Sarah explained, "Well there's six lines this time and it's the fifth one of six." Using either strategy, 27.2% of the students successfully identified the mark as $^{5}/_{6}$ on this number line.

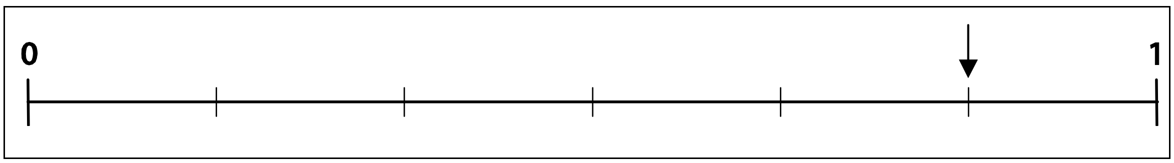


Figure 4.33. Number Line diagram Q. 16e, reading $^{5}/_{6}$.

One misconception observed in the students' explanations to this question, also observed in the previous question (Q. 11d), was reading the marks as tenths as if the number line were a decimal number line. There were two variations of this. Some students read left to right as Jess did as she explained her answer of point five, "'Cause it's not up to one yet. And I thought one, two, three, four, five [points with finger]. And I did zero point five." Other students read right to left, and used fractional language for the decimal as Will did, explaining his answer of nine tenths, "If that's a whole, ten tenths, that's one less". The frequency of this inappropriate decimal reading of number lines was 17%. This misconception would be undetectable on the one decimal number line, Q. 16g, used in the present study.

This reading of the marks on number lines as if they were tenths was not the same as estimating using tenths which Rebecca tried. She ignored the six marks that were there and mentally partitioning the line into different sized parts. Rebecca explained her estimation, "Um I split the line into ten and the arrow's pointing where the eight is." Her estimation was very close to 0.8333, the decimal equivalent to $^5/_6$, but she was still coded as incorrect. Rebecca's strategy was mathematically correct, but she was not accurate enough to be coded correct on this particular question because there were six spaces clearly marked. Jess and Will quoted in the previous paragraph, on the other hand, were using a mathematically incorrect strategy.

The misconception of counting lines not spaces was observed in the students' responses in all three schools. For example in explaining her answer of $^6/_7$, Claire said, "Because that's six and there's seven marks." Students with this misconception started counting "one" at the mark at 0. The frequency of this misconception in the present study was 10.2%.

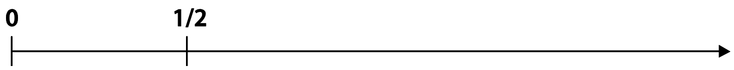
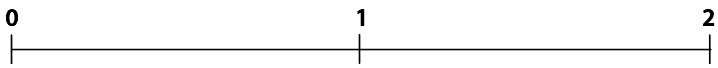
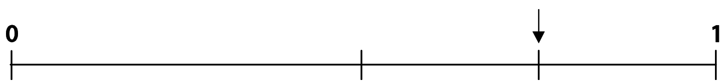
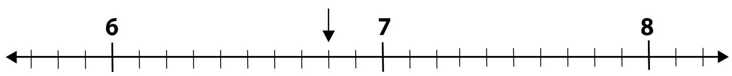
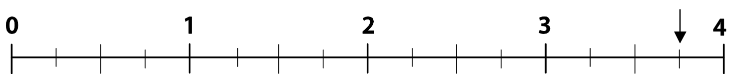
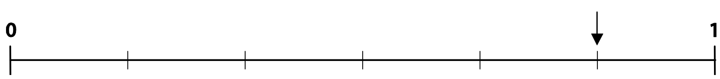
The counting lines not spaces was an error also demonstrated on the Freddo task (Q. 41) (see section 4.2.2.1) and a third (3 out of 9) of the students who used this misconception in the number line task repeated the misconception on the Freddo task. In contrast another third (3 out of 9) were successful on the Footy Card task (Q. 42), and the final third were incorrect on the Freddo task for other reasons. Overall, 46.6% of all the students did not use the counting lines not spaces misconception on any of the three tasks: the number line (Q. 16e), the Freddo task (Q. 41) or the Footy Card task (Q. 42). However, 37.5% (9 out of 24) of the students who had counted the marks correctly on this number line then counted lines not spaces unsuccessfully in one or other of the broken ruler tasks (the Freddo task or the Footy Card task). The use of the misconception in one context did not predict its use in another context.

4.5.2.1.4 Frequency of success on number line questions.

The frequency of success on the eight number line tasks varied from 27.2% for Q. 16e discussed above, to 71.6% for Q. 16b iterating a half to place $1^1/_2$ on a number line (see Table 4.20). The question for Q. 16c was to mark where one quarter would go on this number line.

Table 4.20

Frequency of Success on Number Line Tasks

<table>
<tr><th>Question</th><th>Diagram</th><th>success</th></tr>
<tr><td>16b</td><td>marking one and a half</td><td>71.6%</td></tr>
<tr><td>16c</td><td>marking a quarter</td><td>55.7%</td></tr>
<tr><td>16f</td><td></td><td>55.7%</td></tr>
<tr><td>16g</td><td></td><td>52.3%</td></tr>
<tr><td>16d</td><td></td><td>38.6%</td></tr>
<tr><td>16a</td><td>Students drew their own number line and marked $^2/_3$</td><td>33%</td></tr>
<tr><td>16h</td><td></td><td>31.8%</td></tr>
<tr><td>16e</td><td></td><td>27.2%</td></tr>
</table>

A Number Line score was calculated for each student, being the number of questions correct out of eight (see Table 4.21). These scores illustrated the range of performance level in the group, and all three schools had students with scores of 0 and 8. The most common score was 3 with just over a quarter of the students achieving that level of success.

Table 4.21

Spread of Number Line Questions Correct

	Score								
	0	1	2	3	4	5	6	7	8
Success	6.8%	11.4%	13.6%	28.4%	10.2%	8%	2.3%	6.8%	12.5%

There were 45 different pathways through the eight number line questions; a pathway being the pattern of correct and incorrect responses. One of those pathways was all incorrect and another pathway was all correct and 17 students had either of those patterns of success, see Table 4.22. Excluding those two possibilities, there were 43 different patterns of success for the 71 students with a Number Line score of 2 to 7. Despite the frequency of success suggesting that there was a pathway through the number line tasks, Q. 16b for example having a greater frequency of success than Q. 16g, the students' performance indicated that there were many different paths towards a coordinated understanding of fractions on number lines. These results do not support the idea that there was a pathway through number line questions based on making partitions being more difficult than reading partitions, nor based on number lines labelled 0 to 1 being easier than number lines that are greater than 1.

Table 4.22

Permutations of Correct and Incorrect Response to the Number Line Questions: Pathways

	Score								
	0	1	2	3	4	5	6	7	8
Number of students	6	10	12	25	9	7	2	6	11
Different pathways	1	5	6	14	7	5	2	4	1

4.5.2.1.5 Associations between number line questions and measurement concepts.

The Number Line score was used to calculate correlations between performance on the measure sub-construct of fractions and measurement concepts. Using Kendall's Tau, there was no significant association between students' Number Line scores and the tasks assessing the key concept of attribute (see Table 4.23). There were substantial associations with the conceptual tasks, both length and area contexts, of the key concept of additivity. These broken ruler tasks and area calculations showed a stronger effect size than the tools and procedures category. There was also substantial associations between the students' Number Line scores and the key concept of units in the conceptual length tasks (e.g., measuring the Keyboard), and also the tools and procedures tasks assessing this concept. The concept of proportionality had typical associations with the students' Number Line score. The students' Multiplication score had a substantial association with their performance on the number line tasks.

Table 4.23

Correlations Between Number Line Score and Measurement Concepts

	Correlation		
	Minimal τ > .07	Typical τ > .20	Substantial τ > .34
Attribute			
Additivity		TPADA τ = .324, p = .001	CADL τ = .384, p < .000 CADA τ = .363, p < .000
Unit		CUNA τ = .307, p = .001	CUNL τ = .408, p < .000 TPUNL τ = .354, p < .000 TPUNA τ = .348, p < .000
Proportionality		CPRL τ = .229, p = .010 TPPRA τ = .223, p = .017 TPPRL τ = .210, p = .025	
Multiplication			MULT τ = .364, p < .000

4.5.2.2 The relative size of fractions: the Fraction Pair task.

The Fraction Pairs task (Q. 22) was used to assess the students' understanding of the relative size of fractions.

All fifty-six video records of the students completing the Fraction Pairs task were double coded by another mathematics education researcher who had coded fraction pairs before in another research project. To be correct a student had to give the correct answer with a mathematically correct explanation. There were 15 instances out of 448 where this second coder disagreed with my categorisation of the student being correct or incorrect. Half of the coding discrepancies resulted from a difference in my criteria for gap thinking in the fraction pair $^{3}/_{8}$ and $^{7}/_{8}$ (Q. 22a) from other research projects. Other coding discrepancies were resolved by discussion of a transcript of the students' explanations by the two coders. However, it was agreed that in the fraction pair $^{4}/_{5}$ and $^{4}/_{7}$, some explanations did not provide enough detail to confirm whether a correct or incorrect strategy was used, and for this reason I had decided not to report the frequency of success of that particular fraction pair.

4.5.2.2.1 Successful strategies.

In the fraction pair $^{3}/_{8}$ and $^{7}/_{8}$ (Q. 22a), the *same denominator and compare numerators* strategy was observed. It was the most common correct explanation when comparing $^{3}/_{8}$ and

$^{7}/_{8}$, and was illustrated by Lily's explanation, "Because there's the same denominator and seven's larger."

A description more closely linked to the *visualisation* of area models common in classroom activities, can be seen in Zannah's explanation for the same fraction pair comparison; $^{3}/_{8}$ and $^{7}/_{8}$. After reaching for paper and being told she had to "do it in your head", Zannah chose $^{7}/_{8}$ as the larger fraction and explained: "Because this one is covering more of the shape. Because three eighths is only covering three sections and this one's covering seven sections." This explanation drew on comparisons.

The *complement to one* strategy was additive thinking which was appropriate for fractions between zero and one with the same denominator. This was demonstrated by Sylvie's explanation "Because of the eight [points to the denominators], that one [points to $^{7}/_{8}$] needs one more to get to the whole. And that one [points to $^{3}/_{8}$] needs five more." The same complement to one strategy was used by Kate in her explanation, "Because there's only more piece to make a whole for seven eighths. And for three eighths you'd need another, you'd need another um five, more pieces." Michael illustrated the role of the denominator in the complement to one strategy as the number that represented the whole: "This one's larger [points to 7/8] because it's closer to the [undecipherable] the denominator whatever it's called". In other research projects (see e.g. Clarke & Roche, 2009), these three explanations would have been classified as gap thinking (using the difference between the numerator and the denominator). However, in the present study, on this particular pair, they were classified as correct mathematical reasoning. The term complement-to-one was used by Pearn and Stephens (2004) to describe gap thinking, a mathematically incorrect explanation. But I am using the term in *this* pair to illustrate that it is a correct strategy when used with appropriate fraction pairs, although it cannot not be generalised successfully. The thinking in Sylvie's, Kate's and Michael's explanations used additive word patterns.

There were some explanations that may have been a mixture of two strategies or that would need further probing to determine which of the two strategies the student was using. For example, Ebony chose $^{7}/_{8}$ as the larger fraction and explained, "Well this is near the beginning [points to 3/8]. And this is near the end [points to 7/8]. Just one from the end." She may have been visualising a length or circle diagram, which would have an end. On the other hand, she may have been thinking of the number 8 (the denominator) as being *the end*.

The successful strategies for comparing the fractions $^{2}/_{4}$ and $^{4}/_{8}$ (Q. 22b) have been reported above with respect to the use of equivalence (see section 4.5.1.3).

The fraction pair $^{1}/_{2}$ and $^{5}/_{8}$ (Q. 22c) was successfully compared using benchmarking. For example, Ruby explained "Um, I think that's bigger [points to 5/8]. Because that's half [points to ½] and five is bigger than one half, of eight". The fractions were also compared drawing on the concept of unit-forming. In this approach $^{5}/_{8}$ was thought of as being composed of two unequal sized pieces; $^{1}/_{2}$ plus another piece. Rose illustrated this in her explanation, "Because this [5/8] is like three quarters out of eight and this is only half; and so it's one quarter extra." Rose had an Equivalence score of 1 so I do not believe that she meant that $^{5}/_{8}$ was nearly $^{6}/_{8}$ or $^{3}/_{4}$. She was using *quarters* to indicate *pieces*.

The fraction pair $^{2}/_{4}$ and $^{4}/_{2}$ (Q. 22d) was successfully compared using equivalence understanding; $^{2}/_{4}$ was a half and $^{4}/_{2}$ was two. Tom attempted to rename $^{4}/_{2}$ as a mixed number, "this is a whole and a half – no two wholes". In other words, $^{4}/_{2}$ was a whole plus another piece (in this case, another whole) and bigger than the other fraction which was less than a whole. Some students used benchmarking to one rather than a half by identifying $^{4}/_{2}$ as an improper fraction. For example, Kate explained "Because the four is bigger than two and it's an improper fraction".

The successful strategy *comparing denominators when numerators are the same* was observed in the comparison of the fraction pair $^{4}/_{5}$ and $^{4}/_{7}$ (Q. 22e). This was illustrated in the explanation of why Sarah chose $^{4}/_{5}$ as larger, "Because the top numbers are both four, but there's seven and five on the bottom; and seven means that the pieces are littler. So four of them wouldn't equal four of the fifths."

The successful use of benchmarking or common denominators in the comparison of $^{3}/_{7}$ and $^{5}/_{8}$ (Q. 22f) has been reported above (see section 4.5.1.3).

The *residual* strategy was observed in the size comparison of $^{5}/_{6}$ and $^{7}/_{8}$ (Q. 22g). Sarah provided an example of this residual reasoning: "because if I imagine a pie cut into sixths and you do five of them. And I imagine a pie cut into eight and there's seven of them; that's a little more." When prompted, "How do you know?" she elaborated, "Because eighths are smaller, and like seven of them would be closer to a whole than five sixths."

Common denominators was a strategy used successfully in the comparison of the fraction pair $^3/_4$ and $^7/_9$ (Q. 22h). This was illustrated in Lily's explanation of how she decided that $^7/_9$ was larger: "I tried to get thirty six [waves finger across both denominators]. And I times that by nine [points to 3], and that by four [points to 7]."

One student used an operator sub-construct approach in the fraction pair $^3/_4$ and $^7/_9$ (Q. 22h) which could be rephrased as whether seven was more or less than three quarters of nine. One student, Seth, did this successfully, reasoning "'cause that one was three quarters [points to $^3/_4$] and this one [points to $^7/_9$] was just a little over three quarters because a quarter of nine, because a quarter of nine would be one point- no, like two point three. Yeah, umm, two point; two point two five. And three, two point two fives would be- is six point seven five." Summarising Seth's explanation: $^1/_4$ of 9 was 2.25 (using partitioning); $^3/_4$ was equivalent to $^{6.75}/_9$ (drawing on equivalence); seven was 6.75 and another bit (drawing on unit-forming). So $^7/_9$ was larger than $^3/_4$.

The frequency of success on seven of the pairs is reported in Table 4.24. For 78.4% of the students the order asked was also the sequential order of difficulty.

Table 4.24

Frequency of Success on Fraction Pair Task

	Pair							
	$^3/_8$ $^7/_8$	$^2/_4$ $^4/_8$	$^1/_2$ $^5/_8$	$^2/_4$ $^4/_2$	$^4/_5$ $^4/_7$	$^3/_7$ $^5/_8$	$^5/_6$ $^7/_8$	$^3/_4$ $^7/_9$
Success	90.9%	72.7%	54.5%	39.8%	not reported	13.6%	12.5%	6.8%

A fraction pairs score was calculated as the number correct out of seven (Q. 22e being excluded). There was a spread of performance, with about 25% of the students scoring 0 or 1, about 60% scoring 2, 3 or 4, and about 15% scoring 5 or above (see Table 4.25). Around 85% of the students were successful on four pairs or less.

Table 4.25

Percentage of Students with each Fraction Pairs Score from 0 to 7

	Score							
	0	1	2	3	4	5	6	7
Frequency	4.5%	22.7%	15.9%	20.5%	22.7%	4.5%	2.3%	6.8%

4.5.2.2.2 Misconceptions: gap thinking and whole number strategies.

Three main misconceptions are reported; gap thinking, higher or larger numbers, and bigger denominator indicates bigger fraction thinking. The double coding revealed that a highly detailed list of strategies was unwieldy. However, there was broad consensus between the two coders about the general types of strategies used and their frequency. All explanations which were identified as gap thinking by either coder were transcribed from video or audio files and analysed, along with some non-examples. A third coder, who was a mathematics education lecturer, looked at these transcripts and triple coded them and any discrepancies were resolved through discussion.

The most instructive place to discover all the variations of gap thinking explanations was in the students' responses to the fraction pair $^5/_6$ and $^7/_8$ (Q. 22g). This was because the gap answer was distinctive: "they're the same". The students were shown the fraction pair $^5/_6$ and $^7/_8$ on a card and instructed, "please point to the larger fraction or tell me if they are the same". Both a gap answer and a gap explanation were needed as evidence of gap thinking. Gap thinking was observed in students from all three schools on this fraction pair. Meg used the complement to one strategy, "one more to become a whole" (see Table 4.26). Tony used the same strategy, explaining "Because one up on the five sixes is a whole. And one up on the seven eights is a whole." With this fraction pair, a complement to one strategy was not mathematically correct.

Table 4.26

Variations of Gap Thinking in Explanations of why $^5/_6$ and $^7/_8$ are Considered the Same

gap thinking variation	transcript of students' explanation
complement-to-one	Meg: They're the same because five sixths has got one more to become a whole. And seven eighths it also has one more to become a whole.
complement to one	Jade: They're the same. Interviewer: And how did you work that out? Jade: Because five out of six is one piece left and seven out of eight is one piece left.
numerical comparison of numerator and denominator	Claire: They're the same. Interviewer: How do you know? Claire: Because there's both, because the top numbers are both one less than the bottom numbers.
equivalence string: $^2/_3$, $^3/_4$, $^9/_{10}$	Hannah: They're the same. Interviewer: And how did you decide? Hannah: 'Cause they're both two thirds, that's another way to say them. 'Cause seven plus one is eight and five plus one is six.
fractional language (sixth, eighth) indicating complement to one	George: They're the same. Interviewer: And how did you decide? George: Because they're like. Five sixths there's one more. There's one more sixth to make a whole. And it's one more eighth.
counting and shading	Courtney: They're the same. Interviewer: And how did you decide? Courtney: Because they both need one more to be coloured in.

Jade used the gap as a "bit" strategy, and described "one piece left". Brad used the same strategy but with the word "spaces", "Because you've got. They're the same way. Um how much spaces to go."

Claire showed attention to the numerical difference between numerators and denominators, "the top numbers are both one less than the bottom numbers". Patrick's explanation used the word "plus" to compare the difference between numerator and denominator, "Because one plus five equals six and one plus seven equals eight [points to 5/6 and 7/8]".

Hannah offered one of the string of equivalences that was offered by the students in the present study $^5/_6$ is $^7/_8$ or $^2/_3$ or $^3/_4$ or $^9/_{10}$, "they're both two thirds". No one child offered all of these equivalences in one answer, but it would appear from their specific responses that any fraction one less than the whole was equivalent to any other fraction one less than the whole. Hannah also used the word "plus" in a numerical comparison of the difference between numerator and denominator in a further explanation of her equivalence approach. The equivalence of fractions one less than the whole could explain Jordan's explanation of why $^5/_6$ and $^7/_8$ was the same, "Three quarters is almost the whole".

The fraction pair $^5/_6$ or $^7/_8$ also revealed that the use of the fractional language of sixths and eighths did not automatically rule out gap thinking. If the students who used this fractional language used "sixth" to indicate a part with a size and an "eighth" to indicate a part of another size, then they could not have concluded that $^5/_6$ and $^7/_8$ were the same. A "sixth" had the same status as a "piece" in these gap thinking explanations. This was illustrated in George's explanation of why the fractions were the same, "Five sixths there's one more. There's one more sixth to make a whole. And it's one more eighth." Explanations that used fractional language and did take into account the size of the pieces could be correct residual thinking or incorrectly executed residual thinking or incorrectly executed reasoning about the number and size of the parts.

Courtney used a counting and shading explanation, "they both need one more to be coloured in".

Gap thinking was used by 50% of students for the fraction pair, $^5/_6$ and $^7/_8$. An answer of "the same" was almost exclusively due to gap thinking. Only one student, Freya, claimed the fractions $^5/_6$ and $^7/_8$ were "the same", but was not using gap thinking. Instead she used a faulty residual explanation. When gap thinking was present, this fraction pair, with its distinctive gap answer, enabled the full range of gap thinking explanations to be elaborated. With these six variations of gap thinking identified, other fraction pairs can be examined.

In the fraction pair $^2/_4$ and $^4/_8$ in which both fractions were equivalent and hence the correct answer was "the same", 73% of the students were successful. Although some of the incorrect answers were $^2/_4$, the fraction with the smaller gap, none of the explanations indicated gap thinking. Of the just over a quarter of students who were incorrect on this task, most errors were whole number based. For example, Ricky used the higher or larger numbers misconception, explaining that $^4/_8$ was larger: "That one, because there's a bigger number on the bottom and a bigger number on the top." Jonno used the whole number misconception that the bigger denominator indicates the bigger fraction, explaining that "because there's four over eight and eight's the larger number at the bottom."

The improper fraction $^4/_2$ in the fraction pair $^2/_4$ and $^4/_2$ (Q. 22d), caused confusion for some students. For example, Patrick chose $^2/_4$ as the larger fraction, explaining "Well, I knew that if you add two more it will equal four, so the closest number. I still [points to 4/2], I don't know this one still." The gap between the fraction was the same, and some students offered the explanation that there was the same numbers in both. For example Simon explained that the fractions were the same "'cause they're. On this one the two's at the top and the other one the two's at the bottom and they've both got the four. On the other one the four's at the bottom and the other one's the top." This explanation was specific to improper fractions, and was similar to other strategies where students flipped the improper fraction, making a proper fraction. Further probing would be needed to rule out gap thinking. However, none of the 13 children who offered the same numbers in both explanation used a linguistic structure similar to any of the gap thinking examples for the fraction pair $^5/_6$ and $^7/_8$ where the difference was also the same. There was no complement-to-one (with or without fractional language), no numerical difference as addition or subtraction, no third equivalent fraction, and no colouring in analogies. I have concluded that no gap thinking was evident in Q. 22c, the fraction pair $^2/_4$ and $^4/_2$.

It was uncommon, but possible, for some students to use gap thinking linguistic structures, particularly the numerical difference between numerator and denominator but choose the larger gap. Only four students did this and it was most noticeable in the fraction pair $^3/_8$ and $^7/_8$ (Q. 22a) because it caused the student to pick the smaller fraction.

The rest of the pairs divide into two groups, those in which gap thinking would give the wrong answer, and those in which the student would get the right answer for the wrong reason. I will start with the former, as they are slightly easier to "hear" and code. In some fraction pairs the gap thinking answer resulted in an incorrect answer with incorrect mathematical thinking. Both a gap answer and a gap explanation were needed as evidence of gap thinking.

It was in fraction pair $^3/_7$ and $^5/_8$ (Q. 22f) that the fine distinctions between the gap thinking strategy and not convincing enough size comparisons become apparent. The close examination of gap thinking would not be complete without examples of the boundaries between gap thinking, possible gap thinking, and other correct or incorrect strategies that could be mistaken for gap thinking. In Table 4.27, I provide examples of these distinctions between strategies from the transcripts of students' explanations of the size comparison of $^3/_7$ and $^5/_8$. In the example of gap thinking, Lara used fractional language, "sevenths" and "eighths". However, further probing revealed that, like some of the fractional language examples in comparing $^5/_6$ and $^7/_8$, this fractional language did not indicate an understanding of parts of different size. Another example of gap thinking (not in Table 4.27) was Patrick's explanation that $^5/_8$ was the larger fraction "Because um three and five is eight [points to 5/8] but three and four is seven [points to 3/7] so three is less than."

Table 4.27

Fraction Pair $^3/_7$ and $^5/_8$ and Distinctions Between Gap Thinking, Possible Gap Thinking, and Non Gap Thinking

Strategy	Explanation from transcript
gap thinking	Lara: [pointing to $^5/_8$] I: And how did you decide? Lara: Because three sevenths isn't that close compared to five eighths. I: How do you know? Lara: Because [mumbled]. Because if you were going three to seven it would be four. And five to eight would be three.
not gap thinking (insufficient explanation of number and size of parts)	Nicky: That would be the smaller [points to 3/7] I: Ok, and how did you decide? Nicky: 'Cause it's that, is four eighths [points to 5/8], no that is three eighths off. That is four sevenths off [points to 3/7]. One seventh is smaller. No, that would be smaller [points to 5/8] I: OK Nicky: No that would be, wait, which one are we doing, bigger or smaller? I: The larger number. Nicky: Yeah. That one will be bigger [points to 5/8], because that's only three eighths off and one eighth is smaller than one seventh. So that's only four sevenths off [points to 3/7], that's five eighths off [points to 5/8]. No, that's three eighths off so that would be bigger but, no smaller. That would be bigger [points to 3/7].
possible gap thinking	James: That is [points to $^5/_8$] I: Which one, sorry? James: That one I: And how did you decide? James: [undecipherable] because that's three sevenths, it's not bigger than that. and that's [points to $^5/_8$] 3 more to one whole I: And tell me about the $^3/_7$ then. James: It's not near one whole, and that's [points to $^5/_8$] bigger

The middle example in Table 4.27 is an illustration of an explanation that was not sufficiently detailed enough to be coded as correct. Nicky used fractional language and tried to grapple with the size of the pieces and the number of the pieces. He almost used residual thinking to compare three eighths and more than three sevenths as the amount away from the whole, but was coded as incorrect because he chose the smaller fraction in the end.

James' explanation was an example of a new category devised in the present study – possible gap thinking. Some mathematically correct strategies "sounded" like gap thinking. Careful listening was required to distinguish between the strategies, even with the benefits of video footage and transcripts. James may have had the right answer for the right reason; benchmarking, where $^{3}/_{7}$ was less than a half and "not near one whole". Or James may have been explaining gap thinking, describing $^{5}/_{8}$ as having three more to go to get to one whole. A possible gap thinking coding indicated that there was not enough evidence to classify the strategy as gap thinking, but there was certainly a strong suspicion that it could be.

Gap thinking was observed in the fraction pair $^{4}/_{5}$ and $^{4}/_{7}$ (Q. 22e). Gap thinking would produce the correct answer, but with an unsatisfactory explanation, for all fraction pairs between zero and one with the same numerator. For example, Lara used the complement-to-one version of gap thinking, "'Cause it's only one away from being a whole" (see Table 4.28). She got the right answer, $^{4}/_{5}$, for the wrong reason. But it was further probing that confirmed that her reasoning was gap thinking, "And this is three away from being a whole".

Table 4.28

Fraction Pair $^{4}/_{5}$ and $^{4}/_{7}$ and Distinctions Between Gap Thinking, Possible Gap Thinking, and Benchmarking

Strategy	Explanation from transcript
Gap thinking	Lara: This one [points to $^{4}/_{5}$] I: And how did you decide? Lara: 'Cause it's only one away from being a whole. I: Mmm? Lara: And this is three away from being a whole
Not gap thinking: correctly benchmarking to $^{1}/_{2}$ and to 1	Chris: [points to $^{4}/_{5}$] I: How did you decide? Chris: Well, five, ff; four fifths is almost a whole I: Mmm? Chris: And four sevenths is um, a bit higher than half
Not gap thinking: correctly benchmarking to $^{1}/_{2}$ and to 1	Adam: This one. [points to $^{4}/_{5}$] I: And how did you decide? Adam: Um four is closer to five. I: Can you tell me a bit more about that? Adam: Um. Four. The four and the seven, there's more less, like, um close to a half, but this one's like almost a whole.
Possible gap thinking	Meg: Four fifths are the same, I mean are the larger, 'cause four is closer to five and four isn't really close to seven.

Chris and Adam's explanations, on the other hand initially sounded like a complement-to-one gap thinking, but their further explanations demonstrated that they were benchmarking to just over a half and nearly a whole. They were coded as correct, providing a correct answer and using the mathematically correct strategy of benchmarking. The fraction pair $^{4}/_{5}$ and $^{4}/_{7}$ also lent itself to benchmarking because $^{4}/_{5}$ was close to one and $^{4}/_{7}$ was just over a half. It was difficult to hear the difference between an alternative correct strategy (benchmarking) and a mathematically incorrect strategy (gap thinking).

Meg's explanation demonstrated that further probing was sometimes needed to confirm the use of gap thinking. In this case, that was not carried out and so she was coded as possible gap thinking. Meg could have been using a complement-to-one variety of gap thinking, "four is closer to five". However, Meg may have been benchmarking, "four is closer to five". The examples of Chris, Adam and Meg illustrate how coding this fraction pair proved difficult.

The gap thinking answer in the fraction pair $^3/_4$ and $^7/_9$ (Q. 22h) was $^3/_4$ and sometimes a student's explanation made this clear, for example Brad said "because it's less pieces, it's less numbers to get from three to four than seven to nine." However, many answers of $^3/_4$ were not based on gap thinking because an explanation grappling with the size of the denominators and the number of pieces could easily conclude (incorrectly) that $^3/_4$ was the larger fraction because for students who were reasoning qualitatively, the difference between the fractions ($^1/_{36}$) was very small.

This question was the last of eight fraction pair questions and if the student gave the explanation, "like the other ones" and their previous reasoning had been gap thinking, I assumed gap thinking to be the reasoning during the interview. However, a transcript of such an explanation did not provide evidence of gap thinking, and so I have coded these explanations as possible gap thinking. It would have been better to prompt the student for further explanation during the interview. This affected 8% of the explanations to this pair $^3/_4$ and $^7/_9$. However, some students clearly offered gap thinking explanations. For example, Patrick explained that $^3/_4$ had the smaller gap, "Well this one I just had to add one more [3/4] and this one you just add two."

The gap thinking answer for the fraction pair $^1/_2$ and $^5/_8$ was $^1/_2$ because it had the smaller gap. In this pair the gap thinking strategy produced the wrong answer with incorrect mathematical thinking. For example, Brad chose $^1/_2$ as the larger fraction because "It takes. There's less to get from one to two than from five to the eight."

Some fraction pairs were more difficult than others to compare (see Table 4.29). But not every pair elicited gap thinking. The highest proportion of gap thinking occurred on the pair $^5/_6$ or $^7/_8$ where half of the students demonstrating this strategy.

Table 4.29

Frequency of Success on Fraction Pair Questions and the Incidence of Incorrect Strategies

	Fraction Pair							
	$^3/_8$ $^7/_8$	$^2/_4$ $^4/_8$	$^1/_2$ $^5/_8$	$^2/_4$ $^4/_2$	$^4/_5$ $^4/_7$	$^3/_7$ $^5/_8$	$^5/_6$ $^7/_8$	$^3/_4$ $^7/_9$
Success	90.9%	72.7%	54.5%	39.8%	NR	13.6%	12.5%	6.8%
Gap Thinking	2.3%	0%	6.8%	0%	NR	21.6%	50%	23.9%
Possible Gap	0%	0%	0%	0%	11.4%	3.4%	0%	8%
Whole Number	0	15.9%	15.9%	3.4%	NR	23.9%	18.2%	18.2%
Other (incorrect)	6.8%	11.4%	22.8%	56.8%	NR	37.5%	19.3%	43.1%

Whole Number: higher or larger numbers, or bigger denominator indicates bigger fraction strategies

NR: not reported

Individual students demonstrated gap thinking on none, some, or many of the fraction pairs (see Table 4.30). A score for gap thinking was calculated by how many fraction pair questions (excluding Q. 22e) elicited a gap thinking explanation. Possible gap thinking was excluded from this score, as was the correct complement-to-one strategy in the fraction pair $^3/_8$ and $^7/_8$. Fewer than half the students did not use a gap thinking explanation on any of the seven fraction pairs. Just over fifth offered a gap thinking explanation once, and a further fifth used gap thinking two or three times. Fewer than 10% of the students repeatedly used gap thinking four or five times. Overall, 52.3% of the students demonstrated gap thinking one or more times during the seven fraction pair questions reported. Choosing the larger gap was uncommon and only four students did this.

Table 4.30

Intensity of Gap Thinking Usage

	Gap Thinking score					
	0	1	2	3	4	5
Frequency	47.7%	22.7%	14.8%	6.8%	6.8%	1.1%

Excludes results from Q. 22e

In the present study, I examined gap thinking separately from two whole number thinking strategies: the higher or larger numbers, and the bigger denominator indicates bigger fraction misconceptions. Rebecca compared $^5/_6$ and $^7/_8$ by choosing the fraction with the larger denominator, "Eight is a larger number than six". This was an example of the bigger denominator indicates bigger fraction strategy. Jasmine compared the same fraction pair by choosing the fraction that had the larger denominator and larger numerator, "Um, I did five is higher than three, and eight is higher than seven." This was an example of higher or larger numbers. Both students gave the right answer but for wrong reasons.

All explanations which were identified by either coder as higher or larger numbers, or, bigger denominator indicates bigger fraction were transcribed. The two coders conferred and the coding was agreed upon. However, Q. 22e was excluded from the results reported here because it had been omitted for gap thinking frequencies.

Whole number strategies had been identified by the researchers in the Rational Number Project and elaborated by recent research (see section 2.1.5.1.2) and included the misconceptions higher or larger numbers, bigger denominator indicates bigger fraction thinking, gap thinking, and the addition strategy (adding the same number to the numerator and denominator). However, the presentation of these misconceptions was not uniform (see Figure 4.34). The presentation of the higher or larger numbers and the bigger denominator indicates bigger fraction misconceptions was more prevalent in the students who had low Fraction Pair scores, whereas gap thinking was less prevalent in students with low Fraction Pair scores. Each student is represented by a line or a space on the x-axis, which is numbered 1 to 88. The students have been ordered by increasing Fraction Pairs score; those with a score of 0 are numbered 0 to 4 and are closest to the y-axis and those with a score of 7 are numbered 83 to 88 and are furthest away from the y-axis (see Table 4.31). The students have the same position (and number) in both graphs. The height of the line represents how many explanations (out of seven) by an individual student used the strategies. If these strategies were not used by the student, then there is no vertical line. The top graph shows the intensity (how many explanations) of the misconceptions of higher or larger numbers and/or the bigger denominator indicates bigger fraction. The bottom graph shows the intensity of gap thinking.

Table 4.31

Ordering of Students' with each Fraction Pairs Score from 0 to 7

	Score							
	0	1	2	3	4	5	6	7
frequency	4	20	14	18	20	4	2	6
number on graph	1-4	5-24	25-38	39-56	57-76	77-80	81-82	83-88

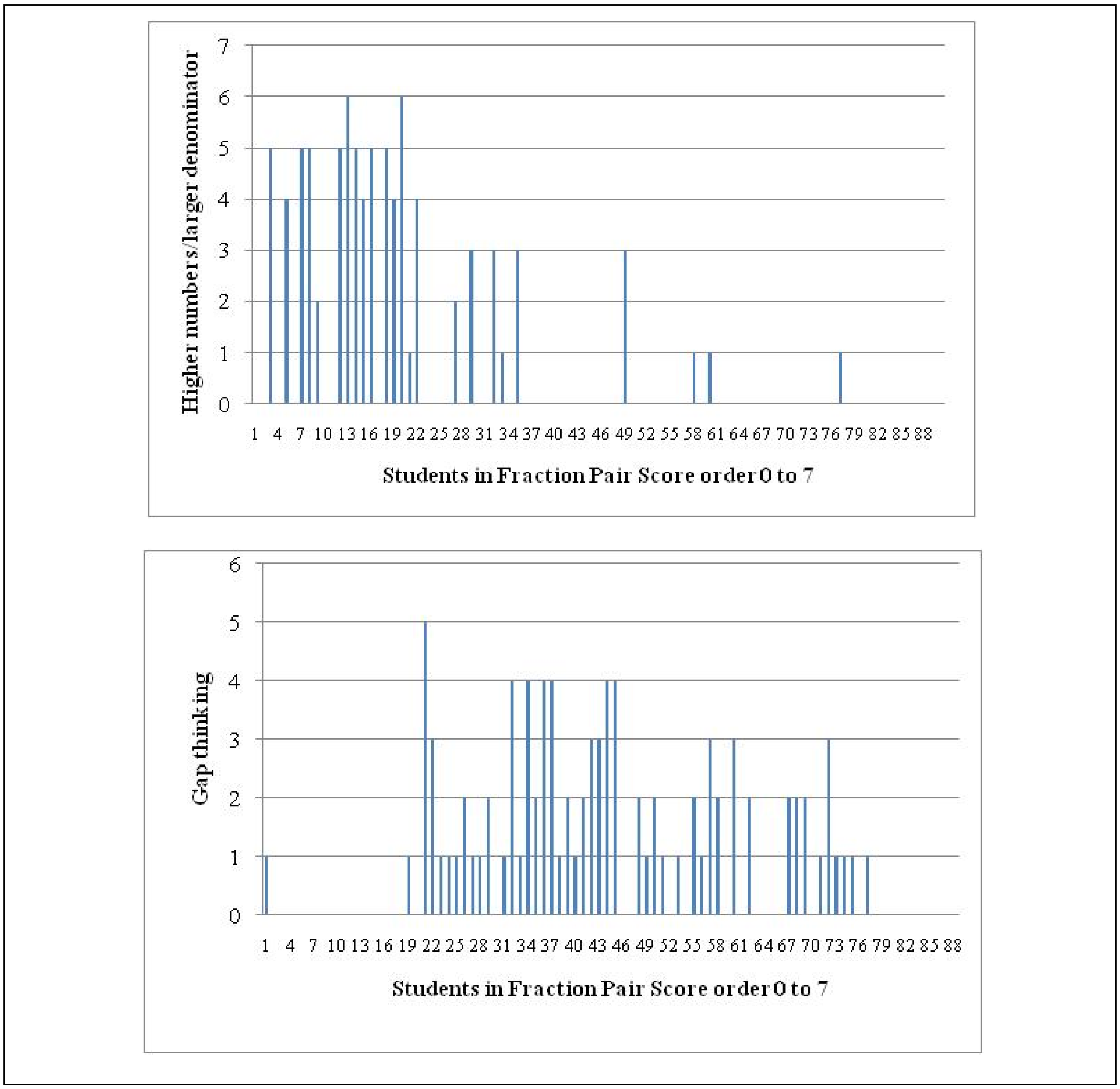


Figure 4.34. Higher or larger numbers and bigger denominator is bigger fraction (top), and gap thinking (bottom) in low to high Fraction Pair score order.

There were just over a quarter of students with a Fraction Pair score of 0 or 1, and they are represented by numbers 1 to 24 on both graphs. The graph of the incidence of higher or larger numbers and/or bigger denominator indicates bigger fraction thinking shows a high frequency (15 out of 24 students) and a high intensity (many explanations per student) of these whole number thinking strategies in the 24 students who had no success or only one correct fraction pair. The students with a Fraction Pair score of 0 or 1 accounted for all the instances of these strategies being used four, five or six times on the seven fraction pairs. Only four students used higher or larger numbers, or bigger denominator indicates bigger fraction thinking, had a Fraction Pair score of 2 or higher. Two of the students with a Fraction Pair score of 3 and two of the students with a Fraction Pair score of 4 used the two misconceptions.

On the other hand, only six of these same 24 students with a Fraction Pair score of 0 or 1 demonstrated gap thinking. Gap thinking lingered for students with a Fraction Pairs score 2 to 4, (numbered 25 to 76) and there was a range of intensity (number of explanations per student), from no use of gap thinking to four gap thinking explanations, in these middle performers. One student with a Fraction Pair score of 5 gave a gap thinking explanation, but gap thinking was not used by students with Fraction Pair scores of 6 or 7. Gap thinking presented differently to the other two whole number strategies.

The difference in presentation of the three misconceptions, higher or larger numbers and/or bigger denominator indicates bigger fraction, and gap thinking also occurs with respect to the students' Equivalence scores. A third of the students had an Equivalence score of 0 or 1 so it was possible to describe the incidence of these misconceptions in students with a range of performance on equivalence tasks. The frequency and intensity of the occurrence of the misconceptions higher or larger numbers and/or bigger denominator indicates bigger fraction is represented in the top graph in the figure below (see Figure 4.35). The students are in a different order to the graphs above which were ordered by Fraction Pair score (see Figure 4.34) because in both the graphs below they have been ordered by increasing success at equivalence questions. The students have the same position on the x-axis in both the top and bottom graphs (see Table 4.32). Students with an Equivalence score of 0 are numbers 1 to 15 along the x-axis in both graphs. There were no students with an Equivalence score of 13, so the five students with an Equivalence score of 12 are numbers 84 to 88 on the x-axis on both graphs. In the both graphs below, the intensity (number of explanations) of the misconceptions is represented by the height of the vertical line. If an individual did not use of the strategies then there is no vertical line from their position on the x-axis.

Table 4.32

Ordering of Students' with each Equivalence Score from 0 to 13

	Equivalence Score													
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Frequency	15	14	4	8	2	3	9	10	5	6	2	5	5	0
Number on graph	1-15	16-29	30-33	34-41	42-43	44-46	47-55	56-65	66-70	71-76	77-78	79-83	84-88	

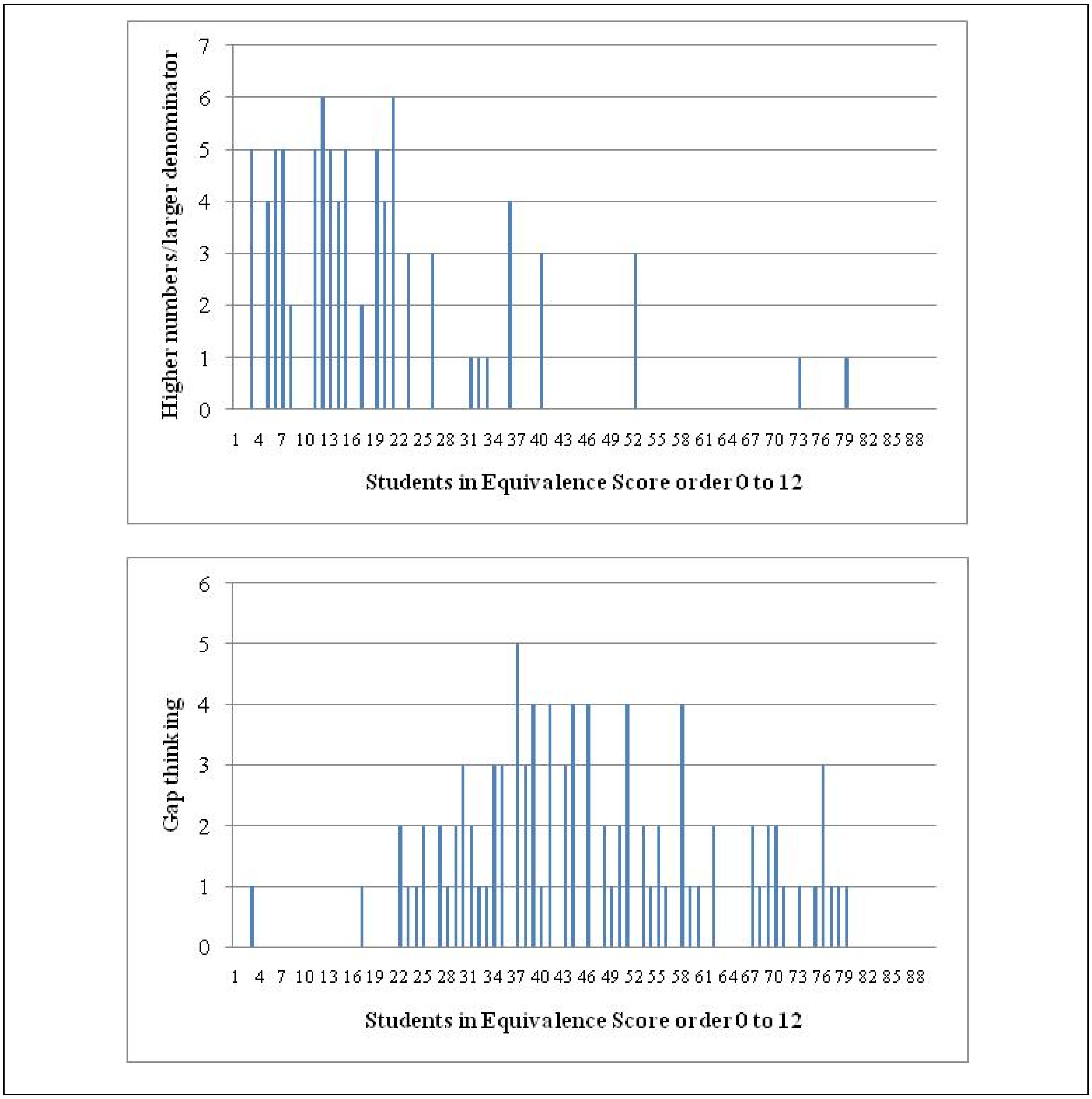


Figure 4.35. Use of higher and larger numbers and/or bigger denominator indicates bigger fraction misconceptions (above) and gap thinking (below) in low to high Equivalence score order.

The incidence of the higher or larger numbers and/or bigger denominator indicates bigger fraction misconceptions was most prevalent in the students who had success on three or less equivalence questions, represented by students numbered 1 to 41. Only three students used these strategies out of the students who had Equivalence scores of 4 to 12.

Gap thinking, on the other hand, presented differently to the other two whole number misconceptions. Two students used gap thinking who had an Equivalence score of 0. One of these students, Shelby, while getting none of the questions correct that contributed to the Equivalence score, did correctly identify $^5/_8$ as larger than $^1/_2$ in the Fraction Pair task (unlike all of the other 14 students who also had an Equivalence score 0). Her awareness of equivalence was just beginning. The other student, Cath, chose the bigger gap in the fraction pair $^3/_8$ and $^7/_8$ which was an unusual presentation for gap thinking. Both of these students gave one gap thinking explanation out of seven explanations. The other 13 students with an Equivalence score of 0 were giving many incorrect explanations for fraction pair comparisons but none of these strategies were gap thinking.

There was a higher frequency of gap thinking in the students with Equivalence scores 1 than of 0 and gap thinking was used by students with Equivalence scores up to 10. The highest intensity was in the students with Equivalence scores of 3 to 7.

The number of students at each Equivalence score was not the same (see Table 4.14). The percentage of students using gap thinking at each Equivalence score (see Figure 4.36) revealed that gap thinking was not common for students with an Equivalence score of zero, emerged at the same time that equivalence knowledge emerged (Equivalence score of 1), increased in intensity (number of explanations used by individual students) as early equivalence knowledge developed, and then was resolved when students had an Equivalence score of 11 and 12. Of the four students who had success with exactly two of the equivalence questions, and so were not yet competent with all of the contexts for equivalence, all demonstrated gap thinking on at least one fraction pair. Gap thinking affected at least 50% of the students in each Equivalence score from 2 to 9. There were no instances of gap thinking by students who had an Equivalence score of 11 or 12. Intensities of 2 and 3 instances of gap thinking have been shaded the same colour, and intensities of 4 and 5 have been shaded the same colour. The higher intensity (four or five gap thinking explanations out of seven) occurred across a range of Equivalence scores, from 3 to 7.

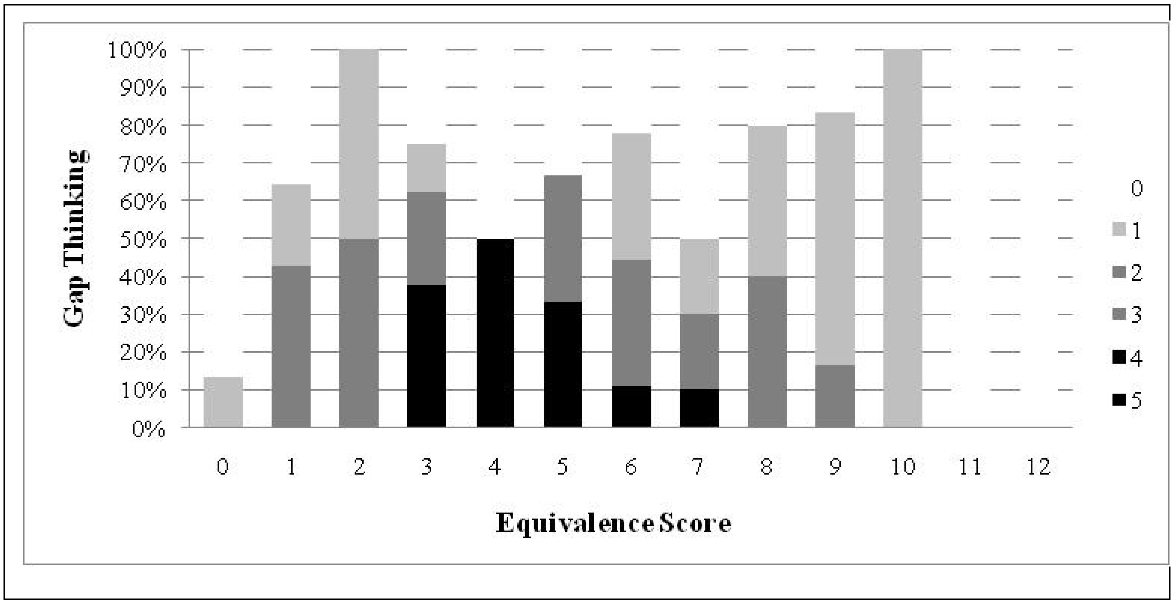


Figure 4.36. Percentage of gap thinking students at each Equivalence score.

A non-linear association between gap thinking and equivalence has been described above. Using Kendall's tau, the only linear correlation between students' Gap Thinking scores and measurement concepts was a negative typical association with CADL, the broken ruler tasks ($\tau = -.239$, $p = .009$). The negative correlation indicated that as performance on broken ruler tasks increased, the intensity of gap thinking decreased, and vice versa. The only linear correlation between students' Gap Thinking scores and other fraction tasks was a negative typical association with the Fraction Pie Part B task ($\tau = -.214$, $p = .028$) and this task is described in section 4.5.2.4.

Students with an Equivalence score of 2 to 9 (48 students), gave a sub-sample to analyse for any linear correlations between gap thinking and other fraction and measurement tasks. The percentage of gap thinkers in this sub-sample was much higher than the overall rate, at 70.8%. Gap thinking in this sub-group (Equivalence score 2 to 9, with intensity was not factored in) had:

- a negative substantial correlation with Part B of the Fraction Pie ($\tau = -.370$, $p = .011$),
- a negative substantial correlation with using a ruler to measure a streamer (TPADL, $\tau = -.344$, $p = .014$),
- a negative typical association with offering standard units for area measures, TPUNA, ($\tau = -.306$, $p = .029$), and
- a negative typical association with broken ruler tasks, CADL, ($\tau = -.288$, $p = .033$).

There was no significant association between this gap thinking in this subgroup (with Equivalence scores of 2-9) and the students' Multiplication score.

4.5.2.2.3 Association between Fraction Pair score and measurement concepts.

There were substantial associations between the students' fraction pairs score and additivity measurement tasks in both length and area contexts: broken ruler tasks (CADL), calculating the area of half rectangles (CADA) and calculating the area of a rectangle (TPADA) (see Table 4.33). There was also a substantial association between the students' Fraction Pair score and conceptual units measurement tasks in both length and area contexts (measuring the Keyboard (CUNL), and calculating the area of an array with leftovers (CUNA) and offering standard units for area measurements (TPUNA)). There was also a substantial association between the students' Fraction Pair score and their Multiplication score.

Table 4.33

Correlations Between Fraction Pair Score and Measurement Concepts

	Correlation		
	Minimal	Typical $\tau > .20$	Substantial $\tau > .34$
Attribute		TPATA $\tau = .253, p = .007$	
Additivity		TPADL $\tau = .232, p = .010$	CADL $\tau = .342, p < .000$ CADA $\tau = .398, p < .000$ TPADA $\tau = .346, p < .000$
Unit		TPUNL $\tau = .277, p = .003$	CUNL $\tau = .461, p < .000$ CUNA $\tau = .347, p < .000$ TPUNA $\tau = .400, p < .000$
Proportionality		CPRL $\tau = .254, p = .005$ TPRL $\tau = .234, p = .014$ TPPRA $\tau = .283, p = .003$	
Multiplication			MULT $\tau = .430, p < .000$

The fraction pair $^2/_4$ and $^4/_2$ (Q. 22d) included an improper fraction. The Keyboard task (Q. 39) also included an improper fraction in the answer, three and three quarters. There were 71.6% of the students either correct on both tasks or incorrect on both tasks (see Table 4.34). Of the six students who correctly named $^4/_2$ as the larger fraction, one gave an answer with four instead of three as the whole number in the Keyboard measurement task. There were 19 students who correctly named the improper fraction in the Keyboard task as three and three quarters (or other acceptable answer), but 14 of them gave answers to the fraction pair question that indicated their unfamiliarity with the symbolic inscription of improper fractions:

- some students flipped the $^4/_2$ to make $^2/_4$,
- some students said $^4/_2$ was not a fraction, and
- some students said there were the same numbers in both.

Table 4.34

Students Performance on the Fraction Pair $^2/_4$ and $^4/_2$ and the Keyboard Task

	Fraction Pair $^2/_4$ and $^4/_2$ correct	Fraction Pair $^2/_4$ and $^4/_2$ incorrect
Keyboard task correct	29	19
Keyboard task incorrect	6	34

4.5.2.3 Non-congruent area diagrams: Fold Me a Quarter task.

The Fold Me a Quarter task (Q. 13) was chosen to assess another aspect of the measure sub-construct of fractions; the use of fraction understanding to quantify non-congruent parts of an area diagram. The student was asked to fold a kinder square piece of paper into quarters, and when they had done that, to fold another kinder square into quarters another way (see Figure 4.37). I then used their pieces of paper to ask them to compare the area of one part (square/triangular/rectangle quarter) of the paper with one part on their other kinder square. After this area comparison, I showed them another kinder square that I had folded into quarters another way (I had all three foldings prepared and showed whichever they had not used) and asked them to compare the area of one of those parts with the parts on their folded kinder squares. This made three comparisons: square with triangle, square with rectangle, and rectangle with triangle. Students were given a score out of 3 according to the number of successful comparisons that they made. A score of 3 was achieved by 61.4% of the students, 5.7% achieved a score of 2, and 33% achieved a score of 0 or 1.

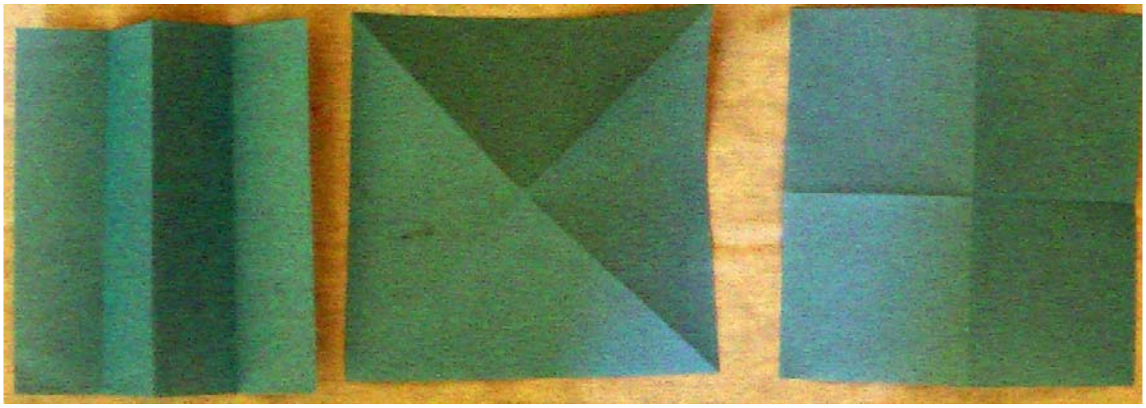


Figure 4.37. Non-congruent quarters in Fold me a Quarter task.

Successful explanations used fraction reasoning, dynamic imagery, or comparisons by eye. Claire used fraction reasoning when comparing the triangle and square parts that she had folded, saying, "No they're the same, because they are the same pieces of paper [touches both kinder squares] and.", and concluding after prompting, "And each piece of paper has the same amount um sized quarters." When shown a new kinder square folded into rectangle quarters she successfully explained that "Um that one's the same size, but it's just in a different shape." Sylvie also explained correctly for her own folding "Because they are same size of paper and I just folded them in quarters but different ways" and similarly when comparing my folded kinder square with her own "Because they are the same size of paper and both cut into four". The dynamic imagery strategy was similar to that used in the Similar Shapes task (Q. 36h) (see Figure 4.4), with students mentally breaking the triangle quarter and rearranging it to make the square.

Incorrect explanations included noticing that the perimeter was longer on the triangle and concluding that that would indicate a bigger area. Some of the students who had a score of 0 or 1 had explanations revealing this misconception. For example, Tyler explained that the triangle part was larger (I had used the word area in the question) "'Cause the outside is longer." Tyler identified that the perimeter of the triangle was longer (correctly), but extrapolated (incorrectly) that that also indicated a bigger area. Other incorrect answers were less clear, such as Alex's: "Because this, oh hang on. It's because it's a bit more wider [touches triangle] than this one [touches square]. Because this is all equal [points to square] edges, and not here [points to triangle]". He may have been using perimeter comparisons or area comparisons but it was difficult to tell from his explanation.

Despite the Fold Me a Quarter task and the Similar Shapes task (see Figure 4.4, Q. 36h) both assessing the comparison of non-congruent areas, there was not a significant association between them (using Kendall's tau). Of the 54 students (61.4%) who had scored 3 on the Fold Me a Quarter task (see Figure 4.38), 49 (43 and 6) also explained that the shaded shapes in the Similar Shapes task had the same area. However, six of the 49 who stated that the areas were the same (incorrectly) reasoned specifically that this was because the perimeters were the same. A further 32 did not articulate this same area equals same perimeter reasoning in the area comparison (Q. 36h) but they had identified incorrectly that the perimeters were the same during the perimeter comparison (Q. 36g). Of the 54 students who made three correct area comparisons in the Fold Me a Quarter task, 11 of them also correctly identified the shaded areas in the Similar Shapes task as the same (using fraction reasoning, dynamic imagery or global visual comparisons) and that the triangle had the larger perimeter. Hence overall, 12.5% of the 88 students successfully compared all non-congruent areas in the Fold Me a Quarter task and the shaded shapes in the Similar Shapes task (Q. 36h), and successfully compared the perimeters in the shaded shapes (Q. 36g). These 11 students used dynamic imagery to rearrange the area or fraction reasoning on all three of the Fold Me a Quarter area comparisons.

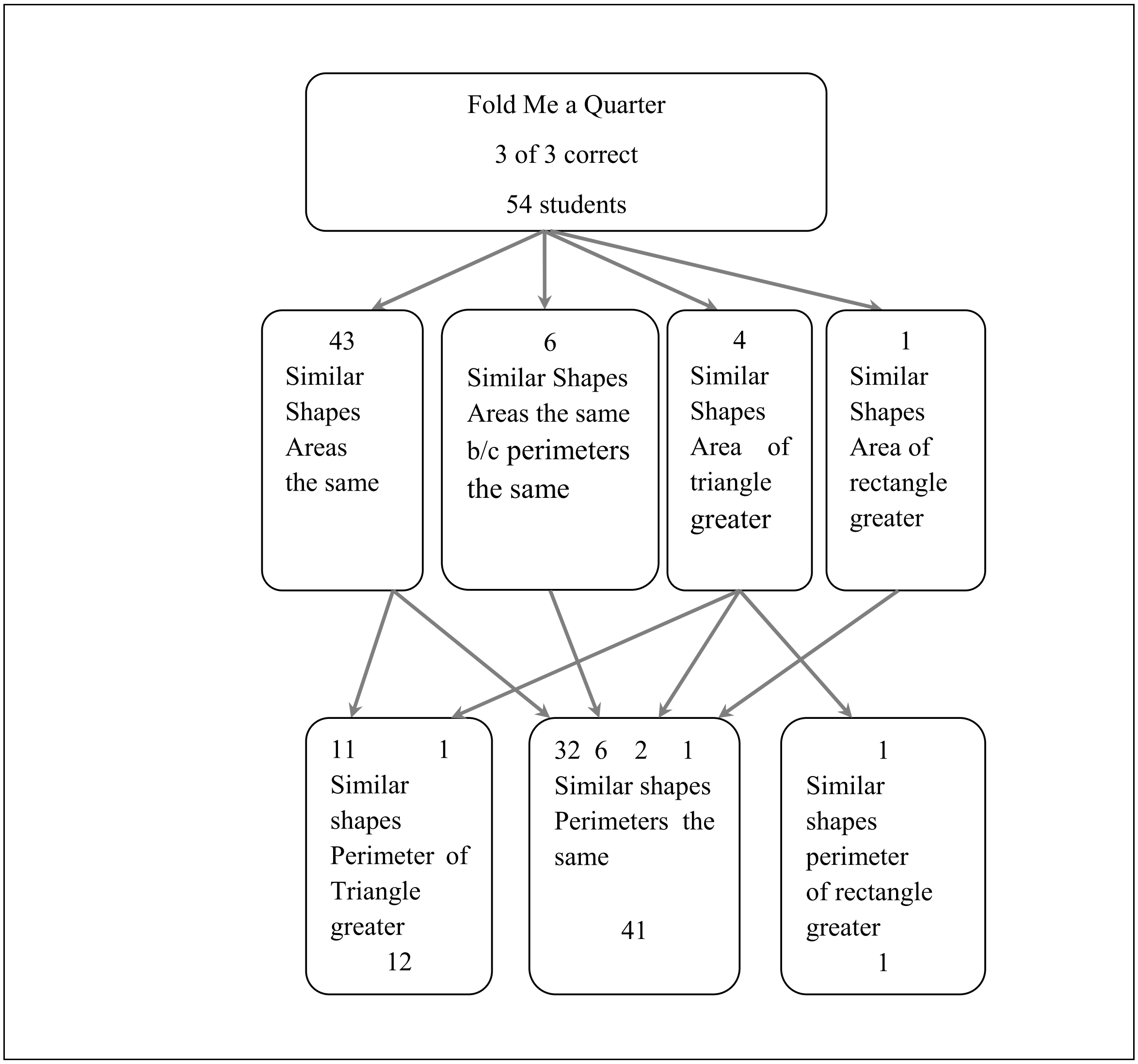


Figure 4.38. Comparison of students' with Fold Me a Quarter score of 3 and performance on Similar Shapes shaded shapes questions.

Five students made two correct comparisons in the Fold Me a Quarter task. Three of them explained that the areas of the shaded shapes were the same in the Similar Shapes task (Q. 36h) and that the perimeter of the shaded triangle was longer (Q. 36g).

There was a much higher frequency in the low performing group (Fold Me a Quarter score of 0 or 1) (see Figure 4.39) of correctly identifying the shaded triangle as having the longer perimeter, than there was in the high performing group (Fold Me a Quarter score of 3). However, for five of the six students, this then led to the conclusion (incorrectly) that the shaded triangle also had the larger area.

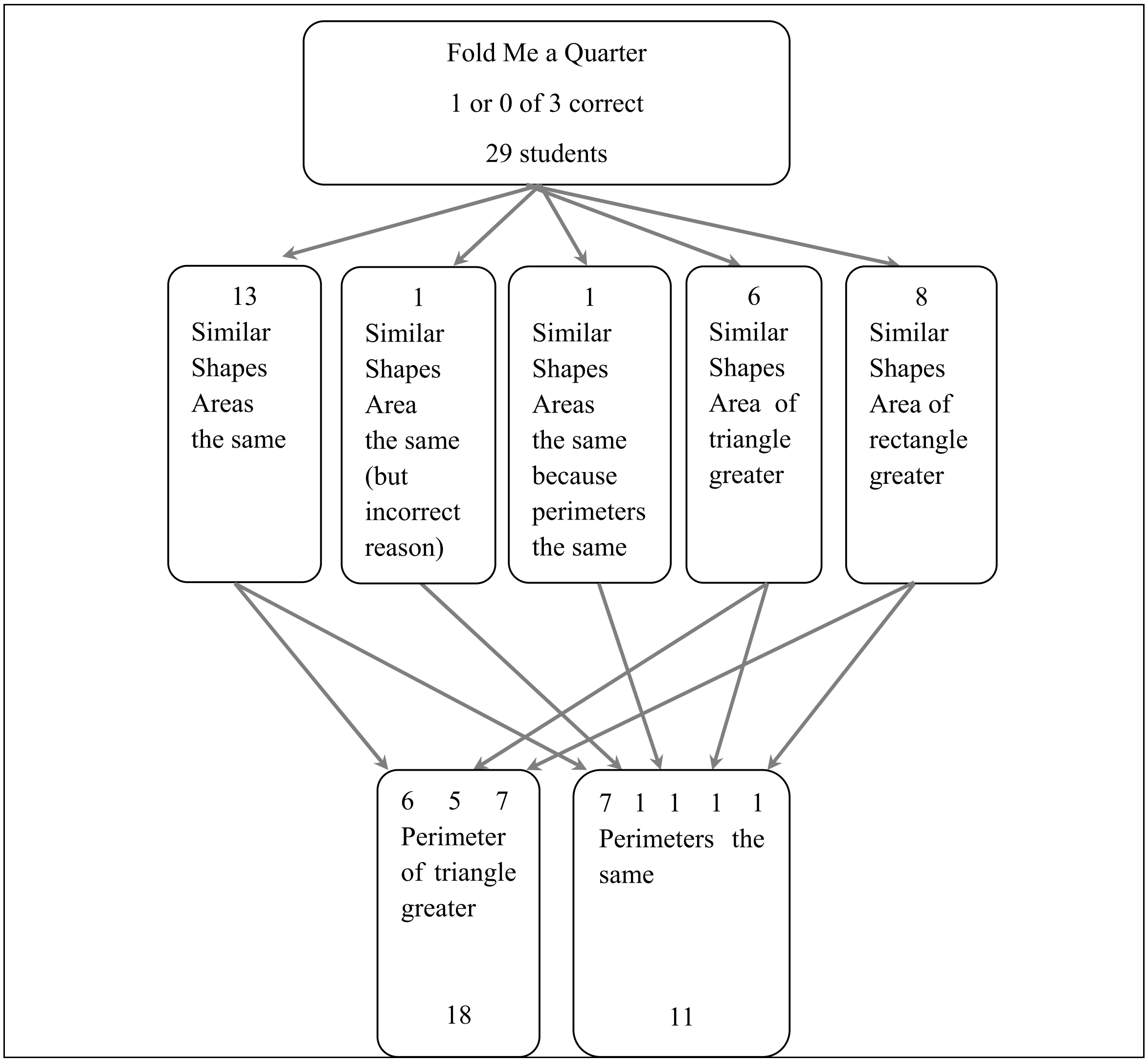


Figure 4.39. Comparison of students' with Q. 13 Fold Me a Quarter score of 0 or 1 and performance on Similar Shapes shaded shapes questions (Q. 36g and h).

Perimeter comparisons were more successful in the students with weaker area knowledge but increased skill at area comparisons appeared to disrupt perimeter knowledge rather than add to it. There was no probing about perimeters in the Fold Me a Quarter task, so the misconception that perimeter and area were always related was not revealed in this task.

The Puzzle task (Q. 57) required geometric reasoning or dynamic visualisation to rearrange three shapes into a square. More students who made none or one successful area comparison in the Fold Me a Quarter task were unsuccessful at the Puzzle task than successful (see Table 4.35). However, of the 54 students who gave three correct explanations for the area comparisons of the three non-congruent quarters in the Fold Me a Quarter task, 26 were successful on the Puzzle task and 28 were unsuccessful.

Table 4.35

Comparison of performance on the Puzzle task and the Fold Me a Quarter task

	Fold Me a Quarter non-congruent area comparisons		
	3 correct	2 correct	0 or 1 correct
Puzzle correct	26	3	7
Puzzle incorrect	28	2	22

4.5.2.4 Non-equal-parts diagrams.

Two tasks are reported in this section, the Fraction Pie task and three questions from the Fraction Sort task.

4.5.2.4.1 The Fraction Pie task.

The Fraction Pie task (Q. 14) was a non-equal-parts diagram and the student was asked what fraction of the circle was Part A and what fraction of the circle was Part B (see Figure 4.40). Initial double coding using transcripts and video footage of 58 of the 88 students completing the task was carried out by a practicing secondary mathematics teacher who also had a Masters by Research degree in mathematics education. An abbreviated coding protocol was used with 18 descriptors. These descriptors and the task were not familiar to the double coder. One error of my coding of correct/incorrect was picked up by the double coder in a student's self corrected response to Part A. Given that this was the only error in 116 answers, I am confident about using my coding of correct and incorrect answers. A second, detailed coding and double coding of all explanations for answers of one third, one fifth, two fifths, one seventh, and two sevenths using transcripts (including from audio files) was completed. Any discrepancies in coding were resolved through discussion.

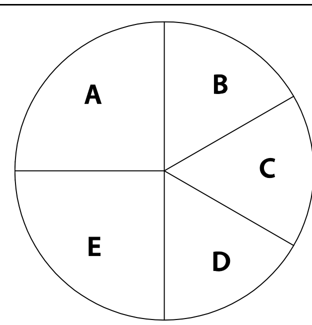


Figure 4.40. Fraction Pie task diagram.

The frequency of success on Part A of the Fraction Pie task, recognising one quarter, was 70.5%. The correct answer to Part B was a sixth, and 27.3% of students in the present study were successful at both offering this answer and a mathematically correct explanation. All students who were successful on Part B had been successful on Part A.

Some students appeared to recognise the shape of the quarter in Part A of the Fraction Pie task as a prototype. For example, Noah explained "'cause it's a circle and. I just know that because if I see a circle and I see that. I just know it's a quarter, I just do." Other students mentally broke the left hand side of the diagram into a half of a half and Alec described this as "well A and E take up half [touches A and E], and they have the same area, so you cut that in half and A's a quarter." Some students imagined the radius between A and E and mentally extended it across the right hand side. This explanation might also be a justification after recognising a prototype but there is no specific evidence of that in explanations such as Felix's: "'Cause if you did four quarters, it would be half and half [traces real diameter and imaginary diameter extending from radius between A and E]".

Successful strategies in Part B included either imagining the left hand side divided symmetrically as a mirror image of the right hand side, or numerically calculating that three on one side would mean six altogether. For example, Jack explained "There's three of the same size on this side. So that means there would be able to fit, uh, three of the same shape on the other side. And that's six. And that would be one of them." No student expressed their strategy specifically in an operator context; one third of a half is one sixth.

Mathematically correct strategies that were not executed correctly in Part B included comparing the size of Part B to a quarter (the size of Part A). This appeared to use either an operator approach or a unit-forming approach. However, no children were coded as correct using these approaches as they either gave an incorrect answer or could not explain their answer of one sixth with sufficient detail. In all, 23.1% of the students approached quantifying Part B by comparing it to Part A (a quarter).

The operator approach was used by Matthew who used geometrical reasoning to state correctly that Part B was *two thirds of a quarter*, but was unable to name this as one sixth and so was coded as incorrect. Zak gave the answer of point seven, explaining "It would be. I don't really know how to say it. I think it would be, maybe zero point seven." When asked to explain he added, "That's a quarter [points to A] and if [lays pen across imagined diameter], that would be zero, under one [points to B], so." Zak appeared to be describing Part B as 0.7

of a quarter but may have lost track of the unit because he left his answer as point seven and was coded as incorrect.

Other students may have been using a unit-forming approach, imagining Part B plus a small part equalled a quarter. The correct calculation would have been *a sixth plus a twelfth equals a quarter* and in this context, answers of one fifth, one seventh, one eighth, and one third (thought to be the unit fraction smaller than one quarter) were estimates of the part that was "nearly a quarter". Some students used faulty partitioning and thought that Part B was half a quarter. It is not clear from explanations such as Elsie's, that Part B was "Um, slightly smaller than a quarter", or Lara's, that Part B was "I don't know, it's a quarter of a quarter, yeah, three quarters of a quarter", whether these were a unit forming (correct additive) or an operator (correct multiplicative) approach.

Two students gave the correct answer of one sixth but their explanations were not sufficiently precise to be coded as correct. However, these two students were using a mathematically correct strategy, they compared the size of Part B to the quarter (Part A), but could not fully explain it:

- Claire said "Because a quarter is four and half is two so that's half of a quarter, [half?] is going to be a sixth." and
- George said "Because half a quarter. I think it's half of a quarter so it might be a sixth."

The frequency of the incorrect answer of one third on Part B was 12.5%. Eight of the 11 explanations concentrated on the right hand side (three parts). Some did not mention the left hand side, but others explicitly said that there were parts of different sizes in the whole circle. For example, Emma explained her answer of a third "'Cause in this half they're in thirds. There's three things and it's one, there's one of them so that makes it one third." The other three answers of one third were accompanied by explanations that compared Part B to Part A (but the students thought incorrectly that $^1/_3$ was smaller than $^1/_4$.)

The answer of one fifth could be due to the double count misconception because there were five parts. However, there were a variety of explanations that were offered for an answer of one fifth or two fifths. Four students used the unit forming or operator approach described above (and are represented by strategy 1 on Table 4.36). One other student did not demonstrate any double counting behaviour, but gave the answer of one fifth (Strategy 2). Alex tried to make equal parts by tracing imaginary lines on the fraction pie and then touching five imaginary pieces clockwise around the circle. It was a geometrical and almost iterative approach, but unsuccessful. Strategies 1 and 2 were mathematically correct approaches but executed inaccurately.

Table 4.36

Different Explanations of an Answer of $^1/_5$ in the Fraction Pie task

	Strategy	Frequency
1	Part B is $^1/_5$ because it is almost a quarter (Part A); estimating using operator thinking or unit forming thinking.	4
2	Part B is $^1/_5$; iterating inaccurately.	1
3	Parts A and B are both a quarter and a fifth depending on whether it is area or number that is relevant ("how much room" or "how many pieces").	1
4	Part A and B are both fifths, but not equal fifths.	1
5	Part A is a quarter and Part B is a fifth, but not an equal fifth.	1
6	Part A is two fifths because there are five parts but A is twice as big as B; and Part B is one fifth because the piece is half the size of A.	1
7	Part A is one fifth because there are five pieces, but Part B is two sevenths because there are two Bs in an A (seven parts).	1
8	Part A is one fifth and Part B is two fifths because there are non equal parts.	1
9	Part A is a fifth because there are five pieces, but Part B is a third using operator thinking or unit forming thinking.	1
10	Part A is one quarter and Part B is one fifth because there are five pieces.	2
11	Part A is one fifth as is Part B because there are five pieces.	8
12	Part A is one fifth and Part B is two fifths because A is already coloured; or because it is the second fifth.	2

The next nine students, Strategies 3-10, offered variations of double counting behaviour but indicated that the size difference between the pieces had been noticed. For example Ruby explained why Part B was a fifth but not an equal fifth "because there are five pieces in it. Um, it would be an equal fifth if they were a bit smaller and they were the same size, but they're not. Because those two are bigger than those three; so technically they're fifths, they're just not equal fifths." Shannon also offered a conditional double counting explanation for Part A "it is one quarter though because how much room it has. But still it's one fifth 'cause how many pieces there is." Students qualified their double counting behaviour by offering different answers for Part A and Part B. For example, Ebony argued that Part A is one fifth because "it's split up into fifths, but not equal" but Part B was two fifths. In Ebony's explanation, the difference in size between Part A and Part B had been noticed and a different name offered for the second part. This was not always the case for a second answer of two fifths (see Strategy 12).

Many successful students had imagined the entire right hand side of the fraction pie flipped back to make a left hand side, and could see this would make six equal parts. However, all the students who imagined the radii on the right hand side of the diagram extending back through A and B, made seven parts, (see the left hand side image of Figure 4.41). I believe that they thought that these parts were equal parts (not noticing that there would be four in one half and three in the other half). Other students visually estimated Part B to be half of Part A, and for some this was confirmed (mistakenly) by this same extending back of the radii from the right hand side, (see the right hand side image of Figure 4.41). For some of these students, Part A was two sevenths because it was made of two pieces of Part B which was one seventh because it was one out of seven (supposedly) equal parts. Sometimes this was overlaid with (conditional) double counting behaviour. For example, while Part A was one fifth because there were five parts, Part B was two sevenths because there were seven parts imagined. For example, Cameron explained "if you've got two Bs or two Ds, if you add them up together then it should make one A. So it's two and the total of the other ones. One, two, three, four, five, six, seven." Trying to make equal parts was a mathematically correct partitioning strategy, but in these examples it was not executed successfully.

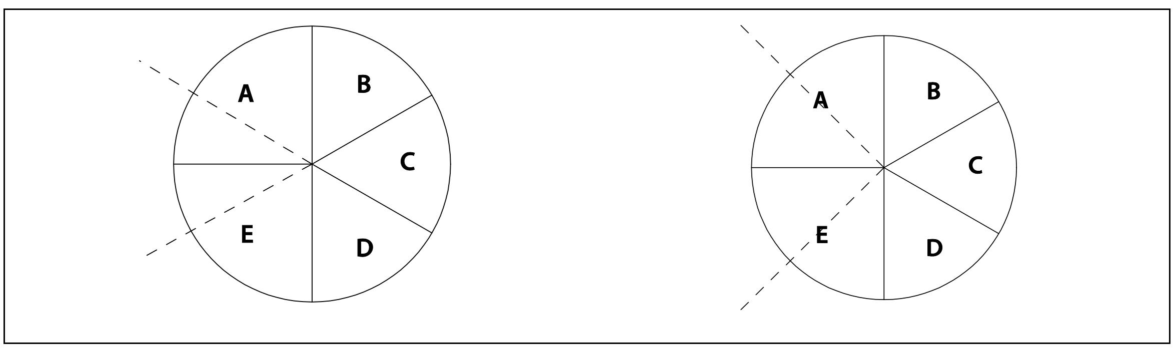


Figure 4.41. Lines imagined by students attempting to make equal parts on the Fraction Pie task.

The final two strategies, (11 and 12), were true double count behaviours with Mia explaining "In fractions, how many is coloured in goes on top, yeah. And then you count the squares of the whole thing and you put that underneath." Unconditional double counting on both Part A and Part B was used by 9.1% of the students and both parts were named one fifth. Two students gave different answers to Part A and Part B, naming Part B two fifths, but there was no evidence in their explanation that they did this because they had noticed that the parts were unequal. Josh visualised double counting and so had already mentally coloured in Part A before mentally colouring in Part B, explaining his answer of two fifths as "I just did that one. And then that one would be coloured. So that's one, two." Tamika named Part B two fifths because it was the second fifth, explaining that she worked it out "The same as A, except that it's the second".

An answer of one fifth for Part B was offered by 21.6% of the students (this excludes Strategies 7, 8, 9 and 12 because another answer was offered for Part B). Under half of these answers of one fifth were accompanied by unconditional double counting behaviour. While 28.4% of students in the present study gave an answer of one fifth in Part A and/or Part B, five of the 24 were attempting to use mathematically sound strategies; nine of the 24 qualified their double counting explanations; and ten of the 24 used double counting without qualification in both Part A and Part B, explaining that both parts were one fifth because there were five pieces (or two fifths because it was the second fifth).

4.5.2.4.2 The Fraction Sort task.

The Fraction Sort task consisted of 24 cards (fraction diagrams) that the students had to sort into piles labelled $^1/_4$, $^1/_6$, $^2/_3$ and *other*. Diagrams were used that assessed students' knowledge of unit ($^1/_4$ and $^1/_6$) and non-unit ($^2/_3$) fractions in discrete, length, and equal parts and non-equal-parts area diagrams. Some of these cards were used to assess students' knowledge of equivalence (see section 4.5.1.1). Seven of the cards used to assess students' understanding of area representations of unit and non-unit fractions (see Figure 4.42) were a circle divided into six equal parts (Q. 19a), a circle divided into quarters (Q. 19g), a square divided into quarters (Q. 19h), a circle divided into three equal parts with two shaded (Q. 19q), a rectangle divided into three parts with two shaded (Q. 19p), a rectangle divided into four unequal parts, one of which was a sixth (Q. 19b) and a triangle divided into three unequal parts (Q. 19r). The triangle area diagram was actually an equivalence task and while two students mentally restructured and named the shaded part six ninths, they put the card in *other,* instead of $^2/_3$. However, it could also be used to explore students' understanding of non-equal-parts diagrams. One of the cards used to assess students' understanding of length representations of unit fractions was a line with a middle quarter shaded (Q. 19l).

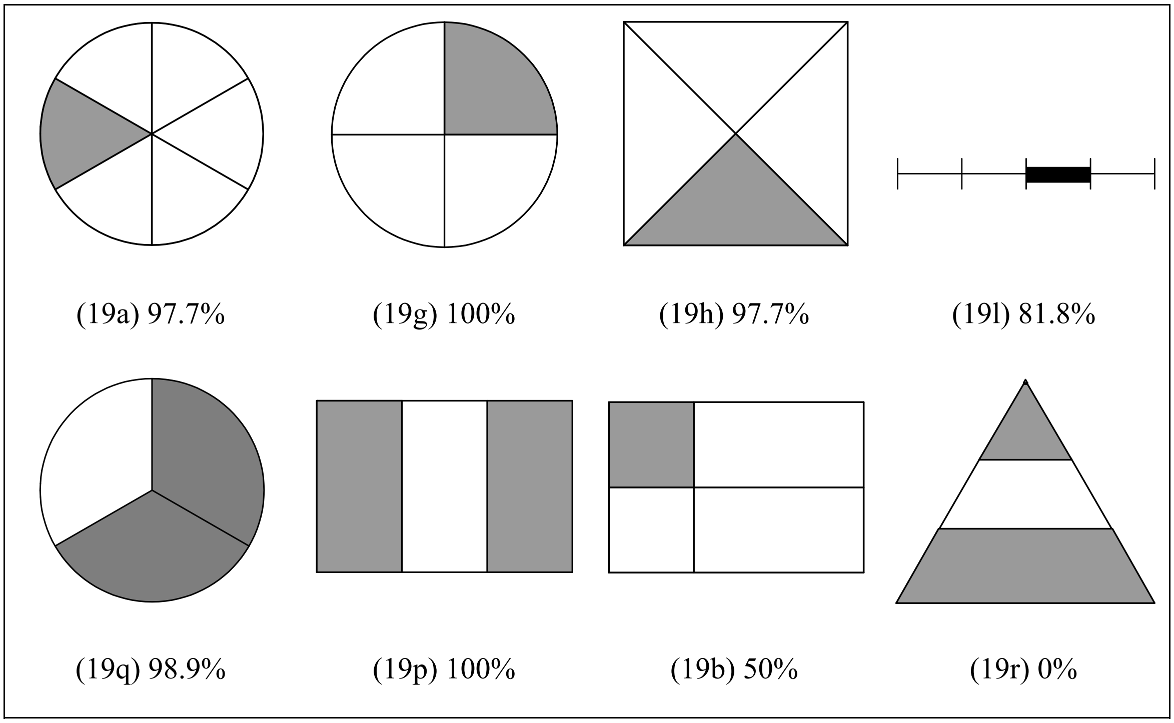


Figure 4.42. Frequency of success on some Fraction Sort cards.

Due to the size of the task, a sub-group of ten students' answers and explanations for all 24 cards were double coded by a mathematics education lecturer. There were no disagreements over correctness of the students' responses. Using simplified coding descriptions, all discrepancies in coding were resolved through discussion.

Students' correct strategies for Q. 19b, the non-equal-parts diagram of $^{1}/_{6}$, included imagining an extra line to make equal parts (see Figure 4.43). The frequency of success on this card was 50%, while 25% gave an incorrect answer but noted the unequal parts, and 25% gave an unconditional double count explanation.

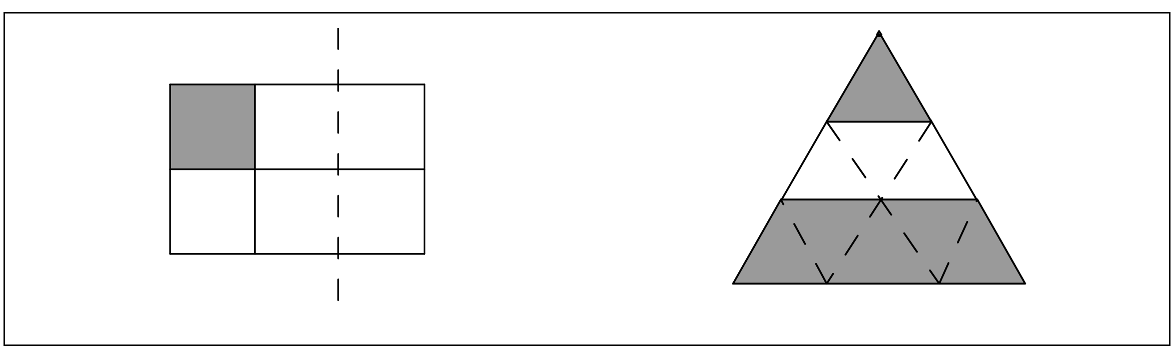


Figure 4.43. Students' imagined lines for the Fraction Sort cards.

While no students were able to see the equivalence of $^{6}/_{9}$ as $^{2}/_{3}$ (Q. 19r), two students were able to see that six out of nine equal parts were shaded after mentally restructuring the diagram (see Figure 4.43). Including these two students, and those who answered *other,* and those who answered two thirds but noted that the pieces were not the same, 37.5% of students noted that that the diagram did not have equal parts. However, 62.5% either stated the answer was $^{2}/_{3}$ or gave an unconditional double count explanation.

Double counting behaviour was observed in the rectangle sixth (Q. 19b) and the triangle (Q. 19r) as well as the non-equal parts Fraction Pie task (Q. 14) (see Table 4.37). The Fraction Pie task had the lowest rate of double counting behaviour, but this was partly due to students being asked about both parts which drew their attention to the fact that the parts were different sizes. The highest rate of unconditional double counting behaviour was observed in the triangle $^{2}/_{3}$. Overall, 69% of students demonstrated double counting (including Strategies 7, 8, 9, 10, 11, and 12 in Table 4.36) in one or more of the four questions: Fraction Pie Part A, Fraction Pie Part B, Q. 19b, and/or Q. 19r. Increasing geometrical complexity of diagrams increased the rate of double counting, but 31% of students did not use this strategy in any of these four questions. This 31% was made up of students from all three schools in the study.

Table 4.37

Frequency of Unconditional Double Counting Explanations in the Fraction Pie Task and Two Non-Equal Part Fraction Sort Cards

Fraction Pie (Q. 14)	rectangle $^{1}/_{6}$ (Q. 19b)	Triangle $^{6}/_{9}$ (Q. 19r)
11.4%	25%	62.5%

Clearly, mentally restructuring the triangle was the geometrically the most difficult for the students. The students found it easier to imagine the equal parts in the rectangle than they did in the circle (both sixths) as demonstrated by the higher frequency of success of Q. 19b (see Figure 4.3). All four students who partitioned the Fraction Pie into seven parts by extending the radii on the right hand side back through the left hand side (see Figure 4.41) correctly made equal parts on the non-equal parts rectangle (Q. 19b). It was easier to make six equal parts successfully in the rectangle (Q. 19b) than the circle (Q 14b). However, the fraction $^{1}/_{6}$ in a circular inscription, in itself, was not an impediment for 97.7% of the students who placed the equal parts sixth card (Q. 19a) correctly (see Figure 4.42). Similarly, all students correctly identified the quarter in Q. 19g, so for the 29.5% of the students who could not identify the quarter in the Fraction Pie Part A, it was not the fraction $^{1}/_{4}$ in a circular inscription, in itself, that was the problem.

It was also observed that students did not necessarily offer the same explanation across the three tasks. In fact, Mia who is quoted above defining double counting behaviour (Strategy 12 in Table 4.36), successfully restructured the rectangle to see one sixth. She was one example of students who used different explanations on different tasks and 71.6% of the students did this. Only four students (4.5%) demonstrated unconditional double counting on Part A and B of the Fraction Pie task, the rectangle $^{1}/_{6}$, and the triangle $^{6}/_{9}$. Another student gave the double count answer for each of the questions, but explained each times that the parts were not equal. Of the 22.7% of students who gave correct responses for Part A and Part B of the Fraction Pie and the rectangle one sixth, just less than a third of them resorted to double counting when confronted with the triangle (Q. 19r). Similarly, of the 29.5% of students who correctly explained that Part B of the Fraction Pie was a sixth, not all of them correctly restructured the rectangle (Q. 19b) to see a sixth, but none of them gave an unconditional double counting explanation.

The students' who used double counting across the fraction sort tasks had similar sounding explanations for each question, indicating that they were not attending to the size of the parts of the diagrams. However, many students who were successful on non-equal-parts diagrams also had double count sounding phrases in their explanations for cards that were equal-parts diagrams (see Table 4.38).

Table 4.38

Explanations for Fraction Sort Cards Illustrating the Double Count Phrasing

Task	Explanations
19g circle one quarter equal parts	Jess: I put that there because there's four and only one shaded in and that's a quarter. Jade: A quarter. 'Cause, um, there's one coloured in and [mumble] there are four
19g circle one sixth equal parts	Jess: [touches each segment with little finger as a count. Thinks. Then places in sixth pile] I put it there because there's six there and then there's one um shaded, and there's six. Jade: One sixth because there's six pieces and one's coloured
19b rectangle sixth non-equal parts	Jess: Yeah I put that there [quarter] because there's four shaded [sic] and one shaded and it's a quarter. Jade: One sixth, because if you put a line down there [indicates with finger] there would be six spaces; one coloured.
19r triangle $^6/_9 = ^2/_3$ non-equal parts	Jess: That's kind of the same of all of those [indicates other cards in $^2/_3$]. And that would be really weird if I did a different decision... Jade: Same as that one [points to rectangle $^2/_3$ in two thirds pile]. Oh wait. *Other* because that space's smaller [touches shaded tip]

4.5.2.4.3 Association between Part B of the Fraction Pie task and measurement concepts.

There was a substantial association between students' performance on Part B of the Fraction Pie task and the conceptual tasks of the units concept in a length context (CUNL) (see Table 4.39). The entry-level task for this category was the Keyboard task.

Table 4.39

Correlations Between the Part B Of The Fraction Pie Task and Measurement Concepts

	Correlation		
	Minimal τ > .07	Typical τ > .20	Substantial τ > .34
Attribute			
Additivity		CADL τ = .201, p = .043 CADA τ = .269, p = .007	
Unit		CUNA τ = .231, p = .031 TPUNA τ = .324, p = .002	CUNL τ = .413, p < .000
Proportionality		CPRL τ = .207, p = .042	
Multiplication		MULT τ = .334, p = .001	

The double count misconception in the Fraction Pie task (Part A and Part B) had two presentations: unconditional (both parts were one fifth because there are five pieces) and conditional (either including a coda such as "but not equal fifths", or indicating that the two parts had different fraction names) (see Table 4.36). Of the ten students who gave unconditional double count responses to the Fraction Pie task, three gave the correct answer of three and three quarters to the Keyboard task (Q. 39). One student offered the imprecise estimation three point nine. The other incorrect answers were four and a third, four and a half, three and a quarter, three and a bit, three and two halves, and twelve centimetres. Of the nine students who gave conditional double count explanations for the Fraction Pie task, five of them gave the correct answer to the keyboard.

4.5.3 The quotient sub-construct.

The Sharing Custard Tarts task (Q. 20) was designed to assess the students' understanding of the quotient sub-construct of fractions. Each of the four questions was presented with the task card and figurines (e.g. Q. 20a, see Figure 4.44) and the children could use pen and paper to work out their solutions. Some students drew lines from the people to their shares to keep track. There were successful strategies mostly at the intuitive level using partitioning and unit forming understandings and 22.7% of the students could successfully partition three strips of liquorice to share equally between five people and name the share as three fifths, or six tenths. A further 39.8% of the students could divide the liquorice so that every person had an equal share, either as fifths or a half and a tenth, but could not name the share successfully.

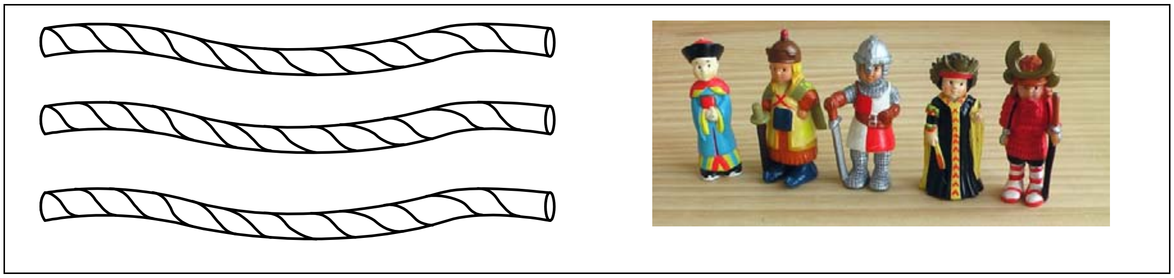


Figure 4.44. Sharing Liquorice, three pieces between five people (Q. 20a).

In the area context (Q. 20b, see Figure 4.45), 26.1% of students could successfully make five equal shares and name it correctly, and a further 36.4% could make five equal shares, either as fifths or a half and a tenth, but were unsuccessful at naming this correctly. Harry found the partitioning difficult with a circular area diagram. He divided a circle into six parts and after puzzling over more drawings he thought aloud, "there's no way you can cut it up into five pieces". Harry was able to articulate the need for partitioning into five parts but could not execute it and appeared unable to use an abstract conjecture, if it was cut into five pieces I would... In contrast, he had mentally partitioned the liquorice strip into five and marked the divisions with a sweep of the back of a pen. He had then drawn the partitioning on his own diagram adding the abstract explanation, "it's even"; indicating to the interviewer that for the purposes of sharing, the pieces were equal sized even if his drawn pieces were not quite even.

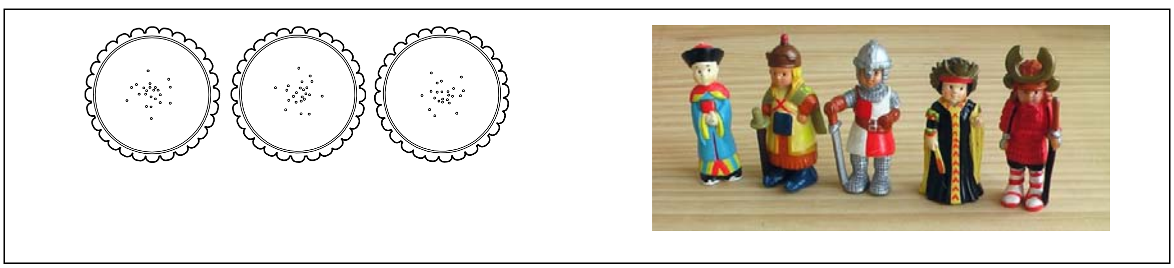


Figure 4.45. Sharing Custard Tarts, three pieces between five people (Q. 20b).

Two further questions were asked with similar task cards and figurines. An improper fraction answer ($^7/_5$ or $1^2/_5$) was generated by the problem of five people sharing seven custard tarts (Q. 20c) and 20.5% of the students could successfully share and name the share in this problem. A further 35.2% of the students could create or imagine equal shares but could not name one person's share successfully. An easier division, but still resulting in an improper fraction answer, was needed to calculate nine pieces of liquorice shared equally between four people (Q. 20d). Either repeated halving, or dividing into quarters led to the correct answer of $2^1/_4$ and 61.4% of the students did this and named one person's share correctly, while a further 13.6% could do the sharing but not name the share correctly.

One of the 88 students successfully offered a fraction as division explanation, three shared between five is three fifths, and only on one of the four questions. So as a test of the technical-symbolic level of understanding of the quotient sub-construct, this understanding was used successfully on only 1 out of 352 occasions (0.003%).

4.5.4 The operator sub-construct.

The Simple Operators task (Q. 18) was used to assess the operator sub-construct. Three initial questions were solved mentally, but students were not asked to explain their answers. As reported in the description of the baseline performance of the group (see section 4.1 above), 98.9% of the students said that three was half of six. However, only 39.8% could calculate that two and a half times six was 15. Incorrect answers included: thirteen, thirteen and a half, and eighteen. These answers indicated that the students could double six but were unsure about what to do with the half, despite having just been asked, half of six. Two thirds of nine was calculated mentally by 68.2% of the students. For the last two questions of this task the students could use pen and paper. The fraction problem a third of a half was successfully solved by 25% of the students.

4.5.5 The ratio sub-construct.

The Bookworms task (Q. 12) required the students to quantify the proportional change in books eaten, given different amounts for different bookworms (see Figure 4.46). The bookworms ate six times, three times, and twice as much (left to right) as the one on the far right. In the first question, one book was placed under the bookworm on the far right (Q. 12a) and 71.6% of the students successfully explained that the other bookworms would eat (right to left) two, three, and six books. Four books were placed under the bookworm second from the right (Q. 12b) and 51.1% of the students successfully explained that the bookworms would eat (right to left) two, four, six, and twelve books. Nine books were placed under the bookworm third from the right (Q. 12c) and 46.6% of the students successfully explained that the bookworms ate (right to left) three, six, nine, and eighteen books. Overall, 40.9% of the students were successful on all three questions. There were many different incorrect answers offered for the three scenarios presented, some additive rather than proportional, but only eight which did not have four amounts from smallest to largest (right to left) and six of those had a repeated value.

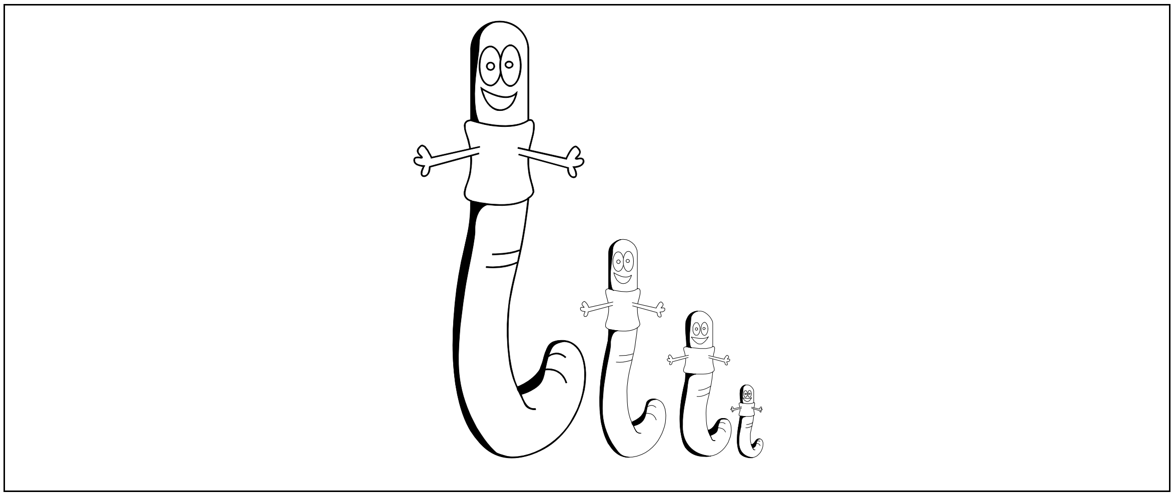


Figure 4.46. Fractions ratio sub-construct, Bookworms task Q. 12.

4.5.6 Correlations between Equivalence score and other fraction sub-constructs.

The students' Equivalence score (0-12) had substantial associations with their performance on other fraction concepts (see Table 4.40). The measure sub-construct was represented by the students' Number Line score (see section 4.5.2.1.4), the Fraction Pair score (see section 4.5.2.2.1), and Part B of the Fraction Pie task (see section 4.5.2.4.1). Using Gilpin's conversion table (1983), there was a common variance of 51% (and a substantial association) between the students' Equivalence score and their Number Line score. However, there was no significant association (p = .167) between the students' Equivalence scores and Q. 16g, the decimal number line representing 6.8. It was not surprising that there was a substantial relationship between the students' Equivalence score and their fraction pairs score as there were two questions in common (Q. 22b and Q. 22f). However, there was no significant association (p = .096) between the students' Equivalence score and comparing the fraction pair $^3/_8$ and $^7/_8$ (Q. 22a). There was also a substantial association between the students' Equivalence scores and Part B of the Fraction Pie task.

Table 4.40

Correlations between Equivalence score and other fraction concepts

	Correlation		
	Minimal $\tau > .07$	Typical $\tau > .20$	Substantial $\tau > .34$
Measure (Q. 16) Number Line Score			$\tau = .510, p < .000$
(Q. 22) Fraction Pairs Score			$\tau = .674, p < .000$
(Q. 14) Fraction Pie Part B			$\tau = .418, p < .000$
Quotient (Q. 20a) Liquorice		$\tau = .262, p = .002$	
(Q. 20b) Custard Tarts		$\tau = .336, p < .000$	
Operator (Q. 18b) $2\ ^1/_2$ x 6			$\tau = .447, p < .000$
Ratio (Q. 12) Bookworms			$\tau = .390, p < .000$

The operator sub-construct was represented, in these correlation calculatons, by just one question, what is two and a half times six? There was a substantial correlation between the students' Equivalence score and the operator sub-construct. There was a substantial association between the students' Equivalence score and the ratio sub-construct represented by the students' performance on the Bookworms task (Q. 12), although it had the weakest effect size of the substantial correlations presented here.

Summary

In the first section of this chapter I reported tasks with 100% frequency of success and 0% frequency of success to describe the base line and upper limits of the students' performance in the present study. In the second section of the chapter I presented the frequency of success on measurement concepts, and described students' strategies, evident in their explanations, to some of the tasks. In the third section of this chapter I reported on the frequency of success of the tasks designed to distinguish between geometric thinking and dynamic imagery, but was unable to provide frequencies of the use of each strategy. In the fourth section I reported the students' Multiplication score calculated from their performance on four of the multiplication questions included in the multiplication and division section of the interview. The last, and largest section of this Results chapter reported students' frequencies of success, pathways through tasks, and strategies evident in their explanations on the fraction concept of equivalence and the sub-constructs of measure, quotient, operator, and ratio. The associations between fraction concepts and measurement concepts were reported.

Quotations of students' explanations of successful strategies and misconceptions, transcribed from audio or video recordings, provided illustration and evidence of specific strategies. Examination of the data by six different double coders enabled me to report on frequencies of success and frequencies of specific strategy use with confidence.

The Results chapter has been structured by tasks, categorised under constructs. The data can now be analysed in terms that interrogate the three research questions that developed out of the literature review. The following Discussion and Implications chapter draws together the threads of common understandings or observations across different tasks in order to elaborate on these three key questions.

Chapter 5: Discussion of results and implications of the present study

This chapter has been structured by the research questions. These research questions emerged from a review of the literature as did the criteria for task selection for the one-to-one task-based interview.

- What strategies are evident in students' explanations of their thinking in a one-to-one task-based interview?
- Is there an association between performance on measurement tasks and performance on fractions tasks? Is there an association between the use of the use of dynamic imagery on visualisation tasks and performance on fractions tasks?
- Can we use Kieren's four-three-four model of fraction understanding (1988, 1992, 1993, 1995) to describe the fraction understandings of students in the present study?

I discuss the strategies children's explanations reveal, the association between performance on measurement tasks and performance on fraction tasks, and the explanatory power of Kieren's four-three-four model. The implications arising from the discussion of the findings have been presented with the discussion of each question.

5.1 Research Question 1: students' strategies and explanations

The first research question addressed is:

- What strategies are evident in students' explanations of their thinking in a one-to-one task-based interview?

Correct strategies and misconceptions were evident in the students' explanations. These included:

- the *perimeter indicates area* misconception, and the *same area indicates same perimeter* misconception;
- correct *unit-forming* and correct *operator* thinking (not successfully executed);
- *dynamic imagery* and *geometric reasoning*;
- the *gap thinking* misconception; and
- correct *benchmarking* thinking (sounding like the *gap thinking* misconception).

The literature on misconceptions as a perfectly good rule misapplied was the starting point of the interpretation of the students' responses to interview tasks (see e.g. Cockburn, 2008; Ginsberg, 1997; Thompson, 1982). A misconception may be sometimes a partial understanding along a path to a more correct or a fuller understanding. All the strategies are discussed and connected to the research literature from the Literature Review chapter to both confirm or refine descriptions of correct strategies and misconceptions, and to note the prevalence of individual misconceptions.

A straightforward understanding of answer and explanation based on work by M. Clements and Ellerton (1995; 2005) has been described in the Methodology and Methods chapter:

- correct answer, correct reasoning
- correct answer, incorrect reasoning
- incorrect answer, mathematically correct/partially correct reasoning
- incorrect answer, incorrect reasoning

The analysis of strategies in this chapter shows that these categories could be used to describe the answers and explanations of students in the present study and further elaboration of these categories is illustrated by the students' explanations.

Some research literature from the study of classroom interactions has been used in the interpretation of results and to frame the implications of the findings of the present study and includes:

- sociomathematical norms: "what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical solution" (Cobb & Yackel, 1996);
- teacher listening behaviours (Davis, 1997; Empson & Jacobs, 2008);
- specialised content knowledge (Hill, Ball, & Shilling, 2008).

5.1.1 The *perimeter indicates area*, and the *same area indicates same perimeter* misconceptions.

The perimeter/area misconceptions were illustrated in the students' explanations of their reasoning in the Similar Shapes task (Q. 36g and h) and the Fold Me a Quarter task (Q. 13). There were instances in both the Fold Me a Quarter task and the shaded shapes pair of the Similar Shapes task (Q. 36h) of students using the magnitude of the perimeter of the shapes as a justification for their decision about the magnitude of the area of the shapes. In the Fold Me a Quarter task (see Section 4.5.2.3), students such as Tyler concluded that the triangle part (one way he had folded into quarters) was larger than the square part (the other way that he had folded into quarters) because "the outside's longer". This explanation was evidence of the perimeter indicates area misconception. The power of "topological intuitions" had been described by Piaget et al. (1960, p. 279) and noted in the intervening 50 years (see e.g. Barrett & D. Clements, 2003; Doig, Groves, & Fujii, 2011; Kidman, 2001).

It seems perceptually counter-intuitive that a longer perimeter does not correspond to a larger area because in the case of similar shapes, the magnitude of the perimeter can in fact indicate the magnitude of the area. This was illustrated by the first two pairs of similar shapes in the present study (see Figure 4.4, Q. 36a and b, Q. 36b and c). Perimeter indicates area thinking was a mathematically correct strategy in these two pairs. One attribute (length) could be used to describe another (area). However, the magnitude of the perimeter did not indicate the magnitude of the area in the last two pairs of the Similar Shapes task (Q. 36e and f, Q. 36g and h). In the third pair of the Similar Shapes (Q. 36e and f), the perimeters of the non-similar rectangles were the same but the areas were different. And in the last pair, the perimeter of the shaded triangle was longer than the shaded rectangle but the areas of the shaded shapes were the same.

Using perimeter indicates area thinking in the last two pairs of shapes was a perfectly good rule misapplied or "faulty extensions of productive prior knowledge" (Smith et al., 1993, p. 152). In the wider field of mathematics education, the incorrect approaches of students on tasks were usually coherent and logical (Jenkins, 2010), and reasonable from the student's perspective (Cobb, 2011a). In the present study, seven students (8%) correctly explained that the perimeter of the shaded triangle was longer (Q. 36g) but incorrectly concluded that the shaded triangle had a larger area (Q. 36h). These students were inappropriately generalising, in the context of triangles and squares, that the magnitude of the perimeter of a shape was directly related to the magnitude of its area. The research on misconceptions suggested that students recognised patterns but generalised them to mathematical contexts where they were not appropriate (Cockburn, 2008).

Students could generalize the premise that perimeter indicates area into other inappropriate contexts. For example, Bella used geometric reasoning, matching edges and parts of edges, to conclude that the perimeters of the tall rectangle and the fat rectangle were the same (see Q. 36f, Figure 4.4). She then demonstrated the perimeter indicates area misconception when comparing the areas, explaining that "Because if the perimeter would be the same, the area would be the same too." The explanatory power of the misconception, which had been correct in the previous two questions, was strong enough to prevent her from using a visual geometric check in the area comparison; dynamic imagery could be used to place the taller rectangle inside the fatter rectangle and the part that was too tall would not take up the other half of the fat rectangle. This particular instance of the perimeter indicates area misconception required the overriding of topological intuitions about area and shape. This phenomenon is supported by Smith, diSessa, and Roschelle's observation that "Some misconceptions are powerful enough to influence what students actually perceive" (1993, p. 162).

The misconception *same area indicates same perimeter* was more prevalent in the shaded pair of shapes (Q. 36g and h) than the perimeter indicates area misconception. Thirty nine students (44.3%) explained (incorrectly) that the perimeters of the shaded shapes were the same (Q. 36g) and used the word half or halves in their explanation, but an eighth of them (5 of 39) did not then describe the areas of the shaded shapes as the same (Q. 36h). All of the students who used dynamic reasoning (dynamic imagery or geometric reasoning about area) to explain that the perimeters of the shaded shapes were the same also described the areas as being the same in the following question. Students did not look at the perimeter to gauge the magnitude of the area, they used geometric and/or fraction reasoning about the area to make an incorrect deduction about the perimeter. The same area indicates same perimeter misconception was not confirmed by topological intuitions because the longer perimeter (the triangle) was not connected to the larger area. An explanation of why the perimeters were the same using the word half or halves or dynamic reasoning in conjunction with an explanation of why the two the areas were the same was offered by 47.7% of the students. It was possible to unsuccessfully use geometric reasoning about length or to unsuccessfully consider the length of the perimeters and decide that they were the same.

The areas of the shaded shapes (Q. 36h) were identified as the same by 76.1% of students but not all explanations were mathematically correct. Some verbalised that the misconception that perimeter and area are related (explaining that the areas were the same *because* the perimeters were the same) and were coded incorrect. However, of the 67% coded as correct, two thirds of them (39 of these 59 students) had also explained incorrectly that the perimeters of the shaded shapes were the same in the previous question. Most (35 of 39) of the successful students used fraction reasoning and/or dynamic imagery or geometric reasoning in both explanations comparing the perimeters and the areas.

Only 22% of all the students correctly explained why the areas of the shaded shapes were the same and why the perimeter of the shaded triangle was longer. For example, Cameron explained that the perimeter of the triangle was longer in the first comparison (Q. 36g) and noted how compelling the perimeter was in suggesting the magnitude of the area (Q. 36h), explaining, "this might look bigger [points to triangle] but it's actually, they're both halves" (see Table 4.3). Only 4.5% of the students correctly compared the four perimeters and the four areas in the Similar Shapes task (Q. 36a, b, c, d, e, f, g, h).

The same area indicates same perimeter misconception was detected in the Similar Shapes task because both area and perimeter questions were asked, but was not as obvious in the Fold Me a Quarter task (Q. 13) because no question about perimeter was asked. Of the 54 students who had correctly compared the area of three non-congruent quarters in the Fold Me a Quarter task (see Figure 4.38) 49 of them explained why the areas of the similar shapes were the same (Q. 36h). However, 6 of those 49 explained (incorrectly) that the areas were the same *because* the perimeters were the same, and 32 of the 49 had explained that the perimeters were the same in the previous question (Q. 36g). Only 11 of the 54 supposedly high performing students had a coordinated area and perimeter understanding.

The word "area" was specifically used in both the Fold Me a Quarter task and the Similar Shapes task. Previous research (Mitchell, 2005), had highlighted that the word "bigger" could refer to either the attribute of length (perimeter) or area. Researchers advised that to understand an incorrect answer the researcher had to determine the question that the student was answering correctly (Ginsberg, 1997; Greer, 2009). The use of the word area in the interview question (does one have a bigger area than the other or are they the same, rather than which shape is bigger) was an attempt to distinguish between students who had misconceptions about attributes (length and area) and students who were interpreting the question differently to the intent of the questioner (as a question about "bigger" perimeter).

Most students did not verbalise a systemic understanding of attributes of measure in the measurement tasks. Although all students identified a length dimension on the Blocks of Ice task (Q. 54), their conceptual knowledge of the attribute did not appear to coordinate:

- that length can be a straight path, a bent or curved path, or a perimeter;
- that the spatial measures are length, area, volume and angle (Lehrer, 2003);
- that width, depth and height measure different dimensions but the same attribute (length); and

- that the perimeters of shapes can be used to compare shapes if the shapes are mathematically similar (e.g., two circles or two squares) but not if the shapes are non-similar (e.g., non-similar rectangles or different shapes).

5.1.1.1 Implications.

Only 4.5% of the students correctly explained all eight perimeter and area comparisons in the similar shapes task (Q. 36 a, b, c, d, e, f, g, h), with 22% successful on both the perimeter and area comparisons of the shaded shapes (Q. 36g and h). This has implications for measurement instructional trajectories. Some researchers described knowledge of attributes as a first stage (see e.g. Outhred et al., 2003; Wilson & Rowland, 1993) with the suggestion that this stage was completed in early measurement learning. Other researchers cautioned that the attribute of length presented with increasing complexity (Barrett, et al., 2006). Frameworks of ideal knowledge, such as Lehrer's eight key concepts for measurement, enable the revisiting of measurement concepts such as attribute throughout primary school and into secondary school. The results of the present study (22% of Grade 6 students successfully compared perimeters and areas of non-congruent halves) suggest that the measurement concept of attribute needs revisiting in primary schools as increasingly complex contexts are introduced.

The perimeter indicates area misconception appears to originate in correct mathematical thinking: the perimeter of similar shapes indicates the magnitude of the area. This intuitive topological consideration is then inappropriately generalised to a different context; non-similar shapes. Piaget and others would argue that the perimeter indicates area misconception was linked to topological intuitions; the explanations of the students in the present study suggest that those topological intuitions become firmly connected to the logical deduction that area indicates perimeter. Students could "see" the perimeters of the non-congruent shaded halves as the same because of reasoning about the areas despite the fact that the perimeter of the triangle was longer. The perceptual information was discounted because of the importance of the inappropriately applied area reasoning. An implication of these findings for teachers is that intervening to try and resolve this misconception will be more complicated than a pre-emptive simple example to prevent the generalisation to inappropriate contexts. Just separating the topological intuitions (consideration of the perimeter) from the consideration of area will not prevent the misconception. Smith et al. (1993) suggested learning was concerned with "learning to use what you already know in either wider or more restricted contexts" (p. 136). The many contexts of perimeter and area comparisons and the meaning attached to words like "bigger" will need to be reflectively explored.

In the present study, the successful use of fraction reasoning when comparing non-congruent halves or quarters did not indicate that the same area indicates same perimeter misconception had been resolved. In fact, 44.7% of the students believed that area and perimeter were directly related in the shaded shapes (Q. 36g and h). Of the 54 high performing students (who had made three correct area comparisons of non-congruent quarters) only 11 were successful on both the area comparison and the perimeter comparison of the non-congruent shaded halves in the Similar Shapes task (Q. 36h and g). The use of fraction reasoning in area comparisons did not cure the dominance of topological perceptions (perimeter indicates area) nor its inductive reverse (same area indicates same perimeter) but obscured these misconceptions. The observation that fraction reasoning can mask these misconceptions rather than resolve them has implications for how teachers think about correct answers with correct explanations in constructivist classrooms: just because a student can use a more sophisticated strategy correctly does not mean that he or she has resolved earlier misconceptions. The focus of the similar shapes tasks was length and area knowledge. Fraction reasoning obscured a misconception in the measurement domain.

5.1.2 Unit forming thinking and operator thinking.

The use of operator thinking or unit-forming thinking was demonstrated in Part B of the Fraction Pie task (Q. 14b). None of the 26.1% of students who used the operator approach or unit-forming approach in Part B of the Fraction Pie task (Q. 14b) were able to provide both a correct answer and a sufficiently precise explanation, even though either of these approaches were mathematically correct. If the students had been able to execute their strategy they would have been successful. Their answers, one eighth, one seventh, one sixth, and one fifth were reasonable estimates of either two thirds of a quarter (operator thinking), or of Part B as a part that when added to a smaller amount would equal one quarter (unit-forming thinking). This was an example of a wrong answer but with full or partial mathematical understanding (see e.g., M. Clements & Ellerton, 1995).

The incorrect answers given by these students in the present study did not indicate that they had poor partitioning skills; they were attempting the more difficult interpretations *two thirds of a quarter is a sixth,* or *a sixth plus a twelfth equals a quarter*. Thompson had described the research question "what is the problem that this student is solving, given that I have attempted to communicate to him the problem in my mind" (1982, p. 154) as a legitimate field of investigation (for a constructivist as opposed to an environmentalist). This question is still relevant almost thirty years later and enabled me to interpret the students' incorrect answers coupled with partially correct explanations. Only Matthew's operator thinking explanation was precise (two thirds of a quarter) but he was unable to calculate that two thirds of a quarter was a sixth. Zak's answer of zero point seven for Part B was obtained by placing his pen horizontally across the diameter to make four quarters and then explaining that Part B was "point seven". This indicated operator thinking but it was not clear whether he had maintained the unit correctly; there was not enough elaboration of his answer to determine whether he meant point seven of a quarter [not articulated] or whether Part A had become the whole and his answer was point seven (of one). Kelly used unit forming and explained that Part B was a fifth "because it's got a smaller area, it's not as big as a half or a quarter". However, in the other students' less precise explanations it was difficult to hear the difference between (correct) additive thinking and (correct) multiplicative thinking: half a quarter, three quarters of a quarter, just less than a quarter. Their answers were estimates: one eighth, one fifth, one sixth, one seventh. The explanations were ambiguous: did "just less than a quarter" indicate the result of operator thinking, or an addend in unit-forming thinking. This made it possible to categorise the students as using operator or unit forming thinking, but not which of those strategies they had used.

One of the answers given by students who were using operator thinking or unit forming thinking was one fifth. The answer of one fifth, however, did not automatically signal the students' use of the incorrect double count misconception. I had noted this possibility in earlier research (Mitchell, 2005) and it was again confirmed in the present study in which four of 24 of the students who answered one fifth were not double counting, but instead were using either (correct but not fully executed) operator thinking or unit forming thinking. One fifth was offered as an *answer* by 27.3% of the students and this compares with 13.6% in a larger study by Clarke et al. (2007).

The frequency of success on Part A (one quarter) was 70.5% and this compares to 83% in a larger study (Clarke at el., 2007) and 76.5% in a smaller study (Mitchell, 2005). The frequency of success on Part B (one sixth) in the present study was 27.3% and this compares to 42.7% in a larger study of Grade 6 students (Clarke et al., 2007), and 35.3% (Mitchell, 2005) and 10% (Stewart, 2005) in smaller studies of Grade 6 students. These frequencies suggested that Grade 6 students in many different schools found this restructuring of non-equal-parts diagrams difficult.

All of the students in the present study successfully identified a quarter of a circle in an equal-parts diagram (Q. 19g, see section 4.5.2.4.4) but only 70.5% identified the quarter in the Part

A of the Fraction Pie task. The non-equal parts aspect of the task appeared to be the difficulty, not the fraction or the circle diagram. All the students who were successful on Part B of the Fraction Pie had been successful on Part A. The restructuring needed for the sixths appeared more difficult than the restructuring for quarters. Although 42.7% of the students successfully explained why Part B was a sixth, 97.7% of all the students could explain why the one shaded part out six on an equal parts circle diagram was one sixth in (see Figure 4.42). This suggests that it was the non-equal parts context, not the fraction (one sixth) or circle diagram that was difficult for students.

Researchers had reported that pre-service teachers focussed on the correctness of the answer (Jansen & Spitzer, 2009) and equated correct answers with mathematical understanding, and equated incorrect answers with carelessness or confusion (Crespo, 2000). *Directive listening* by teachers focussed on whether a child's answer matched an expected response (Empson & Jacob, 2008). The term *directive listening* corresponded to the term *evaluative listening* used by Davis (1997). Teachers who used this type of listening in classroom contexts were *listening for* something, not *listening to* the students (Even, 2005) and this could result in teachers overestimating what students knew (Empson & Jacobs, 2008) by assigning understanding to correct answers with vague explanations (Even, 2005). In the present study, two students gave the correct answer of one sixth with the explanation that Part B of the Fraction Pie was a quarter of a quarter; directive listening would have focussed on the correct answer and possibly attributed the explanation to vaguely explained partitioning, and missed the operator approach or unit-forming approach altogether.

Observational listening (Empson & Jacobs, 2008), on the other hand, was a term used to describe teachers listening to students and trying to work out what the students were actually thinking. Davis had described this as *interpretive listening* (1997) and noted that it occurred in his case study of a teacher when she began to acknowledge the differences in individual children's responses to mathematical tasks. Empson and Jacobs (2008) specified one-to-one task-based interviews as contexts for the use of observational listening. In research on mental computation, the change in focus from speed and accuracy to verbalising strategies required this type of listening (Sparrow & McIntosh, 2004). Observational listening underpinned the methodology of the present study and enabled the discussion of the students' strategies and misconceptions. This made it possible to identify the (correct) operator thinking and unit forming thinking used by students despite their incorrect answers. This was further illustrated in the classification of the strategies that generated the answer of one fifth (see Table 4.36):

- (correct but not fully executed) operator or unit forming thinking (4 students),
- (correct but not fully executed) physical iteration (1 student),
- conditional double counting (9 students), and
- the unconditional double count misconception (10 students).

Some of the students made the quantification of Part B of the Fraction Pie task difficult by their approach. For example, ignoring the radius between the quarters and using a mirror image of the right hand side to get six parts was the easiest approach and was an example of partitioning. Equivalence knowledge was used to double (additively or multiplicatively) three to get six parts. No student in the present study explained their reasoning as one third of a half, which was the easier of the two operator approaches. Two thirds of a quarter was the other operator approach, but the execution proved too difficult for the one student who quantified his operator thinking. Other students were attempting the same operator approach but were not precise in specifying the calculation; "nearly a quarter." The unit forming approach to Part B was simple in first principles, possible to estimate, but difficult to calculate, for example one six plus one twelfth was one quarter. Students' unsuccessful attempts at this task using either the operator approach or a unit-forming approach did not mean that they could not partition. Careful observational listening is needed to distinguish between an inability to partition or an inability to fully execute the more difficult unit forming or operator approaches to this task.

Responsive listening (Empson & Jacobs, 2008) by teachers encompassed trying to understand individual student's approaches and responding to them individually and instantaneously, whilst keeping 25 children engaged and included, in the group dynamic of a single lesson. Davis had termed this *hermeneutical listening* (1997). Observational listening was possible and manageable in the one-to-one task-based interview used in the present study. All the different understandings were coded and classified and interpreted but I did not have to respond to the students' understandings in a classroom context.

5.1.2.1 Implications.

The comparison of the frequencies of success of the present study and the other empirical studies reported in the research literature suggest that the present study is not an outlier result. The original Fraction Pie diagram (Cramer et al., 1997) was used in the previous studies but a mirror image of the diagram was used in the present study. Although the results of the present study cannot be generalised because a representative sample was not used, the findings of the present study have implications for Victorian students and teachers.

The implications of these findings are for responsive listening by teachers. If a student offers an answer of one fifth to Part B of the Fraction Pie task in the classroom, the teacher will need to respond to the strategy that the student is using to effectively help the student towards a successful answer and strategy. If a student has used a unit forming approach and estimated that one addend is a fifth they will not be helped towards correctly executing this strategy if the teacher responds to them by talking about double counting (which also gives an answer of one fifth). If the teacher wants to encourage a student to consider several strategies, it might be more effective to explain how the student's preferred strategy can be executed correctly before the student can focus on an alternative strategy. For example, the teacher may wish to respond to Matthew by validating his operator thinking on a separate equal-parts diagram (dividing each quarter into thirds and demonstrating that two thirds of a quarter is a sixth), and only then asking him to either consider another student's approach to the task (e.g., partitioning) or to come up with another strategy himself that would generate an answer of a sixth.

5.1.3 Dynamic imagery.

The visualisation tasks were selected to determine students' preference for dynamic imagery or geometric reasoning. The research literature described a difference between dynamic imagery and geometric thinking using the terminology visual processing/spatial ability for dynamic imagery (Bishop, 1983; M. Clements, 1983). It was possible to distinguish between dynamic imagery and geometric reasoning in some of the students' responses to the Wattanawa Block task but it was not possible to do this with all the students' explanations. For some students, the lack of a vocabulary to talk about dynamic imagery and geometric reasoning, made it impossible for the interviewer to differentiate between the two strategies on some of the visualisation tasks. The students may not have been aware that there were several different strategies for attempting visualisation tasks, and so they did not make the distinctions clear in their explanations. The authors of literature on interview methodology had cautioned that children's responses to tasks gave researchers only their explanations not their thinking (Ginsberg, 1997) and this was echoed by authors interested in mathematics education and linguistics (Barwell, 2009). This distinction between thinking and explanation was evident in the interpretation of the students' responses because determining the strategy was difficult.

The students' responses to clarifying questions in the interview demonstrated the impact that their perception of directive and observational listening had upon their participation in a one-to-one task-based interview. The use of confirmatory questions that were intended to reflect back to the student their own reasoning were often unsuccessful and appeared to prompt the student to agree with the interviewer. Alex demonstrated this in her ready agreement to both a dynamic imagery rephrasing of her strategy and a geometric rephrasing of her strategy. Alex deferred to me as the interviewer signalling that she no longer believed me to be engaged in observational listening. Her second-guessing of which strategy she thought I wanted to hear suggested that she had positioned me as being engaged in directive listening.

Previous research had identified that querying an answer signalled to a child that the answer was incorrect (Ginsberg, 1997). Barwell (2003) had elaborated that the linguistic structure of a question determined that what was said in response was by definition an answer. Students were sensitised to interpret a query as a suggestion that they should offer another answer. The query operated as a *repair*, a positioning of the students' response as inconclusive and needing elaboration or as inappropriate and needing reframing (Woofitt, 2005). In the interview used in the present study, students quickly accepted that every answer was followed by a query about strategy, and that this was not a repair because they were told this at the beginning of the interview: "I won't tell you whether you get an answer right or wrong. But I will probably always say, and how did you work that out?" as this was stated before the tasks were offered. However, the visualisation tasks illustrated that confirmatory questions could be interpreted as a repair by students who then responded to observational listening as if it were evaluative listening.

5.1.3.1 Implications.

It was not possible in the interview format to determine what visualisation strategy the students used. In order to assess students' visualisation skills it may be necessary for future research projects to include a preparatory classroom based intervention in which students investigate different strategies for tasks and establish a shared vocabulary to discuss these strategies (which may impact positively on their visualisation skills). The knowledge of geometric reasoning, dynamic imagery, fraction reasoning and the intersections between them may also enable students to offer clearer explanations to fraction tasks that use diagrams.

5.1.4 The gap thinking misconception.

The gap thinking misconception was evident in the Fraction Pairs task (Q. 22) and emerged at the same time as equivalence knowledge emerged. There was a lower frequency of success on the fraction pairs in the present study than in another Victorian research project (Clarke & Roche, 2009; Clarke et al., 2007; 2011; Clarke, Roche, Mitchell, & Sukanic, 2006), but the order of difficulty (by frequency of success) was the same. In the fraction pair, $^5/_6$ and $^7/_8$, in the present study, gap thinking was used by 50% of the students, which was greater than the 29.4% who gave the answer of "the same" in the larger study (Clarke & Roche, 2009). Gould (2011) reported that 8% of Year 7 and 8 students in another study argued that a residual pair of fractions were the same. In the fraction pair $^3/_7$ and $^5/_8$, in the present study, gap thinking was used by 21.6% of the students which was similar to the 21.2% reported by Clarke and Roche (2009). Clarke and Roche (2009) reported that 4.3% of students were correct on all eight pairs which was similar to the 6.8% of students who had a Fraction Pair score of 7 (Q. 22e was excluded from frequencies of success).

In order to describe how gap thinking is a perfectly good rule misapplied, I will elaborate on several contexts where additive thinking is correct when comparing fraction pairs. Sylvie explained that $^7/_8$ was larger than $^3/_8$: "Because of the eight, eight [points to the two denominators]. That one [points to 7/8] needs one more to get a whole and that one [points to 3/8] needs five more." This additive word pattern, to get to a whole, was used correctly in this comparison. A variation of this was Patrick's explanation for the same fraction pair, "This one's larger [points to 7/8] because it's closer to the [d(undecipherable)], the denominator, whatever it's called." Kate's explanation correctly used additive thinking, "Because there's only one more piece to make a whole for seven eighths. And for three eighths you'd need another, you'd need another um five, more pieces". Fraction pairs between 0 and 1 with same denominator were a context in which additive thinking was appropriate when making comparisons of the size of fractions.

Of the 20.4% of the students in the present study who used doubling or halving explanations for the fraction pair $^2/_4$ and $^4/_8$ it was difficult to distinguish whether some were demonstrating a ratio understanding or whether they were using correct additive thinking in this context of a half. Some students clearly used multiplicative language. For example, Nicky used the term "simplify" indicating a ratio understanding when explaining "if you simplify four eighths you make it go down to two quarters, you simplify that again it would be one half." Jai used multiplicative language, "this is times by two to get that, and this is times by two, so it's both equal". Some students used additive language. For example, Emma explained that "four plus four is eight and two plus two is four." Some students used both additive and multiplicative language. For example Hannah explained "'cause if you do two plus two is four, and four plus four is eight. So they are pretty much times two." However, some students explanations of doubling or halving did not clearly indicate whether their thinking was additive or multiplicative (both of which would be correct). For example, Julia said "four is half of eight and two is half of four, so they're both half of what the whole is" but it was not clear whether this was proportional or additive halving. In the context of equivalences to a half, additive thinking would be a strategy that would give a correct answer and be mathematically appropriate.

Gap thinking used incorrect additive explanations (see Table 4.26). For example, gap thinking in the fraction pair $^5/_6$ and $^7/_8$ could sound like "one more to become a whole" or "one piece left" (complement-to-one thinking); "because the top numbers are both one less than the bottom numbers" (numerical comparison of numerator and denominator); "Cause they're both two thirds...seven plus one is eight and five plus one is six" (equivalence string); "there's one more sixth to make a whole. And it's one more eighth" (fractional language as complement-to-one); and "they both need one more to be coloured in (counting and shading). This residual context for gap thinking, where both fractions are one away from the whole, was the first to appear and the last to be resolved in the present study.

Gap thinking explanations from other fraction pairs illustrated additional variations of additive language: Tony explained (incorrectly) why 5/8 was larger than 3/7, "Because you count up from that, six seven eight [points to 5 of 5/8] and it's a whole. And you have to count up, that's six seven eight, that's three. And this one is four five six seven and that's four". Brad explained (incorrectly) why 1/2 was larger than 5/8, "It takes. There's less to get from one to two than from five to the eight". Gap thinking used additive language.

Although gap thinking had several variations, it involved calculating the difference between numerator and denominator rather than using proportional reasoning to rename or create a new fraction. It was possible that gap thinking was a perfectly good strategy misapplied. Additive thinking could be used correctly to compare the size of fractions with the same denominator between 0 and 1. Additive thinking also could be used to correctly recognise equivalences to $^1/_2$: numerator plus numerator equals denominator.

The use of the word *plus* in the explanation of the equivalence to a half of $^2/_4$ and $^4/_8$ by four students illustrated the extent of the additive language. Hannah had early equivalence knowledge (an Equivalence score of 3). She used gap thinking for five of the seven fraction pair questions although she picked the fraction with the larger gap which was an unusual presentation, although this strategy was observed in Cramer and Wyberg (2009) and used the word "plus" in four of those explanations and "one number to add to" in the other. Patrick used the word "plus" in his correct explanation of why $^2/_4$ and $^4/_8$ were both "half numbers", but used various addition words in his three gap thinking explanations: "three and five is eight" (Q. 22f), "one plus five equals six" (Q. 22g), and "this one I just had to add one more" (Q. 22h). He, like Hannah was a very early equivalence learner with an Equivalence score of 2. On the other hand, Jordan had an Equivalence score of 7 and was in Equivalence Band E, and had explained that "two plus two is four and four plus four is eight" to correctly identify that $^2/_4$ and $^4/_8$ were "like half". His only gap thinking error was on the fraction pair $^5/_6$ and $^7/_8$ where he used an equivalence-string explanation: "'Cause five over six is nearly the same and seven over eight is nearly the same and it's like three quarters." Emma also had an Equivalence score of 7 and was in Equivalence Band D, but did not offer any gap thinking explanations for the fraction pairs. She used addition to expand her explanation of halving: "they're halved, so they would be the same. So there's four plus four is eight and two plus two is four". Jordan and Emma's use of the word "plus" suggested that students' multiplicative understandings of the terms double or half also had an additive resonance and although they used an additive word pattern, they could co-ordinate the additive and multiplicative contexts correctly.

Gap thinking was not used by any student as a strategy for the fraction pair $^2/_4$ and $^4/_8$; no student compared a difference of two and a difference of four. If gap thinking were present it appeared that it was trumped by the additive word pattern for a half.

Gap thinking emerged at the same time as early equivalence understanding (see Figure 4.35). This was not a linear association. The first equivalence tasks that students were successful at in the present study involved equivalences to a half. All of the students in Equivalence Band B had an Equivalence score of 1 and all but one of the 14 either recognised $^2/_4$ and $^4/_8$ as equivalent in the Fraction Pairs task, or had rolled three gold and three white beans in the Golden Beans task (Q. 22b) and successfully offered two names for the fraction, three sixths and a half. Gap thinking was resolved in students with an Equivalence score of 11 and 12.

It seems counter intuitive that a misconception that used incorrect additive language could be connected to equivalence which is multiplicative and proportional. However, early equivalence knowledge could sound additive because the word pattern for equivalences to a half could use the word "plus" as Emma's explanation demonstrated. The quotations from students above demonstrated that the additive language lingered for some students. The additive nature of gap thinking did not sound like the equivalence explanations of experts but highlighted the additive sound of early equivalence knowledge. Cockburn (2008) suggested that misconceptions revealed children's thinking. The gap thinking misconception reveals the additive base of early equivalence understanding.

The subgroup of students with an Equivalence score of 2 to 9 had a higher frequency of gap thinking (see section 4.5.2.2.2). The students in this sub-group who did not use gap thinking performed well on

- making a new zero-point in broken ruler tasks,
- realigning a ruler when measuring a streamer that was longer than 30 cm,
- using cm^2 to describe units that have length and width, and
- not double counting in Part B of the Fraction Pie task.

The association between success on Part B of the Fraction Pie task and non-presentation of gap thinking by students with an Equivalence score of 2-9, suggested that the double count misconception was worth investigating with respect to this subgroup of 48 students This interpretive analysis used a finer grained analysis than the dichotomous variables of the statistical analysis (correct/incorrect at Part B of the Fraction Pie task and use/non-use of the gap thinking strategy if Equivalence score 2-9) and categorised the students into three groups:

a) no gap thinking (fourteen students),
b) one instance of gap thinking in the seven fraction pairs (fourteen students), and
c) two to five instances of gap thinking (twenty students).

Double counting behaviour was also considered in two other tasks: the triangle $^2/_3$ (Q. 19r) and the rectangle $^1/_6$ (Q. 19b) in the Fraction Sort task. The double counting behaviour in the three tasks was also classified as conditional or unconditional (see Table 4.37 and Table 4.38).

The fourteen students in category (a) did not use the gap thinking misconception in any of the seven fraction pairs, nor did they use double counting (0%) in Part B of the Fraction Pie task. In the two Fraction Sort questions, however, five of the 14 (35.7%) used double counting once or twice.

The fourteen students in category (b) used gap thinking once in the Fraction Pair task, and three of them (21.4%) used conditional double counting in Part B of the Fraction Pie task. A further 42.9% used double counting in either or both the two Fraction Sort tasks.

The twenty students in category (c) used gap thinking for two or more fraction pairs, and four of them used unconditional double counting (20%) and one used conditional double counting (5%) in Part B of the Fraction Pie task. A further 50% used double counting in either or both of the two Fraction Sort questions.

The subgroup made up of students with an Equivalence score of 2-9 was created in order to target gap thinking by removing students with an Equivalence score of 0 (who may or may not *become* gap thinkers) and students with an Equivalence score of 10-12 (who may or may not *have been* gap thinkers). Students who did not use gap thinking did not use double counting for Part B of the Fraction Pie and had the lowest frequency of double counting in the two other tasks. The use of double counting in the three questions increased in frequency as the intensity of gap thinking increased (evident in the number of gap thinking explanations) from 35.7% for category (a) to 64.3% for category (b) to 75% overall for Category (c). Unconditional double counting in Part B of the Fraction Pie task was only seen in students with Gap Thinking scores of 2 and above (in this sub group of students with Equivalence scores from 2 to 9).

The students who imagined an area diagram in the residual pair $^5/_6$ and $^7/_8$, and were gap thinkers, used double counting language to describe "one more to be coloured in" for this pair and so had folded back to an image whose interpretation was limited (double counting) and not generalisable (partitioning and comparisons). Double counting was appropriate for limited contexts (equal-parts diagrams) but became a misconception when applied to other examples: non-equal-parts diagrams or residual partitioning comparisons. Inappropriate double counting appeared to (incorrectly) confirm gap thinking. In this particular pair, it would seem that the additive language was compelling. The original term residual thinking had been coined by researchers in the Rational Number Project to describe a student invented strategy, which the researchers attributed to the use of circle models in fraction instruction (Post et al., 1986; Post & Cramer, 1987). Cramer and Wyberg (2009) reported that instruction using fraction charts provided Grade 4 students with strong mental images when comparing residual fractions.

The data in the present study suggested that gap thinking resolved itself: if students had an Equivalence score of 11 or 12, gap thinking was not present. On the other hand a coordinated knowledge of partitioning (not having the double count misconception) might offer some protection against the development of gap thinking. There was no proof of cause and effect in the present study, however, this detailed analysis offers some direction for further research.

5.1.4.1 Implications.

Gap thinking is a misconception of equivalence. If teachers thought of gap thinking as one of the misconceptions of equivalence, they might target their responses to this misconception with more precision. They could choose examples carefully to illustrate the thinking, and revisit the images of unit forming and partitioning that students need to fold back to when thinking about mathematical symbols. The traditional part-whole interpretation has labelled gap thinking as one of several whole number thinking strategies but gap thinking and two other whole number thinking strategies (higher or larger numbers, and bigger denominator indicates bigger fraction) did not present in the same way. The findings of the present study indicate that the *difference* (additive) language of gap thinking may be a hangover from an early additive understanding of equivalence. Attention needs to be drawn to the appropriate contexts for additive thinking:

- equivalences to $^1/_{2,}$ and
- when comparing fraction pairs between 0 and 1 with the same denominators.

Attention also needs to be drawn to inappropriate contexts for additive thinking, particularly residual pair comparisons.

The 50% of students in the present study who used gap thinking when comparing the fraction pair $^5/_6$ and $^7/_8$, did not fold back to an image of partitioned areas and use residual thinking to compare the parts needed to make the whole. Just as $^1/_6$ and $^1/_8$ are different sized pieces in the comparison of partitioned unit fractions so too the residual $^1/_6$ and $^1/_8$ are different sized pieces in the comparison of $^5/_6$ and $^7/_{8.}$ Residual comparisons were the first gap thinking type to emerge and the last to leave. The use of gap thinking when comparing the fraction pair $^5/_6$ and $^7/_8$ was demonstrated by students in each Equivalence score from 1 to 10 (except by the four students with an Equivalence score of 2). It is possible that the double counting misconception reinforces the use of gap thinking in a residual pair. Students need to be able to fold back to the understanding that non-equal-parts area diagrams are not named using the double count misconception, so that residual pairs are not incorrectly confirmed as equivalent.

A highly detailed knowledge of gap thinking is needed for improvements in teachers' pedagogical content knowledge. Specialised content knowledge (Hill et al., 2008), a knowledge of mathematics and a knowledge of students, is necessary for pedagogical content knowledge. The link between the importance of specialised content knowledge and Japanese lesson study was made by Knapp, Bomer, and Moore, (2011), who, along with Doig, Groves and Fujii (2011) highlighted the study of constructs as an integral part of Japanese lesson study.

5.1.5 Benchmarking/Gap thinking.

Two strategies (correct/incorrect) with similar initial explanations were evident in the Fraction Pairs task (Q. 22e, $^4/_5$ and $^4/_7$). Students' answers, initial explanations, and follow up explanations designed to elicit the gap thinking misconception demonstrated that observational listening involved careful listening. The similarity of initial explanations of misconceptions and correct mathematical thinking was revealed by observational listening and further interpretation was possible using the terminology and concepts of classroom interaction research: Hermeneutical or responsive listening by teachers (Davis, 1997; Empson & Jacobs, 2008), and calculational explanations, equivalent explanations and parallel explanations of peer conversations (Cobb, Yackel, & Wood, 1992).

Experience with students had taught me that confirmatory questions had not been successful in the visualisation tasks and therefore in the Fraction Pair task I used non-directive probes. Despite this, there were still 11.4% of responses to the fraction pair $^4/_5$ and $^4/_7$ which were *possible gap thinking*, and which further questioning had not elicited the distinction between gap thinking and benchmarking (see Tables 4.28 and 4.29). For example, when comparing the fraction pair $^4/_5$ and $^4/_7$ Lara chose $^4/_5$ as the larger fraction "'Cause it's only one away from being a whole." The prompt "mmm?" encouraged her to elaborate "And this is three away from being a whole" (see Table 4.28). Lara was using the gap thinking misconception, calculating the complement to one for each fraction and choosing the fraction with the smaller gap. In contrast, Adam explained his answer of $^4/_5$ with "four is closer to five." The prompt, 'Can you tell me a bit more about that?", encouraged him to elaborate "The four and the seven, there's more less, like um close to a half, but this one's like almost a whole." Adam was benchmarking to a half and one and using a correct mathematical strategy. Chris was also benchmarking (see Table 4.28) explaining his answer of $^4/_5$ with "four fifths is almost a whole." When prompted, "mmm?", he added, "And four sevenths is um, a bit higher than a half." These paired strategies, benchmarking and gap thinking, were difficult to distinguish in fraction pairs because the answers were the same and the initial explanations were similar:

- "'Cause it's only one away from being a whole."

- "four is closer to five."
- "four fifths is almost a whole."

Both strategies had been identified in the research literature. Benchmarking had been reported in Australia (Clark et al., 2007; Clarke & Roche, 2009), and was called the transitive or reference point strategy in the United States (Behr & Post, 1986; Behr et al., 1984; Post et al., 1986; Post & Cramer, 1987). Gap thinking had been described in Australia (Clark et al., 2007; Clarke & Roche, 2009; Gould, 2011; Pearn & Stephens, 2004) and was one of four whole number dominance strategies described by Post and Cramer (1987).

Observational listening (Empson & Jacobs, 2008), or interpretive listening (Davis, 1997) was used to understand strategies that children were using. In another context, I had identified that careful listening was needed to correctly interpret students' strategies (Mitchell, 2004). Cockburn (2008) suggested that one way of tackling misconceptions was to check a student's response even when he or she answered correctly in case the student and the teacher were focusing on different angles of the task. Observational listening and non-directive prompts to elaborate further explanations were needed in the fraction pair examined here.

The fact that the similarity of initial explanations (and the same answers) for paired strategies obscures the distinction between correct mathematical reasoning and incorrect mathematical reasoning has implications for how teachers respond to students and how students explain their thinking to each other. For example, if the teacher were explaining the benchmarking strategy for the fraction pair $^4/_5$ and $^4/_7$ and said "four fifths is nearly a whole", Adam might hear his benchmarking strategy (four is closer to five) confirmed but Lara would also hear her gap thinking strategy confirmed (it's only one away from being a whole). Lara might not experience cognitive conflict between the teacher's strategy and her own, because if the difference was difficult enough to hear as a researcher with access to transcripts, it would also be difficult for Lara to hear the distinction between the mathematically correct reasoning of the teacher and her own mathematically incorrect reasoning in the classroom. Researchers had suggested that teachers using directive listening interpreted vague explanations as correct mathematical reasoning if the answer was also correct (Even, 2005). It is possible that students listening to classroom conversations react in the same way: if the answer was the same as theirs and the explanations were similar, they would assume their strategy was the same as the teachers.

In some classrooms *calculational* explanations counted as an acceptable mathematical argument despite the fact that calculational explanations made it difficult for students to recognise whether they had *equivalent* strategies or *parallel* strategies (Cobb, 2011b). Calculational explanations retell the calculational steps of a strategy rather than describe the purpose of the calculations (Cobb et al., 1992). This should not be confused with Skemp's distinction between relational and instrumental thinking (Cobb, 2011b). For example, if we imagine that Lara, Adam and Chris were working together to solve the fraction comparison task $^4/_5$ and $^4/_7$, the terminology of peer conversation would enable us to describe their initial explanations as calculational: "'Cause it's only one away from being a whole" and "four is closer to five" and "four fifths is almost a whole." All three children describe a difference calculation and none explain why they are doing this. At this point they might imagine that they are agreeing on the strategy (that they have equivalent strategies). Even if Lara added "And this is three away from being a whole", Adam might still not realise that she was not benchmarking like he was, unless he knew to listen for gap thinking. Parallel interpretations have the same answer and the same initial calculational explanation, but are actually different strategies. Lara and Adam have parallel strategies. Adam and Chris who are both benchmarking have equivalent strategies. Cobb, Yackel and Wood's (1992) examples of calculational, parallel, and equivalent explanations were of addition by Grade 2 children. The explanations are used here in an imaginary classroom interaction to elaborate this phenomenon in a fraction context in Grade 6.

5.1.5.1 Implications.

Observational listening by teachers may require interpretations not only of answers and initial explanations but also prompting for further explanations and/or consideration of responses to other carefully chosen tasks. The students' responses when comparing the fraction pair $^4/_5$ and $^4/_7$ demonstrated that these initial answers were considered acceptable mathematical answers by the students in the interview context. The establishment of sociomathematical norms around acceptable mathematical explanations could include the valuing of a why statement with calculational explanations.

The implication of the illustration of (apparent) parallel explanations in the present study of Grade 6 students is that students may also need to acquire a repertoire of descriptions of strategies and misconceptions at this level. Cobb et al., (1992) were working with Grade 2 students and excellent teaching by the classroom teacher enabled students to increase their knowledge of addition strategies. Grade 6 teachers have access to a repertoire of descriptions of strategies and misconceptions to draw on when responding to student explanations. However, for students to recognise parallel explanations, they may need to acquire a similar sophisticated repertoire of possible strategies in order to make sense of other students' explanations.

5.1.6 Summary of answer, explanation types, and teacher and student conversations.

The discussion of the students' correct strategies and misconceptions revealed finer distinctions in the right answer/wrong answer, partially correct/incorrect reasoning unpacking of students' responses to mathematical tasks (see Table 5.1). This is an elaboration of the description by M. Clements and Ellerton (1995, 2005). The perimeter indicates area misconception in the Similar Shapes task highlighted an undetected conceptual conflict in a correct answer for a correct reason. The Fraction Pie task illustrated that students could get the wrong answer and still have partial mathematical understanding and be using a strategy which if executed correctly would demonstrate correct mathematical reasoning. Gap thinking could produce the wrong answer for the wrong reason or the right answer for the wrong reason. The importance of further probing was highlighted in the discussion of the paired strategies benchmarking and gap thinking in which a student could have the right answer for the right reason (benchmarking), the right answer for the wrong reason (gap thinking), be assumed to have the right answer for the right reason (benchmarking) when in fact they had the right answer for the wrong reason (gap thinking), or be assumed to have the right answer for the wrong reason (gap thinking) when in fact they had the right answer for the right reason (benchmarking).

Table 5.1

Answer, explanation and further explanation types

Coded	Answer	Initial explanation	Further explanation	Question
correct	correct	correct: dynamic imagery, or geometric		Wattanawa Block
correct	correct	correct: close to one	benchmarking	$^4/_5$ $^4/_7$
correct	correct	correct, but listener unsure if dynamic imagery, or geometric reasoning used	probing caused agreement	Cubes
correct	correct	correct: areas the same because both half, or dynamic imagery	undetected conflict with perimeter	Similar Shapes
incorrect	correct	assumed correct: close to one	gap thinking	$^4/_5$ $^4/_7$
incorrect	correct	incorrect: gap thinking		$^3/_7$ $^5/_8$
correct	correct	assumed incorrect: gap thinking, close to one	benchmarking	$^4/_5$ $^4/_7$
incorrect	incorrect	partial correct strategy: operator or unit-forming		Fraction Pie, Part B
incorrect	incorrect	incorrect: gap thinking, "the same"		$^5/_6$ $^7/_8$
incorrect	incorrect	assumed gap thinking		other tasks

5.2 Research Question 2: Associations between performance on fraction and measurement tasks

The second research question addressed is

- Is there an association between performance on measurement tasks and performance on fractions tasks? Is there an association between the use of the use of dynamic imagery on visualisation tasks and performance on fractions tasks?

As I could not confidently distinguish between the students' use of dynamic imagery in the visualisation tasks and the use of other geometric reasoning (see sections 4.3 and 5.1.3), this aspect of the question remains unexamined. The following section discusses the association between the performances on measurement and fraction tasks.

The linear associations between the students' Number Line score and their performance in measurement categories and multiplication have been reported in the Results chapter (Table 4.23). A Number Line score could be 0 to 8 (see Table 4.21) but there were 45 different pathways through the eight number line tasks (see Table 4.22). There were substantial associations between the students' Number Line score and the categories CUNL, CADL, Multiplication, CADA, TPUNL, and TPUNA (in order of effect size).

The linear associations between the students' Equivalence score and their performance in measurement categories and multiplication have been reported in the Results chapter (see Table 4.17). An Equivalence score could be between 0 and 13 (see Table 4.15) although no student scored 13. There were substantial associations between the students' Equivalence score and the categories CUNL, Multiplication, TPUNA, CADL, and CADA (in order of effect size).

The linear associations between the students' Fraction Pair score and their performance in measurement categories and multiplication have been reported in the Results chapter (see Table 4.33). A Fraction Pair score could be between 0 and 7 (see Table 4.25) because Q. 22e was excluded from the results. There were substantial associations between the students' Fraction Pair score and the categories CUNL, Multiplication, TPUNA, CADA, CUNA, TPADA, and CADL (in order of effect size).

The linear associations between the students' performance on Part B of the Fraction Pie task and their performance in measurement and multiplication categories have been reported in the Results chapter (see Table 4.39). In the present study, 27.3% of the students were successful on Part B of the Fraction Pie task. There was a substantial association between the students' performance on Part B of the Fraction Pie task and the category CUNL. The categories with the next highest effect sizes were Multiplication and TPUNA but they had a typical association with the students' performance on Part B of the Fraction Pie task.

5.2.1 Substantial associations with fractions: units and additivity categories of measurement.

All the substantial associations between fraction categories and measurement categories were with additivity and units tasks. The measurement category of conceptual tasks assessing the units concept in a length context (CUNL) had substantial associations to all four fraction categories. The effect size of the association to CUNL was larger than the students' Multiplication scores. The measurement category of conceptual tasks assessing the additivity concept in a length context (CADL) also had a substantial association with the students' Number Line scores (which was a larger effect size than their Multiplication scores), their Equivalence scores, and their Fraction Pair scores.

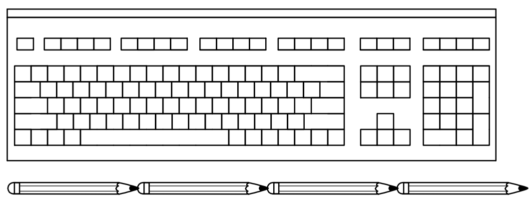
The linear associations were calculated using Kendall's *tau* because the scores were non-parametric. However, using Pearson's *r* produced similar results (this test assumed interval data and so was not appropriate for the present study). The associations between the students' Number Line score and CADL, CUNL, CADA and TPUNA had a larger effect size than their Multiplication score but they were all typical not substantial associations. The substantial correlations between the students Equivalence score and their performance on other tasks, using Pearson's r, were in the same order as reported above using Kendal's tau. Both the Students' Multiplication score and their performance in the CUNL category had substantial associations with their Fraction Pair scores, with .002 separating them. TPUNA had the highest typical association with the students' Fraction Pair scores. The categories CUNL, Multiplication score and TPUNA were in the same order but all had typical associations with the students' performance on Part B of the Faction Pie task. While a larger sample size would give greater statistical power to these calculations, the results from the descriptive statistical tests suggest that these tasks can be analysed further.

The measurement category units had conceptual links to Kieren's measure sub-construct. Measuring contexts generated a leftover and quantifying this needed rational numbers. Researchers in both measurement (Brown et al., 1995; Lehrer 2003) and fractions (Kieren, 1995; Lamon, 2007; Sophian, 2002) had proposed a conceptual link between measurement and fractions. Russian researchers had suggested that measurement should be taught before number (Davydov & Tsvetkovich, 1991) and the pedagogical suggestion to teach number through measurement developed from this research (Dougherty & Venenciano, 2007). The units category had a fractional component to three of the four task series (CUNL, CUNA, and TPUNL) so it is not surprising that students' fraction knowledge was linked to their performance on tasks in these categories. However, rather than using descriptive statistics to position fraction knowledge as a prerequisite for fraction tasks in other domains (or vice versa), I have used the descriptive statistics as a guide to focus my qualitative analysis on examining the conceptual links between fraction and measurement tasks.

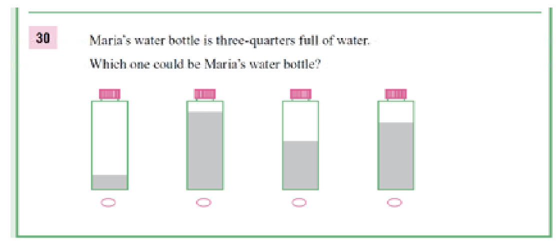
5.2.1.1 The Keyboard task.

The Keyboard task (Q. 39) was the entry-level task for the CUNL category. It was offered to every student and 54.5% were successful. Maria's Water Bottle task with a frequency of success of 52%, a simpler task with the same partial unit $^{3}/_{4}$, was an item on the Grade 3 Assessment Improvement Monitor (AIM) test used in Victorian State-wide testing (Victorian Curriculum and Assessment Authority, 2007) (see Figure 5.1). Taking into account that the

Grade levels of the students were different, but that the Maria's Water Bottle task did not have whole units to co-ordinate and had multiple choice answers, frequency of success was similar (for frequency of success see VCAA, personal communication). The performance of students in the present study suggests that they are not an outlier sample.



54.5% the present study Grade 6



52% Maria's Water Bottle task, AIM Grade 3

Figure 5.1. Comparison of frequencies on success on measurement context of three quarters.

The Keyboard task and the Number Line question 16d both were representations of $3^{3}/_{4}$. The research literature had suggested that some children may work out the leftover correctly but misrepresent the count (Brown et al., 1995). The answer four and three quarters to the keyboard task would fit this description and was offered by only two students. Similarly, the answer three quarters correctly quantified the leftover part but left out the whole number part of the count of units. Eight students had omitted the whole number part in their response to the Number Line task (and had not self-corrected during their explanation) but six of them answered successfully three and three quarters on the Keyboard task. The measurement context would appear to cue these students into the improper fraction context more readily than the number line context. Transfer between a measurement context and a number line context had not occurred for these six students. However, the results suggest that folding back to measurement contexts may be useful for students when thinking about improper fractions.

Some misconceptions that appeared in the number line questions did not appear in the Keyboard task. For example, although eight students used the *assuming decimal number lines* misconception in the number line question (reading the mark at $3^{3}/_{4}$ as 3.3), five of them correctly identified the length of the keyboard as $3^{3}/_{4}$ or $3^{2}/_{3.}$ These six students did not transfer their understanding from the measurement context to the number line question, but measurement inscriptions may provide an image to fold back to when interpreting number lines.

The pair fraction pair $^2/_4$ and $^4/_2$ (Q. 22d) included an improper fraction. Of the 35 students correct on this fraction pair, 29 were correct on the keyboard (see Table 4.34). However, of the 19 students correct at the Keyboard task who also incorrectly compared the fraction pair, 14 of them gave explanations for the fraction pair that demonstrated their unfamiliarity with the symbolic inscriptions of improper fractions. Some flipped the $^4/_2$ to make it $^2/_4$, some pointed to $^4/_2$ and said it was not a fraction, and some said there were the same numbers in both. All the students who answered 22g and 22h correctly were successful on the Keyboard. A length inscription of an improper fraction appeared easier to interpret than the symbolic inscription of an improper fraction.

There were 12 answers to the Keyboard task in which students gave four (instead of three) as the whole number part of the answer (as discussed above two of these students gave the answer of four and three quarters). Two thirds of those answers (8 out of 12) were given by students with an Equivalence score of 0 and 1. Only one of the 15 students with an Equivalence score of 0 was able to describe the Keyboard correctly as three and three quarter pencils long. The successful students were able to use the quarter of the pencil past the edge of the keyboard to iterate back along the leftover part, to mentally break the line into tenths and give a decimal estimate or to mentally halve the fourth pencil and halve again. These actions draw on partitioning (iterating and breaking into tenths) and unit-forming ($^3/_4$ is $^1/_2$ and $^1/_4$). Measurement tasks with leftovers to quantify create a context in which students have to use partitioning and unit-forming concepts without referring to pre-marked divisions. These actions of dividing into equal pieces (partitioning) and non-equal pieces (unit-forming) are also needed in the development of equivalence understanding.

Students with unconditional double counting in Part B of the Fraction Pie task were less likely (3 out of 10) to be successful on the Keyboard task than students with conditional double counting behaviour (5 out of 9) (see section 4.5.2.4.5). It is possible that the double count misconception may interfere with students' understanding of partitioning and unit forming in the Keyboard task

The results from the present study demonstrated that success with an improper fraction in the measurement context did not automatically transfer to recognition of the improper fraction in the Number Line task nor to the symbolic inscriptions of the Fraction Pair task. Measurement contexts can be a useful context for understanding improper fractions but transfer across domains is not automatic. Conceptual links between fractions and leftovers in measurement can be made in the classroom. Yanik, Helding, and Flores (2008) used fraction bar kits in their length measurement activities. Using these bars to measure classroom objects, the researchers encouraged the students to describe their measure in terms of wholes and then a fractional part, determined by how many fraction bar wholes had been iterated and what fraction bars had been used to quantify the leftover. The researchers then found that the students were less likely to assume a number line was a "whole" and paid more attention to the scale. This classroom activity was designed to help resolve the misconception of treating a number line marked from 0 to more than 1 as one whole. This misconception had also been observed in the present study in Q. 16a and Q. 16c (see section 4.5.2.1.1). In a description using mixed units, the whole units are the same size and the partial units are the same size. The double count misconception is analogous to the non-specifying of mixed units. Both improper fractions and the double count misconception have counterparts in measurement activities and may be easier to resolve in those contexts. This would then provide an image to fold back to when students are thinking about improper fractions and double counting in the fractions domain.

5.2.1.2 Broken Ruler tasks.

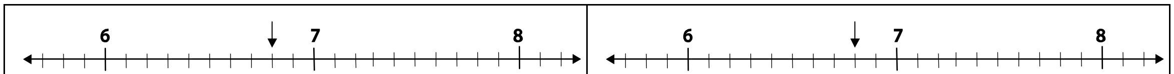
The measurement category CADL had an association with a larger effect size than the students' Multiplication score to their Number Line score. The broken ruler task, Freddo (Q. 41) was the entry-level task and 56.8% of the students were successful. The harder task, Footy Card (Q. 42), was also a broken ruler task. A comparison between the frequencies of success (see Table 5.2) of students in the present study on the Freddo task and student performance in the American National Assessment of Educational Progress (NAEP) testing (47% success on a similar broken ruler task and 41% on a broken ruler task with less friendly numbers) shows that the students in the present study are not an outlier sample. A similar result had been obtained in the United Kingdom on a pen and paper test with a line drawn over 1 to 7 on a ruler with 49.1% of 12-year olds successful ($n > 500$) (Hart, 1981). In a separate interview study of 89 New South Wales Grade 6 students, 69% of students successfully identified the length in a similar broken ruler task (Bragg & Outhred, 2004).

Table 5.2

Comparison of Frequency of Success on Broken Ruler Tasks

Task and Test	Success	Grade level
Freddo: an object on a diagram over 3 to 8 cm on a ruler. Q. 41, interview, the present study.	56.8%	6
A diagram of a bar over 3-8 inches on a ruler. NAEP 1984, multiple choice, cited in (Kamii & Clark, 1997).	47%	7
A diagram of a toothpick over 8 to $12^{1}/_{2}$ cm on a ruler. NAEP 2003, multiple choice, cited in Nguyen (2010).	41%	8

The frequency of success in the present study (52.3%) on a decimal number line (see Figure 5.2) was very similar to the success in the New Zealand numeracy project (50.2%, Vince Wright, personal communication, January 23, 2008). However, the frequency of success of 38.6% on Q. 16d ($3^{3}/_{4}$) was less than the 87.2% of Grade 6 students correct on the same number line in a multiple choice format (Lesh et al., 1983).



52.3% (the present study Grade 6)

50.2% (New Zealand Year 7)

Figure 5.2. Comparison of frequency of success on decimal number line tasks.

The misconception of counting lines not spaces had been reported in the literature on broken ruler tasks in junior primary school children (Lehrer et al., 1998) and in senior primary school children (Bragg & Outhred, 2004). In Bragg and Outhred's Australian study, 53% of the Grade 6 students used the same misconception on a task using a ruler with marks but no numerals to measure a length. This was similar to the 48.9% of students in the present study who demonstrated the counting lines not spaces misconception in either the Footy Card or Freddo task. The counting lines not spaces misconception had also been reported in the literature on number lines (Pearn & Stephens, 2007) and 10.2% of the students in the present study used this incorrect strategy on the number line marked at $^{5}/_{6}$ (Q. 16e), calling it $^{6}/_{7}$.

The use of the misconception in one context, number lines or broken ruler tasks, did not predict performance in the other (see section 4.5.2.1.3). Only 33% (3 out of 9) of the students who used the misconception on the number line task (Q. 16e) did so on the Freddo or Footy Card tasks. Only 37.5% (9 out of 24) of the students who had been successful on the number line question (Q. 16e) used the counting lines not spaces misconception on the broken ruler tasks (either the Freddo or the Footy Card task). This misconception was not the cause of the association between performance in the measurement category and performance in this particular fraction number line question. A misconception may be a perfectly good rule misapplied, in this case that the verbal counting sequence begins at one, but it is not applicable to every possible context. Further research could target this specifically and ask the students why they counted from 1 in one context and not another.

The responses to the broken ruler tasks highlighted a distinction between two strategies used when students' drew their own number lines (Q. 16a). Both the ratio misconception (using other ratios such as 6 out of 9 to represent $^2/_3$ on a number line) and unsuccessful partitioning (placing $^2/_3$ two thirds of the way along the line but labelling 1 incorrectly) demonstrated that the students had strategies that showed correct mathematical thinking but did not accommodate the specific conventions of number lines (see section 4.5.2.1.1). However, the difference in performance by students with these two misconceptions, on broken ruler tasks, suggests that the unsuccessful partitioning was a less co-ordinated understanding of scales than the ratio misconception. A similar percentage of students with the two misconceptions (69.2% and 70%) were successful on the Freddo task (Table 4.19). Of the students who used the ratio misconception, 53.8% (7 of 13) were successful on the Footy Card task. In contrast, 10% (1 of 10) of the students who used partitioning unsuccessfully on the number line task were successful on the Footy Card task.

There was a linear association between the measurement category CADL and the students' fraction Number Line scores. A higher CADL score was often paired with a higher Number Line score, while a lower CADL score was often paired with a lower Number Line score. On individual tasks, however, the relationships were more complex. There was no clear trajectory through the number line questions. The results did not suggest that marked partitions or marking partitions was easier, nor whether number lines from 0 to 1 were easier than 0 to 2. There was no significant association between the students' Equivalence scores and Q. 16g, the decimal number line representing 6.8. The counting lines not spaces misconception in a number line context or a broken ruler context was not predictive of its use in the other context. However, despite the similar overall frequency of the misconceptions of unsuccessful partitioning (11.4%) and ratio representations (14.8%) (see Table 4.19) when drawing their own number line (Q. 16a), the contrast in the students' performance on the broken ruler tasks suggests that the students using the ratio misconception, while incorrect, had a greater understanding of length scales than those using unsuccessful partitioning.

5.2.1.3 CUNA and CADA.

The construct validity of some tasks was undermined by novel representations (concrete materials and inscriptions). This lack of familiarity was evident in the interview in the students' handling of the length representations in the Fraction Sort task (Q. 19m, l and t) and the pattern blocks in the Pattern Block task (Q. 17). Alternative tasks would be needed to further investigate associations between these aspects of the measure sub-construct and measurement categories. The Cuisenaire rods in the Cuisenaire Units task (Q. 48) were unfamiliar to the students, and the difficulty, not predicted by the pilot study, of the Array with Leftovers task (Q. 46) may explain why the CUNA category did not have strong associations with other tasks.

It appeared that the tools and procedures task for the additivity construct in an area context (TPADA) was a prerequisite task for the tasks used to assess the conceptual aspect of the additivity concept (see Table 4.5). However, the conceptual category, CADA, had a larger effect size than the tools and procedures category, TPADA, in associations with the students' Number Line scores, Equivalence scores, Fraction Pair score, and performance on Part B of the Fraction Pie task. The area of half rectangles was a measurement context in which a simple fraction component could illustrate conceptual links across domains.

5.2.1.4 Attribute.

There were no significant associations between CATA, CATL, or TPATA (TPATL had 100% frequency of success) and the students' Number Line score (see Table 4.23), nor their performance on Part B of the Fraction Pie task (see Table 4.39). There was a typical association between TPATA and the students' Fraction Pair score (see Table 4.33) and a minimal association with their Equivalence score (see Table 4.17). The concept of attribute has been seen as a foundational understanding in measurement instructional trajectories (see e.g., Outhred et al., 2003; Wilson & Rowland, 1993). The concept of attribute has also been seen as being necessary for the understanding of fraction diagrams (see e.g., Steinle & Price, 2008). However, in the present study, the misconceptions of an understanding of the key concept of attribute appeared to be quarantined. This may be due to construct validity problems with the tasks chosen to assess attribute. Or it may be because fraction knowledge masks rather than cures the perimeter indicates area/ same area indicates same perimeter misconceptions (see discussion in section 5.1.1).

5.2.1.5 Proportionality.

There was a typical association between the measurement categories CPRL, TPPRL and TPPRA (CPRA had 100% frequency of success) and the students' Equivalence score (see Table 4.17), their Number Line score (see Table 4.23) and their Fraction Pair score (see Table 4.33). There was a typical association between the measurement category CPRL and the students' performance on Part B of the Fraction Pie task (see Table 4.39). Since the conceptual link between order, equivalence, ratio and proportion had been elaborated in the research literature (see e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Confrey, 2008; Lesh, Post, & Behr, 1988; Lamon, 1993), the small effect sizes found in the present study were unexpected.

In the present study 80.7% of the students successfully chose Tim as having the biggest steps in the Steps task (Q. 44). In a pen and paper version of this task (National Center for Educational Statistics, 2007), 41% of Australian Grade 4 students were successful on the 2003 Trends in International Mathematics and Science Study. Age and the testing format may have influenced the contrasting frequencies of success. In present study 47.7% of the students chose and used an equal interval ruler (or imagined equal spaces on a non-equal marked ruler) in the Choosing Rulers task (Q. 45). Similarly, Pettito (1990) had found that 42.9% of children in late Grade 3 (n = 21) chose the equal spaced ruler.

The tasks assessing the measurement construct of proportionality were not quantified; the students only had to nominate the direction of change. In developing the interview tasks, if a proportionality task included quantification I classified it as a (fraction) ratio task. The results suggest that there is not a strong association between the concept of proportionality in an unquantified context and fraction constructs. If the proportionality (measurement) tasks had included a fractional component as the units category did, there may have been stronger associations.

There were analogous tasks in length and area at the easiest level: Part B of the Paperclips task (Q. 40b) and Part B of the Draw Your Own Array task (Q. 38b). It would have been possible to develop an analogous task in the area context to the Steps task and it is a limitation of the present study that this was not done. However, a task analogous to the Choosing Rulers task suitable for Grade 6 students was not developed for the present study.

5.2.1.6 TPUNA.

In general, conceptual categories had stronger associations than tools and procedures categories to fraction constructs. The categories of CUNL and CADL have been analysed in the previous section. However, the category of TPUNA consistently had substantial associations with the Fraction constructs (Equivalence scores, Number Line scores, and Fraction Pair scores) and, like students' Multiplication score, a typical association with Part B of the Fraction Pie task. This was an unexpected result.

In developing tasks for the interview, I had thought that knowing to offer cm as the unit for length tasks, and to offer cm^2 as the unit for area tasks was rote learned. Therefore, I did not specifically ask the students to explain why they had offered the units that they did, hence I do not have verbal explanations to interpret.

Around 70% of students in Equivalence Bands A, B and C had TPUNA scores of 0 (see Table 4.18). Around 30% of students in Equivalence Bands D and E had TPUNA scores of 0. No students in Equivalence Band F had TPUNA scores of 0. Equivalence Band E represents the first band where equivalence knowledge was used as part of another strategy, such as benchmarking or creating common denominators. However, while there was an association between low Equivalence scores and the use of incorrect units (cm) in area tasks, and high scores and the use of correct units (cm^2), the specific difference between Equivalence Band D and Equivalence Band E was shown not to be important.

Arrays were used in tasks to assess area knowledge. All students in the present study could use coordinates on a grid on the Treasure Map task (Q. 50, see Figure 4.1) thus demonstrating that they could see an array structure. Battista et al. (1998) had noted that when younger children have to physically draw units on arrays, some drew each square in its entirety rather than using one line to show the edge of two adjacent squares. This did not occur in any of the 88 inscriptions that the students made in the present study (see section 4.2.4.4). Outhred and Mitchelmore (2000) had developed the Draw Your Own Array task and 71% of the Grade 4 students in their study were successful at restructuring the rectangle. In the present study, 89.8% of the students created an acceptable array or indicated an array with row and column hash marks. Of the nine students who were unsuccessful on Part A of the Draw Your Own

Array task (Q. 38a), seven had a TPUNA score of 0 (offered cm incorrectly), one had a TPUNA score of 1 (offered informal units but not cm^2 nor cm), and one had a TPUNA score of 2 (offered cm^2 and not cm). If the TPUNA category was in fact a conceptual category, then the units aspect of the task may be an extension to traditional trajectories of array understandings. Battista's array trajectory (2007) and Outhred and Mitchelmore's trajectory (2000) had been based on children in Grades Prep to 4 (see section 2.1.2.2). Offering correct formal units may need to be incorporated into trajectories for array understanding that extends to Grade 6.

5.2.1.7 Implications.

The *measurement to partitioning/equivalence/unit-forming to measure sub-construct* conceptual pathway was demonstrated by the substantial associations between the measurement categories, additivity and units, and the fraction constructs, equivalence and the measure sub-construct (Number Line, Fraction Pair, and Fraction Pie tasks). This pathway was also evident in the analysis of students' explanations to particular tasks. This pathway has been examined in reports of classroom interventions (see e.g. Cobb et al., 2011; Nguyen, 2010; Yanik, Helding, & Flores, 2008) and in theoretical models (Confrey, 2008). The understanding of key concepts of measurement matter, and can be conceptually connected to fraction understanding. The *multiplication to equivalence to ratio* pathway (see e.g., Confrey, 2008) is also an important pathway, as demonstrated by the substantial association between students' Multiplication scores and fraction constructs, but it is not the only pathway to fraction understanding. The present study has shown that there are opportunities for making conceptual connections between fractions and length and area measurement.

Conceptual measurement tasks are important for the development of the measurement to partitioning/equivalence/unit-forming to measure sub-construct pathway. Using broken rulers is as important as using a ruler. Measuring with items with units that leave a leftover is as important as co-ordinating the iteration of a whole number of units. Although it was quarantined in the present study, a systemic knowledge of attribute could be part of the curriculum. Spatial measures are length, area, volume and angle. Length presents in increasingly complex ways from straight paths to perimeters. Area means tiling or restructuring in two dimensions not "what's inside" in a three dimensional context.

The use of formal units appears to be more than just tools and procedures knowledge. An implication of this is that further research is required to extend descriptors of array understandings in the elaboration of these trajectories.

5.3 Research question 3: Kieren's four-three-four model

The third research question investigated was:

- Can we use Kieren's four-three four model of fraction understanding (1988, 1992, 1993, 1995) to describe the fraction understandings of students in the present study?

There were two key structures to Kieren's four-three-four model. Firstly, there were *four* sub-constructs, measure, quotient, operator, and ratio, supported by *three* concepts, partitioning, equivalence, and unit forming. Secondly, while the constructs described different contexts for rational numbers, a child's engagement with each construct could be at *four* levels: ethnomathematic, intuitive, technical-symbolic or axiomatic deductive (1988, 1992, 1993, 1995). Pirie and Kieren (1994a; 1994b) had described the process of dynamical learning, and highlighted the role of folding back to earlier images rather than permanently moving past earlier understandings, and this can be connected to Kieren's four levels of engagement with fractions tasks.

Kieren's earlier five-part model had many citations in the research literature both in its own right (Kieren & Nelson, 1978; Kieren & Southwell, 1979; Leung, 2009; Norton & Wilkins, 2010) and as the framework for the Rational Number Project research (see e.g. Behr, Harel, Post, & Lesh, 1992; Behr, Post, & Silver, 1983). Researchers citing the Rational Number Project research often use the five-part model (see e.g. Charalambous & Pitta-Pantazi, 2006; Clark et al., 2003; DeWindt-King & Goldin, 2001; Tepylo & Moss, 2011). Some researchers using the five-part model noted the double count misconception which could develop from the part-whole sub-construct, but they continued to use the model while cautioning against simplistic interpretations of part-whole (see e.g. Gould, 2005; Lamon, 2007; Mosely, 2005). Research by radical constructivists with other frameworks has continued (Confrey, 2008; Nguyen, 2010; Steffe, 2003). Other research with less prominent theoretical frameworks did not cite Kieren's work (see e.g. Duzenli-Gokalp & Sharma, 2010; Petit et al., 2010; Rayner, Pitsolantis, & Osana, 2009). Kieren's four-three-four part model remains rarely cited in the research literature. On the other hand, the Pirie-Kieren model describing learning as a dynamical recursive process had been taken up by researchers who examined children's learning (Martin 2008) and pre-service teachers (Borgen, 2006).

5.3.1 Measure sub-construct.

In selecting and designing tasks for the present study, I focussed on the measure sub-construct and the concept of equivalence. The measure sub-construct included three aspects. Some tasks (bolded) were reported in the Results chapter and have been discussed earlier in this chapter:

- area and length diagrams or contexts (Tightrope Walker, Q.15; Puff Machine, Q. 23; **Fraction Sort, Q. 19; Fraction Pie, Q. 14; Fold Me a Quarter, Q. 13**),
- comparisons of the relative size of fractions (**Fraction Pairs, Q. 22**; Density, Q. 25),
- **Number lines** (**Q. 16**).

Kieren had included length and area diagrams as part of the measure sub-construct in his reframing of the part-whole sub-construct (1992). The research literature had categorised the comparison of the relative size of fractions (order) as part of a broad part-whole sub-construct (see e.g Behr, Wachsmuth, Post, & Lesh, 1984), as part of the measure sub-construct (Lamon, 1999; Ni, 2000), as part of equivalence and ratio understanding (Behr et al., 1992), or as understanding fractions as numbers (Clarke & Roche, 2009). Following Lamon (1999), I have classified the Fraction Pair task as a measure sub-construct task. Number lines have been accepted as an example of the measure sub-construct by researchers using Kieren's (1980) five-part model (see e.g, Clarke, Roche, & Mitchell, 2011; Lamon, 1999; Pearn & Stephens, 2007) and by researchers in the Rational Number Project (see e.g., Bright et al., 1988).

Different strategies for individual tasks illustrate different aspects of the measure sub-construct, links between sub-constructs, or different levels of response. Hence the Fraction Sort task, the Fraction Pie task, the Fold Me a Quarter task, the Fraction Pairs task, and the Number lines task will be discussed in this section with respect to the specific concepts as appropriate:

- partitioning concepts drawn on in the measure sub-construct
- partitioning concepts drawn on in length measurement tasks
- equivalence concepts drawn on in the measure sub-construct
- unit-forming concepts drawn on in the measure sub-construct

For example, there were several correct strategies that were evident in the students' responses in the present study to the Fraction Pair task (see Table 5.3) but the different strategies will be reported in the partitioning (5.3.1.1), equivalence (5.3.1.2), and unit-forming (5.3.1.3) sections. The misconception of gap thinking will be discussed in the equivalence section. Seth's strategy for the fraction pair $^3/_4$ $^7/_9$ which used operator thinking will be discussed in the links across the four sub-constructs section (5.3.5).

Table 5.3

Strategies for Fraction Pairs

Fraction Pair	Possible Correct Strategy	Fraction Concept Drawn On
$^3/_8$ $^7/_8$	same denominator and compare numerators; visualisation of area complement to one	partitioning partitioning unit-forming
$^2/_4$ $^4/_8$	equivalence	equivalence
$^1/_2$ $^5/_8$	benchmarking	unit forming or equivalence
$^2/_4$ $^4/_2$	compare $^1/_2$ and $^4/_2$ benchmarking to one mixed number	equivalence and partitioning equivalence unit forming and equivalence
$^4/_5$ $^4/_7$	same denominator, compare numerator benchmarking	partitioning equivalence and unit forming
$^3/_7$ $^5/_8$	benchmarking	equivalence and unit forming
$^5/_6$ $^7/_8$	residual thinking	partitioning
$^3/_4$ $^7/_9$	equivalence and residual thinking; common denominators	equivalence and partitioning equivalence and partitioning

5.3.1.1 Partitioning concepts drawn on in the measure sub-construct.

Kieren's description of the concept of partitioning was linked to children's actions of making equal parts and exploring the multiplicative relationships between different sized parts: the "folding space" (1995). Partitioning privileged the actions of making or imagining equal parts rather than double counting pre-shaded area diagrams (Confrey, 2008; Kieren, 1983, 1995). However, making equal parts and comparing the results of two different partionings is also an aspect of the partitioning concept. In regard to the measure sub-construct, three aspects of partitioning are discussed:

- making equal parts (including using visualisation),
- the double count misconception, and
- comparing different sized parts or different numbers of equal sized parts.

When faced with non-equal-parts area diagrams some students, who understood that partitioning was based on equal parts, imagined equal parts. For example, in the Fraction Sort task the students had to choose the fraction represented by a diagram of a rectangle divided into four non-equal parts (a sixth, see Figure 4.43, Q. 19b). This task had a frequency of success of 50%. For example, Jade described imagining an extra line to make equal parts "because if you put a line down there [indicates with finger] there would be six spaces; one coloured."

Making equal parts was also evident in the non-equal-parts diagram of the Fraction Pie task (see Figure 4.41). Some students recognised the need to make the same number of partitions in both halves. For example, Jack explained how he made the same sized parts on both halves of the circle, "There's three of the same size on this size. So that means there would be able to fit, uh, three of the same shape on the other side. And that's six. And that would be one of them."

The students who imagined lines back through the left hand side from the radii on the right hand side of the Fraction Pie diagram (see Figure 4.41) were attempting to make equal parts. They either forgot to exclude the line dividing the quarters, or they believed that the imaginary lines divided the quarters in halves and therefore incorrectly made sevenths. It would appear that no cognitive conflict was generated by these imagined lines creating four parts in the left half and three parts in the right half. The initial premise, making equal parts, was a mathematically correct strategy but not executed successfully.

Partitioning was also utilised in the Number Line task Q. 16a, in which the students had to draw a number line and mark two thirds on it. Some students combined successful partitioning and successful measure sub-construct knowledge by marking $^2/_3$ two thirds of the way between 0 and 1 on the number line. Students partitioned by eye, by iterating with their fingers, or by using a ruler. The 33% frequency of success on this question was lower than the 51% frequency of success on the same question in a larger study (Clarke et al., 2007). One misconception observed in the present study was the incorrect labelling of 1 on the number line. Some students marked $^2/_3$ two thirds of the way along the line but then labelled 1 at the position of $^1/_3$. These students had drawn upon partitioning concepts but they did not co-ordinate the labelling conventions of number lines with their partitioning, possibly using 1 to indicate $^1/_3$.

This misconception was related conceptually to marking ¼ on a number line marked 0 to 2 at a half (a quarter of the way along the line) by partitioning the whole line into quarters. This misconception had been noted in the research literature (Clarke et al., 2007, 2011; Kieren, 1993; Petit et al., 2010). This response was observed in the present study on Q. 16c. However, not all students making this error had done this in both questions (Q. 16a and 16c). This indicated that the different contexts affected whether the partitioning misconception would emerge.

The double count misconception was associated in the literature with the use of limited part-whole definitions of fractions (Kieren, 1988; Lamon, 2007). In the measure sub-construct tasks, the Fraction Pie task and the Fraction Sort task in the present study, the double count misconception is classified as a misconception of partitioning. The unconditional double count misconception answer of one fifth, because there were five parts, to both Part A and Part B of the Fraction Pie task was given by 9.1% of the students. This was less than half of the 21.6% of students who gave the answer of one fifth to either Part A or Part B of the Fraction Pie (see Table 4.36). The offering of conditional double counting explanations, such as Ruby's, "because there are five pieces in it. Um, it would be an equal fifth if they were a bit smaller and they were the same size, but they're not. Because those two are bigger than those three; so technically they're fifths, they're just not equal fifths" was observed in the present study and had also been reported in previous research (Post et al., 1985).

Although the concept of part-whole in the original research by Kieren (1980) and the Rational Number Project researchers (Behr et al., 1983; Behr et al., 1992) was broader than a static double count, researchers suggested (see e.g., Lamon, 2007) than teachers' instruction on the part-whole concept was limited. Resolving the double count misconception may be complex. Simple explanations may confirm a misconception because of the similarity of the linguistic structure of correct and incorrect calculational explanations. For example, both Jess and Jade gave almost indistinguishable explanations as to why the Fraction Sort card diagram of one of six equal parts shaded was one sixth (see Table 4.38):

- "because there's six there and then there's one um shaded, and there's six."
- "because there's six pieces and one's coloured."

One of these students then verbalised the double count misconception when sorting the card with a diagram of one shaded part out of four non-equal parts (a sixth):

- "Yeah I put that there [quarter] because there's four shaded [sic] and one shaded and it's a quarter."
- "because if you put a line down there [indicates with finger] there would be six pieces; one coloured."

If a teacher used a correct explanation like Jade's, "because there's six pieces and one's coloured" to explain the strategy of partitioning on an equal-parts diagram, then Jade's correct partitioning thinking would be confirmed. The difference between the explanations to the card with the non-equal-parts diagram (a sixth) showed that Jess' explanation was still calculational, "Yeah I put that there [quarter] because there's four shaded [sic] and one shaded and it's a quarter". Jade's explanation on the other hand, included

- a calculational aspect, "there would be six pieces; one coloured."
- how she arrived at the calculation, "because if you put a line down there [indicates with finger] there would be six pieces"
- but not an explicit statement of why she did this (to make equal parts).

In the Fraction Pairs task (Q. 22) the comparison of the relative size of fractions was presented as symbolic inscriptions. Some, but not all, of the strategies used for comparisons drew upon simple partitioning concepts (see Table 5.3):

- comparing different numbers of equal parts, or
- comparing different sized parts.

For the fraction pair $^{3}/_{8}$ and $^{7}/_{8,}$ the comparison could involve folding back to an image of partitioning into eight equal parts and then comparing different numbers of these parts. This was expressed in the technical symbolic language of Lily, "Because there's the same denominator and seven's larger." Students who visualised a diagram drew on images of partitioning actions to decide that $^{7}/_{8}$ was the larger fraction. The classification of these strategies as *partitioning* in a fraction comparison context corresponds to their classification as *part-whole* in research based on other models (see e.g., Cramer & Wyberg, 2009). In the present study, improper fractions were an extension of this use of comparison after partitioning..Some students called $^{4}/_{2}$ an *improper* fraction signalling that four haves was more than one half. For example, Kate explained "Because the four is bigger than two and it's an improper fraction". These students also co-ordinated their knowledge of equivalence with this partitioning: either renaming $^{2}/_{4}$ as a half or benchmarking to 1.

The fraction pair $^{4}/_{5}$ and $^{4}/_{7}$ was analogous to unit-fraction pairs. Comparison were based on the size of the pieces because the number of pieces was the same. This strategy had been described in the research literature (Behr et al., 1984; Clarke & Roche, 2009; Clarke et al., 2008; Cramer & Wyberg, 2009; Post & Cramer, 1987; Post et al., 1985) and was observed in the present study. For example, Sarah chose $^{4}/_{5}$ as the larger fraction "Because the top numbers are both four, but there's seven and five on the bottom; and seven means that the pieces are littler. So four of them wouldn't equal four of the fifths." In classroom activities the actions of partitioning generate unit fractions which are then compared: a fifth is bigger than a seventh. This partitioning and comparison action can be extended to the same number of pieces of different sizes: four fifths is bigger than four sevenths because if you have the same number of pieces you want the bigger pieces. I argue that these comparisons are part of the partitioning concept and are drawn upon in the measure sub-construct context of fraction pair (presented with symbolic inscriptions) comparisons.

Residual thinking is the third strategy that is part of the broad understanding of the actions of partitioning. Residual thinking had been described in the literature as comparing the unit fraction leftover in fraction pairs such as $^5/_6$ and $^7/_8$ (Post, Behr et al., 1986; Clarke & Roche, 2009; Cramer & Wyberg, 2009). In classroom activities, the action of partitioning produces fractional parts of kits or diagrams which are then compared: in this case the residual pieces are compared.

An aspect of partitioning not investigated in this thesis was the multiplicative relationship between pieces. For example Kieren describes paper folding problems such as what folds are needed to get from thirds to twelfths (1995).

The measure sub-construct encompassed the contexts of number lines, area diagrams and relative size comparisons. The concept of partitioning involved many understandings. The making of equal parts on number lines or diagrams, or the imagining of equal parts on non-equal-parts diagrams, was one aspect of partitioning. Linked to this was resolving the double count misconception. Another aspect of partitioning was comparing the results of partitioning. Three strategies used in the Fraction Pair task illustrate this: comparing the number of pieces if the pieces are the same size, comparing the size of the pieces if there are the same number of pieces, and comparing two residual pieces. In the present study, all these aspects of partitioning were used by students explaining their responses to measure sub-construct tasks.

5.3.1.1.1 Partitioning concepts drawn upon in length measurement tasks.

Partitioning was drawn upon in length measurement contexts. Visualising equal partitions was a feature of the responses to the length measurement task, Q. 39 Keyboard (see Figure 4.9). Some children worked mentally left to right and divided the final pencil into tenths, or split it in half and half again. Other children worked from right to left with the part left over the end of the keyboard and iterated backwards to discover it was a quarter. These explanations indicated that partitioning was also used in measurement contexts. Subsequent research has confirmed a conceptual link between equi-partitioning and length and area tasks (Nguyen, 2010).

The perimeter comparisons of the shaded shapes in the Similar Shapes task (Q. 36h) revealed that students could drawn upon partitioning and not recognise that their fraction partitioning knowledge was masking a misconception about the relationship between perimeter and area. Students recognised that the shaded shapes were both partitioned in halves. However, they used this knowledge to incorrectly deduce that the perimeters must be the same. This use of fraction partitioning may mask the geometric misconception and this was not flagged by Kieren's model which is a framework of ideal knowledge in fractions.

5.3.1.2 Equivalence concepts drawn on in the measure sub-construct.

In Kieren's reframing of part-whole, in the four-three-four model (1988, 1992, 1993, 1995) the concept of equivalence was given its own category which had conceptual connections to not just the ratio sub-construct, but also the measure, quotient and operator sub-constructs. In the five-part model for fraction understanding the part-whole sub-construct included order and equivalence (Post et al., 1985), and equivalence was linked to ratio (Post et al., 1985; Wong & Evans, 2007).

In the present study, equivalence understanding was drawn upon in the measure sub-construct number lines tasks. This was suggested by the common variance of 51% (and a substantial association) between the students' Equivalence score and their Number Line score (see Table 4.40). However, there was no significant association ($p = .167$) between the students' Equivalence scores and Q. 16g, the decimal number line representing 6.8 suggesting that the use of only decimal number lines to assess the measure sub-construct may not have content validity and that co-ordinating an understanding of fraction number lines may be important.

The ratio misconception was observed in the first number line question in the present study and 14.8% of the students placed $^{2}/_{3}$ incorrectly when drawing their own number line (for example, at 2 out of 3, 6 out of 9, 6.66 out of 10, or 8 out of 12). These students drew on equivalence knowledge, for example, $^{2}/_{3}$ is the same as 8 out of 12, but they could not co-ordinate this successfully with the conventions of a number line (that $^{2}/_{3}$ is two thirds of the way between 0 and 1). This behaviour had been observed in earlier research on the same task (Clarke et al., 2007).

Equivalence understanding was drawn upon in the measure sub-construct tasks using area and length diagrams. There were eight cards in the Fraction Sort task (Q. 19) that represented equivalent fractions (see Figure 4.20). For example Rohan explained why the length ${}^{4}/_{6}$ was ${}^{2}/_{3}$, "And this is two thirds 'cause it's four sixths and if you halve it it will be two thirds". Only two students could successfully restructure the triangle divided into three non-equal parts into ${}^{6}/_{9}$ (Q. 19r) but neither could rename this ${}^{2}/_{3}$. Equivalence understanding, in the length and area diagram context of the measure sub-construct, required both visualisation and numerical skills. There was also a substantial association between the students' Equivalence scores and Part B of the Fraction Pie task (see Table 4.40) suggesting that partitioning and equivalence concepts were co-ordinated by the students who could correctly name Part B of the Fraction Pie as one sixth.

It was not surprising that there was a substantial relationship between the students' Equivalence score and their Fraction Pair score as there were two questions in common (Q. 22b and Q. 22f) (see Table 4.40). However, there was no significant association ($p = .096$) between the students' Equivalence score and comparing the fraction pair ${}^{3}/_{8}$ and ${}^{7}/_{8}$ (Q. 22a). This supports the argument that this particular fraction pair drew on partitioning rather than equivalence knowledge.

The comparison of fractions could require the recognition of equivalent fractions such as ${}^{2}/_{4}$ and ${}^{4}/_{8}$ (Q. 22b). Previous research had demonstrated the importance of one half as a first step to equivalence understanding (see e. g. Callingham & Watson, 2004). In the present study, the (in this instance correct) additive language was revealed in some students' explanations of one half. For example, Emma explained why ${}^{2}/_{4}$ and ${}^{4}/_{8}$ where the same, "Well 'cause, you just, they're all, they're halved, so they would be the same. So there's four plus four is eight and two plus two is four." Similarly Jack's explanation illustrated that the relationship could be additive or multiplicative but that this was not clear in students' use of the word half, "they're both half. Of the bottom number". Multiplicative thinking was evident in Nicky's explanation using the word *simplify*, " 'cause two quarters and four eighths; if you simplify four eighths you make it go down to two quarters, you simplify that again it would be one half".

Another strategy that drew on equivalence was benchmarking where fractions were compared to a third fraction such as to ${}^{1}/_{2}$. For example, Lily chose ${}^{5}/_{8}$ as larger than ${}^{3}/_{7}$ explaining, "Because half of eight is four and means that's gone more, that's more than a half. And that one's three point five. And to go over a half, that has to be four." Adam used benchmarking to explain why ${}^{4}/_{5}$ was larger than ${}^{4}/_{7}$, "Um. Four. The four and the seven, there's more less, like, um close to a half, but this one's like almost a whole." In both of these examples benchmarking is not possible without knowing that $^{1}/_{2}$ is equivalent to $^{4}/_{8}$, $^{3.5}/_{7}$ or $^{2.5}/_{5}$. A related strategy drawing on equivalence was using common denominators. This was illustrated in Lily's explanation of how she decided that $^{7}/_{9}$ was larger than $^{3}/_{4}$, "I tried to get thirty six [waves finger across both denominators]. And I times that by nine [points to 3], and that by four [points to 7]." Common denominators are used to make friendly equivalent fractions.

The partitioning concepts (correct strategies and the double count misconception) drawn upon in the comparison of the residual fraction pair $^{5}/_{6}$ and $^{7}/_{8}$ have been discussed above in the partitioning section (5.3.1.1.1). Gap thinking, a misconception of equivalence (see section 5.1.4), was used on this residual pair by 50% of the students in the present study. A co-ordinated knowledge of equivalence includes understanding why residual pairs are not equivalent. A classroom activity that addressed equivalence was the Japanese lesson study activity the *Janken* game (paper, scissors, rock). Students received different pattern blocks for a win with paper (green sixth), scissors (blue third), or rock (yellow half) (Yamamoto, 2007). The practical problem that the children discovered needed solving was how to give "change" if their opponent had run out of one sort of pattern block. Trading (equivalence) solved this (Yamamoto, 2007). In addition, I would argue, the activity provided students with an image of residual fraction comparisons. $^{2}/_{3}$ (two blue blocks or four green blocks) as change for a whole, given for a win with scissors from a child with no blue blocks left, was not equivalent to the $^{5}/_{6}$ (five green blocks) given as change for a whole given for a win with paper by a child with no green blocks left. The transactions in this game give students a concrete image to fold back to of the equivalence of $^{2}/_{3}$ and $^{4}/_{6}$ and the non-equivalence of the residual pair $^{2}/_{3}$ and $^{5}/_{6}$.

I have argued in this thesis that gap thinking could be thought of as a misconception of equivalence. Kieren's four-three-four model is explanatory, not in the sense of describing a mechanism, but in the sense of naming the key concepts and their connections. Fraction size comparisons can be classified as measure sub-construct tasks. Different fraction pairs draw on partitioning, equivalence or unit-forming. The *as much as* meaning of equivalence is drawn upon in additive initial understandings of equivalences to one half, illustrated in Emma's use of the word *plus*. This meaning is related to unit-forming: a fractional part can further be made up of parts added together (Kieren, 1999). The *as many as* meaning of equivalence is drawn upon in multiplicative understandings of equivalences to one half, illustrated by Nicky's use of the word *simplify*. This aspect of equivalence is related to partitioning: if a fractional part is divided into equal parts, this action is replicated on all parts so that the whole is made of, simultaneously, these smaller equal parts and equal parts the size of the original fractional unit. This understanding was explored by Clark and Kamii (1996).

The gap thinking misconception is a misconception of equivalence. Additive calculational explanations are inappropriately generalised. They are (incorrectly) confirmed by the partitioning or unit forming concepts drawn upon when comparing the fraction pair $^3/_8$ and $^7/_8$ and the unit forming aspect of equivalence drawn upon in recognising equivalences to one half. Gap thinking is reinforced by the double count misconception (a partitioning misconception), in the residual fraction pair $^5/_6$ and $^7/_8$. Drawing on (correct) partitioning concepts in this fraction pair, grounded in concrete materials or inscriptions, may help resolve the gap thinking misconception. Students who consistently use equivalence as part of strategies such as benchmarking and common denominators did not exhibit gap thinking indicating that this may have contributed to the resolution of this misconception. Kieren's terminology of unit-forming, partitioning, and equivalence, and the measure sub-construct enable the framing of this discussion of the gap thinking misconception. It would be possible to use the sub-construct part-whole as an overarching term for partitioning, equivalence and unit-forming, and propose the same explanation for the gap thinking misconception within the framework of the five-part model: either Kieren's model (1980) or the Rational Number Project framework (Behr et al., 1992; Behr et al., 1983). However, the elaboration of the four-three-four model (Kieren, 1988, 1992, 1993, 1995) enabled the fine-grained analysis and focuses attention on possible strategies to assist in resolution of misconceptions.

Some research used the five-part model to demonstrate student proficiency in each of the five categories, finding that students performed best in part-whole and weakest in measure (Charalambous & Pitta-Pantazi, 2006; Leung, 2009). However, equivalence was not a separate category in these two projects and I have argued that equivalence could be drawn upon in the four sub-constructs (e.g., measure) and thought of a concept in its own right (or part of the Rational Number Project's construct of part-whole). The results of the present study showed that equivalence was an important facet of students' fraction understanding with substantial correlations to other fraction constructs. Including or excluding an equivalence component in a task would have changed the construct validity of the category in those two research projects that used the five-part model.

5.3.1.3 Unit-forming concepts drawn on in the measure sub-construct.

Confrey (2008) defined equi-partitioning as multiplicative and different from "breaking, fracturing, fragmenting, or segmenting in which there is the creation of unequal parts". Kieren had also made this distinction, calling this other sort of restructuring into unequal parts unit-forming or the combining space (Kieren, 1995). This concept was drawn upon in students' explanations of measure sub-construct tasks. The number line context and unit-forming has not been analysed in this thesis. How the unit-forming concept is drawn upon in the other measure sub-construct contexts of non-equal parts diagrams and fraction pair comparisons is elaborated in the following paragraphs.

In non-equal-parts diagrams, the concept of unit forming reinforced that double counting would be inappropriate. The research literature had reported classroom activities in which students worked with non-equal-parts diagrams approaching the naming of parts from a unit-forming perspective not with the partitioning misconception of the double count (Kieren, 1995; Kieren et al., 1996). Students' attempts to compare Part B of the Fraction Pie to Part A have been discussed at length in section 5.1.2. For example some answers of a fifth were an estimation of a fifth plus a-small-fraction equals a quarter.

It was the students' choice of strategy that determined whether the Fraction Pie task assessed simple partitioning (three on one side so six altogether), unit forming (a sixth plus a twelfth is a quarter), or the harder operator (two thirds of a quarter), which were observed in the present study; or equivalence (a half is three sixths), or the easier operator (a third of a half) which were not observed in the present study. Being unsuccessful on the Part B of the Fraction Pie task did not mean that they could not partition. Some had attempted the much more difficult operator calculation (two thirds of a quarter) or the more difficult unit-forming calculation (a sixth plus a twelfth is a quarter). Presmeg (1985) had cautioned against assuming that a child's preferred strategy was their only strategy. Thus a false negative, analogous to the false negatives for visualisation described by Bishop (1983), may result if observational listening is not used.

Unit-forming was drawn upon in the measure-sub-construct context of the comparison of fraction pairs (Q. 22). The complement to one strategy used in the fraction pair ${}^{3}/_{8}$ and ${}^{7}/_{8}$ restructured the whole into two unequal added parts:

- ${}^{3}/_{8}$ plus ${}^{5}/_{8}$, and
- ${}^{7}/_{8}$ plus ${}^{1}/_{8}$

The concept of unit forming enabled the description of improper fractions as mixed numbers. A mixed number is made up of two unequal parts added together, a whole number part and a fractional part. This premise appeared to underpin Tom's explanation for the fraction pair $^2/_4$ and $^4/_2$, "this is a whole and a half – no two wholes".

In the fraction pair $^5/_8$ and $^1/_2$ it was possible to use unit forming to think of $^5/_8$ as made of two unequal pieces, $^1/_2$ and another piece. Rose used *quarters* to indicate pieces in her explanation of why $^5/_8$ was the larger fraction, "Because this [5/8] is like three quarters out of eight and this is only half; and so it's one quarter extra." Of course it was possible to use partitioning and equivalence to see $^5/_8$ as $^4/_8$ plus $^1/_8$ (benchmarking), which some other students did.

Although unit-forming can be seen as one aspect of the part-whole sub-construct, as a term for a single concept, it can categorise some classroom fraction activities and highlight the correct additive thinking required in a co-ordinated understanding of fractions. For example, the place value approach to decimals (whole numbers plus tenths plus hundredths plus thousandths) was reported as beneficial for students (Desmet, Gregoire, & Mussolin, 2010; Roche & Clarke, 2011; Steinle & Stacey, 2011). Using a decimat to represent the additive connection between the decimal parts (Roche, 2010, 2011) could be described as using unit-forming understanding.

In another classroom activity, Colour in Fractions, children roll two dice (one with a numeral for the numerator and one with $^*/x$ (x was 2, 3, 4, 6, 8, or 12) for the denominator. They then coloured in the resulting area on a fraction wall (Clarke et al., 2008; Clarke & Roche, 2010). The roll of the dice enabled improper and proper fractions. Equivalence was encountered additively: four one-eighths was as much as one half. The activity was framed as exploring equivalence and improper fractions but could also be classified as using unit-forming and equivalence concepts. For example having rolled $^3/_4$, a child can colour in three one-quarter pieces; a half, a sixth and a twelfth; or any combination of fraction wall pieces that add up to the roll of $^3/_4$. The child also records this early addition symbolically. The mathematics is similar to the unit forming activities that Kieren described in his combining space such as making $^3/_4$ out of fraction kit parts (1995).

Tasks such as Construct a Sum (see Figure 5.3, image supplied by Doug Clarke & Anne Roche), developed by the Rational Number Project researchers (Behr et al., 1986; Behr, Wachsmuth, & Post, 1985) where students had to make two fractions that would add to close to but not equal to one (given a choice of specified numerals), were the types of activities that helped students move from an intuitive understanding of unit-forming to a technical-understanding in the measure sub-construct in the context of the relative size of fractions. Results from a later study showed that 25.4% of Grade 6 students created a combination between 0.9 and 1.1 (Clarke et al., 2008, 2011).

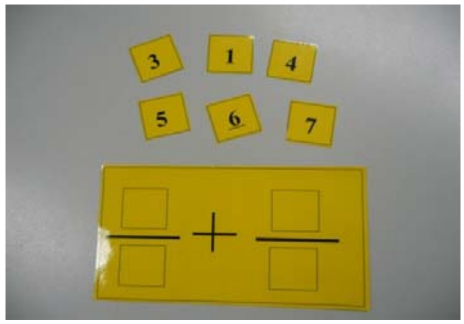


Figure 5.3. Construct a Sum

5.3.2 Quotient sub-construct.

The quotient sub-construct described sharing or fractions as division contexts (e.g., Clarke, 2006; Kieren, 1992; Lamon, 1999). For example three pizzas shared between five people yielded a fractional share.

5.3.2.1 Partitioning concepts drawn on in the quotient sub-construct.

In the Sharing Custard Tarts task (Q. 20), some children elected to draw a diagram of five children sharing three custard tarts. Some used French division, as the Dutch called it (Streefland, 1991), and divided each tart into five equal parts. Some had the partitioning knowledge to name this correctly as three fifths. Other students' physical partitioning was successful but they named the share incorrectly as three fifteenths. Harry's partitioning of circles was less developed and after trying halving and quartering, he said, "there's no way you can cut it up into five pieces." The ability to keep track of the whole when more than one whole was used, which Lamon (1999) called unitising, was connected to the partitioning knowledge called upon in this quotient task. Only one student used the technical symbolic understanding that three shared between five was three fifths.

The present study demonstrated that partitioning and unit-forming were drawn upon in quotient sub-construct tasks. Researchers had advocated the use of the sharing context as an introduction to fractions instead of part-whole (Empson, 1999; Mamede et al., 2005). The Dutch fraction curriculum had an emphasis on fair sharing (Streefland, 1991). The connection between fractions and division was shown to be hampered by the use of part-whole as the dominant form of fraction instruction (Lamberg & Middleton, 2009). In the present study, Kieren's four-three-four model quotient has framed the elaboration of the connection between the quotient sub-construct and the underpinning concepts of partitioning, equivalence (not analysed here), and unit-forming. The model does not advocate a trajectory but has explanatory power of why a quotient sub-construct context can develop early fraction understanding.

5.3.2.2 Unit-forming concepts drawn on in the quotient sub-construct.

Some students started their attempts at sharing using halving. This had been noted in the literature as an early partitioning strategy (Pothier & Sawada, 1983). In the problem of five people and three custard tarts this resulted in five halves with one half left over. Some students then partitioned this left over half into five parts. This process produced five equal *shares* (of a half and a tenth) but not five equal *parts*. Re-unitising was difficult for some students who incorrectly named the share as a half and a fifth. Unit-forming concepts were drawn upon in this (correct) strategy of making a share out of two unequal pieces. This correct strategy was often only partially executed as naming the share correctly as six tenths or three tenths required complex partitioning knowledge.

Confrey (2008) used the term equi-partitioning to encompass sharing problems, such as 15 coins between 3 pirates, or 1 cake between 4 people, which would be classified as quotient tasks in Kieren's five part-model (1980) or four-three-four model (1993). The unit-forming aspect of naming non-equal shares that arise in students' attempts to apportion equal amounts in the quotient sub-construct context is clearer in Kieren's four-three-four model than in Confrey's equi-partitioning model.

Much of the research on fractions as division contexts suggested the inclusion of the quotient sub-construct alongside a part-whole sub-construct (Clarke et al., 2007; Lamon, 2007). The use of partitioning and unit-forming concepts by students in attempting these tasks supports Kieren's classification of these sharing contexts as the quotient sub-construct which draws on partitioning, equivalence, and unit-forming concepts making clear the contribution of the various concepts that make up "part-whole" knowledge.

5.3.3 Ratio sub-construct.

In the design of tasks to investigate ratio, discrete contexts were chosen in the Fraction Sort task (Q. 19f and 19x), in the Golden Beans task (Q. 21a and 21c), and in the Show Me Thirds task (Q. 27). It was assumed that most of the instruction prior to the data collection would have focused on a part-whole interpretation of these types of tasks because the curriculum documents were framed in this construct, however as the present study was not a classroom study no evidence was collected to support this.

5.3.4 Levels of response.

Kieren's four-three-four model differentiated between levels of understanding as well as different constructs (1993). However, his four levels, ethnomathematic, intuitive, technical-symbolic, and axiomatic- deductive, did not fit seamlessly with empirical data and combined approaches were observed (Kieren, 1988).

Kieren had identified an ethnomathematic response to a sharing task as "each gets a bite and Mom puts the rest in the fridge" (1988, p. 172). In the similar Sharing Custard Tarts task (Q. 20b), no Grade 6 student in the present study offered an ethnomathematic response. Most students in the present study operated at the intuitive level using strategies framed by school mathematics. The success of these responses varied and included strategies such as French division, repeated halving, and non-exhaustive sharing (a share is a half and a bit), but all were attempting the task at an intuitive level. One student used the technical-symbolic approach and did not need to tie their understanding of the task to a real context or diagram. Classroom activities such as the Sharing Chocolate Game (Clarke, 2006) are designed to elicit the fractions as division response or technical symbolic level approach.

The Pirie and Kieren model of dynamical learning (1994a, 1994b) with its emphasis on the movement between levels with the notion of folding back, is a better explanatory model than the static four level descriptors in Kieren's four-three-four model. In the present study, folding back, has been a more useful way of elaborating one of the ways that partitioning, unit-forming and equivalence are drawn upon in the measure and quotient sub-constructs, than categorising students responses using the four levels of the four-three-four model. The Pirie Kieren model also has more purchase in the research literature (see e.g. Borgen, 2006; Martin, 2008).

5.3.5 Implications.

A strength of the four-three-four model was its explanatory power to describe the underlying concepts of partitioning, equivalence and unit-forming and link them to the four sub-constructs measure, quotient, operator and ratio. A co-ordinated understanding of "part-whole" in other models would include the three concepts of partitioning, unit-forming and equivalence but not use these terms (see e.g. Behr, Harel, Post, & Lesh, 1992; Lamon, 2007). In that way, the four-three-four model could be seen as an elaboration of the framework used by the Rational Number Project researchers in which the part-whole sub-construct was an underlying construct that developed into the four sub-constructs (Behr et al., 1983). However, it was not substitutable for the five-part model often attributed to the Rational Number Project by other researchers (see e.g., Clark et al., 2003), nor the six-part model (including decimals) in which part-whole was one of six sub-constructs (Behr & Post, 1992).

The adoption of the four-three-four model would not be incompatible with the research already conducted using variations of the five-part model. The four-three-four model reframes the part-whole sub-construct and its relation with the other sub-constructs. Researchers have called for the broadening of teachers' pedagogical repertoire of part-whole instruction (see e.g., Cramer & Wyberg, 2009; Lamberg & Middleton, 2009; Lamon, 2007). However, elaborating the categories of partitioning, equivalence (both already in use) and unit-forming would be one way to do it, while still working in the overall research framework influenced by Kieren (1980) and the Rational Number Project (Behr et al., 1992; Behr et al., 1983).

One way to begin this elaboration of the underlying concepts of partitioning, equivalence and unit-forming would be in the classification of classroom activities. Teachers know that games such as Colour in Fractions (Clarke & Roche, 2010), problems such as Construct a Sum (Behr et al., 1986), and inscriptions such as decimats (Roche, 2010) are the types of activities that develop fraction understanding. Being able to classify the partitioning, equivalence and unit-forming aspects of these activities adds to their pedagogical power. In addition, the concepts could be elaborated through developing teachers' pedagogical content knowledge of strategies for fraction tasks. The interplay of partitioning, equivalence and unit-forming in the misconception of gap thinking provides an example. In addition the distinction between fraction pairs that are compared using partitioning rather than equivalence can help teachers focus on the use of specific aspects of tasks to elicit specific mathematical thinking.

The present study was not a comparative study. I have not evaluated Kieren's model with respect to Steffe's learning trajectories (2002) or Confrey's trajectories (2008) for fraction understanding.

Kieren's four-three-four model for rational number knowing (1988, 1992, 1993, 1995) was an ideal model. Lehrer's eight key concepts of spatial measurement were also an ideal model of measurement understanding. Neither of these models offered hypothetical learning trajectories, unlike Steffe's work on fractions (Steffe, 2002) or Confrey's work on rational number (Confrey, 2008; Nguyen, 2010). The results of the present study, particularly the spread of Equivalence scores from 0 to 12 demonstrated that the underlying concepts are still relevant in upper primary school. However, the four-three-four model does not provide detail of instructional sequences, it can only serve to frame interventions developed by teachers or researchers.

It is not clear from Kieren's explanation of his four-three-four part model (1995) whether the terminology is intended for use in instruction or only for teachers' and researchers' conversations. Anecdotally, the introduction of the Early Numeracy Interview in Victoria was accompanied by the use of terminology such as "counting on" in professional pedagogical conversations and also in instruction, and this led to the children using the terminology in their explanations in class. For example a six year old might explain, I *counted on* from the bigger number. If the names of constructs were used in classrooms, as equivalence and ratio already have been, then this may be a powerful organising conceptual tool for students.

The four-three-four model explicitly links the measure sub-construct with length and area measurement contexts. My elaboration of the measure sub-construct included number line tasks, tasks using non-equal diagrams, and fraction pair comparisons which had precedence in the research literature (Kieren, 1992; Lamon, 2007; Ni, 2000). The associations between the students' Number Line scores, Fraction Pair scores and performance on Part B of the Fraction Pie task, and their performance on measurement tasks illustrated that the conceptual links between the measure sub-construct of fractions and the measurement categories of additivity and units are worth further investigation.

Chapter 6: Conclusions

This study investigated three things:

- What strategies are evident in students' explanations of their thinking in a one-to-one task-based interview?
- Is there an association between performance on measurement tasks and performance on fractions tasks? Is there an association between the use of the use of dynamic imagery on visualisation tasks and performance on fractions tasks?
- Can we use Kieren's four-three-four model of fraction understanding (1988, 1992, 1993, 1995) to describe the fraction understandings of students in the present study?

Data on 88 Grade 6 children's performance on measurement, visualisation, multiplication, and fractions tasks, along with their explanations of the strategies they used to attempt those tasks, were collected using a one-to-one task-based interview. Each interview lasted on average two and a half hours. Notes on record sheets and transcripts from audio and video data supported my interpretive analysis of the students' explanations.

Dimensional sampling was used to select three schools with different socio-economic categorisations. The present study did not have a representative sample and so the results are not generalisable. However, the comparison of the frequency of success on several tasks used in the present study with the frequencies of success of the same or similar tasks in state or national tests (see e.g., Figure 5.1, Table 5.2, Figure 5.2) demonstrated that the sample is not an outlier group. The baseline tasks on which all students were successful (see Figure 4.1) and the ceiling task on which no student was correct (see Figure 4.3) show that all the students had some understanding of unit fractions, the array structure, and simple multiplication (or repeated addition). Similarly errors and related strategies demonstrating misconceptions such as gap thinking was observed in all three schools and, despite only a 12.5% frequency of success on the Array with Leftovers task (Q. 46), there were students who answered correctly in all three schools. The misconceptions and sophisticated strategies were not the result of the individual instruction of one teacher. Therefore the findings of the study have implications for other students in other schools.

6.1 Research Question 1: Strategies

The students' explanations of five strategies in particular, some of which were misconceptions, were examined in depth.

The *same area indicates same perimeter* misconception was not evident in the Fold Me a Quarter task (see Figure 1.2) but was evident in the comparison of the perimeters of two non-congruent halves in a length measurement task. The more sophisticated fraction reasoning, that the areas were the same because they were both half, was successful in the area comparison, but obscured the measurement misconception that perimeter and area are always related.

As identified in the Introduction chapter, observational listening was required to distinguish between the double count misconception and the mathematically correct but only partially executed operator (or unit-forming) approach in the Fraction Pie task (see Figure 1.1). The answer given might imply a particular misconception but asking the students for an explanation revealed that some students were using quite sophisticated strategies which, had the students been able to execute them fully, would have led to a correct answer.

It was not possible in the present study to distinguish between the use of dynamic imagery or geometric reasoning on visualisation tasks because any probing questions positioned the interviewer as using evaluative listening. Students did not seem to have the language to explain dynamic imagery or geometric reasoning.

Gap thinking in fraction pair comparisons was shown to emerge at the same time as early equivalence understanding. Students whose equivalence understanding was strong exhibited no gap thinking, but nor did nearly all of the students who had an Equivalence score of 0. Gap thinking is additive in nature and this suggests that early equivalence understanding may also be additive in nature. Regarding this strategy as a misconception of equivalence (but confirmed by the double counting misconception) may broaden teachers' specialised content knowledge.

The correct strategy of benchmarking could present with an answer and calculational explanation that was initially indistinguishable from the misconception of gap thinking. Both gap thinking and benchmarking generated the correct answer for one fraction pair comparison and the initial explanation sounded similar. It was only after a prompt for further explanation that the differences could be established.

In the introduction, one aspect of the significance of the study that related to this investigation of children's strategies, was the complex task that teachers face responding to individual children and the whole class in a constructivist environment. The analysis of these five strategies only illustrates just how complex that task is. Teachers cannot assume that a correct answer and explanation using one domain (fractions) indicates misconception-free thinking in another (length measurement).

Responsive listening includes responding to the student's strategy (Empson & Jacobs, 2008). For example, if a child has used a unit-forming approach to the Fraction Pie task, but gives the answer (estimate) of one fifth it will be no use if the teacher talks about the double count misconception (which can also generate the answer of one fifth). Instead the teacher has to determine if the child is using unit forming (one sixth (Part A) plus one twelfth is one quarter (Part B)) or operator thinking (two thirds of a quarter is a sixth) and respond to the specific strategy. They must be ever alert that students do not perceive them to be using evaluative listening.

A belief in constructivism also involves anticipating misconceptions. If teachers are to tackle the gap thinking misconception, then they will need to provide experiences of residual fractions that form a strong image for students to fold back to. And it is not just teachers who will increase their knowledge of strategies and misconceptions. If students are to learn through peer conversation then they must establish the classroom norm that calculational answers are only partly acceptable mathematical answers. They will also have to develop their own knowledge of strategies, such as gap thinking and benchmarking so that they recognise when they have equivalent explanations or parallel explanations.

6.2 Research Question 2: Associations

The findings of the present study show that there is an association between fractions and measurement understandings. It is strongest between the measurement categories of additivity and units, and the fraction sub-construct of measure. One aspect of the significance of the present study that relates to this investigation of a conceptual link between fractions and measurement, is the development of curriculum that makes these associations evident. Some research is already underway in this field using fraction strips in length measurement activities to quantify partial units (Yanik et al., 2008). The findings of the present study indicated that it was the conceptual tasks more than the tools and procedures tasks that were co-ordinated in the students' understanding and the curriculum should specify this. For example, broken rulers are a simple addition to the standard curriculum on length measurement. The present study also had some implications for area trajectories. An unexpected result was the substantial association of volunteering formal units in area problems (cm^2) and success on fraction tasks. This suggests that the coordination of formal units might be an extension of array trajectories already developed for younger children.

6.3 Research Question 3: Kieren's four-three-four model

Kieren's four-three-four model was shown to have significant explanatory power in describing students' strategies for tasks in the measure sub-construct. The broadening of the measure sub-construct to include number lines, fraction pair comparisons and area diagrams was possible in this model. The present study was able to analyse in fine detail the way partitioning, equivalence and unit-forming concepts were drawn upon in response to measure sub-construct tasks. At its simplest, partitioning, equivalence and unit-forming just elaborate the part-whole construct. However, the use of these concepts and their connections to all the sub-constructs enabled analysis of students' performance on fraction tasks to be classified in a more informative way than the use of part-whole.

The significance of this finding relates back to the big picture presented in the introduction. If one-to-one task-based interviews are used in Victorian schools as formative assessment tools (see e.g. Department of Education & Training, 2001; Department of Education and Early Childhood Development, 2009b) then clear theoretical frameworks are needed to enable teachers to interpret the data that they collect. Kieren's four-three-four part model is particularly relevant to primary school children's performance and strategy use because it explicitly details both underlying concepts (partitioning, equivalence, and unit-forming) and levels of understanding with the four sub-constructs. The four-three-four part model for fraction understanding does not detail a learning trajectory, and so cannot provide a developmental path, unlike the Early Numeracy Project's growth points for number knowledge (Clarke et al., 2002). However, it could frame teachers' interpretation of a fraction interview, and could be used to extend teachers' pedagogical content knowledge of the fractions domain.

Equivalence has been a term in use with both teachers and students and so the concept has a recognised space in the curriculum. A strength of the four-three-four model was its categorisation of unit-forming. This gives prominence to the many correct additive aspects of fraction understanding. The concept can be used to describe students' strategies but also to describe the mathematical focus of a task. Students and teachers both use the word equivalence in classroom conversations. Extending the vocabulary to include partitioning and unit-forming would extend the explanatory power of the model to the students themselves.

6.4 Directions for further research

The present study has opened up a range of possibilities for further research. It was a study framed using observational listening. However, all of the findings could be investigated in a classroom study of responsive listening. Understanding of the strategies, the associations between measurement and fractions and the explanatory power of Kieren’s and other models would benefit a classroom based longitudinal study. Some research possibilities are given below.

- A classroom study of gap thinking
    - Thinking of gap thinking as a misconception of equivalence,
    - Focusing on the correct contexts for additive strategies: pairs with the same denominator between 0 and 1, and equivalences to a half,
    - distinguishing between double counting and correct partitioning strategies in fraction area diagrams,
    - including a longitudinal component to track gap thinking and equivalence knowledge.

- Evaluation of another model for fraction understanding. For example, the present study did not compare Kieren’s model (1995) to Steffe (2002) to determine if this had the explanatory power to describe the data found in the present study.

- Investigating the volunteering of cm^2 as a unit for area measurement and linking this to models of the understanding of arrays.

- A classroom study investigating the effects of the students using the terminology of Kieren's four-three-four part model.
    - Linking partitioning and fraction size comparisons,

- Naming unit-forming activities,
- Introducing additive (as much as) equivalence and ratio (as many as) equivalence,
- Using measurement contexts such as measuring length with fraction strips (Yanik et al., 2008), the Keyboard task (Q. 39) or broken ruler tasks and linking these to the measure sub-construct,
- Using sharing contexts in early fraction work.

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Appendices

Appendix A: Data Collection Interview

Introduction

This appendix contains the interview script, task cards, and images of materials used in the data collection interview. In this interview protocol:

- *Italic text described what the interviewer did*.
- Plain text was what the interviewer said.
- Students were not told whether their answers were correct or incorrect. But they were usually asked, "and how did you work that out?"
- If students answered "don't know" or did not offer an answer, they were not asked "How did you work that out?", instead they were prompted to elaborate if they could, for example, "If you knew what to do, what do you think you might do?" or "What are you thinking?"
- Students were praised and encouraged by being thanked for sharing their thinking, or helping pack up tasks. If students were having obvious difficulty with many questions, while they remained willing to participate, they were given the option of saying "I don't know" to tasks that they felt that they could not attempt.
- All task cards in this appendix are reproduced at a similar size to the actual materials used during data collection. Most were laminated. Task cards for the pen and paper test, Q16a Number line and 16b Number line, and Q38a Draw Your Own Array, were printed on plain paper and were consumable, that is, the students wrote or drew on them.
- The actual script used by the interviewer during the interview did not include masters of task cards, nor references, but these have been added to this reference appendix version for readers of the thesis. These images have not been labelled as figures.
- A prepared record sheet was used during the interview to note students' responses and this follows at the end of the interview tasks. It was slightly larger, but has been reduced to fit the margins of the pages in this thesis.
- The order of the interview tasks was: multiplication and division, four pen and paper measurement tasks, fractions, measurement, geometry and visualisation.

- If tasks were used or adapted from other sources an acknowledgement in italics appears after the question. If there is no acknowledgement, it can be assumed that the task was developed by me in conjunction with other ACU staff.

Materials list

The content of the task cards were developed or adapted by me and many were digitally produced by Rikki Bochow, at Acornweb.com.au, using Adobe Illustrator. I produced tasks Q. 22, Q. 26, Q. 44, Q. 46 and Q. 48 using Microsoft Word 2003 tools. The task cards were printed on coloured paper and cut and laminated as necessary. Paper and pen and a 30cm ruler, marked in cm but not mm, were on the interview table and available if the student chose to use them except in tasks where the interviewer specifically requested that they not be used, for example, Q. 11 Missing Number or Q. 22 Fraction Pairs. Some other objects were used in the tasks and they are detailed below:

- Multiplication and division
  - Tennis balls in a packet of three.
- Fractions
  - Kinder squares (12.5cm x 12.5cm coloured paper squares)
  - Pattern blocks: two yellow hexagons, two red trapeziums, two blue rhombi, one 2cm orange square, one green triangle
  - Figurines of people (6cm high)
  - Golden beans (lima beans spray-painted gold on one side)
  - Doll for tightrope walker (artist's doll with movable joints)
  - String tightrope
- Measurement
  - 93cm streamer
  - Nine 33mm and four 50mm paper clips
  - DVD case (rated G)
  - Cuisenaire rods
  - Sixteen 2cm square, orange pattern blocks
  - Kinder squares (12.5cm x 12.5cm coloured paper squares)
- Spatial visualisation and geometry
  - Flag on stick
  - 2cm wooden cubes and models of Q. 58 Cube Rotations multiple choice answers

- 2cm wooden cubes and model of Q. 60 Blocks

Interview script

The interview begins with the following words by the interviewer:

Are you happy to do some maths with me today?

I am interested in how you think when you are doing maths. I have a whole lot of tasks to do with you here. I won't tell you whether you get an answer right or wrong. But I will probably always say, and how did you work that out? You can tell me what you were thinking while you were working out the problem. Or, sometimes you just know an answer, so then you can explain how you know that you are right. If you change your mind about an answer while you are explaining it, that's fine, you just tell me your new answer.

Some of the questions might be easy. Some might be hard. Some of the things you might not have been taught yet, so just do your best.

When resuming an interview, repeat part or all of the above as necessary.

Multiplication and division

All tasks in the multiplication and division section were from the Victorian Department of Education and Early Childhood Development (DEECD) *Early Numeracy Interview* booklet, (Department of Education & Training, 2001). The task numbering was different to the DEECD interview as some of those tasks were omitted. This interview can be found in Clarke et al., (2002) and can also be downloaded as the mathematics online interview from http://www.eduweb.vic.gov.au/edulibrary/public/teachlearn/student/mathscontinuum/onlineinterviewbooklet.pdf

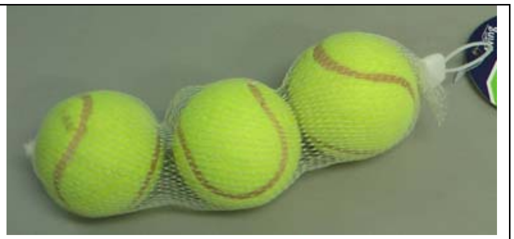
Q. 1 Tennis Ball task**.**

Put out 1 packet of 3 tennis balls.

Here is a packet of tennis balls. How many balls would there be in four packets? How did you work that out?

If the child appears to be counting all, ask

Could you do that another way, without counting them one by one?



Q. 2 Dots Array task.

Here are some dots. *Show card for an instant, in the orientation shown.*

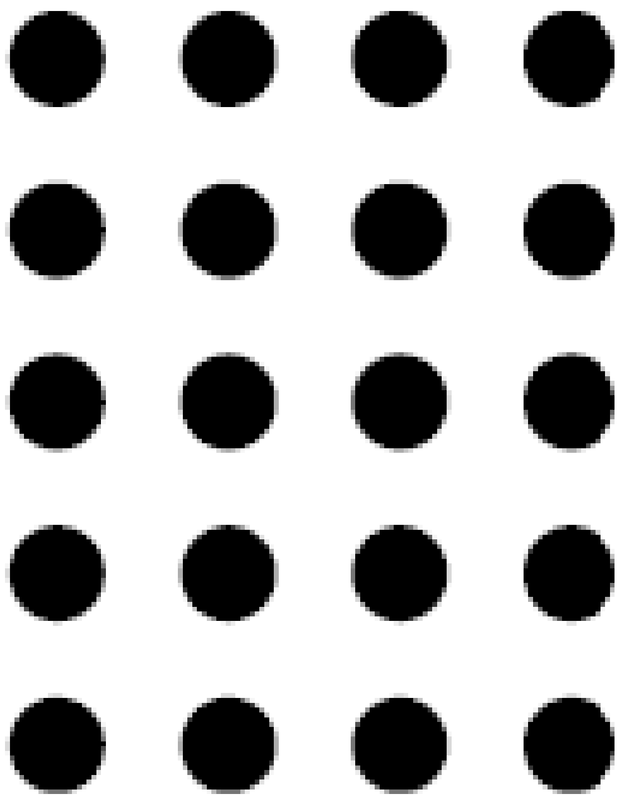
I'm going to hide some.

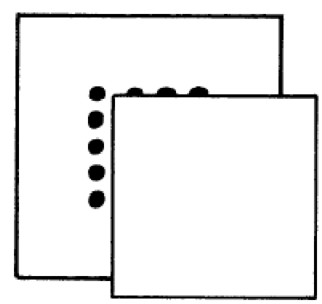
Cover the bottom 4 × 3 section and the bottom half of the 3 dots above it

a) How many dots are there altogether on the whole card? How did you work that out?

If the child appears to be counting all, ask:

b) Could you do that another way, without counting them one by one?





Q. 3 Teddy Cars.

Not offered.

Q. 4. Sharing Teddies on the Mats.

Not offered.

Q. 5 Children at the Movies.

There are fifteen children altogether at the movies. They are sitting in three equal rows. How many children are in each row? How did you work that out?
Adaption for this data collection, if the student answers 45, prompt with, "there's fifteen altogether."

Q. 6 Multiplication Problems.

Show the child the card 3×10. Establish what the child prefers (e.g., do you say three times ten or do you say three tens) Remove the card.
Tell me the answers to these questions
Read the problems one at a time.

a) 3×*10*

b) 2×*7*

c) 10×*7*

d) 3×50

e) 4×30

f) 5×7

3 x 10

Q. 7 Division Problems.

Show the child the card 16 ÷ 2. Establish what the child prefers (e.g., do you say "sixteen divided by two?" or do you say "sixteen how many twos?" or do you say "how many twos in sixteen?") Remove the card.

Tell me the answers to these questions

Read the problems one at a time.

a)16 ÷ 2

b) 60 ÷ 10

c) 80 ÷ 4

d) 24 ÷ 3

e) 35 ÷ 5

f) 35 ÷ 7

16 ÷ 2

Q. 8 Off to the Circus.

Ninety-seven people are going to the circus. Twenty people can ride in each bus. How many buses will be needed to get all ninety-seven people to the circus?

Q. 9 Sharing Our Money.

Pen and paper methods are acceptable for this task.

Show the child the card $52

Share fifty-two dollars evenly between four people.

How much does each person get? How did you work that out?

$52

Q. 10 In Your Head.

Show the child the card with the expression 23 × 4.

Please tell me the answer for *23 × 4*.

How did you work that out?

23 x 4

Q. 11 Missing Number.

Show the child the orange card with 54 × __ = __ __ 2

a) The answer to fifty-four times something ends in 2.What can you tell me about this missing number? …Pointing to the space after the multiplication sign.

How did you work that out?

b) Could it be any other number? How do you know?

$$54 \times \_\_ = \_\_\ \_\_\ 2$$

Pen and paper measurement tasks

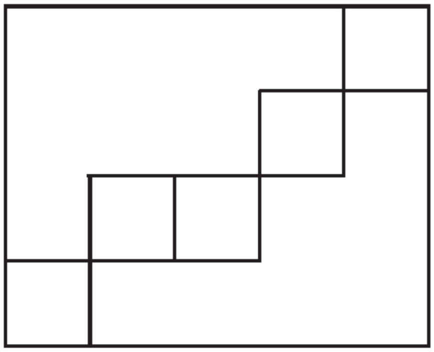
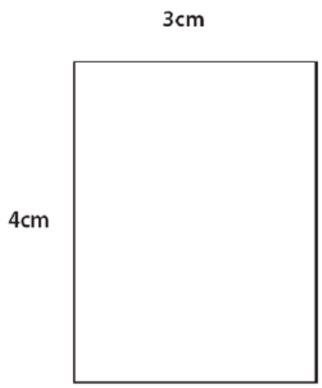
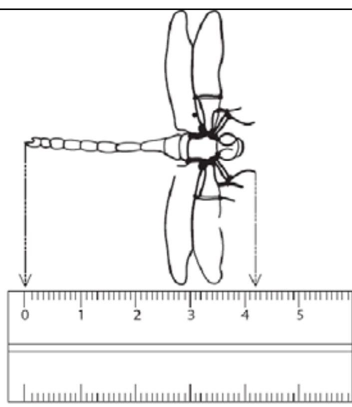
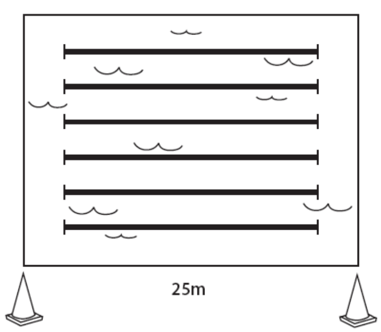
Q. 62 Staircase Array, (Battista, et al., 1998),

Q. 63 Area Calculation of a Rectangle,

Q. 64 Dragonfly,

Q. 65 Witch's Hats. **(**Adapted from Presmeg, 1985).

The pen and paper test on the following page was read aloud if necessary. The tasks were numbered Q. 62 to Q. 65 in my coding, although they were completed after Q. 11 in the multiplication and division section, and before Q. 12 in the fractions section.

	The area of the big rectangle is being measured using 1cm square tiles. But there are not enough tiles to cover the whole shape. What is the area of the big rectangle?
	What is the area of this rectangle?
	This ruler measures in centimetres. What is the length of the dragonfly?
	A child put a row of witch's hats along the side of a swimming pool. The pool was 25 metres long. The child put one witch's hat at each end of the pool, and one every five metres in between. How many witch's hats did he put out altogether?

Fractions

Q. 12 Book worms.

Place pictures of four bookworms and a pile of books in front of child.

This bookworm ...*point to third bookworm*…eats twice as much as this bookworm…*point to smallest*

This one….*point to second*…eats three times as much as this one....*point to the smallest.*

This one….*point to first*…eats six times as much as this one....*point to the smallest*

This bookworm…*point to third*…eats twice as much because it is twice as tall.

And this one…*point to second*…eats three times as much because it is three times as tall.

And this one…*point to first*…eats six times as much because it is six times as tall.

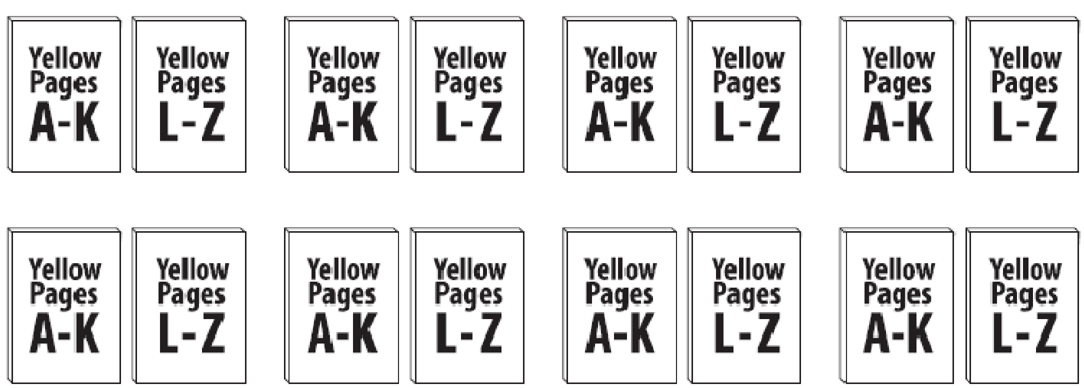
a) If this one eats one book...*place one book in front of smallest*…how many books do the other bookworms eat? *If necessary, repeat introduction…*

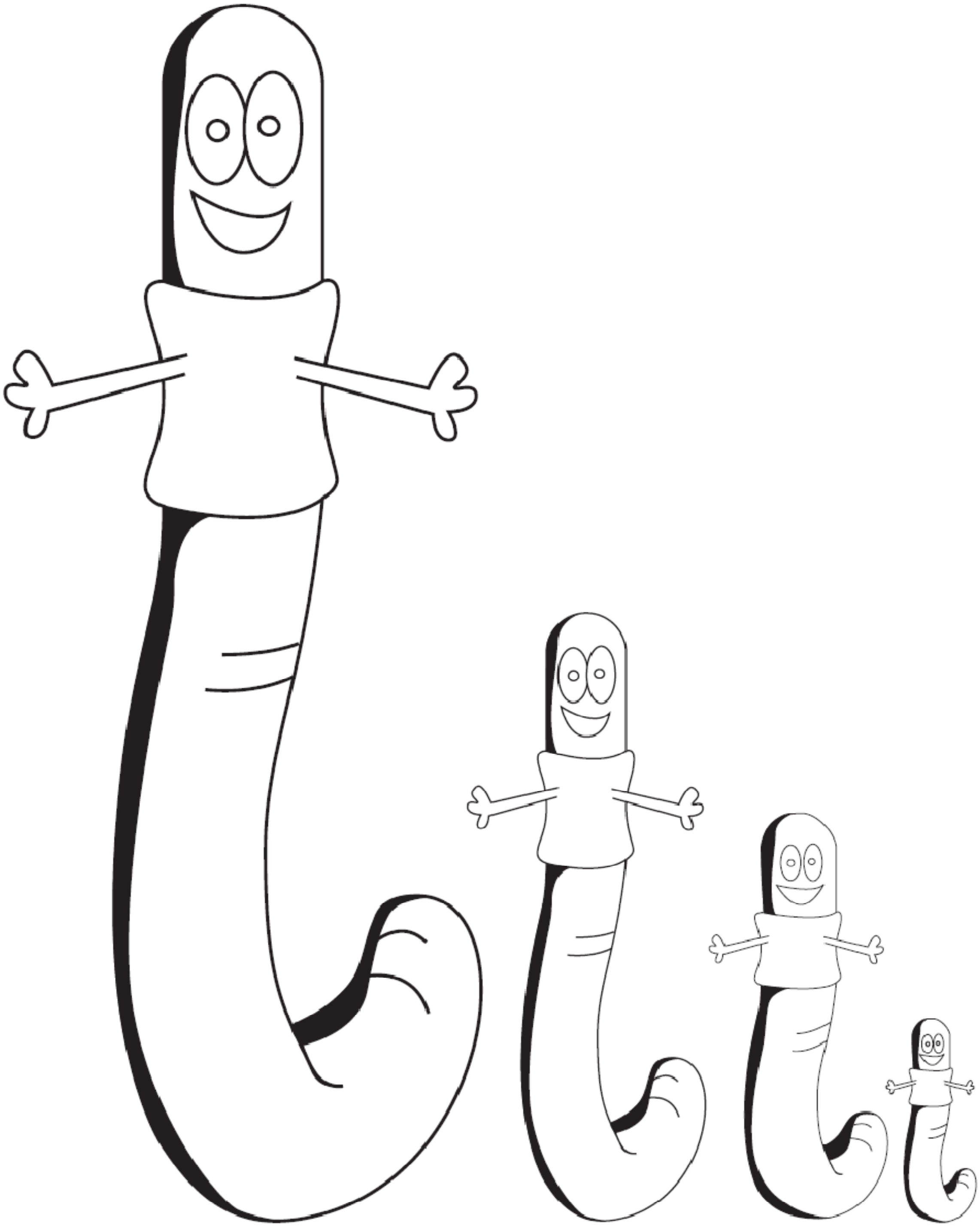
How did you work that out?

b) If this one eats four books…*place 4 books in front of third*…how many books do the other bookworms eat? How did you work that out?

c) If this one eats this many books…*place 9 books in random arrangement in front of second bookworm*…how many books do the other bookworms eat? How did you work that out?

Adapted from Clark and Kamii (1996)





Q. 13 Fold Me a Quarter.

Hand the child one square piece of paper (kinder square).

a) Please fold the square into quarters.

Hand the child a second square piece of paper.

Please fold this into quarters a different way.

b) Let's look at these two parts *indicate a quarter on each of the child's folded pieces of paper.* What can you tell me about the area of these two pieces? *Point to non-congruent quarters.* Does one piece have a larger area than the other or are they the same?

How did you work that out?

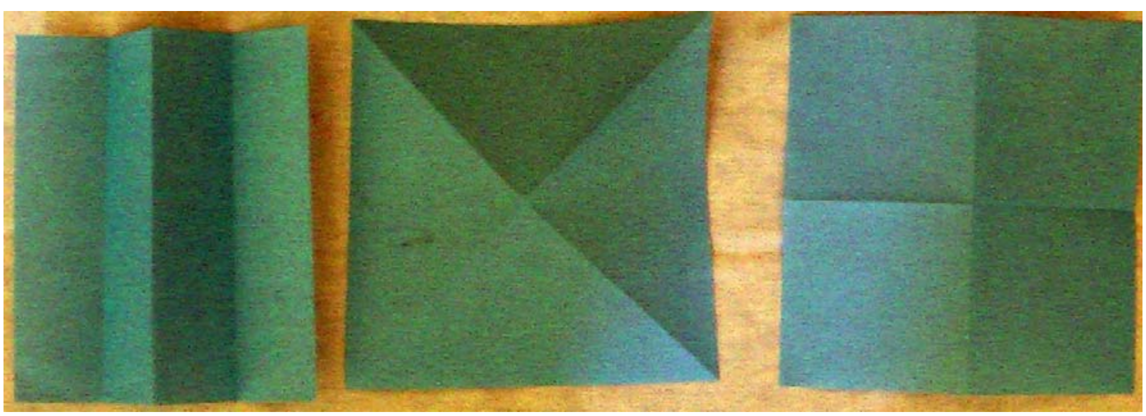
c) This is another square that has been folded. *Show child third kinder square, folded into quarters a different way (squares, triangles, or sticks prepared earlier).*

What can you tell me about the area of these pieces? *Point to a quarter on their square and a shaded quarter on third prepared kinder square.*

How did you work that out?

Repeat with other comparison – their second quarter and the third prepared kinder square.

Developed by Anne Roche and Doug Clarke, Australian Catholic University (ACU).



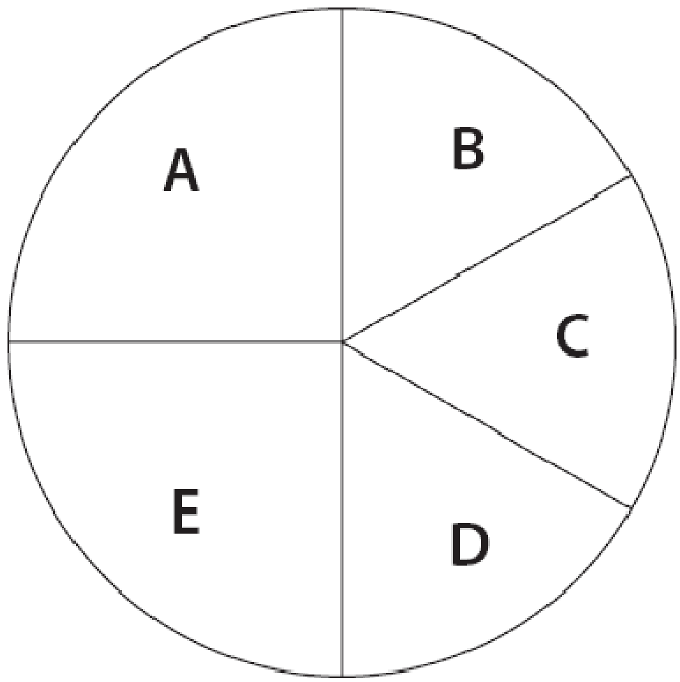
Q. 14 Fraction Pie.

Show the child the fraction pie diagram. Point to the region as it is named in the question.

a) What fraction of the circle is part A? How did you work that out?

b) What fraction of the circle is part B? How did you work that out?

Adapted from fraction assessment interview in Cramer et al. (1997).



Q. 15 Tightrope Walker.

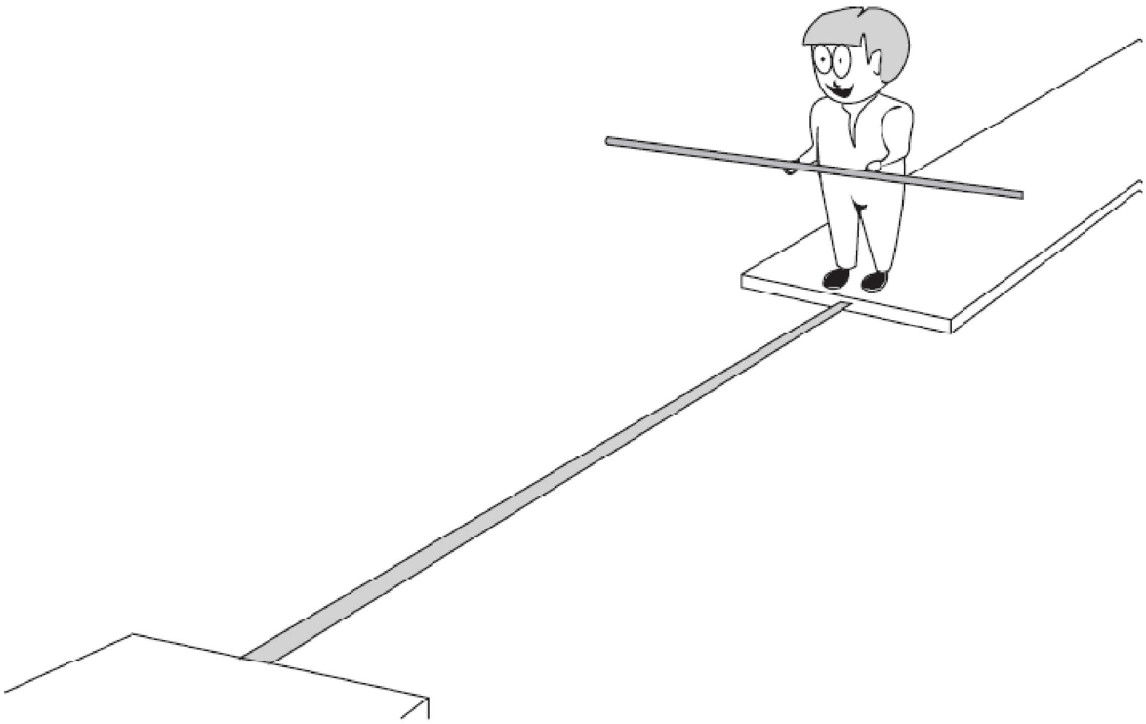
Place pegs and tightrope walker in front of child, and show picture of tightrope walker. Use gender of child in asking question- he/she

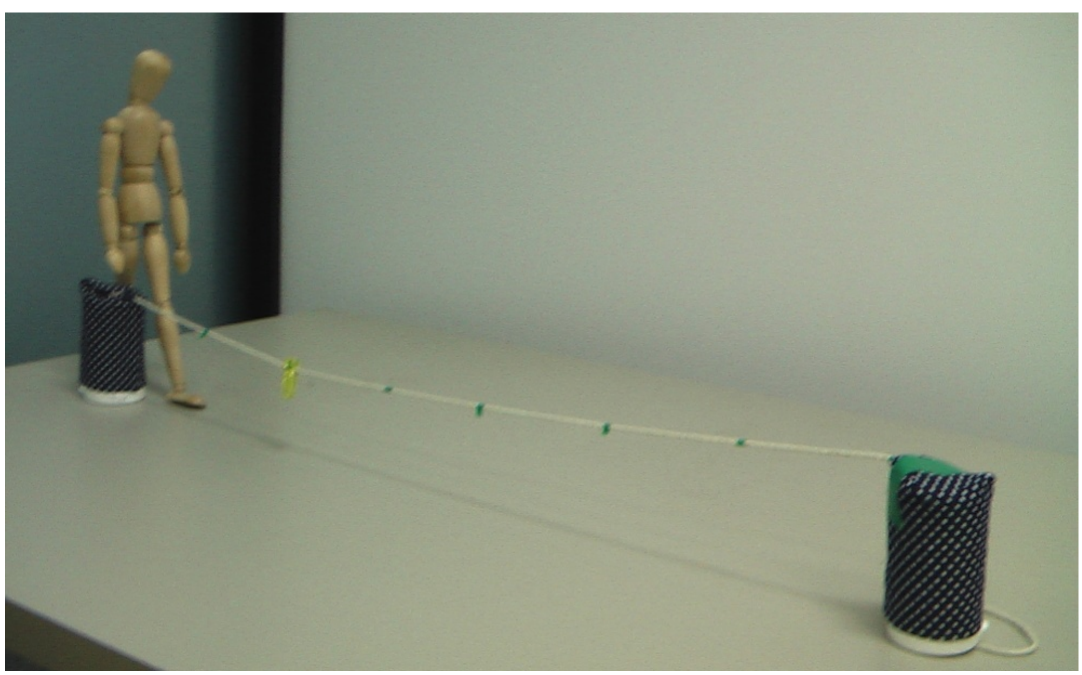
This is a tightrope walker. He/she uses this rope to practise. Because he/she falls off, he/she has put some marks along the rope so that he/she knows how far along he/she got before falling off.

a) Please put a peg on the rope to show where half way across would be. How did you work that out?

b) Roughly, where would nine-tenths of the way across be?

c) *Indicate second mark* If he/she fell off here, how far across would he/she have got?





Q. 16a Number Line.

Give the child a blank piece of paper and pen.

Please draw a number line and mark two thirds on it.

If child does not mark 0 or 1, ask where does zero go? … Where does one go?

How did you work that out?

Clarke et al. (2007).

Q. 16b Number Line.

Place number line B (0, ½ marked) in front of child

If this is half, point to half, please mark where one and a half would be.

If child does not label fraction say, please label it one and a half.

How did you work that out?

Bright et al. (1988).

Q. 16c Number Line.

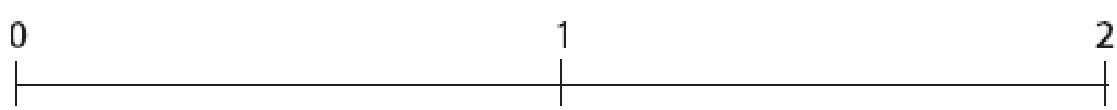
Place number line C (0, 1, 2 marked) in front of child

Please mark where one quarter would be.

If child does not label fraction say, please label it one quarter.

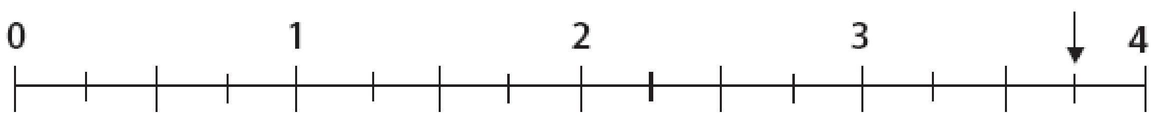
How did you work that out?

Pearn and Stephens (2007).



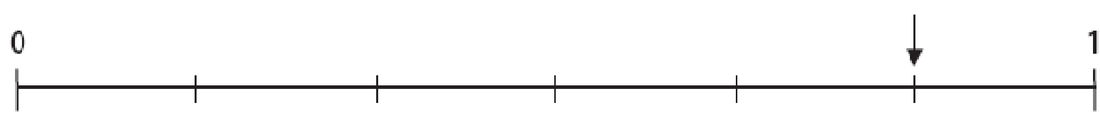
Q. 16d Number Line.

Place number line D (0-4 marked) in front of child. Indicate hash mark pointed to by arrow
What number or fraction is that point on the number line? How did you work that out?
Adapted from Lesh, Landau, and Hamilton (1983).



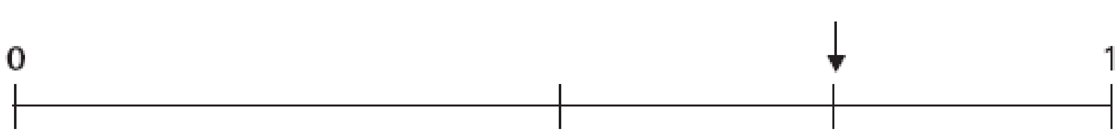
Q. 16e Number Line.

Place number line E (0, 1 marked, with evenly spaced hash marks) in front of child. Indicate hash mark pointed to by arrow
What number or fraction is that point on the number line? How did you work that out?
Adapted from Novillis (1976).



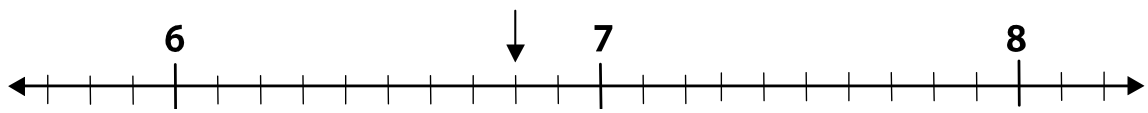
Q. 16f Number Line.

Place number line F (0, 1 marked, with non-evenly spaced hash marks) in front of child. Indicate hash mark pointed to by arrow.
What number or fraction is that point on the number line? How did you work that out?
Adapted from Pearn and Stephens (2007).



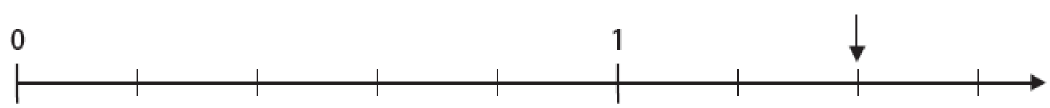
Q. 16g Number Line.

Place number line G (6-8 marked) in front of child. Indicate hash mark pointed to by arrow.
What number or fraction is that point on the number line? How did you work that out?
Ministry of Education (2007) (New Zealand).



Q. 16h Number Line.

Place number line H (0-1 and beyond marked in even spacings with arrow on improper fraction) in front of child.
What number or fraction is that point on the number line? How did you work that out?



Q. 17 Pattern Blocks.

Place pattern blocks on the table (2 yellow hexagons, 1 red trapezium, 2 blue rhombi, 1 green triangle, and 1 orange 2cm square).

You can move the blocks and pick them up if you need to.

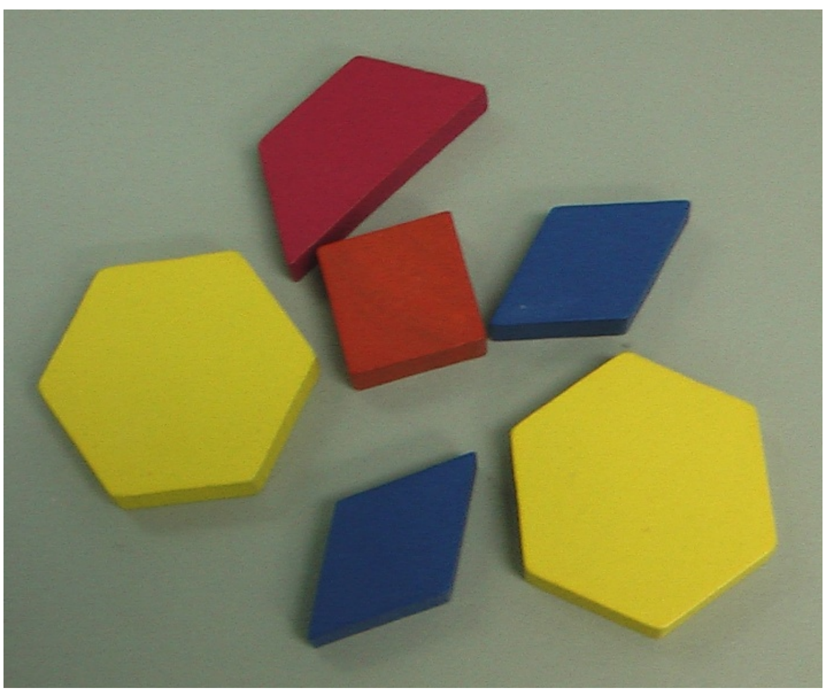
a) A blue is what fraction of a yellow? How did you work that out?

b) A blue is what fraction of a red? How did you work that out?

c) This time the blue is a whole. What could you call the red, if the blue is one? How did you work that out?

d) If the green is a half, what would you call the yellow? How did you work that out?

Adapted from tasks developed by Doug Clarke, ACU.



Q. 18. Simple Operators.

Administer verbally, first three to be done in child's head:

a) What is one-half of six?

b) What is two and a half times six?

c) What is two-thirds of nine?

d) What is one third of a half? *If no answer given or child is thinking, ask*

Would you like to try that with pen and paper? How did you work that out?

e) What is one half of a third? *If immediate answer given, ask* How did you work that out? *Otherwise offer paper and pen and then ask* Can you tell me about what you've done?

If diagrammatic methods used for d and e, ask did you think about the other questions –half of six, a fifth of ten, two thirds of nine – as a picture as well? How did you think about them?

Adapted from Clarke et al. (2007).

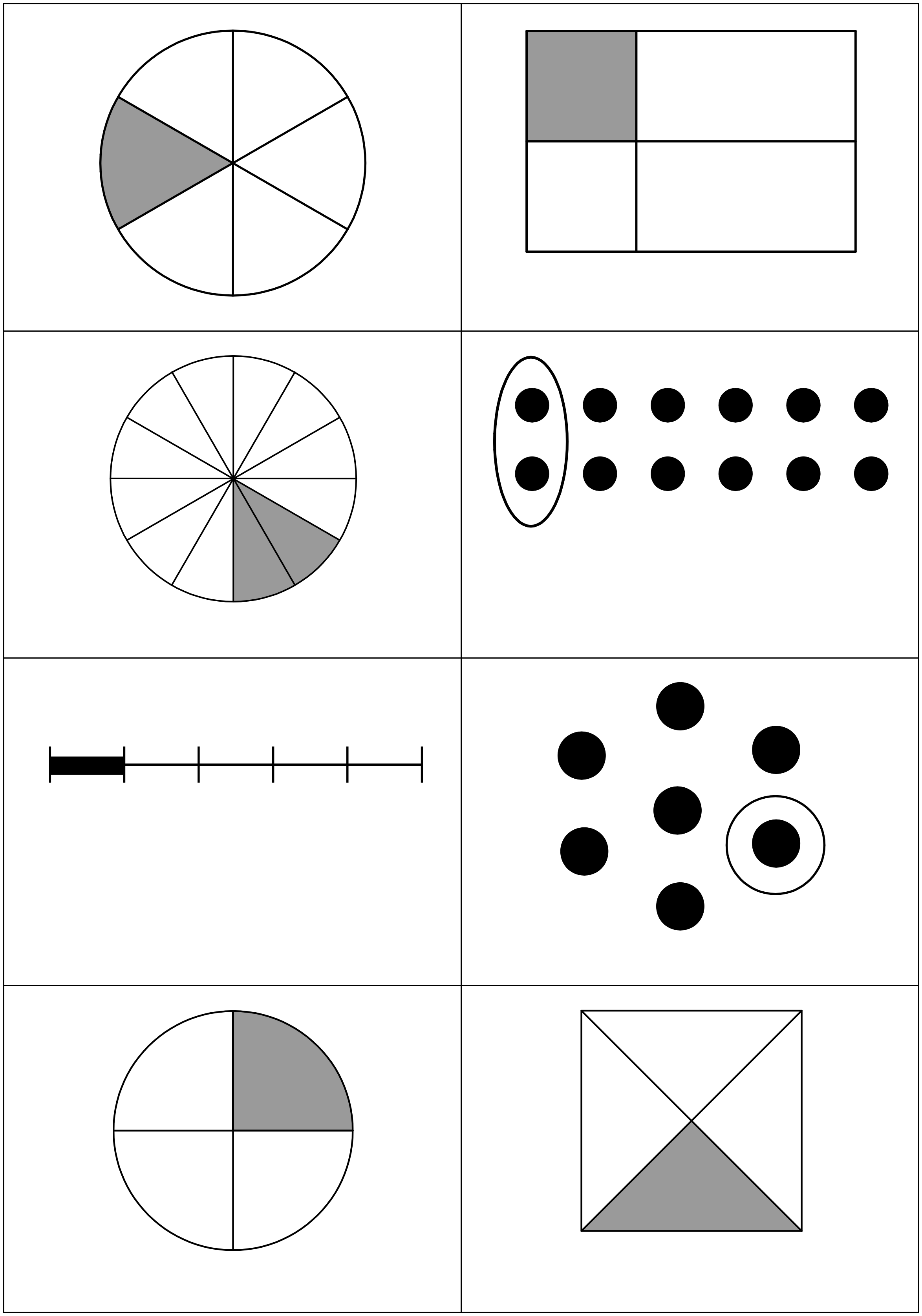
Q. 19 Fraction Sort.

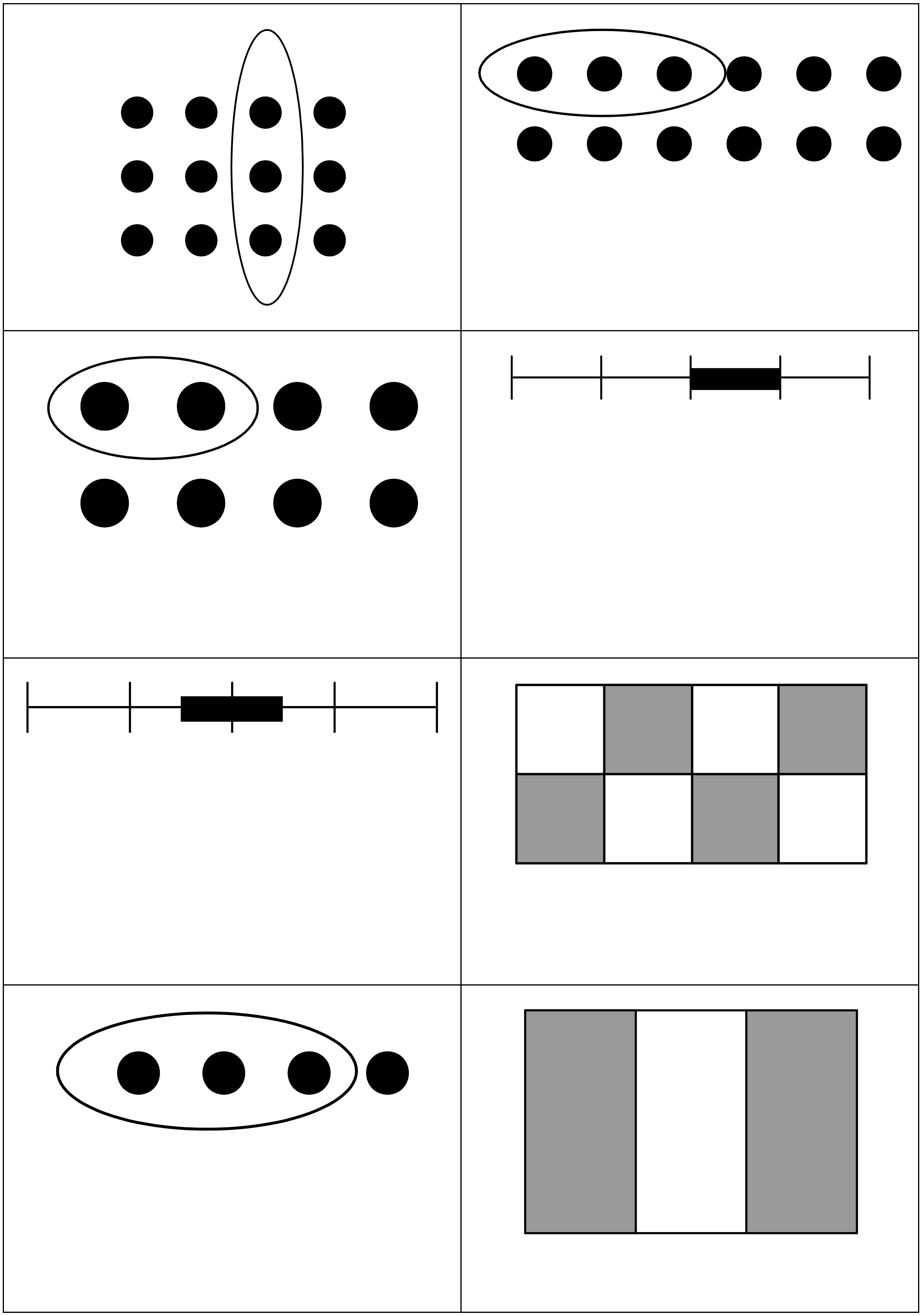
Place cards with "$^1/_4$", "$^1/_6$", "$^2/_3$" and "others", and fraction sort cards, in front of child. Point to cards when saying fraction name. Start with simple circle quarter.

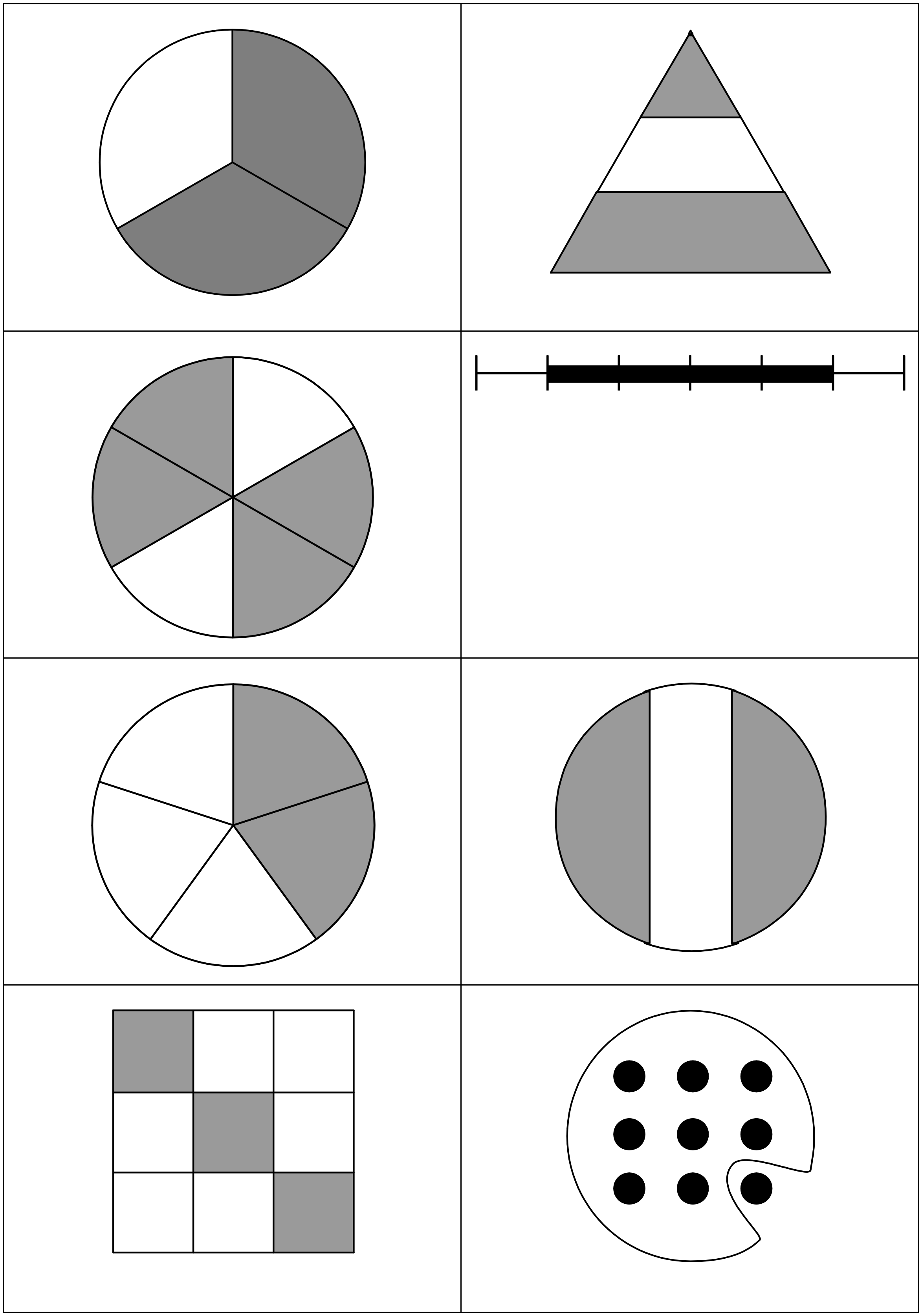
Here are some cards that represent different fractions. Some are one quarter, some are one sixth, some are two thirds and some are none of those so we can call them others *indicate cards.*

Please sort the fractions into the correct group. If the picture isn't one of these fractions, you can put it near "others". Tell me what you're thinking as you go.

Some representations adapted from Baturo (2004); National Center for Educational Statistics (2007)







Q. 20. Sharing Custard Tarts and Liquorice.

a) *Place 5 people and the picture of the 3 pieces of liquorice in front of child.*

Five people are sharing three strips of liquorice equally.

The liquorice can be cut anywhere. How much of a strip does each person get?

Provide the child with pen and paper to draw if necessary.

b*) Place 5 people and the picture of the 3 custard tarts in front of child*

Five people are sharing three custard tarts equally.

The tarts can be cut anywhere. How much custard tart does each person get?

Provide the child with pen and paper to draw if necessary

c) *Place 5 people and the picture of the 7 custard tart in front of child*

This time, seven custard tarts were shared equally between the five children.

How much custard tart does each child get?

Provide the child with pen and paper to draw if necessary.

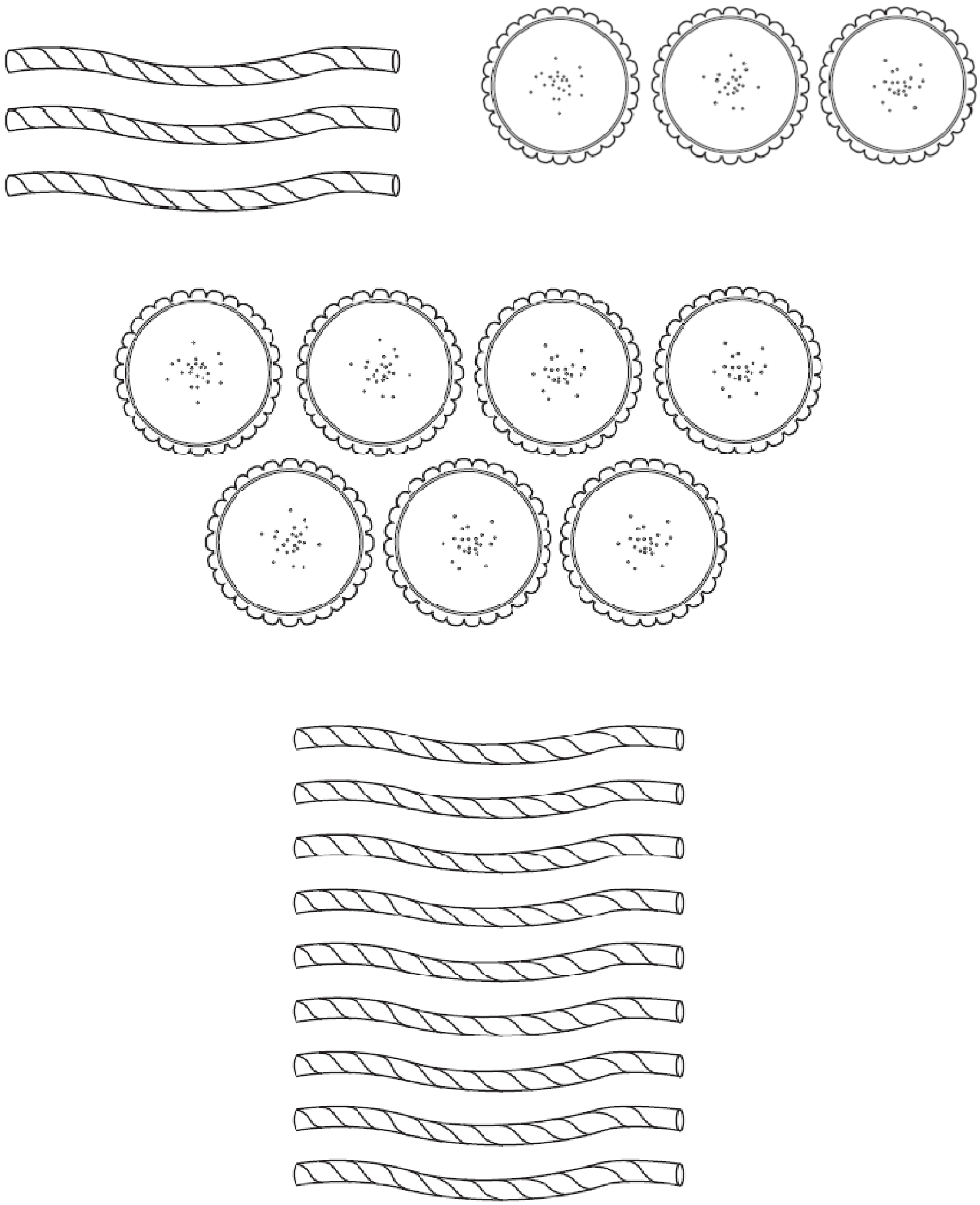
d) Place 4 people and the picture of the 9 pieces of liquorice in front of child.

Now there are only four children. They are sharing nine strips of liquorice equally.

How much of a strip does each child get?

Provide the child with pen and paper to draw if necessary.

Adapted from Clarke et al. (2007). Similar tasks appear in Keijzer and Terwel (2002); Lamon (1999)



Q. 21 Golden Beans.

Place on the table 6 beans that are gold on one side and white on the other

a) Here are six beans that are gold on one side and white on the other *demonstrate as you say this* Please toss the beans like this *(demonstrate)…….*

Now you have a turn. What fraction of the beans have landed gold side up? How did you work that out?

(If 6 white or 6 gold land face up, then ask the child to toss them again)

b) Is there another name for that fraction?

If necessary prompt, Can you think of an equivalent fraction?

c) This time I'm going to add some beans. Please toss the beans again.

Add three beans so that 9 beans are on the table with 3 gold/white and 6 white/gold facing up in a mixed up arrangement.

If nine beans had landed like this, what fraction of the group of beans is gold/white *(the colour of the three beans)?*

How did you work that out?

d) Is there another name for that?

If necessary prompt, Can you think of an equivalent fraction?

How did you work that out?

Task adapted from one developed by Doug Clarke, ACU.

Q. 22 Fraction Pairs.

Show the child each fraction pair card, one at a time, a-g Please point to the larger fraction, or tell me if they are the same……. How did you decide? *Don't allow use of pen and paper* h) *If successful on at least 6 of 7 pairs, place selection of fractions on individual cards in front of child.* These are some of the same fractions as before. Please put them in order from smallest to largest. Tell me about the order that you have put them in. Where would zero go? Where would one go? *Adapted from Clarke et al. (2007).*

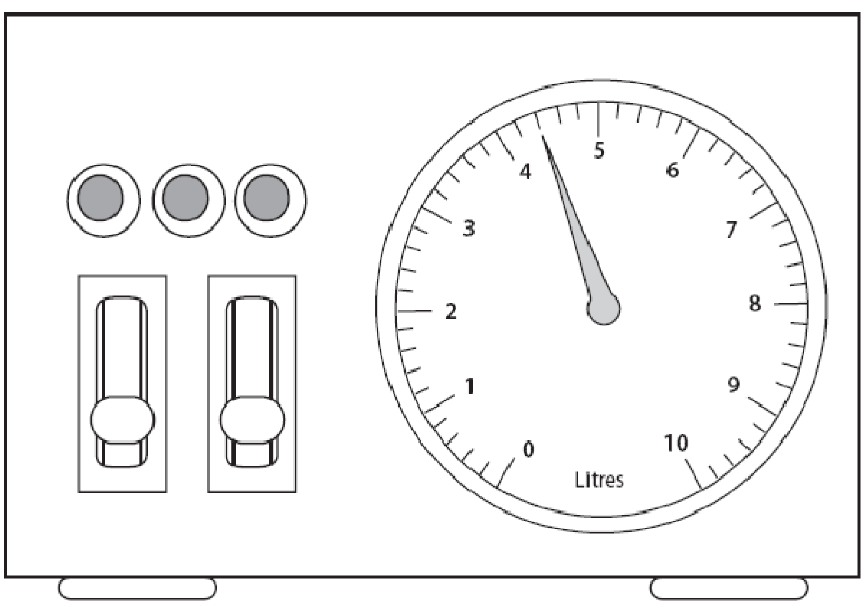
$\frac{3}{8}$ $\frac{7}{8}$	$\frac{4}{5}$ $\frac{4}{7}$
$\frac{2}{4}$ $\frac{4}{8}$	$\frac{3}{7}$ $\frac{5}{8}$
$\frac{1}{2}$ $\frac{5}{8}$	$\frac{5}{6}$ $\frac{7}{8}$

$\frac{2}{4}$ $\frac{4}{2}$	$\frac{7}{9}$

$\frac{3}{8}$	$\frac{7}{8}$	$\frac{2}{4}$	$\frac{1}{2}$	$\frac{4}{2}$	$\frac{3}{7}$

Q. 23 Puff Machine.

Show the child picture of the puff machine This machine measures how much air, in litres, I can blow out of my lungs. What is the reading on the machine?



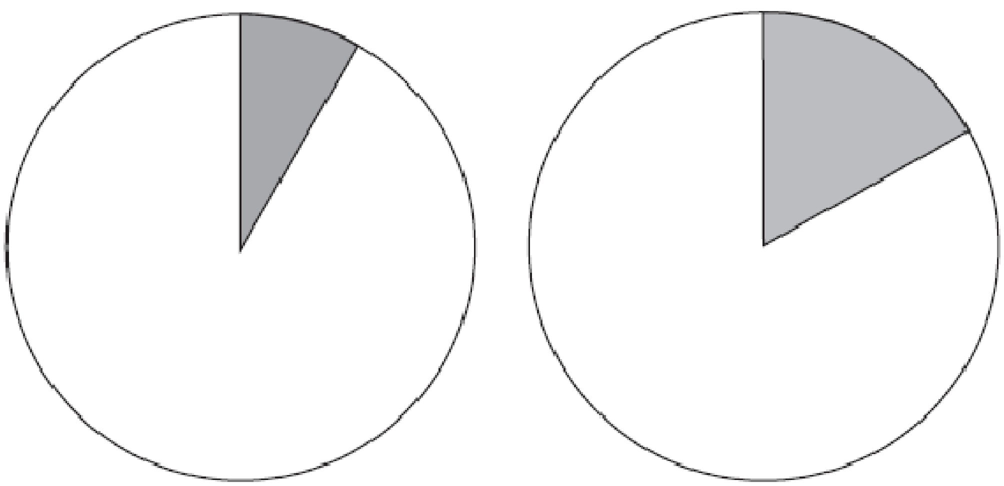
Q. 24 Arc, Angle, Area.

Place card with circle unit fractions in front of child. Point to shaded pieces

a) Which shaded part is bigger? How do you know?

b) I asked some other children how they knew that it was the bigger part. One said, because the angle here *(point to angle)* was bigger. One said, because you could cover this space with more tiles point to shaded parts. And the last one said, because this line is longer *(trace arc of larger shaded part with finger)*. What do you think of their explanations?

c) Did you use any of those ideas to help you decide which shaded part was bigger?



Q.

25. Density.

Place the card 2/5 to 3/5 in front of the child.

a) Is there a fraction between two-fifths and three-fifths? If child says yes, ask, what is it?

If the child says two and a half-fifths ask, What is another name for that?

How did you work that out?

b) Are there any other fractions between two-fifths and three-fifths?

If child answers yes, ask How many are there?

How do you know?

Task developed by Doug Clarke, ACU. Similar tasks in appear in Lamon (1999).

Q. 26 Fraction Algorithms.

Place pen and paper and algorithm cards in front of child.
Here are some fraction problems. These are addition, this is take-away and this is multiplication. Can you work out the answer to these? You can do them in any order you like.

$\frac{3}{4} + \frac{1}{2} =$	
$4\frac{1}{4} - \frac{2}{4} =$	
$\frac{5}{6} + \frac{1}{6} =$	$\frac{1}{3} + \frac{1}{2} =$

Q. 27 Show Me Thirds.

Place picture of cupcake arrays (4) and pen in front of child

These are some cupcakes that have been iced.

a) Without counting them all one by one, can you see half of the tray? Please use the pen to show me the halves.

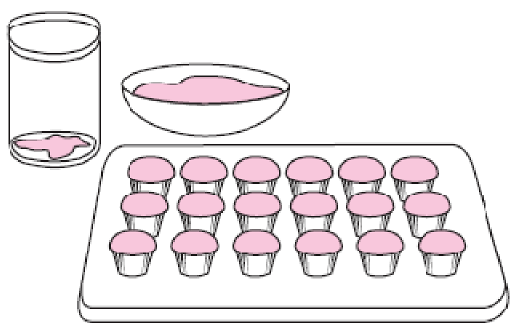
*If necessary….*Can you do it without counting every cupcake?

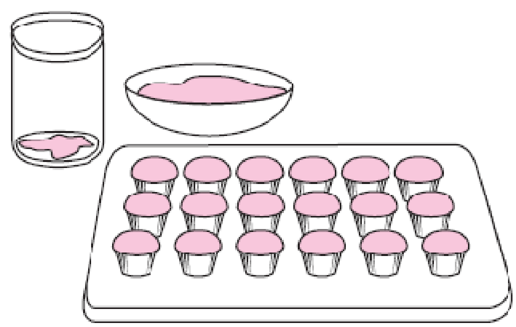
How did you work that out?

b) Can you see thirds? (Use the pen to show me). How did you work that out?

c) Can you see sixths? (Use the pen to show me). How did you work that out?

Adapted from Lamon (2002).



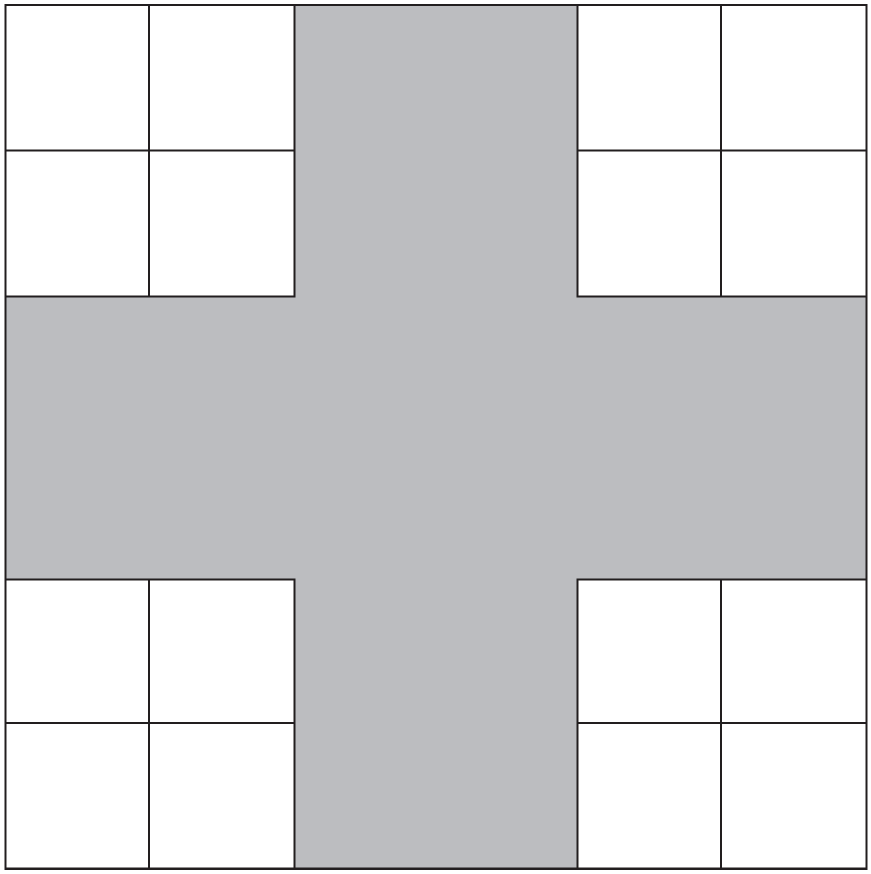


Q. 28 Crossroads.

Place diagram of crossroads in front of the child.

This is the whole. Indicate the whole shape. What fraction of the whole shape is shaded? How did you work that out?

Adapted from Victorian Curriculum and Assessment Authority (2006).



Q. 29 Ending on a Positive: Fraction.

Place Cuisenaire rods in front of child. Show me any fraction you like using these rods. *If child is correct, tell them so. If not, modify task or offer new task until they get one correct.*

Q. 30 Off the Record.

You have done lots of different tasks about fractions today. Which did you like best? Why was that? When you have done fractions in class, what sort of activities have you done? Have you seen any of the questions we did today before? Did you think the tasks we did today were easy or challenging or both? *Reflect back child's experience.*

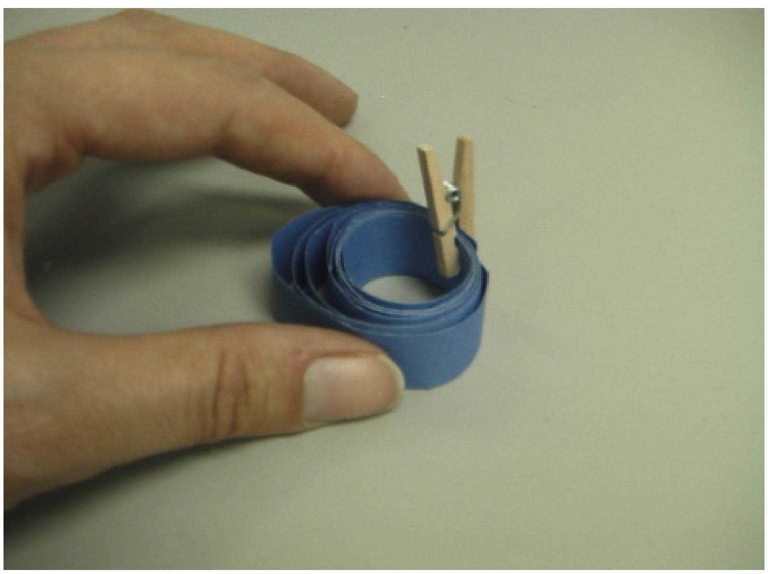
Measurement

This section of the interview is introduced with:

For some of the tasks you might use pen and paper or a ruler.

Q. 31 Streamer.

I tried to cut one meter piece of streamer. Please measure how long the streamer is *(allow the child to use a pen to mark the streamer if necessary, but without prompting)….* What did you find?
If correct number but no units given in answer, prompt for units, eg ninety what?
How far out was I? (*If child is unclear, ask* how far off one meter was I?)
Adapted from the Early Numeracy Interview, Department of Education & Training (2001).



Q. 32. Measure a DVD with a Ruler

(if incorrect on Q. 31).
Here is a DVD. Here is a ruler. Please measure this length of the DVD with the ruler *(indicate longer side).* What did you find?
If correct number but no units given in answer, prompt for units, e.g. nineteen what?
Adapted from the Early Numeracy Interview, Department of Education & Training (2001).

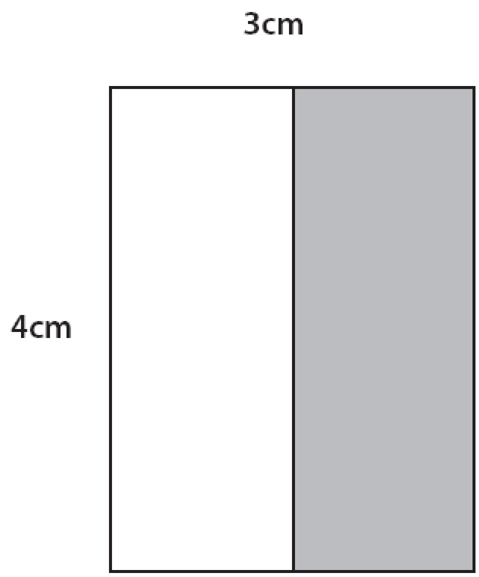
Q. 33 Area Calculation – Half Rectangle.

Place 3 by 4cm rectangle diagram, shaded in halves, in front of child.

What is the area of the shaded part?

If correct number but no units given in answer, prompt for units, e.g. twelve what?

How did you work that out?



Q. 34. Square to Triangle Sequence –Cutting

(if incorrect on Q. 13 and/or Q. 33).

Give child two kinder squares and scissors.

Please cut the square in half… indicate diagonally. Now move the pieces around to make a new shape. Does this square have a bigger area, does this shape have a bigger area or do they have the same area?

How did you work that out?

Task developed by Catherine Trethowan (Watsonia Heights Primary School).

Q. 35 Missing Oval

(if incorrect on Q. 33, or incorrect area units on Q. 62).

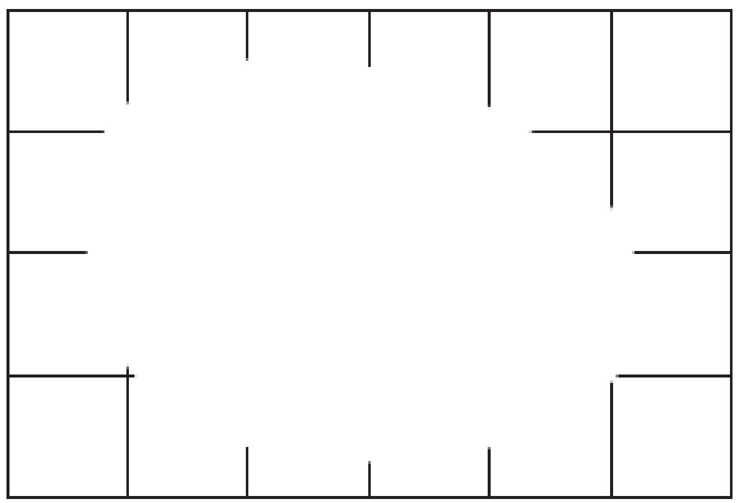
Show array diagram with oval missing grid lines.

This rectangle was covered with one centimetre square tiles. *Point to complete tile* But some of them have been rubbed out. What is the area of this rectangle?

If correct number but no units given in answer, prompt for units, eg 20 what?

How did you work that out?

Battista, et al. (1998).



Q. 36. Similar Shapes.

Show child shape comparison pictures one at a time.

The perimeter is the length around the outside of a shape. *Trace perimeter with finger.*

a) What can you tell me about the perimeter of these two shapes.

If necessary, does one have a larger perimeter than the other or are they the same?

How do you know?

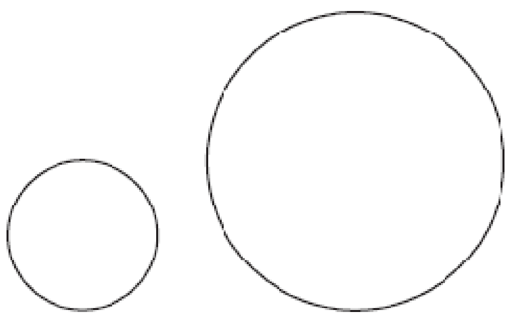
b) The area is how many units fit inside the shape. *Trace inside shape with finger*.

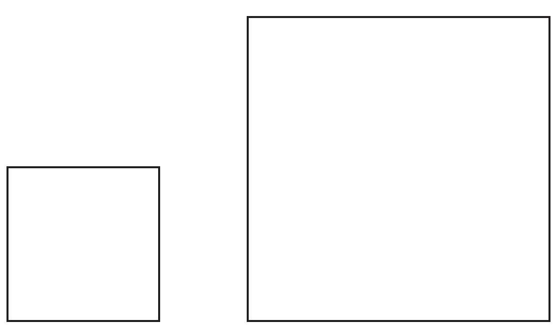
What can you tell me about the area of these two shapes?

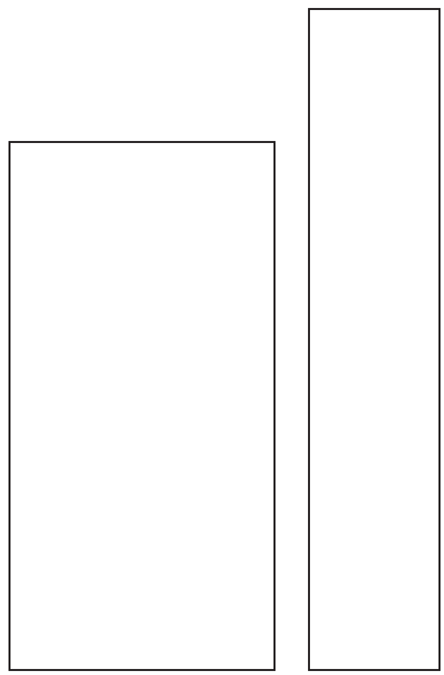
If necessary, does one have a greater area than the other or are they the same?

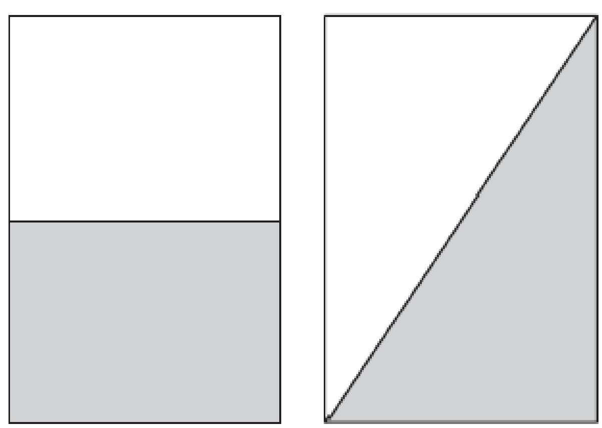
How do you know?

Repeat for each pair: a) and b) square, c) and d) circle, e) and f) rectangles, g) and h) shaded rectangle halved









Q. 37 Four Triangles

(if correct on Q. 36h).

Place task cards in front of child.

Here are four triangles. They are right angle triangles and these lengths are 5cm and this length we don't know (*indicate on diagram*.)

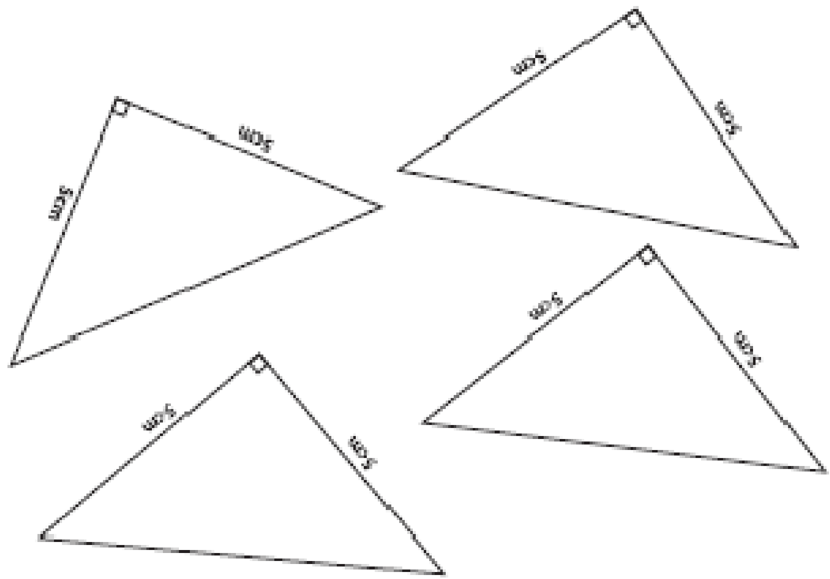
a)These four triangles have been moved to make this shape *(point to rectangle).*

What is the area of this shape? How did you work that out?

b) What is the area of this shape *(point to square)*? How did you work that out?

c) What is the area of this shape *(point to trapezium)*? How did you work that out?

Adapted from National Center for Educational Statistics (2007).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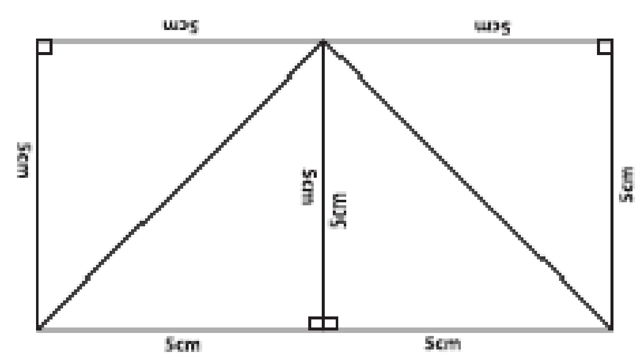
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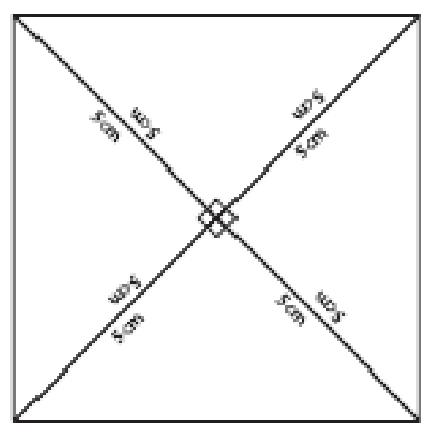
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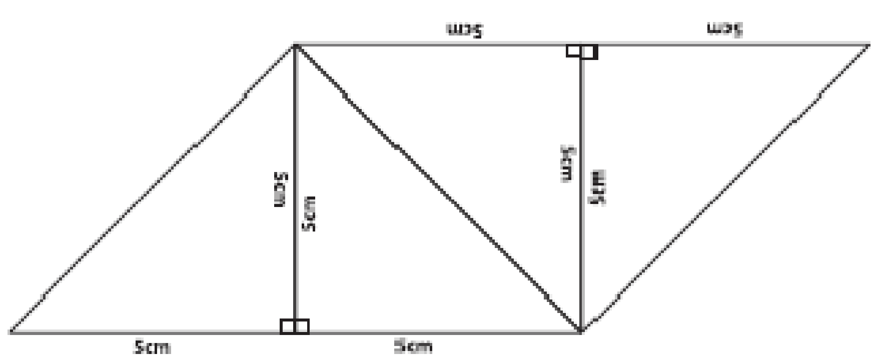
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Q. 38. Draw Your Own Array.

Give child tile and rectangle diagram and show one 2cm square pattern block.

This is one tile, *place block on square tile in diagram*.

a) Please draw the tiles on this rectangle. What is the area of this rectangle?

If correct number but no units given in answer, prompt for units, eg thirty what?

How did you work that out?

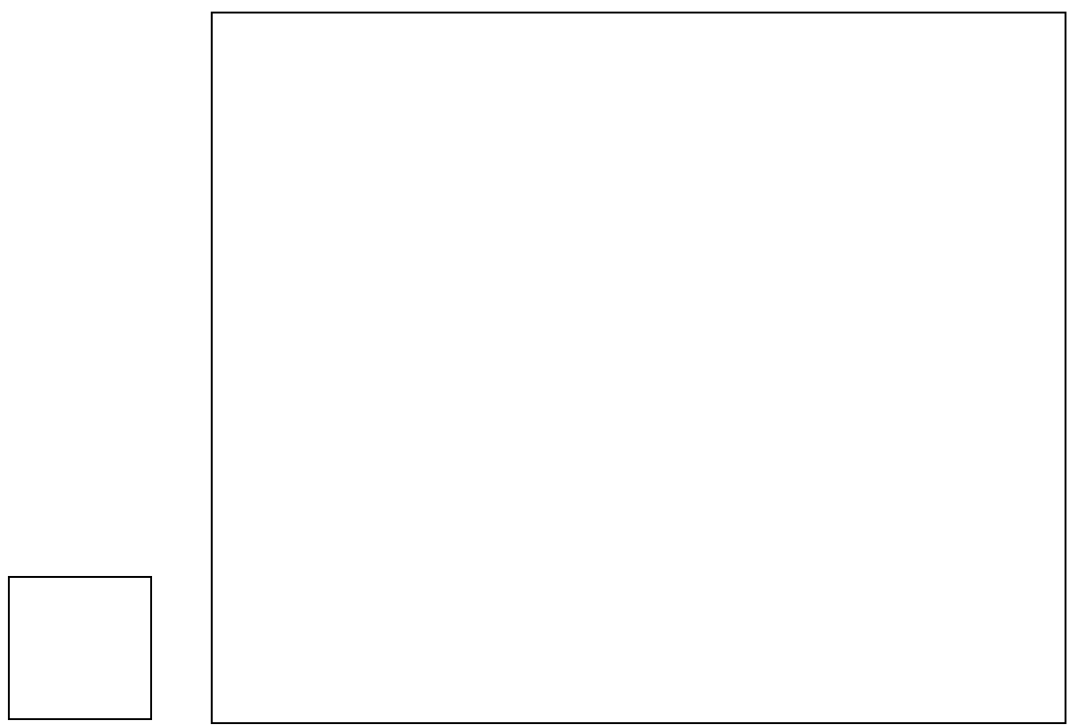
Drawing on diagram is required, ruler permitted

b) *Inverse relationship between size of tile and count (if incorrect on Q. 36h)*

Show child picture of two draw your own array diagrams, one with bigger tile.

This is the same as your diagram. If you used this tile instead, would you need more or less or the same number of tiles? How did you work that out?

Adapted from Outhred and Mitchelmore (2000).



Q. 39 Keyboard.

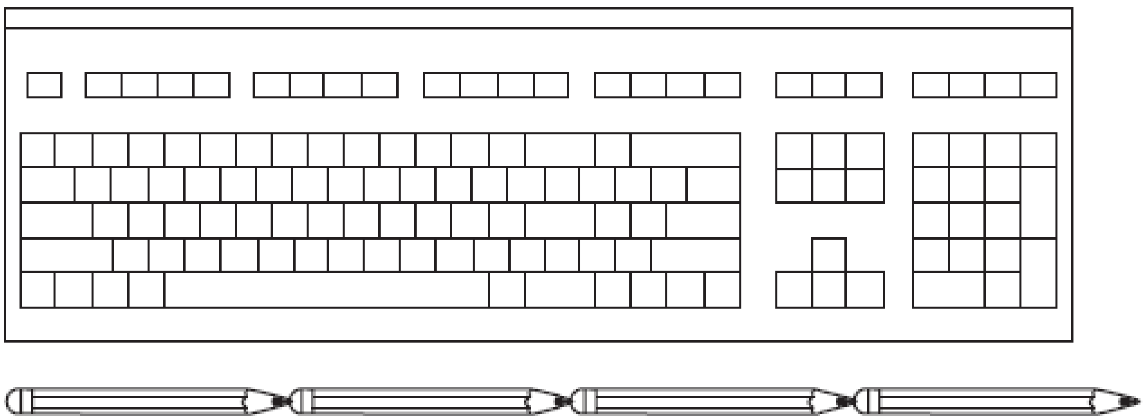
Place picture of keyboard and pencils in front of child.

Some children were measuring things in the classroom using pencils as their units. How long is this keyboard in pencils?

If a whole number given, prompt, e.g. Is it exactly four?

If correct number but no units given in answer, prompt for units, e.g. three and three quarter what?

How did you work that out?



Q. 40. Using Paperclips to Measure

(if incorrect on Q. 39).

Place paper clips (33mm and 50mm) and DVD in front of child.

Please measure the width of the DVD with the paper clips.

(indicate shorter length of DVD).

If child hesitates Use the paper clips to measure the DVD.

What did you find?

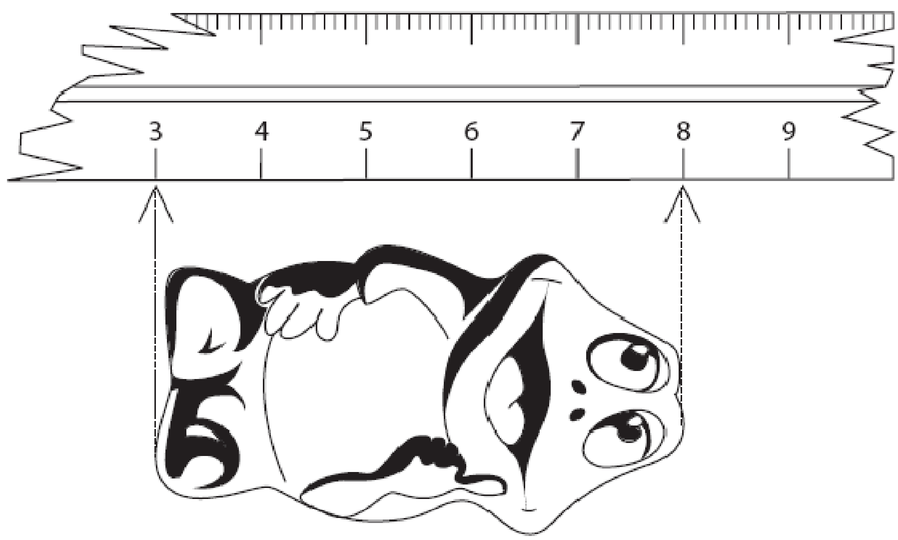
If correct number but no units given in answer, prompt for units, e.g. four what?

How did you work that out?

Adapted from Department of Education & Training (2001).

Q. 41 Freddo Frog.

Place picture of broken ruler measuring Freddo Frog in front of child.
This centimetre ruler is broken. It is measuring a Freddo frog. How long is the Freddo frog? How did you work that out?
Adapted from Bragg and Outhred (2004).
(Permission to use Freddo Frog image in this thesis given by Kraft Foods, see Chapter 3, section 3.2.1.3)



Q. 42 Footy Card

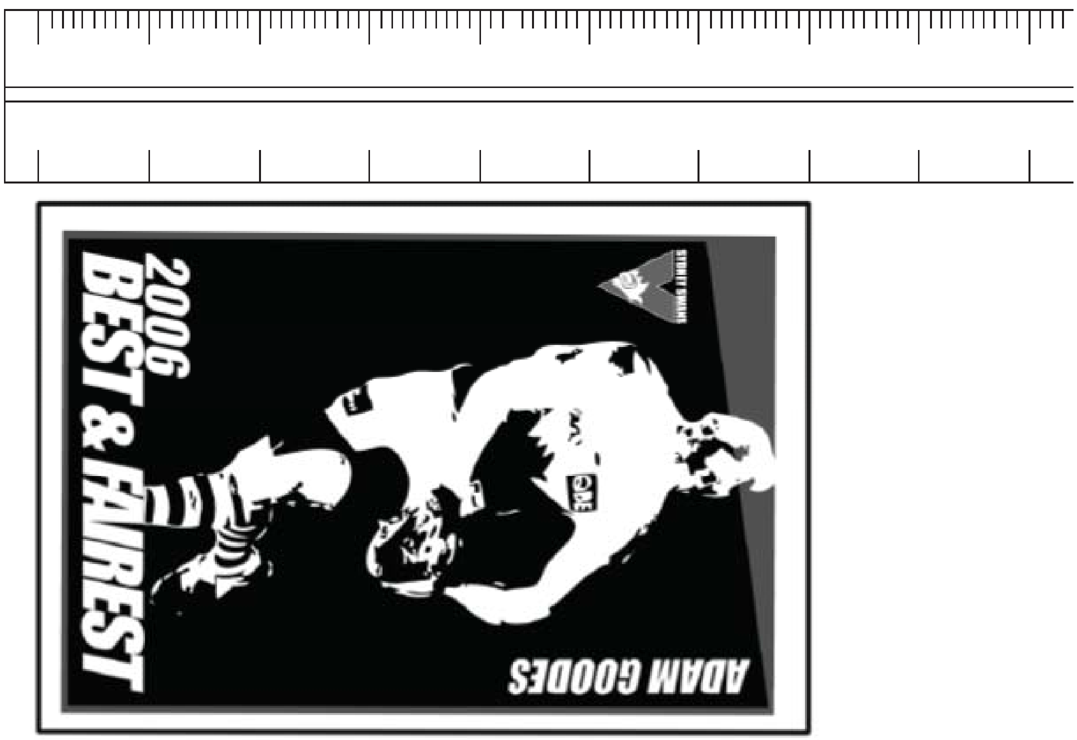
(if correct on Q. 41).

Place picture of ruler with no numerals measuring a footy card in front of child.

This ruler measures in centimetres but there are no numbers on it.

What is the length of the footy card? How did you work that out?

Adapted from Bragg and Outhred (2004).



Q. 43 Straightening Wires

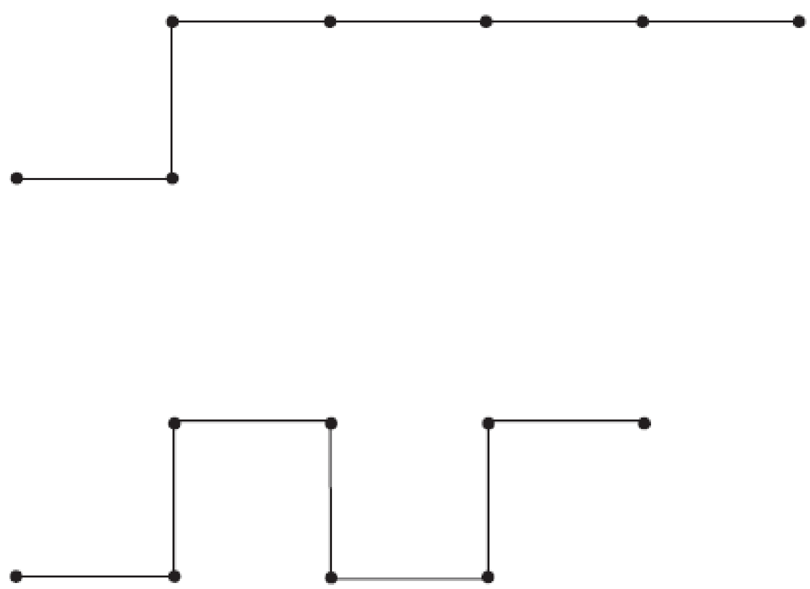
(if incorrect on Q. 41)

Place diagram of two wires in front of child

These are two pieces of wire that can be moved. Between the dots is the same length.

If the wires were straight would they be the same length, or would one be longer than the other? How did you work that out?

Adapted from Battista (2006).



Q. 44 Steps.

Place paces table in front of child
a) Some children were measuring the length of the room by counting how many steps they took. This shows how many steps each child took across the classroom.
Point to name … Jack took 10 steps, Emily took 8 steps, Max took 9 steps, Tim took 7 steps. Who takes the biggest steps? How did you work that out?
Adapted from National Center for Educational Statistics (2007).

Name	**Number of steps**
Jack	10
Emily	8
Max	9
Tim	7

Q. 45 Choosing Rulers

(if correct on Q. 44).

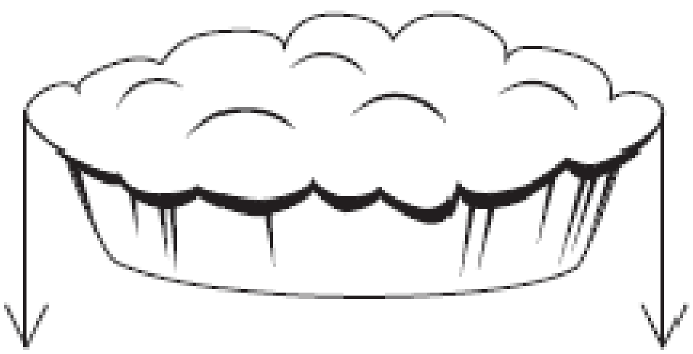
Place rulers and object in front of child.

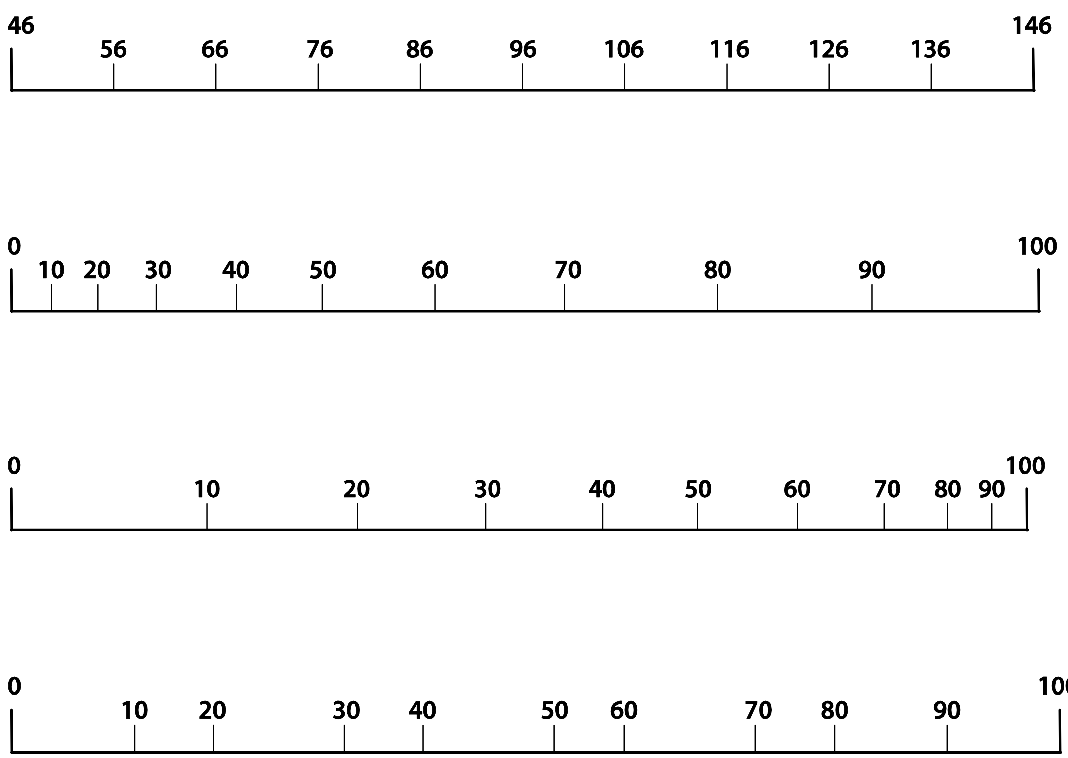
Here are some rulers marked in millimetres.

a) Please choose a ruler that will help you measure the length of the pie. How long is the pie? How did you work that out?

b) Can you use any other of the other rulers to measure? Why/ what did you find?

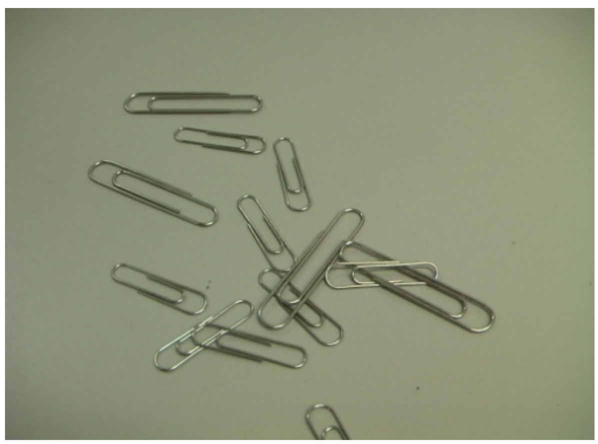
Adapted from Petitto (1990).





Q. 45c Two Sizes of Paperclips.

(if incorrect on Q. 44).
A child was using paperclips to measure the length of this DVD. He/she has these two different sizes. If they used these ones…*indicate group with longer length*... would s/he use more or less or the same number as these …*indicate other pile*.
How did you work that out?



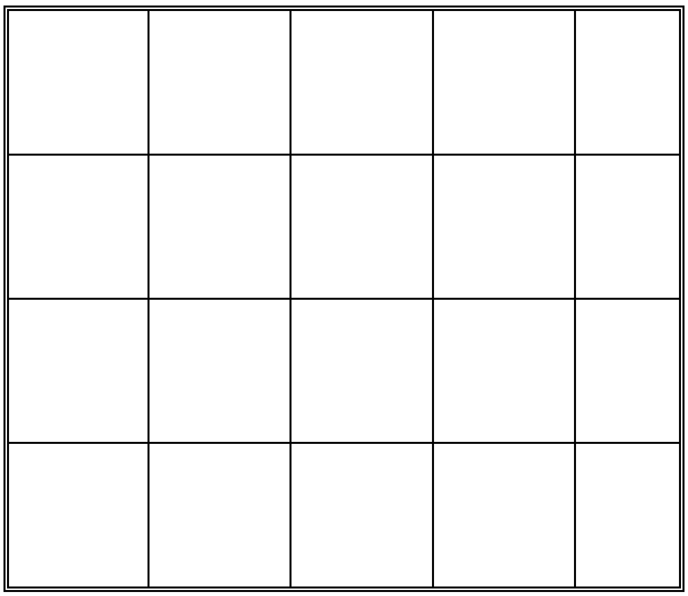
Q. 46 Array with Leftovers.

Place card with array, and 16 orange 2cm square pattern blocks in front of child.

Here is a shape. Trace around outside of shape with finger. Please use the tiles to measure the area of the shape. A tile fits in this square.

Demonstrate that one tile fits in a partitioned square.

What is the area of the shape? How did you work that out?



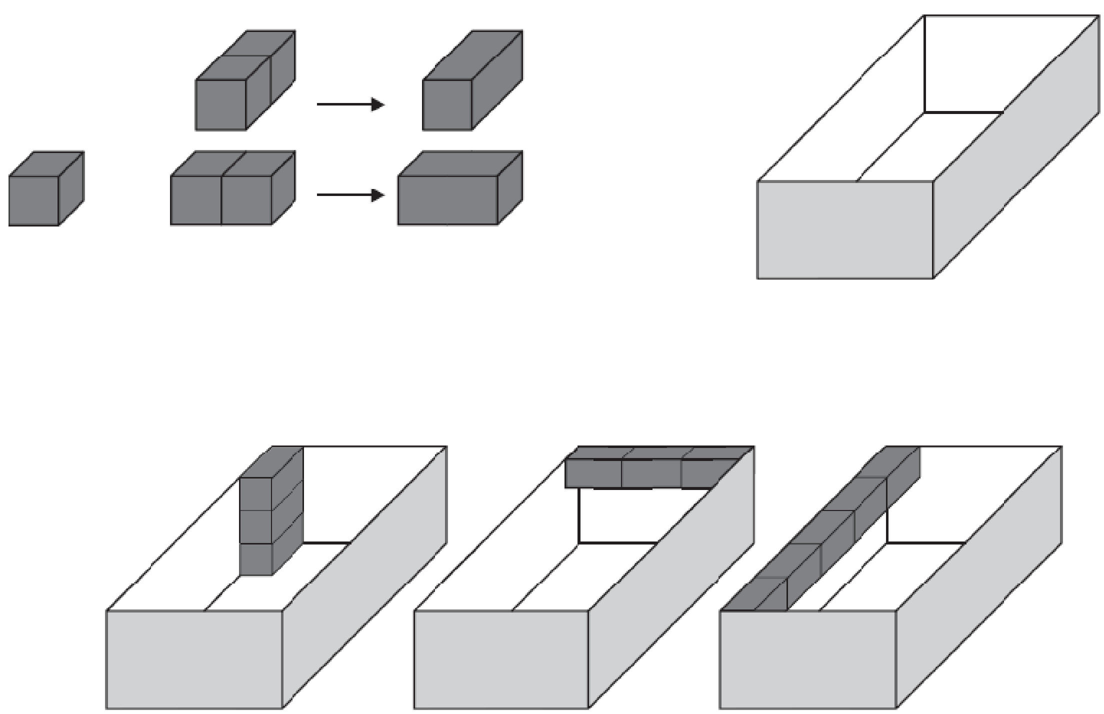
Q. 47. Packing Boxes.

(if correct on Q. 46).

Place diagram in front of child.

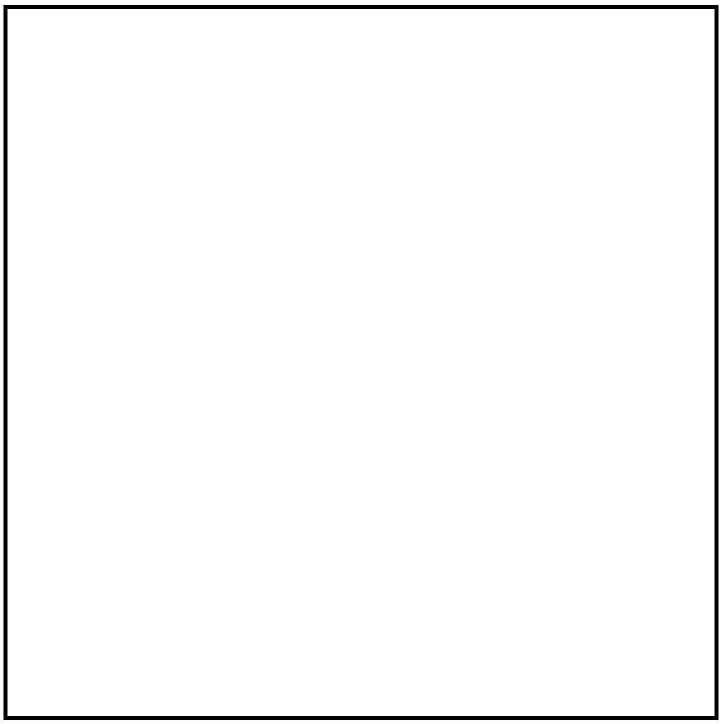
In this cup factory, the cups are always boxed as pairs. So a cup is put in a box, and then two are boxed together like this*….indicate on diagram*. Then those boxes are packed into crates. Three boxes like this fit up the side of the crate. Three boxes like this fit along the end of the crate. And five boxes like this fit down the length of the crate….*indicate on diagram.* How many cups fit into the crate….indicate single box? How did you work that out?

Adapted from Battista (2007).



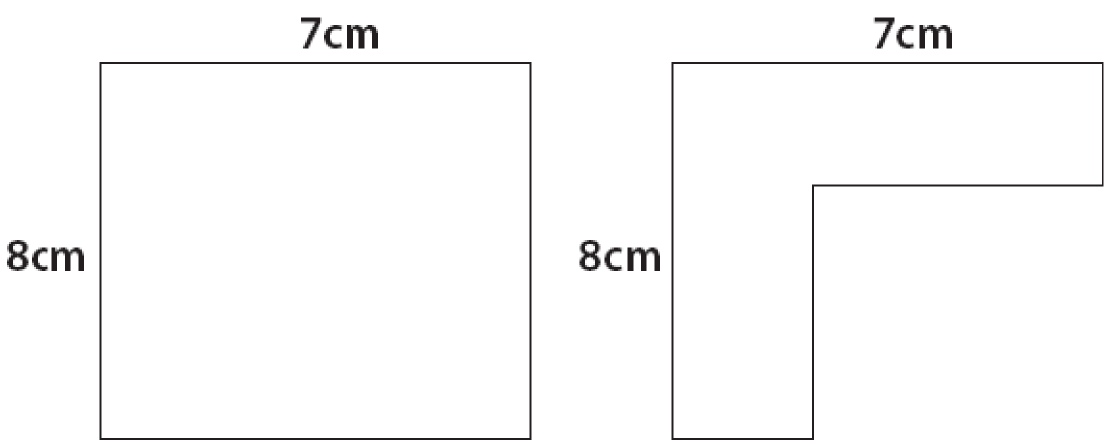
Q. 48. Cuisenaire Units.

Place square and Cuisenaire rods in front of child.
We can use these rods to measure the area of this square. If this is a square unit *indicate a "white one"*, what is the area of the square? How did you work that out?



Q. 49 Perimeter of an L shape.

The perimeter of a shape is the length around the outside. So the perimeter of this rectangle is this length, plus this length, plus this length, plus this length,
run finger along 8 by 7 rectangle sides (8, 8, 7, 7).
What is the perimeter of this shape? *Point to L shape* How did you work that out?
Adapted from Battista (2006).



Q. 50 Pirate Treasure Map.

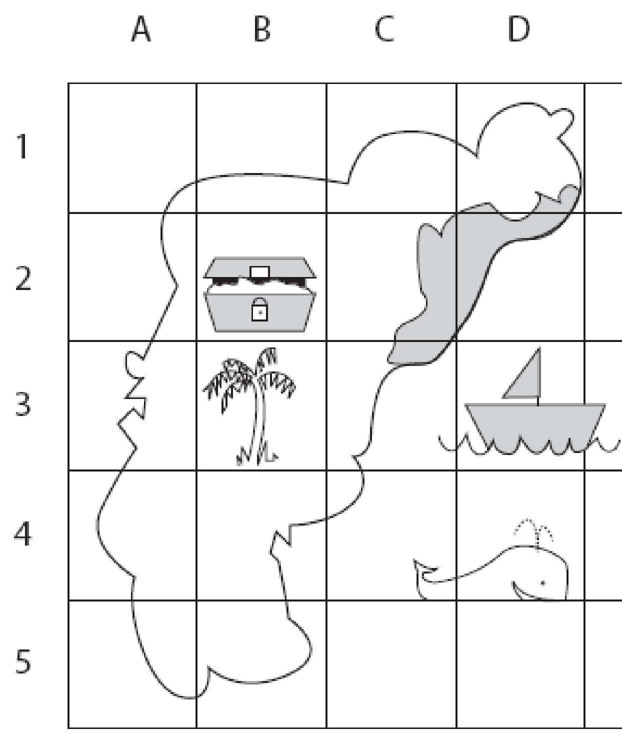
Place diagram of pirate treasure in front of child

This is a treasure map.

a) Please tell me what is on square D3. How did you work that out?

b) What are the co-ordinates for the square with the palm tree in it?

How did you work that out?

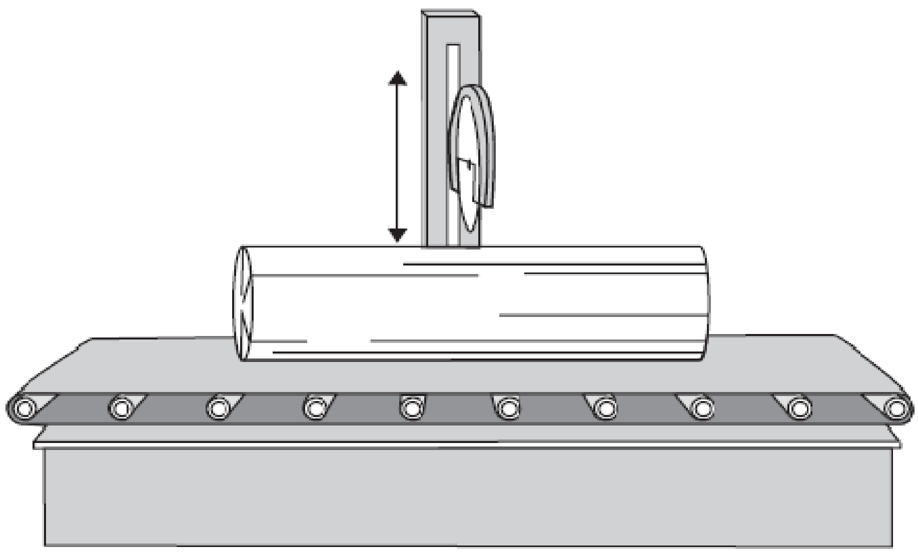


Q. 51 Sawmill.

Place picture of log and circular saw in front of child.

At the sawmill they want to cut this log into eight pieces. They can move it *indicate on diagram* backwards and forwards. The saw goes bzzzt like that to cut it *indicate vertical cut*. How many cuts will they need to make with the saw to get eight pieces? How did you work that out?

Adapted from Presmeg (1985).



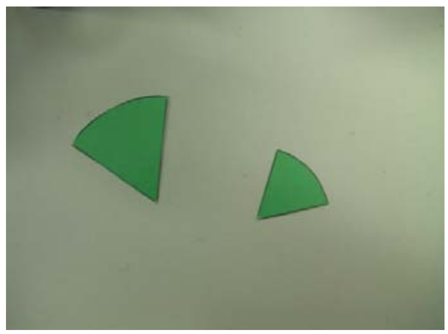
Q. 52 Angle Pieces.

Place pieces of circle in front of child.

Indicate smaller piece These are two fractional pieces from different circles. Can you show me with your finger what the whole circle would look like.

Indicate larger piece Can you show me what the circle that this piece came from would look like.

Indicate smaller piece This is one sixth of a circle. If this is one sixth, what fraction of its circle is this piece? How did you work that out?



Q. 53 Area Calculation – Triangle

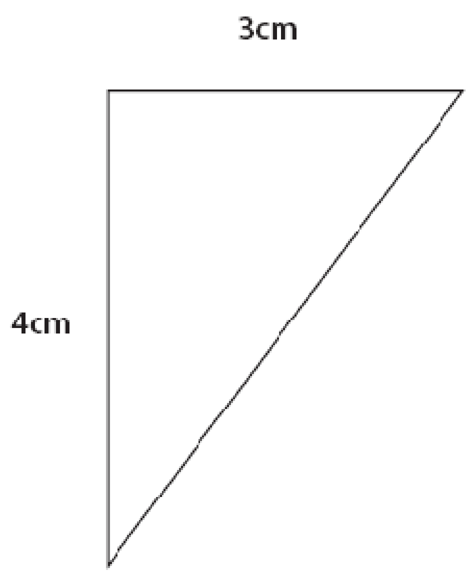
(if correct on Q. 33).

Place 3 by 4 triangle diagram in front of child.

What is the area of this triangle?

If correct number but no units given in answer, prompt for units, e.g. twelve what?

How did you work that out?



Q. 54 Blocks of Ice.

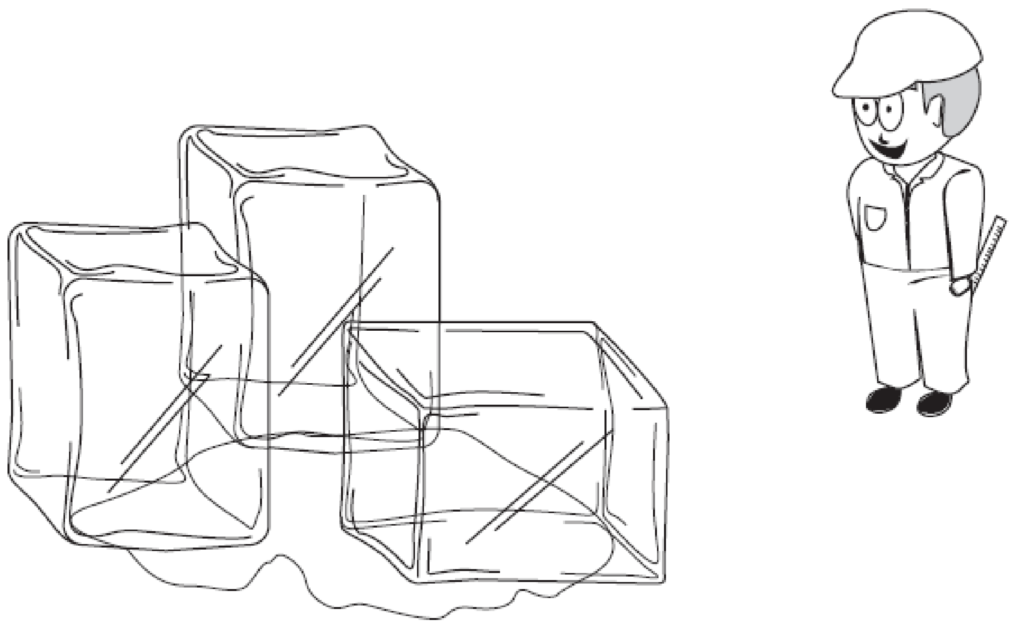
Show child picture of ice hotel. Have you heard about the ice hotel? Everything is made of ice, the chairs, even the roof and the beds. This man is going to build an ice hotel out of ice with these big blocks of ice.

What could he measure about these blocks? Show me on the diagram what that would be.

If necessary, what attributes of the blocks could he measure?

What else could he measure?

Repeat until child can not come up with more attributes. If child does not volunteer a length measure or an area measure, prompt with, could he measure a length? Could he measure an area? Show me on the diagram what that would be.



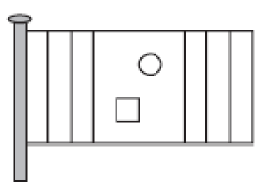
Q. 55 Connections.

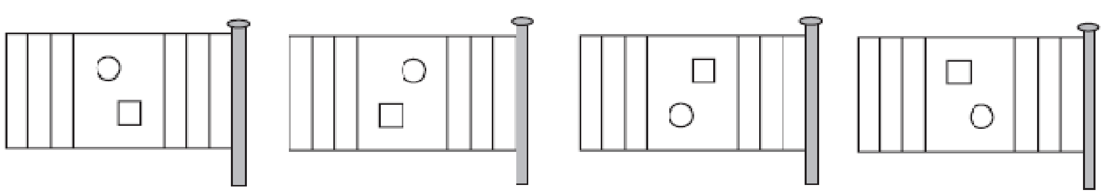
You have done some measurement tasks and some fractions tasks with me. Can you see anything similar about fractions and measuring?

Dynamic imagery

Q. 56 Flags.

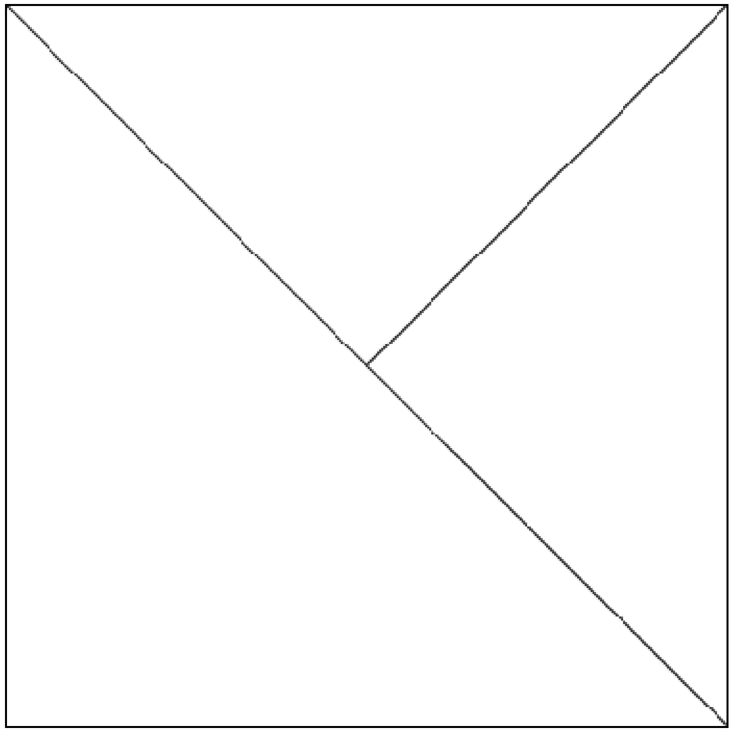
Place plastic flag in front of child, "blowing" to the right.
When this flag is blowing in the wind on top of a flag pole the pattern on it looks like this. The pattern goes through to the other side. When the flag blows in the opposite direction, *move flag through 180 degree rotation*, the pattern looks different.
This is a different flag. *Point to the flag at the top of the diagram*.
One of these is the same as this flag but it is blowing in the opposite direction… *point to row of 4 flags and then model 180 degree rotation with hand.*
Which one is it? How did you work that out? *Ask confirmation question if unclear about whether dynamic imagery or geometric properties of shape reasoning was used.*
Adapted from Australian Council for Educational Research (1978).

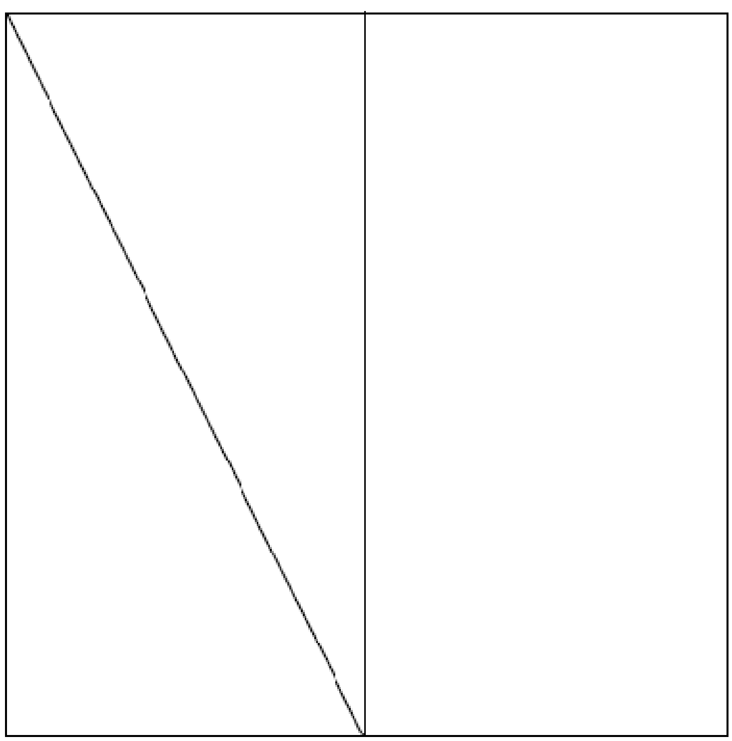


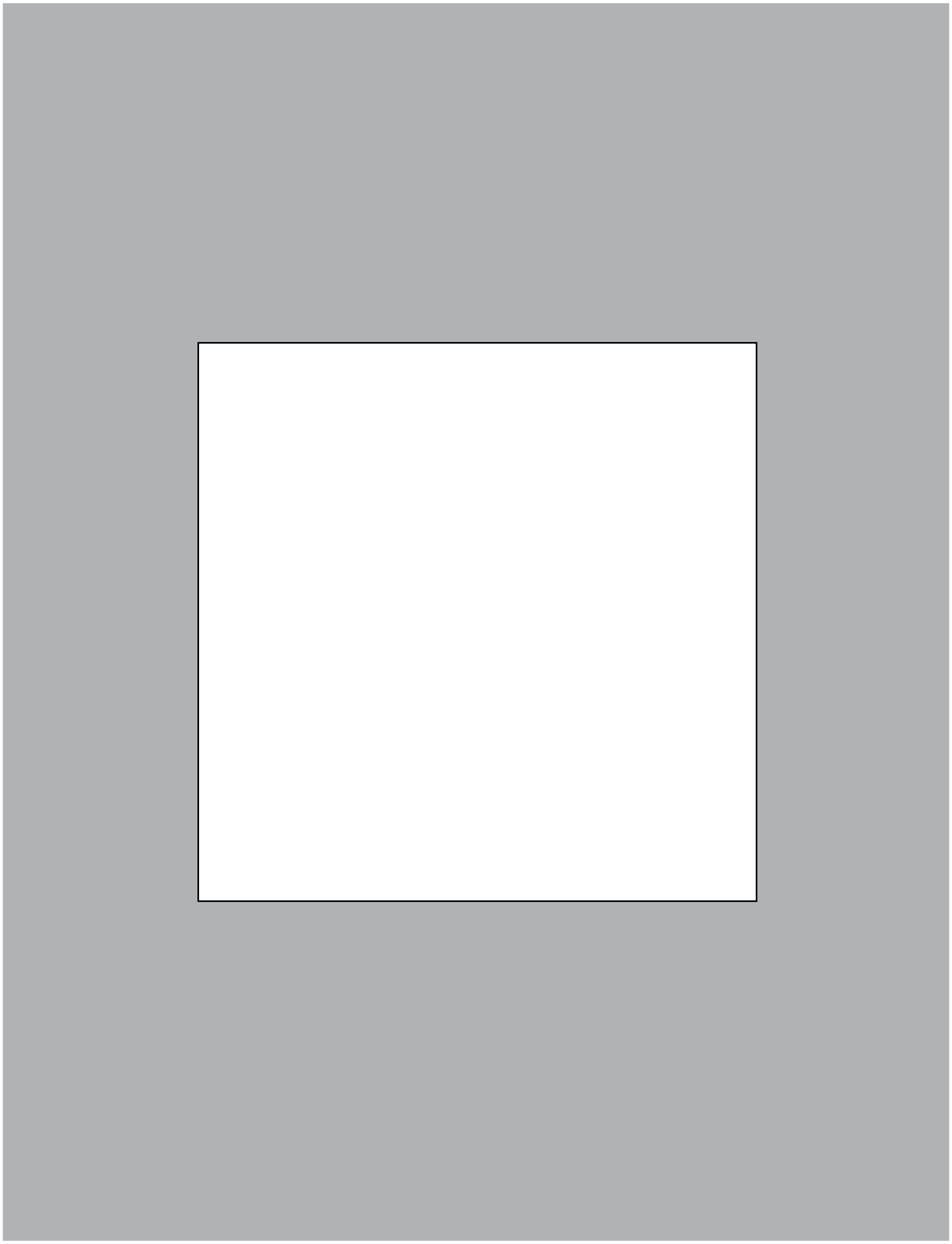


Q. 57 Puzzle.

Place the green card with the shading on the table. ***Randomly*** *place the six mauve shapes beside the green card (mix these up to avoid prompting).*
This is like a jigsaw puzzle. I have this card with a square on it…I'm wondering, **without moving any pieces yet**, if you can find three pieces that you think would fit together like a puzzle to cover the square exactly, Please point to them if you think you know.
If the child suggests three correct pieces, push the others to one side, and ask them to try and show how they fit. Ask confirmation question if unclear about whether dynamic imagery or geometric properties of shape reasoning was used.
Department of Education & Training (2001).







Q. 58 Cube Rotations.

a). *Place task card and real cubes in front of child.*

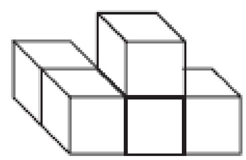
This shape, point to first shape, has been made with cubes like these. Can you please make this shape with the blocks.

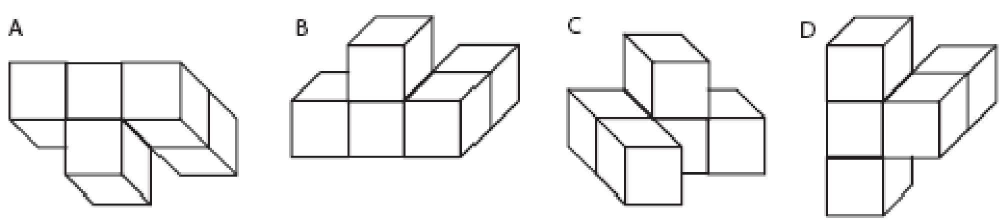
Place fixed models of cube shapes in front of child.

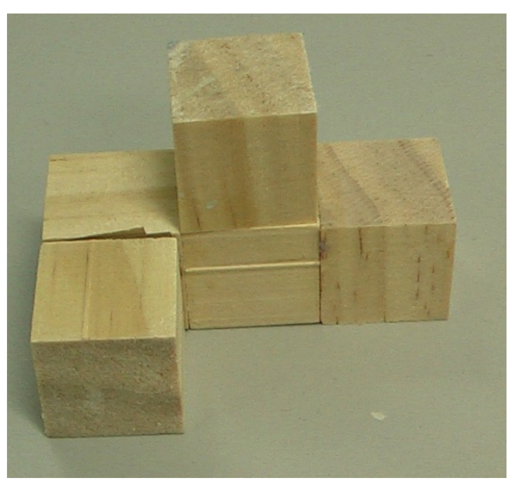
b). One of these four shapes is the same as the top one, but it has been moved around in the air. The other three are different shapes. Which one is the same as the top one? How did you work that out?

Ask confirmation question if unclear about whether dynamic imagery or geometric properties of shape reasoning was used.

Adapted from National Center for Educational Statistics (2007).







Q. 59 Design.

Show the child the white page with the design on it.

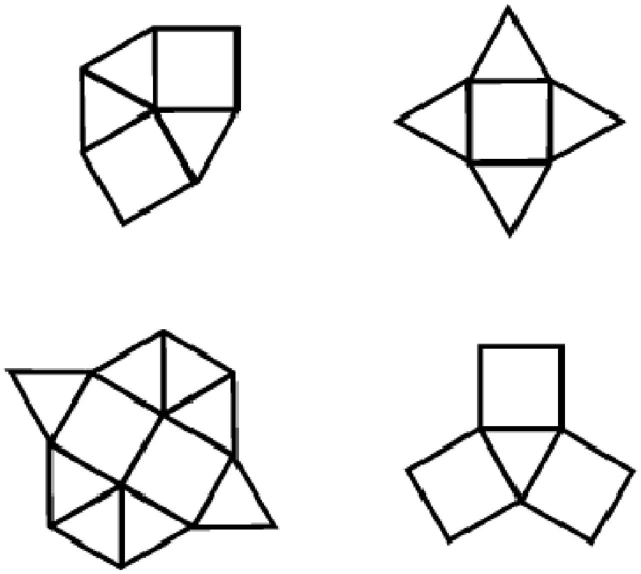
a) *Pointing to the small pieces on the page,* Which piece is **not** part of the design? This may take a little while.

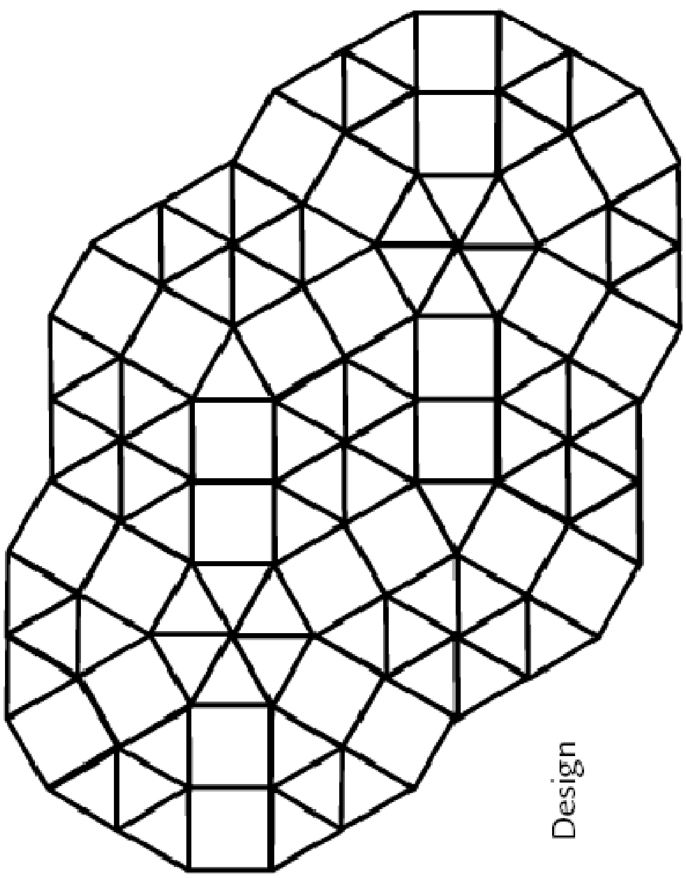
*If some time passes without comment or action...*Would you like to tell me what you are thinking?

*Once the child has decided which piece is not part of the design….*How did you work that out?

Ask confirmation question if unclear about whether dynamic imagery or geometric properties of shape reasoning was used.

Department of Education & Training (2001).





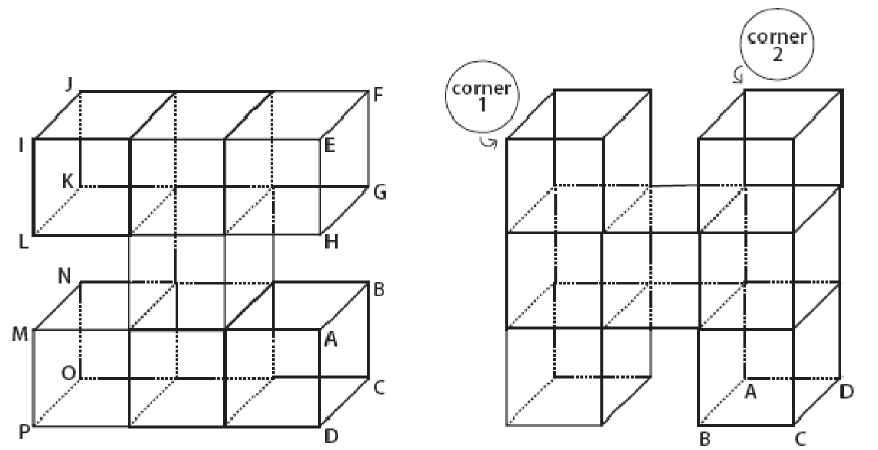
Design

Q. 60 Wattanawaha Block Rotation.

Place H models in front of child.

These two blocks are the same, and they have coloured corners that match.

This one has moved in the air and landed like this. *Arrange blocks as diagram*

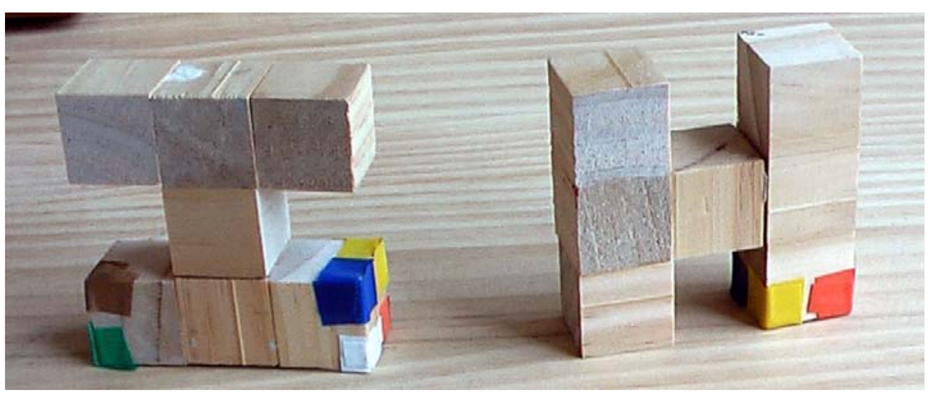


a) What colour would go on that corner (*point to corner 2*)?

How did you work that out?

b) Please describe how the shape moved.

Adapted from M. Clements (1983).



Q. 66 Wet-day Timetable.

Think about all the tasks that we have done and the materials that we have used. If it was wet day timetable, would you play with any of these things?

Record Sheet

Name School Date

□ □	**1.Tennis balls task** a. Answer_________ • Multiplication fact_______ • Skip count • Doubles______________ • Count all by ones • Other________________ b. • Multiplication fact_______ • Skip count/ Doubles • Count all by ones • Other	Imagery – Task • Concrete • Pattern • Memory formulae • Kinaesthetic • Dynamic Imagery explanation C P Mf K D
□ □	**2. Dots array task** a. Answer_________ • Multiplication fact\ • Skip count • Doubles • Count all by ones • Other b. • Multiplication fact • Skip count/ Doubles • Count all by ones • Other	Imagery task C P Mf K D explanation C P Mf K D
□	**3. Teddy cars**	Imagery task
□	**4. Sharing teddies on the mat**	Imagery task
□	**5. Children at the movies** a. Answer • Multiplication/division fact • Skip count • Count all by ones • Other	Imagery task C P Mf K D explanation C P Mf K D
□	**6. Multiplication problems** a. 3 × 10 d. 3 × 50 b. 2 × 7 e. 4 × 30 c. 10 × 7 f. 5 × 7	Imagery task C P Mf K D explanation C P Mf K D
□	**7 Division problems** a. 16 ÷ 2_______ d. 24 ÷ 3 b. 60 ÷ 10______ e. 35 ÷ 5 c. 80 ÷ 4 ______ f. 35 ÷ 7	Imagery task C P Mf K D explanation C P Mf K D
□	**8. off to the circus** Answer	Imagery task C P Mf K D
□	**9. Sharing our money ($52 between 4)** Answer • Mentally • Pen and paper used • Short division algorithm • Share by 1s	C P Mf K D
□	**10. In your head (23 × 4)** Answer	Imagery task C P Mf K D explanation C P Mf K D
□ □	**11. Missing number (54 × _ = _ _ 2)** a. Answer b. Answer	Imagery task C P Mf K D explanation C P Mf K D
□	12. **Bookworms** a.Answer__ __ __ 1 (dot under first) • Comparative: Twice, double, half, third, two/three/six times • multiplication equation language. 2x1, 3x1, 6x1 • skip counting • other	Imagery task C P Mf K D explanation C P Mf K D

□ □	b Answer__ __ 4 __(dot under first response) • Comparative: Twice, double, half, third, two/three/six times • Multiplication/div equation language. • skip counting • other c Answer __ 9 __ __ (dot under first response) • Comparative: Twice, double, half, third, two/three/six times • Multiplication/div equation language. • skip counting • other	Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D
□ □ □ □ □ □ □ □ □	**13. Fold me a quarter** a._______b. _______ c. same / square / triangle / strips • both quarters • looks bigger • has longer side/sides • could fit over/ inside • dynamic re-arrangement • other d. same / square / triangle / strips • both quarters • looks bigger • has longer side/sides • could fit over/ inside • dynamic re-arrangement • other e. same / square / triangle / strips • both quarters • looks bigger • has longer side/sides • could fit over/ inside • dynamic re-arrangement • other **14. Fraction pie** a. . Answer_________ • looks like a quarter • imagined line across L to R • five pieces • half of a half • other 14b. Answer_________ • imagined lines across L to R • imagined RHS as LHS • five pieces • three pieces on RHS • Other **15. tightrope walker** a. half way • 2 dots on each side • 3 spaces on each side • by eye • other b. 9/10ths • by eye • just less than 1 whole • iterates 1/10 • uses marks • other c2nd mark • Double count lines/ spaces • Counts lines and zero-pt. • Other	Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D

Name School Date

☐	**16. number lines** a. two thirds • 0 to 1 and 2/3 marked correctly • prompt for 0, 1 needed • other ratio used eg 0-9 • 0 to 1 non-prompt and 2/3 marked over 1 • 2/3 at • other	Imagery task C P Mf K D explanation C P Mf K D
☐	b. 0, ½ marked, please mark one and a half • iterates ½ • by eye • uses arrow on RHS as number • other	Imagery task C P Mf K D explanation C P Mf K D
☐	c. 0, 1, 2, please mark one quarter • half and half again of one unit • iterates ¼, guess and check • by eye (no other marks) • quarter of 2 • at 1 ¼ • other	Imagery task C P Mf K D explanation C P Mf K D
☐	d. 0- 4, read arrow. Answer • after 3, divided into parts • omitted whole number, parts read correctly • counts lines and zero-pt • counts lines successfully • Counts spaces • Decimalises count • Other	Imagery task C P Mf K D explanation C P Mf K D
☐	e. 0-1 evenly spaced hash marks, read arrow. Answer • counted spaces • double count • counted lines successfully • counted lines and zero-pt • decimalises count • other	Imagery task C P Mf K D explanation C P Mf K D
☐	f. 0-1, non evenly spaced hash marks, read arrow. Answer • imagines missing line • by eye, • one half and half/quarters (geometric reasoning) • counts lines successfully • counted lines and zero-pt • counts spaces • decimalises count • other	Imagery task C P Mf K D explanation C P Mf K D
☐	g. 6-8, read arrow. Answer • units and leftovers, • counted lines successfully • counted lines and zero-pt • omitted whole number leftover part read correctly • other	Imagery task C P Mf K D explanation C P Mf K D
☐	h. 0-1 and beyond, improper Answer • units and leftovers, • counted lines successfully • counted lines and zero-pt • omitted whole number, leftover part read correctly • other	Imagery task C P Mf K D explanation C P Mf K D

☐ ☐ ☐ ☐	**17. Pattern Blocks** a. blue of yellow. Answer strategy b. blue of red. Answer strategy c. red of blue. Answer strategy d. green is half, yellow is. Answer strategy	Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D
☐ ☐ ☐ ☐ ☐	**18. Simple operators** a. half of 6 b. two and a half times six c two thirds of nine d a third of a half • mentally, • drawing, • pen and paper e. a half of a third • automatic commutitivity • mentally, • Drawing • Pen and paper f. confirmatory question for a, b, c	Imagery task C P Mf K D explanation C P Mf K D
☐	**19. Fraction sort**	Imagery task C P Mf K D explanation C P Mf K D
☐		Imagery task C P Mf K D explanation C P Mf K D
☐		Imagery task C P Mf K D explanation C P Mf K D
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Name School Date

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☐		Imagery task C P Mf K D explanation C P Mf K D
☐		Imagery task C P Mf K D explanation C P Mf K D
☐		Imagery task C P Mf K D explanation C P Mf K D
☐	**20. Sharing custard tarts and liquorice** 5 people, 3 strips of liquorice Answer • mentally immediately as division • mentally, • guess and iterate • Drawing- guess and share • Drawing, into fifths • Drawing, other	Imagery task C P Mf K D explanation C P Mf K D

☐	b. 5 people, 3 custard tarts. Answer • Recognizes similar problem type as (a.). • mentally immediately as division • mentally, • guess and iterate • Drawing into fifths • Drawing- guess and share • Drawing other	Imagery task C P Mf K D explanation C P Mf K D
☐	c. 5 people, 7 custard tarts. Answer • mentally immediately as division • mentally, • guess and iterate • Drawing, eliminate wholes and then into fifths • Drawing, all into fifths • Drawing- eliminate wholes then guess and share, • Drawing- guess and share, • Drawing other	Imagery task C P Mf K D explanation C P Mf K D
☐	d. 4 people and 9 pieces of liquorice. Answer • mentally immediately as division • mentally, • guess and iterate • Drawing – eliminate wholes, then into quarters • Drawing, all into quarters • Drawing- eliminate wholes then guess and share • Drawing- guess and share, • Drawing other	Imagery task C P Mf K D explanation C P Mf K D
☐ ☐ ☐ ☐	**21. Golden beans** a. b. equivalent fraction. c. d. equivalence.	Imagery task C P Mf K D explanation C P Mf K D
☐	**22. Fraction pairs** a. 3/8 and 7/8 or same • D the same and compares N • benchmarking to ½ and 1 • residual thinking(1/8<5/8) • residual thinking closer to 1 • other (satisfactory) • compares n only 7>3 • more area • gap thinking (1 < 5) • smaller numbers mean bigger fractions • other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D

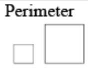
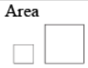
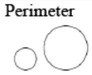
Name School Date

☐	b. 2/4 or 4/8 or same • equivalent • both ½ • 2 is half of 4, 4 is half of 8 • 4 is double 2, 8 is double 4 • Other satisfactory • More area • Higher or larger numbers • Denominator • Gap thinking • Other unsatisfactory	Imagery task C P Mf K D explanation C P Mf K D
☐	c. ½ and 5/8 or same • benchmarks to one half • common denominators 5/8>4/8 • Other (satisfactory) • more area • Higher or larger numbers • Because it's half • Gap thinking (1>3) • Other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D
☐	d. 2/4 and 4/2 or same • equates to ½ and 2 • converts to common denominators • benchmarks to 1 (improper or more than one) • other (satisfactory) • more area • 2 is half of 4 • Compares numerators or denominators • Other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D
☐	e. 4/7 and 4/5 or same • N the same and compared D • size of the pieces different • converts to common D • benchmarks to ½ and 1 • residual thinking(1/5>3/7) • other (satisfactory) • more area • compares denominators only • gap thinking (1 < 3) • other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D
☐	f 3/7 and 5/8 or same • benchmarks to ½ • converts to common D • other (satisfactory) • residual thinking(3/8<4/7) • higher or larger numbers • gap thinking (3<4) • other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D
☐	g 5/6 and 7/8 or same • benchmarks to ½ • converts to common D • other (satisfactory) • residual thinking • higher or larger numbers • gap thinking • other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D

☐	H 3/4 and 7/9 or same • benchmarks • converts to common D • other (satisfactory) • residual thinking • higher or larger numbers • gap thinking • other (unsatisfactory)	Imagery task C P Mf K D explanation C P Mf K D
☐	Ordering numbers	Imagery task C P Mf K D explanation C P Mf K D
☐	**23. Puff machine** Answer • Describes unit and left overs, lines/spaces are fifths, 0.2, (skip counting) • Describes unit and leftovers, lines are… • Uses whole numbers • Other	Imagery task C P Mf K D explanation C P Mf K D
☐ ☐	**24. Arc, angle, area** a. Answer • Uses words Arc/angle/area • Describes arc line, lines meeting, line rotating, space, fit inside • other b. • all are correct • like me in _______ • they are incorrect c Arc, angle, area, other	Imagery task C P Mf K D explanation C P Mf K D
☐ ☐	**25. Density** a Answer • prompt for equivalence explanation b. how many? • infinite number • lots/many • other explanation • density/equivalence/ • other	Imagery task C P Mf K D explanation C P Mf K D
☐	**26. fraction algorithms** 5/6 +1/6 • Mentally • Written equation/answer • Written algorithm • Diagram	Imagery task C P Mf K D explanation C P Mf K D
☐	¾ +1/2 • Mentally • Written equation/answer • Written algorithm • diagram	Imagery task C P Mf K D explanation C P Mf K D
☐	1/3 + ½ • Mentally • Written equation/answer • Written algorithm • diagram	Imagery task C P Mf K D explanation C P Mf K D
☐	4 ¼ - 2/4 • Mentally • Written equation/ answer • Written algorithm • diagram	Imagery task C P Mf K D explanation C P Mf K D

Name School Date

☐	½ x 1/3 • Mentally • Written equation/answer • Written algorithm • diagram	Imagery task C P Mf K D explanation C P Mf K D
☐ ☐ ☐	**27. Show me thirds** a. halves. Answer • count top/bottom line • by eye • count all • other b thirds. Answer • count top/bottom line • count rows • count all • other c. sixths. Answer • count top/bottom line • count rows • count all • half of 1/3 • other	Imagery task C P Mf K D explanation C P Mf K D
☐	**28. Crossroads** Answer • Reunitized to/ “saw” 4 by 4 squares • Counted by ones, fours • Double count • Multiplied rows and columns • Other	Imagery task C P Mf K D explanation C P Mf K D
	29. Ending on a positive fraction	
	30 Off the record Like best Why In class Seen questions Easy/challenging/both	
☐ ☐	**31. Streamer** a. length. Answer • Prompt for units ____ • Iterated 30cm lengths using ruler • Marked iteration, pen/ finger • Iterated whole ruler • Other b. from 1 meter. Answer	Imagery task C P Mf K D explanation C P Mf K D
☐	**32. Measure DVD with a ruler** Answer • Prompt for units ____ • Used zero as starting point • Other	Imagery task C P Mf K D explanation C P Mf K D
☐	**33. Area calculation- half rectangle** Answer • Prompt for units____ • Half of 12 (rectangle) • Not 12 • Multiplied two lengths • Other	Imagery task C P Mf K D explanation C P Mf K D

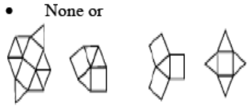
☐	**34. Square to triangle sequence-cutting** Answer • Same b/c nothing added or taken away, just moved • looks bigger • Triangle has longer side/sides • Other		Imagery task C P Mf K D explanation C P Mf K D
☐= ☐#	**35. Missing oval** Answer • Prompt for units ____ • Multiplication, R by C • Skip count rows/columns • Count in R/C by ones • Count by ones		Imagery task C P Mf K D explanation C P Mf K D
	36. Similar shapes		
☐	Perimeter 	• Looks bigger • Line/s longer • Would fit inside	Imagery task C P Mf K D explanation C P Mf K D
☐	Area 	• Looks bigger • Line/s longer • Would fit inside • b/c > perim	Imagery task C P Mf K D explanation C P Mf K D
☐	Perimeter 	• Looks bigger • Line/s longer • Would fit inside • Has no perim	Imagery task C P Mf K D explanation C P Mf K D
☐	Area 	• Looks bigger • Line/s longer • Would fit inside • b/c > perim	Imagery task C P Mf K D explanation C P Mf K D
☐	Perimeter 	• Looks bigger • Lines longer • Dynamic	Imagery task C P Mf K D explanation C P Mf K D
☐	Area 	• Looks bigger • Lines longer • Dynamic	Imagery task C P Mf K D explanation C P Mf K D
☐	Perimeter 	• Looks bigger • Line/s longer • Dynamic • Both half	Imagery task C P Mf K D explanation C P Mf K D
☐	Area 	• Looks bigger • Line/s longer • Dynamic • Both half • Because have same perim	Imagery task C P Mf K D explanation C P Mf K D

Name School Date

□ □ □ □	**37. Four triangles** a. answer b. answer c. answer proportional	Imagery task C P Mf K D explanation C P Mf K D
□= □A □	**38. draw your own array** a. area of reactangle. Answer • Prompt for units ____ • used ruler to mark hash marks, joined rows and columns • used ruler to measure 2cm at a time • used ruler to rule columns, rows • by eye grid structure • individual tiles • other b. inverse. Bigger tile. Answer • b/c it's bigger • other	Imagery task C P Mf K D explanation C P Mf K D
□	**39. Keyboard** Answer • Prompt for units ____ • Described leftovers	Imagery task C P Mf K D explanation C P Mf K D
□ □	**40. Using paperclips to measure** Answer • Long ____short____ • Prompt for units ____ • Used identical units • Named non-identical units • no gap/overlap • Described leftovers **40b. units and count**	Imagery task C P Mf K D explanation C P Mf K D
□	**41. Freddo frog** Answer • 8 take away 3 • Mentally realigned 3 to 0/1 • Counted spaces • Counted hash marks success/y • Counted hash marks inc zero-p • Other	Imagery task C P Mf K D explanation C P Mf K D
□	**42. Footy card** Answer • Counted spaces • Counted hash marks success/y • Counted hash marks inc zero-p • Other	Imagery task C P Mf K D explanation C P Mf K D
□	**43. Straigtening wires** Answer • by eye • counted spaces • counted dots as end of spaces • counted dots • other	Imagery task C P Mf K D explanation C P Mf K D

□	**44. Steps** Answer • the least steps • the smallest/ bigger number • other	Imagery task C P Mf K D explanation C P Mf K D
□ □ □	**45. Choosing rulers** a. ruler: measure: • Chose 1 ruler and measured • Tried and discarded some rulers b. • Chose 1 ruler and measured • Tried and discarded some rulers • Not equal	Imagery task C P Mf K D explanation C P Mf K D
□ □	**46. Array with leftovers** • Whole tiles plus leftovers **47. Packing boxes** • Multiplied 3 by 3 by 5 (by 2) • Multiplied 5 by 6 by 3 by 2 • Skip counted	Imagery task C P Mf K D explanation C P Mf K D Imagery task C P Mf K D explanation C P Mf K D
□	**48. Cuisenaire array**	Imagery task C P Mf K D explanation C P Mf K D
□	**49. Perimeter of an L shape** Answer • Geometric reasoning • Estimated unknown lengths, by eye • Measured unknown/all lengths with ruler – ratio calc • Measured unknown lengths with ruler- added to given lengths • Measured all sides with ruler and added • Other	Imagery task C P Mf K D explanation C P Mf K D
□ □	**50. Pirate treasure map** a. square D3. Answer • array structure • other b. palm tree? Answer • array structure • other	Imagery task C P Mf K D explanation C P Mf K D
□	**51. Sawmill** Eight pieces, # of cuts. Answer • Algebraic • b/c one is at the end • b/c there is 8 • diagram	Imagery task C P Mf K D explanation C P Mf K D

Name School Date

☐	**52. Angle Pieces** • Compare angle • Compare arcs • Compare widths • rotates • other	
☐	**53. Area calculation triangle** Answer • Prompt for units • Half of 12 (rectangle) • Not 12 • ½ base times height • Multiplied two lengths • Other	Imagery task C P Mf K D explanation C P Mf K D
☐L ☐A	**54. Blocks of ice**	Imagery task C P Mf K D explanation C P Mf K D
	55. connections • Both use numbers • Can describe leftovers • No connection	
☐	**56. Flags** Answer • Dynamic imagery • Geometric reasoning	Imagery task C P Mf K D explanation C P Mf K D
☐	**57. Puzzle**  Tick the three selected if any • Successful fit immediately without adjustment • Successful fit after some adjustment • Unsuccessful	Imagery task C P Mf K D explanation C P Mf K D
☐ ☐	**58. Cube rotations** a. Makes model b. Picks rotated cube. Answer • Dynamic imagery • Geometric reasoning	Imagery task C P Mf K D explanation C P Mf K D
☐	**59. Design** Circle the selected piece (if any) • None or  • Dynamic imagery • Geometric reasoning	Imagery task C P Mf K D explanation C P Mf K D
☐	**60. Wattanawaha block rotation** Corner 2 __________ • Dynamic imagery • Geometric –opposite a.	Imagery task C P Mf K D explanation C P Mf K D

	61. Wet Day Timetable	
☐ ☐U	PP staircase answer units	
☐ ☐U	PP area of rectangle answer	
☐	PP dragonfly Answer	
☐	PP swimming pool	

Appendix B: Ethics

Approval from the Australian Catholic University Human Research Ethics Committee.

Approval from the Department of Education and Early Childhood Development.

Information letter for Director of Northern Metropolitan Region, Department of Education and Early Childhood Development.

Information letter to Principals and consent form (principals).

Information letter to participants and consent forms (parents/guardians and children).

Approval from Cadbury to use the Freddo image.

Australian Catholic University
Brisbane Sydney Canberra Ballarat Melbourne

ACU National

Human Research Ethics Committee

Committee Approval Form

Principal Investigator/Supervisor: A/Prof Marj Horne Melbourne Campus

Co-Investigators: Melbourne Campus

Student Researcher: Anne Mitchell Melbourne Campus

Ethics approval has been granted for the following project:
Primary school children's understanding of fraction concepts

for the period: 10th October 2006 - 30th June 2008

Human Research Ethics Committee (HREC) Register Number: V200506 79

The following <u>standard</u> conditions as stipulated in the *National Statement on Ethical Conduct in Research Involving Humans* (1999) apply:

(i) that Principal Investigators / Supervisors provide, on the form supplied by the Human Research Ethics Committee, annual reports on matters such as:
- security of records
- compliance with approved consent procedures and documentation
- compliance with special conditions, and

(ii) that researchers report to the HREC immediately any matter that might affect the ethical acceptability of the protocol, such as:
- proposed changes to the protocol
- unforeseen circumstances or events
- adverse effects on participants

The HREC will conduct an audit each year of all projects deemed to be of more than minimum risk. There will also be random audits of a sample of projects considered to be of minimum risk on all campuses each year.

Within one month of the conclusion of the project, researchers are required to complete a *Final Report Form* and submit it to the local Research Services Officer.

If the project continues for more than one year, researchers are required to complete an *Annual Progress Report Form* and submit it to the local Research Services Officer within one month of the anniversary date of the ethics approval.

Signed: Jo Mushin Date: 12.10.2006

(Research Services Officer, Melbourne Campus)

res ethics
Sent by: Jo Mushin

20/12/2007 10:25 AM

To:Marj Horne/patrick@patrick, Annie Mitchell/patrick@patrick
cc:
bcc:
Subject:Ethics Modification V200506 79

Dear Marj and Annie ,

Thank you for submitting the request to modify form for your project V200506 79 *Primary School Children's Understanding of Fraction Concepts.*

The Chair of the Human Research Ethics Committee has approved the modification, please take this email as confirmation of your approval.

We wish you well in this ongoing research project.

Kind regards,

Jo

****

Jo Mushin
Research Services Officer (Ethics)
Research Services
Australian Catholic University Limited
ABN 15 050 192 660
St Patrick's Campus
(Locked bag 4115)
115 Victoria Parade Fitzroy VIC 3065
Ph: (03) 9953 3158
Fax: (03) 9953 3315
Email: res_ethics@acu.edu.au

Department of Education & Training

Office of Learning and Teaching

SOS003302

Ms Annie Mitchell
Australian Catholic University
115 Victoria Parade
FITZROY 3065

Dear Ms Mitchell

Thank you for your application of 19 June 2006 in which you request permission to conduct a research study in government schools titled: *Primary School Children's Understanding of Fraction Concepts.*

I am pleased to advise that on the basis of the information you have provided your research proposal is approved in principle subject to the conditions detailed below.

1. Should your institution's ethics committee require changes or you decide to make changes, these changes must be submitted to the Department of Education and Training for its consideration before you proceed.

2. You obtain approval for the research to be conducted in each school directly from the principal. Details of your research, copies of this letter of approval and the letter of approval from the relevant ethics committee are to be provided to the principal. The final decision as to whether or not your research can proceed in a school rests with the principal.

3. No student is to participate in this research study unless they are willing to do so and parental permission is received. Sufficient information must be provided to enable parents to make an informed decision and their consent must be obtained in writing.

4. As a matter of courtesy, you should advise the relevant Regional Director of the schools you intend to approach. An outline of your research and a copy of this letter should be provided to the Regional Director.

2 Treasury Place
East Melbourne, Victoria 3002
Telephone: +61 3 9637 2000
DX 210083

GPO Box 4367
Melbourne, Victoria 3001

Victoria
The Place To Be

5. Any extensions or variations to the research proposal, additional research involving use of the data collected, or publication of the data beyond that normally associated with academic studies will require a further research approval submission.

6. At the conclusion of your study, a copy or summary of the research findings should be forwarded to the Research and Development Branch, Department of Education and Training, Level 2, 33 St Andrews Place, GPO Box 4367, Melbourne, 3001.

I wish you well with your research study. Should you have further enquiries on this matter, please contact Chris Warne, Project Officer, Research on (03) 9637 2272.

Yours sincerely

SMahar

Sandra Mahar
A/Assistant General Manager
Research and Innovation Division

13/07/2006

enc

Annie
Thank you for your letter dated 11 January 2008 in which you request approval for amendments to your application to conduct research in schools titled: 'Primary school students' understanding of fraction concepts'.

The Department of Education and Early Childhood Development approves the amendments subject to the conditions outlined in the original letter of approval.

Regards
Chris Warne
Senior Research and Policy Officer
Research Branch
Education Policy and Research Division
Office for Policy, Research and Innovation
Department of Education and Early Childhood Development
Level 2, 33 St Andrews Place
GPO Box 4367
MELBOURNE 3001

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SEPTEMBER 12 2007

Wayne Craig
Regional Director
Northern Metropolitan Region
Department of Education and Early Childhood Development
Locked Bag 88
Fairfield 3078

RESEARCH IN SCHOOLS

TITLE OF PROJECT: PRIMARY SCHOOL CHILDREN'S UNDERSTANDING OF FRACTION CONCEPTS

SUPERVISOR: ASSOCIATE PROFESSOR MARJ HORNE
STUDENT RESEARCHER: MS ANNIE MITCHELL, PhD CANDIDATE

Dear Mr Craig,

We are writing to inform you that we would like to approach two schools in the Northern Region to conduct a pilot study of the tasks and protocols to be used in the research project, Primary School Children's Understanding of Fraction Concepts. We would like to contact the Principals of [school names deleted] to ask their permission to conduct part of this research in their schools and to send information letters to participants, and consent forms, to the families of Grade 6 students.

This research project aims to investigate the relationship between children's understandings of fraction tasks and any connections they may make with measurement concepts and spatial reasoning. This research has ethics approval from the Australian Catholic University Human Research Ethics Committee and the Department of Education and Early Childhood Development. Annie Mitchell is an experienced primary school teacher, currently undertaking her PhD studies in mathematics education. She will be conducting all the interviews, and has current Victorian Institute of Teaching registration.

Each child will complete a pen and paper test on measurement and space concepts. The children will then take part in two one-to-one task-based interviews, similar in format to the Early Numeracy Interview. All sessions would be audio-taped, and the answers to interview tasks recorded on a record sheet, collected and analysed. Permission will be sought for a smaller number of students to be filmed so that a) the recording of strategies and coding of data can be "double checked" and b) footage of children's strategies for attempting fraction tasks can be shown to other teachers and researchers in professional development sessions and at conferences including on the internet.

Fractions is a very important topic in mathematics. We are trying to find out more about students' understandings of fractions so that we can improve opportunities for all students. The students participating in the study may benefit from the opportunity to reflect upon their learning.

Please find attached a copy of the letter of approval from the Australian Catholic University Human Research Ethics Committee. Also attached is a copy of the letter of permission to conduct this study from Chris Warne, Research Branch, Education Policy and Research Division, Department of Education and Early Childhood Development. Copies of information letters to Principals and Participants are also included.

We believe that this work may provide valuable insights into mathematics learning in the middle years, and we look forward to being able to share our major findings with you.

Yours faithfully

Marj Horne Annie Mitchell

Australian Catho ic University
Brisbane Sydney Canberra Ballarat Melbourne

ACU National

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December 12 2007

Wayne Craig
Regional Director, Northern Metropolitan Region
Department of Education and Early Childhood Development
Locked Bag 88
Fairfield 3078

Re: RESEARCH IN SCHOOLS

TITLE OF PROJECT: PRIMARY SCHOOL CHILDREN'S UNDERSTANDING OF FRACTION CONCEPTS

Dear Mr Craig,

We are writing to inform you that we would like to approach two additional schools in the Northern Region to conduct one-to-one task-based interviews to be used in the research project, Primary School Children's Understanding of Fraction Concepts. We would like to contact the Principals of [school names deleted] to ask their permission to conduct part of this research in their schools and to send information letters to participants, and consent forms, to the families of Grade 6 students.

Our letter to you on September 12 of this year outlined the rationale for, and conduct of, the project. This project has ethics approval from the Australian Catholic University Human Research Ethics Committee and from the Department of Education and Early Childhood Development (# SOS003302).

Yours faithfully

Associate Professor Marj Horne Annie Mitchell

CRIOCOS registered provider:
00004G, 00112C, 00873F, 00885B

Australian Catholic University
Brisbane Sydney Canberra Ballarat Melbourne

ACU National

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Facsimile 03 9953 3005
www.acu.edu.au

[date]

[Principal's name deleted]
Principal
[school name deleted]
[School address deleted]
[School address deleted]

PRINCIPAL LETTER FOR STUDENT PARTICIPATION IN RESEARCH PROJECT

TITLE OF PROJECT: PRIMARY SCHOOL CHILDREN'S UNDERSTANDING OF FRACTION CONCEPTS

SUPERVISOR: ASSOCIATE PROFESSOR MARJ HORNE
STUDENT RESEARCHER: MS ANNIE MITCHELL, PhD CANDIDATE

Dear [Principal's name deleted],

We are writing to ask your permission to invite Year 5 and Year 6 students in your school to take part in a research project that aims to investigate the relationship between children's understandings of fraction tasks and any connections they may make with measurement concepts and spatial reasoning. Each child will complete a pen and paper test on measurement and space concepts. Many children will then take part in two one-to-one task-based interviews, similar to the format of the Early Numeracy Interview.

You will be informed about the progress of the research. If a student becomes distressed during an interview, the interview will be terminated, and he/she will be returned to their class teacher or another appropriate staff member for any necessary counselling or support, and you will be informed. Annie Mitchell is an experienced primary school teacher, currently undertaking her PhD studies in mathematics education. She will be conducting all the interviews, and has current Victorian Institute of Teaching registration.

The research would take place during class time. The pen and paper test may take up to one session (one hour) to complete. The interviews may take up to one session (1 hour) each to complete. Some children will complete the assessment tasks in a shorter time and will immediately be able to return to their normal program. Teachers will need to make their students available during class time. They will also be asked to distribute and collect parental consent forms. A quiet room will be required for conducting the interviews. All sessions would be audio-taped, and the answers to interview tasks recorded on a record sheet, collected and analysed. Permission will be sought for a smaller number of students to be filmed so that a) the recording of strategies and coding of data can be "double checked" and b) footage of children's strategies for attempting fraction tasks can be shown to other teachers and researchers in professional development sessions and at conferences including on the internet.

Fractions is a very important topic in mathematics. We are trying to find out more about students' understandings of fractions so that we can improve opportunities for all students. The students participating in the study may benefit from the opportunity to reflect upon their learning. We hope to use the results in Annie Mitchell's PhD thesis, to publish research papers in education journals, and to present papers at education conferences.

Having given your consent for the researchers to work with students in your school, you are free to withdraw your consent or to discontinue participation in the project at any time, without giving reasons. There will be no penalty for you or your school for withdrawing from the study. The participation of students (and their parents' consent) is voluntary and they are free to discontinue participation in the project at any time, without giving reasons.

As the data will be collected by withdrawing students from the class, it will be possible for other students in the grade, and teachers, to know who took part in the research. Written reports from the study will not enable anyone to identify individual children. The data collected throughout this study may be used in publications, used in teaching or shared with other researchers, but confidentiality of teachers' and students' identities will be maintained at all times. The video footage may be used for a) "double checking" our method of recording information on the record sheet and coding those answers and b) in professional development sessions and shown at conferences to other teachers and researchers, including on the internet. The name of your school may be kept confidential if you wish, or you may consent to your school being named in order for the school community to be thanked for participating in this research.

Any questions regarding this project should be directed to the Principal Researcher, Associate Professor Marj Horne, School of Education, St Patrick's Campus, 115 Victoria Parade, Fitzroy 3065 (Tel: 9953 3289).

Following the data analysis, we would be pleased to share a copy of our major findings with you should you indicate an interest in this.

This study has been approved by the Human Research Ethics Committee at Australian Catholic University. Ethics approval for this project has been granted by the Department of Education. Copies of both are attached.

In the event that you have any complaint or concern about this study, or if you have any query that the Principal Researcher has not been able to satisfy, you may write to: the Chair of the Human Research Ethics Committee, C/o Research Services, Australian Catholic University, Melbourne Campus, Locked Bag 4115, Fitzroy VIC 3065 (Tel: 9953 3157; Fax: 9953 3315). Any complaint or concern will be treated in confidence and fully investigated. You will be informed of the outcome.

If you give permission for students at your school to be approached to participate in this project, we would ask that you forward the attached parent information letters and informed consent forms to the students and their families.

Please indicate your response to the questions below and return the coloured copy to the researchers.

I consent to students in my school being approached to take part in this study:	yes / no
I consent to my school being named in publications or presentations:	yes / no
I would like a copy or summary of the research findings:	yes / no

We look forward very much to your response.

Yours sincerely

Marj Horne Annie Mitchell

Please indicate your response to the questions below and return the coloured copy to the researchers.

Name of School …………………………………………………………..

Name of Principal ………………………………………………………..

I consent to students in my school being approached to take part in this study: yes / no

I consent to my school being named in publications or presentations: yes / no

I would like a copy or summary of the research findings: yes / no

Australian Catholic University
Brisbane Sydney Canberra Ballarat Melbourne

ACU National

Australian Catholic University Limited
ABN 15 050 192 660
Melbourne Campus (St Patrick's)
115 Victoria Parade Fitzroy Vic 3065
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Facsimile 03 9953 3005
www.acu.edu.au

JANUARY 29, 2008

PARENT INFORMATION LETTER FOR STUDENT PARTICIPATION IN RESEARCH PROJECT

TITLE OF PROJECT: PRIMARY SCHOOL CHILDREN'S UNDERSTANDING OF FRACTION CONCEPTS

SUPERVISOR: ASSOCIATE PROFESSOR MARJ HORNE
STUDENT RESEARCHER: MS ANNIE MITCHELL, PhD CANDIDATE

Dear Parent/Guardian,

We are inviting your child to take part in a research project that aims to investigate the relationship between children's understandings of fraction tasks and any connections they may make with measurement concepts and spatial reasoning. Each child will complete a pen and paper test on measurement and space concepts. Many children will then take part in two one-to-one task-based interviews, similar to the format of the Early Numeracy Interview. Participation in this research project is voluntary.

If your child becomes upset during the interview, his/her participation in the research will cease, and he/she will be returned to the class teacher or another appropriate staff member for any necessary counselling or support. Annie Mitchell is an experienced primary school teacher, currently undertaking her PhD studies in mathematics education. She will be conducting all the interviews, and has current Victorian Institute of Teaching registration.

The research will take place during class time. The pen and paper test may take up to one session (one hour) to complete. The interviews may take up to one session (1 hour) each to complete. Some children will complete the assessment tasks in a shorter time and will immediately be able to return to their normal program. All sessions will be audio-taped, and the answers to interview tasks recorded on a record sheet, collected and analysed. If you give your permission, the interview may also be filmed.

Fractions is a very important topic in mathematics. We are trying to find out more about students' understandings of fractions so that we can improve opportunities for all students. The students participating in the study may benefit from the opportunity to reflect upon their learning. We hope to use the results in Annie Mitchell's PhD thesis, to publish research papers in education journals, and to present papers at education conferences.

Having given your consent, you are free to withdraw your consent or to discontinue the participation of your child at any time, without giving reasons. There will be no penalty for withdrawing from the study for you or your child.

As the data will be collected by withdrawing students from the class, it will be possible for other students and teachers at the school to know who took part in the research. Written reports from the study will not enable anyone to identify your child. The data collected throughout this study may be used in publications, used in teaching or shared with other researchers, but confidentiality of teachers' and students' identities will be maintained at all times. The names of the schools participating may be made public so that they can be thanked for taking part in the research. If you are happy for the school to receive some feedback on how your child went on the interview tasks, please tick the box on the consent form.

Any questions regarding this project should be directed to Associate Professor Marj Horne, Faculty of Education, St Patrick's Campus, 115 Victoria Parade, Fitzroy 3065 (Tel: 9953 3289).

Following the data analysis, we would be pleased to share a copy of our major findings with you should you indicate an interest in this, and provide an address to which it should be sent.

This study has been approved by the Human Research Ethics Committee at Australian Catholic University and by the Department of Education, Victoria.

In the event that you have any complaint or concern about the way you or your child have been treated during the study, or if you have any query that the Principal Researcher has not been able to satisfy, you may write to: the Chair of the Human Research Ethics Committee, C/o Research Services, Australian Catholic University, Melbourne Campus, Locked Bag 4115, Fitzroy VIC 3065 (Tel: 9953 3157; Fax: 9953 3315). Any complaint or concern will be treated in confidence and fully investigated. You will be informed of the outcome.

If you agree for your child to participate in this project, you should sign both copies of the Consent Form, retain one copy for your records and return the other copy to your child's class teacher which will then be collected by the researcher. We are asking for a small number of children be filmed so that a) the filling in of record sheets with their answers to questions can be "double checked" by another researcher and b) so that footage of children's different strategies for attempting fractions tasks can be shown to other teachers and researchers, and at conferences or professional development sessions for teachers, including on the internet. If you are happy for your child to be part of the sample that may be filmed, please tick the box next to "filming permission" on the consent form. If you would like your child to take part in the project but **not** be filmed, simply do not tick the box, and they will not be filmed. ALL children will be audio-taped during the interview so that their responses to the tasks can be written down.

Yours sincerely

Marj Horne Annie Mitchell

Australian Catholic University
Brisbane Sydney Canberra Balarat Melbourne

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Facsimile 03 9953 3005
www.acu.edu.au

PARENT CONSENT FORM (Parent copy)

Title of Project: Primary School Children's Understanding of Fraction Concepts
Supervisor: Associate Professor Marj Horne
Student Researcher: Ms Annie Mitchell, PhD Candidate

I .... have read (*or, where appropriate*, have had read to me) and understood the information provided in the Parent Information Letter. I am happy for my child to complete a pen and paper test and take part in two mathematics one-to-one task-based interviews. I understand that my child will be audio-taped while taking part in this research. Any questions I have asked have been answered to my satisfaction. I agree for my child to participate in this activity, realising that I can withdraw this permission at any time. I agree that the data collected may be published or may be provided to researchers and teachers in professional development programs.

☐ filming permission
I give my permission for my child to be filmed during the one-to-one task-based interview. This footage would be used a) for checking the researcher's recording of information and b) for showing at professional development sessions and at conferences. I understand that this may include "snippets" of my child's interview being made available on the internet.

☐ I give my permission for the school to receive some feedback on how my child went on the tasks.

☐ I would like a copy of the major findings of this research project

Please forward this to my address below (or care of the school)

NAME OF CHILD:……………………………………………………………………………………

(block letters)

NAME OF PARENT OR GUARDIAN: ....

(block letters)

SIGNATURE .... DATE ....

SIGNATURE OF PRINCIPAL RESEARCHER .... DATE ....

ASSENT OF PARTICIPANTS AGED UNDER 18 YEARS

I ……………………… (*the participant aged under 18 years*) understand what this project is about. What I will be asked to do has been explained to me. I understand that I do a maths test and will be interviewed about my answers to mathematics tasks. I am happy to be audio-taped during this research. Information that I give may be used in publications and shared with other researchers and teachers but my name will not be used. I agree to take part in the project, realising that I can stop at any time without having to give a reason for my decision. If my parents have given permission for me to be filmed, I understand that this will be used to check the researcher's recording of information and could be shown to other teachers and at conferences, or on the internet.

NAME OF PARTICIPANT AGED UNDER 18: ....

(block letters)

SIGNATURE .... DATE...................................……....

SIGNATURE OF TEACHER OR PARENT WITNESS ....

DATE ....

SIGNATURE OF PRINCIPAL RESEARCHER .... DATE ....

Australian Catholic University
Brisbane Sydney Canberra Ballarat Melbourne

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Telephone 03 9953 3000
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www.acu.edu.au

PARENT CONSENT FORM (Researcher copy)

Title of Project: Primary School Children's Understanding of Fraction Concepts
Supervisor: Associate Professor Marj Horne
Student Researcher: Ms Annie Mitchell, PhD Candidate

I .... have read (*or, where appropriate*, have had read to me) and understood the information provided in the Parent Information Letter. I am happy for my child to complete a pen and paper test and take part in two mathematics one-to-one task-based interviews. I understand that my child will be audio-taped while taking part in this research. Any questions I have asked have been answered to my satisfaction. I agree for my child to participate in this activity, realising that I can withdraw this permission at any time. I agree that the data collected may be published or may be provided to researchers and teachers in professional development programs.

☐ filming permission
I give my permission for my child to be filmed during the one-to-one task-based interview. This footage would be used a) for checking the researcher's recording of information and b) for showing at professional development sessions and at conferences. I understand that this may include "snippets" of my child's interview being made available on the internet.

☐ I give my permission for the school to receive some feedback on how my child went on the tasks.

☐ I would like a copy of the major findings of this research project

Please forward this to my address below

NAME OF CHILD:……………………………………………………………………………………
(block letters)

NAME OF PARENT OR GUARDIAN: ....
(block letters)

SIGNATURE .... DATE ....

SIGNATURE OF PRINCIPAL RESEARCHER .... DATE ....

ASSENT OF PARTICIPANTS AGED UNDER 18 YEARS

I ……………………… (*the participant aged under 18 years*) understand what this project is about. What I will be asked to do has been explained to me. I understand that I do a maths test and will be interviewed about my answers to mathematics tasks. I am happy to be audio-taped during this research. Information that I give may be used in publications and shared with other researchers and teachers but my name will not be used. I agree to take part in the project, realising that I can stop at any time without having to give a reason for my decision. If my parents have given permission for me to be filmed, I understand that this will be used to check the researcher's recording of information and could be shown to other teachers and at conferences, or on the internet.

NAME OF PARTICIPANT AGED UNDER 18: ....
(block letters)

SIGNATURE .... DATE................................………...

SIGNATURE OF TEACHER OR PARENT WITNESS ....

DATE ....

SIGNATURE OF PRINCIPAL RESEARCHER .... DATE ....

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21st January 2011

To: Annie Mitchell
1/23 Chapman Street
North Melbourne
Vic 3051

Dear Ms Mitchell

Request to use the Freddo image and FREDDO trade mark in PhD thesis

We understand that you **(You)** wish to use the image of the Freddo chocolate and the FREDDO trade mark (the **Images**) in your doctorate thesis in mathematics education that will be available at the Australian Catholic University (the **Thesis**).

You have asked for Cadbury Pty Ltd's **(Cadbury)** consent to such use.

Cadbury consents to Your proposed use of the Images on the following terms:

1. You will only use the Images:
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   - in Australia.
2. Neither party is required to make payment to the other party pursuant to this letter agreement.
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Please confirm Your agreement to these terms by signing a copy of this letter and returning it to us as soon as possible.

Yours sincerely

Zahra Banatwala
Cadbury Pty Ltd

Signed by:

[signature]

Print name:

ZAHRA BANATWALA

Print title:

BRAND MANAGER PRE-TEENS

Date:

21 January 2011

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Appendix C: Summary Statistics for Victorian Government Schools

Reprinted from Department of Education and Early Childhood Development (2008).

Summary Statistics for Victorian Schools

March 2008 **Next edition July 2008**

This brochure provides a ready reference for the latest statistics on school education. It is reissued when a later set of information becomes available.

Size of the Victorian school education system

1. Numbers of schools, students and teachers

	Government	Catholic	Independent	All Schools
Number of schools by school type, February 2008				
Primary	1,201	379	47	1,627
Primary–secondary	53	13	139	205
Secondary	253	86	21	360
Special	76	8	14	98
Language	4	–	–	4
Total	**1,587**	**486**	**221**	**2,294**
Number of students (FTE) by year level, February 2007				
Preparatory	43,400.1	14,176.2	6,082.4	63,647.7
Year 1	43,378.0	14,169.4	5,677.8	63,225.2
Year 2	43,780.8	13,984.2	5,625.8	63,390.8
Year 3	43,713.6	13,880.7	5,909.0	63,503.3
Year 4	43,813.3	13,933.4	6,083.6	63,830.3
Year 5	43,906.9	14,112.2	6,845.2	64,864.3
Year 6	44,134.3	14,052.0	7,114.6	65,300.9
Ungraded	6.0	-	43.6	49.6
Primary total	306,223.0	98,307.1	43,282.0	447,812.1
Year 7	38,737.7	15,526.4	11,860.2	65,124.3
Year 8	39,417.7	15,370.4	11,642.0	65,430.1
Year 9	39,931.8	15,009.2	11,769.1	65,710.1
Year 10	37,991.8	14,390.6	11,717.6	64,100.0
Year 11	37,523.9	13,526.2	12,304.1	63,354.2
Year 12	29,663.5	11,781.4	10,976.1	52,421.0
Ungraded	24.2	-	24.0	48.2
Secondary total	223,290.6	86,604.2	70,293.1	373,187.9
Special	8,006.3	153.6	445.6	8,604.5
Language	1,339.0	-	-	1,339.0
Total	**538,857.9**	**184,064.9**	**114,020.7**	**836,943.5**

Source: DEECD *February School Census* NB: Figures on student FTE include all changes from the 2007 school enrolment audit process.

Number of teaching staff (FTE) in schools on pay by school type, as at end of quarter

	December 2006	*March 2007*	*June 2007*	*September 2007*	*December 2007*
Primary	18,811.8	r18,908.0	18,929.7	18,916.1	18,794.3
Secondary	17,048.4	r17,221.7	17,306.8	17,288.1	17,066.1
Special/P–12/ Lang/Other	3,997.1	r4,130.5	4,175.2	4,212.8	4,177.5
Total	**39,857.4**	**r40,260.2**	**40,410.7**	**40,416.9**	**40,037.9**

Source: DEECD *Quarterly Workforce Summary* r = revised

2. Historical trend in numbers of government schools & students, February

Year	*Number of Schools*	*Number of Students (FTE)*				
		Primary	*Secondary*	*Special*	*Language*	*Total*
1999	1,635	306,218.0	216,367.4	5,413.6	1,073.0	529,072.0
2003	1,615	312,134.4	218,740.7	5,517.0	917.0	538,309.1
2004	1,610	311,964.0	220,271.7	6,826.2	1,010.0	540,071.9
2005	1,617	309,972.6	221,618.3	7,219.4	1,142.0	539,952.3
2006	1,606	307,576.5	222,626.7	7,756.1	1,184.0	539,343.3
2007	1,594	306,223.0	223,290.6	8,005.3	1,339.0	538,857.9

Source: DEECD *February School Census*

Compiled by **Statistical Information and Analysis Unit: 9637 3225**
http://www.education.vic.gov.au/about/publications/newsinfo/default.htm

State Government Victoria
Department of Education and Early Childhood Development

March 2008

Profile of Victorian government school students

3. Number (FTE) of students by sex in government schools, February

Year	Male	Female	Percentage of male students: Primary	Yr 7–10	Yr 11–12	All Students
1999	271,859.0	257,213.0	51.7	51.7	47.6	51.4
2003	277,491.6	260,817.5	51.7	51.8	48.7	51.5
2004	278,370.3	261,701.6	51.7	52.0	48.4	51.5
2005	278,333.8	261,618.5	51.7	52.0	48.1	51.5
2006	278,111.9	261,231.4	51.7	52.1	48.1	51.6
2007	278,343.8	260,514.1	51.7	52.1	48.6	51.7

Source: DEECD *February School Census*. NB: Yr 7–10 includes secondary ungraded students.

4. Number (FTE) of students with disabilities in government schools, August

Year	In regular schools	In special schools	Total	% of total student cohort
1999	8,262	5,506	13,768	2.62
2003	12,351	6,458	18,809	3.49
2004	13,964	7,180	21,144	3.93
*2005	8,824	7,618	16,442	3.00
*2006	9,419	7,252	16,671	3.10
*2007	9,691	7,707	17,398	3.25

Source: DEECD *August School Census* and records. *Excludes students in the Language Support Program.

5. Number (FTE) of Indigenous students in government schools and per cent of student cohort, August

Year	Primary Number	Primary %	Years 7–10 Number	Years 7–10 %	Years 11–12 Number	Years 11–12 %	Special Number	Special %	Total Number	Total %
1999	3,376.4	1.1	1,387.1	0.9	277.8	0.5	70.9	1.3	5,112.2	1.0
2003	4,019.3	1.3	1,651.4	1.1	348.0	0.5	125.9	1.9	6,144.6	1.1
2004	4,263.3	1.4	1,836.1	1.2	387.7	0.6	138.5	2.0	6,625.6	1.2
2005	4,368.5	1.4	1,983.1	1.3	432.9	0.7	151.9	2.1	6,936.4	1.3
2006	4,502.7	1.5	2,104.1	1.3	r448.7	0.7	r158.0	2.0	r7,213.5	1.3
2007	4,629.0	1.5	2,252.9	1.4	536.9	0.8	176.3	2.2	7,595.1	1.4

Source: DEECD *August School Census* r = revised
NB Years 7–10 includes secondary ungraded students.

6. Number of students – language backgrounds other than English, August

Year	Speak mainly English at home: Yes	No	Total	Most common non-English languages spoken at home
2004	63,090	72,733	135,823	Vietnamese, Arabic, Cantonese, Turkish
2005	61,596	74,945	136,541	Vietnamese, Arabic, Cantonese, Turkish
2006	61,166	76,799	137,965	Vietnamese, Arabic, Cantonese, Turkish
2007	33,423	79,801	113,224	Vietnamese, Arabic, Cantonese, Turkish

Source: DEECD *August School Census* NB: 2007 figures can not be compared to previous years due to a change in definition. From 2007 onwards, a student is from a language background other than English if either the student or one parent/guardian speaks a language other than English at home.

7. Provision of languages other than English – government schools, August

Year	Primary LOTE Students	Primary LOTE %	Secondary LOTE Students	Secondary LOTE %	Languages with highest Enrolments in VCE Unit 4
1999	272,696	88.7	115,015	54.2	French, German, Japanese, Indonesian
2003	267,827	85.5	115,109	53.3	Chinese, French, Japanese, German
2004	271,192	86.7	121,445	56.0	Chinese, French, Japanese, Indonesian
2005	265,989	85.5	117,655	53.9	Chinese, French, Japanese, Indonesian
2006	248,873	80.6	114,498	52.1	Chinese, French, Japanese, Italian

Source: DEECD *August LOTE Surveys*

8. International students (fee paying) in government schools

Year	At June 30th	Main countries of origin
1999	787	Japan, China, Indonesia, Hong Kong, Vietnam
2003	2,212	China, Japan, Korea, Vietnam, Hong Kong
2004	2,378	China, Korea, Japan, Vietnam, Hong Kong
2005	2,148	China, Korea, Japan, Vietnam, Hong Kong
2006	2,128	China, Korea, Japan, Vietnam, Hong Kong
2007	2,589	China, Korea, Vietnam, Japan, Hong Kong

Source: DEECD International Education database.

9. VET in schools programs – certificate enrolments

Year	Government schools	Non-government schools	Total certificate enrolments
1999	10,103	4,773	14,876
2003	21,771	9,804	31,575
2004	25,985	11,976	37,961
2005	28,877	13,352	42,229
2006	30,134	15,433	45,567
2007	31,759	17,008	48,767

Source: VCAA database NB: Excludes students whose home enrolment is a non-school setting. Students may enrol in more than one VET certificate.

Retention and transition rates

10. Apparent retention rates[1] by sex and sector, February (per cent)

Year	Apparent retention Years 10–12			Apparent retention Years 7–12		
	Govt	Non-gov	All schools	Govt	Non-gov	All schools
1999	79.2	89.0	82.8	76.5	89.2	81.2
2003	82.7	93.5	86.9	81.2	93.4	85.8
2004	82.9	94.1	87.2	81.3	94.2	86.2
2005	82.7	92.0	86.4	80.3	92.7	85.1
2006	81.1	93.0	85.8	78.8	93.0	84.4
2007	81.3	91.7	85.5	79.9	91.1	84.4
Male	*74.2*	*87.5*	*79.4*	*72.1*	*85.9*	*77.5*
Female	*88.8*	*95.8*	*91.7*	*88.4*	*96.3*	*91.8*

Source: DEECD *February School Census*. Refer footnote[1] under Table 16 for definition of apparent retention rate and explanation of differences between February and August figures.

11. Transition rates for government schools, February (per cent)

	1999–00	2002–03	2003–04	2004–05	2005–06	2006–07
Year 9–10	97.1	97.2	97.1	97.1	97.6	97.2
Year 10–11	95.4	96.8	96.1	97.2	96.1	97.1
Year 11–12	82.6	86.6	85.6	84.3	83.4	82.8

Source: DEECD *February School Census*

Size of classes

12. Class sizes in government schools, February

	1999	2003	2004	2005	2006	2007
Primary classes						
Average class size – all classes	25.4	22.9	22.8	22.6	22.4	22.3
Percent of all classes over 30 (%)	4.6	0.5	0.4	0.3	0.3	0.2
Average class size – Prep	23.2	19.7	19.7	19.6	19.4	19.4
Average class size – P–2	24.3	21.0	20.9	20.8	20.8	20.7
Percent of P–2 classes 21 or less (%)	17.4	52.7	54.4	55.6	55.0	54.4
Average class size – Years 3–6	26.2	24.3	24.3	24.0	23.7	23.4
Secondary English classes						
Average class size – all classes	22.7	22.0	21.9	21.9	21.5	21.6
Average class size – Year 12	21.0	20.2	20.2	19.9	19.5	19.7

Source: DEECD *February School Census* NB: Comparable interstate data is not readily available as class size is not part of the National Schools Statistics Collection.

Regional summary – government schools

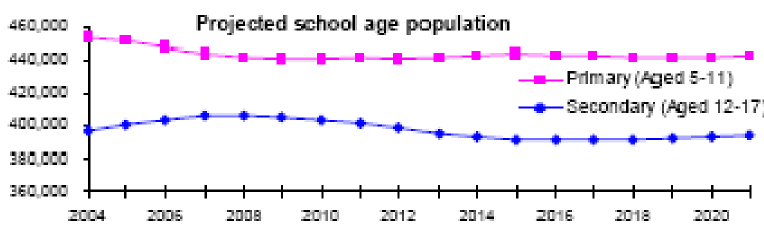
13. Schools, students, apparent retention (ARR) and class sizes by region, February

Region	February, 2007				Feb, 1999		Feb, 2007	
	Schools	Students (FTE)	ARR		Average class size			
			10–12	7–12	P–2	All Prim	P–2	All Prim
Western Metropolitan	144	68,096.6	81.2	82.6	25.1	26.1	21.4	23.0
Northern Metropolitan	201	77,492.8	88.1	87.6	25.1	26.2	21.2	22.6
Eastern Metropolitan	248	105,000.3	86.5	86.3	25.2	26.3	21.0	22.7
Southern Metropolitan	246	117,687.5	82.8	87.2	25.3	26.3	21.2	22.8
Barwon South Western	141	40,887.0	72.8	69.2	23.2	24.3	20.6	22.1
Grampians	131	25,987.5	72.9	67.6	22.7	23.7	19.0	20.5
Loddon Mallee	169	40,192.4	74.9	70.3	22.8	24.0	19.7	21.2
Hume	163	33,207.2	73.7	66.8	22.7	23.9	19.2	21.0
Gippsland	151	30,306.6	72.3	68.4	22.4	23.8	19.6	21.1
Metropolitan regions	839	368,277.2	85.3	86.2	25.2	26.3	21.2	22.8
Non-metropolitan regions	755	170,580.7	73.4	68.6	22.8	23.9	19.6	21.2
Total	**1594**	**538,857.9**	**81.3**	**79.9**	**24.3**	**25.4**	**20.7**	**22.3**

Source: DEECD *February School Census*

Projected growth in school enrolments

14. Projected growth in school-aged population, June (Source: ABS June 2006)



Interstate comparisons including some key indicators

Apparent retention rates

15. Year 10–12 apparent retention rates[1], August (per cent) Full-time students

August	NSW	Vic	Qld	SA	WA	Tas	NT	ACT	Australia
Government									
1999	64.7	**73.5**	73.2	64.8	67.8	67.5	70.0	107.1	69.6
2004	68.6	**77.2**	75.0	62.9	66.7	76.5	90.8	100.8	72.2
2005	68.5	**77.0**	72.7	64.4	66.3	67.2	76.2	99.5	71.3
2006	68.2	**75.8**	72.2	64.6	66.0	64.4	79.0	101.1	70.8
2007	67.8	**76.2**	72.3	65.7	63.7	63.7	75.7	96.6	70.5
All schools									
1999	70.0	**78.7**	78.3	71.4	71.5	68.9	64.7	92.5	74.4
2004	73.2	**83.0**	80.6	71.6	72.4	76.3	75.2	88.4	77.2
2005	73.2	**82.2**	79.3	72.1	72.2	67.8	69.5	88.1	76.5
2006	73.0	**82.1**	r78.6	72.7	71.4	65.0	68.0	88.9	r76.2
2007	72.4	**81.8**	78.6	73.3	69.5	65.3	65.5	85.9	75.6
Male	*67.6*	***76.0***	*74.8*	*67.1*	*64.0*	*57.6*	*63.3*	*85.6*	*70.6*
Female	*77.4*	***87.7***	*82.6*	*79.9*	*75.3*	*73.4*	*67.7*	*86.1*	*80.8*

16. Year 7–12 apparent retention rates[1], August (per cent) Full time students

August	NSW	Vic	Qld	SA	WA	Tas	NT	ACT	Australia
Government									
1999	61.2	**69.8**	71.8	58.1	66.5	65.7	60.0	110.0	66.4
2004	65.8	**74.4**	75.3	58.0	65.9	70.7	72.0	100.5	69.8
2005	65.8	**74.0**	73.0	61.7	65.4	65.5	70.5	99.6	69.4
2006	65.1	**72.6**	71.6	61.9	65.1	63.2	72.3	103.2	68.5
2007	64.5	**73.9**	71.1	64.0	63.1	63.3	69.4	96.6	66.3
All schools									
1999	67.6	**76.2**	77.5	67.0	71.5	66.7	52.9	92.5	72.3
2004	71.1	**81.1**	81.2	68.0	72.6	76.4	59.0	88.5	75.7
2005	71.1	**80.6**	79.9	70.7	72.5	67.1	59.1	87.5	75.3
2006	70.5	**79.9**	r78.8	71.5	71.8	64.8	58.4	88.7	74.7
2007	69.7	**80.1**	78.5	72.7	70.3	65.4	61.7	85.2	74.3
Male	*64.7*	***73.3***	*73.9*	*66.2*	*64.5*	*57.4*	*62.1*	*84.0*	*68.8*
Female	*74.9*	***87.4***	*83.3*	*79.5*	*76.4*	*73.9*	*61.3*	*86.5*	*80.1*

Source: ABS, *Schools Australia* r revised

[1] ABS Year 10–12 & 7–12 apparent retention rates refer to Year 12 enrolment of students in full-time school education expressed as a proportion of Year 10/7 enrolments two/five years earlier. DEECD calculates its retention rate on an FTE basis. Note that apparent retention rates calculated for February are higher than for August as a number of Year 12 students leave during the year.

Participation rates

17. Participation rates – all schools, August (per cent of population)

Age	Year	NSW	Vic	Qld	SA	WA	Tas	NT	ACT	Aust.
14	2004	97.3	**98.8**	93.0	97.7	98.3	99.3	92.0	108.5	98.1
	2005	97.4	**98.6**	97.7	98.2	99.1	98.3	89.4	111.2	98.1
	2006	97.0	**98.8**	97.1	98.6	98.3	100.5	90.1	110.7	97.9
	2007	97.8	**99.4**	97.7	99.2	98.0	99.5	86.0	112.8	98.4
15	2004	92.7	**94.4**	92.1	95.5	92.1	99.1	86.5	104.4	93.4
	2005	93.5	**96.0**	91.9	96.3	92.2	99.5	86.6	107.0	94.1
	2006	93.6	**96.3**	92.6	96.4	95.6	98.9	84.6	110.2	94.7
	2007	93.1	**96.2**	92.8	97.6	94.0	100.9	83.9	109.2	94.5
16	2004	79.8	**88.5**	83.4	87.3	78.0	89.0	75.4	100.2	83.5
	2005	79.1	**88.1**	82.4	87.4	77.7	87.2	73.8	101.0	82.9
	2006	80.2	**90.2**	82.6	88.5	80.3	87.6	72.8	102.7	84.2
	2007	80.2	**90.3**	83.1	90.8	79.7	86.9	71.8	104.4	84.4
17	2004	68.6	**78.8**	51.5	67.8	41.6	69.9	54.8	89.7	65.1
	2005	68.6	**77.3**	50.1	68.0	42.6	67.4	51.6	90.4	64.6
	2006	68.2	**78.2**	49.3	69.9	41.3	67.7	51.2	90.5	64.4
	2007	68.2	**79.6**	43.8	71.0	40.7	67.8	49.0	91.0	64.5
18	2004	15.9	**21.7**	6.3	14.2	4.9	18.8	16.9	21.7	14.3
	2005	15.6	**21.6**	5.7	13.5	4.3	25.7	14.2	24.5	14.2
	2006	15.8	**22.7**	5.3	14.5	4.0	28.1	12.7	23.3	14.5
	2007	15.8	**22.9**	5.1	14.4	3.5	28.0	13.3	22.3	14.4
19	2004	2.2	**2.8**	1.3	4.0	1.3	4.4	4.0	2.5	2.3
	2005	2.1	**2.4**	1.1	3.7	1.2	3.8	4.4	2.1	2.1
	2006	1.8	**2.2**	0.9	3.6	0.9	3.9	4.3	2.1	1.9
	2007	1.6	**2.3**	0.8	3.9	0.8	3.7	2.8	1.6	1.8

Source: ABS *Schools Australia* NB: School participation rates now include both full time and part time students as well as 14 year olds. Previous reports showed full time students only for 15 to 19 year olds.

Year 12 completion rates

18. Victorian young people successfully completing Year 12 or equivalent (per cent)

(a) Per cent of young people (aged 19) who have successfully completed Year 12 or qualification at Australian Qualifications Framework (AQF) 2 or above

	1999	2000	2001	2002	2003	2004	2005	2006
Victoria	73.6	74.4	75.7	75.8	r78.4	r79.5	r79.1	81.0

Source: VCAA & DIIRD certificate completion data and ABS population estimates. r = revised
NB: The rate from 2003 includes actual completions in the VET sector. The rate up to 2002 includes an estimate of VET completions based on the ABS Census.

(b) Percent of 20–24 year-olds who have completed Year 12 or a qualification at AQF 2 or above

	1999	2000	2001	2002	2003	2004	2005	2006	2007
Victoria	82.9	81.8	r82.1	r82.8	r84.9	r84.7	r83.9	85.5	86.1
Australia	80.1	80.3	r79.1	r80.0	r80.4	r81.3	r81.2	81.9	83.5

Source: ABS *Survey of Education and Work* r = revised
NB: In 2008 the ABS issued a revision of this series which has resulted in a change to the historical series.

Number of schools

19. Number of schools by sector, August 2007

Sector	NSW	Vic	Qld	SA	WA	Tas	NT	ACT	Australia
Government	2,190	**1,593**	1,250	602	769	212	149	88	6,853
Non-government	917	**699**	465	201	299	67	36	44	2,728
Total	3,107	**2,292**	1,715	803	1,068	279	185	132	9,581

Source: ABS *Schools Australia*

Number of students

20. Government and non-government full-time enrolments, August

Sector	NSW	Vic	Qld	SA	WA	Tas	NT	ACT	Australia
1999									
Govt	763,169	**624,848**	425,876	176,303	227,232	62,954	28,487	38,804	2,247,674
N-govt	326,423	**269,706**	168,708	73,920	89,377	20,859	8,280	21,704	978,976
Total	1,089,592	**794,664**	594,584	250,223	316,609	83,813	36,767	60,508	3,226,650
2004									
Govt	744,229	**636,216**	448,806	165,866	229,766	60,685	28,335	35,821	2,249,724
N-govt	362,820	**286,084**	190,149	82,656	106,300	21,577	8,695	23,959	1,082,240
Total	1,107,049	**822,300**	638,955	248,522	336,066	82,262	37,030	59,780	3,331,964
2005									
Govt	740,439	**638,636**	450,964	164,714	228,817	60,605	28,554	35,359	2,246,087
N-govt	367,247	**288,312**	196,290	84,711	109,483	21,899	8,819	24,291	1,102,052
Total	1,107,686	**826,947**	647,254	249,425	338,300	82,504	37,373	59,650	3,348,139
2006									
Govt	739,307	**638,117**	455,075	163,848	230,293	60,007	28,506	35,076	2,248,229
N-govt	369,640	**293,718**	r202,722	86,078	112,349	22,447	9,074	24,460	r1,120,488
Total	1,108,947	**928,836**	r657,797	249,926	342,642	82,454	37,580	59,536	r3,368,717
2007									
Govt	737,637	**636,883**	478,883	163,904	229,611	58,926	28,916	34,617	2,268,377
N-govt	371,566	**297,870**	219,020	87,545	114,977	22,933	9,355	24,780	1,148,146
Total	1,109,203	**933,863**	697,903	251,449	344,588	81,859	38,271	59,397	3,416,523

Source: ABS *Schools Australia* r = revised

21. Full-time students in government sector, August (per cent)

Year	NSW	Vic	Qld	SA	WA	Tas	NT	ACT	Australia
1999	70.0	**68.1**	71.6	70.5	71.8	75.1	77.5	64.1	69.7
2004	67.2	**66.2**	70.2	66.7	68.4	73.8	76.5	59.9	67.5
2005	66.8	**66.0**	69.7	66.0	67.6	73.5	76.4	59.3	67.1
2006	66.7	**64.6**	69.3	65.6	67.2	72.8	75.9	58.9	66.8
2007	66.5	**64.3**	68.6	65.2	66.6	72.0	75.6	58.3	66.4
Difference 2006–2007	-0.2	-0.3	-0.7	-0.4	-0.6	-0.8	-0.3	-0.6	-0.4

Source: ABS *Schools Australia* NB: Difference from 2006-2007 reflects the percentage point difference between the 2007 and 2006 rate.

Staffing details

22. Number of teachers (FTE) in government sector, August

Year	NSW	Vic	Qld	SA	WA	Tas	NT	ACT	Australia
1999	50,108.0	**36,167.0**	29,164.0	11,952.0	14,866.0	4,351.0	2,221.0	2,665.0	150,484.0
2004	50,215.0	**37,783.0**	31,191.0	11,542.0	16,129.0	4,252.0	2,286.0	2,757.0	156,156.0
2005	50,704.0	**38,141.0**	31,221.0	11,473.0	15,826.0	4,201.0	2,244.0	2,754.0	156,564.0
2006	51,384.5	**38,808.2**	31,609.3	11,630.1	15,732.5	4,185.2	2,309.1	2,735.2	158,194.2
2007	51,286.0	**38,943.2**	33,203.0	11,608.8	16,635.7	4,148.2	2,293.1	2,673.1	160,791.1

Source: ABS *Schools Australia* NB: The ABS has specific definitions for counting teachers. This excludes teachers not in schools or ancillary education establishments, those on more than four weeks leave and casual relief teachers. The above teacher data are used in the national calculations of student–teacher ratios.

Student - teacher ratios

23. Student–teacher ratios in government schools, August

Government schools		*NSW*	*Vic*	*Qld*	*SA*	*WA*	*Tas*	*NT*	*ACT*	*Aust.*
Primary	1999	17.7	**17.2**	16.0	16.9	17.6	15.7	13.8	17.1	17.0
	2004	17.0	**16.2**	15.4	16.2	16.2	15.9	13.5	14.2	16.2
	2005	16.7	**16.1**	15.5	16.1	16.3	15.9	13.6	13.8	16.1
	2006	16.2	**15.9**	15.5	15.7	16.2	15.8	13.3	13.8	15.8
	2007	16.2	**15.7**	15.5	15.6	15.4	15.6	13.7	13.6	15.7
Secondary	1999	12.7	**12.0**	12.7	12.4	12.6	13.7	11.6	12.3	12.7
	2004	12.5	**12.1**	13.0	12.5	11.7	13.2	11.0	11.8	12.4
	2005	12.4	**12.0**	13.0	12.5	12.0	13.2	11.6	11.8	12.4
	2006	12.4	**11.9**	13.0	12.5	12.5	13.2	11.2	11.9	12.4
	2007	12.5	**11.8**	12.9	12.7	11.7	13.1	10.9	12.2	12.3

Source: ABS *Schools Australia*

Expenditure on school education

24. Real[#] (in-school) per student expenditure by state ($) – government schools

	NSW	*Vic*	*Qld*	*SA*	*WA*	*Tas*	*NT*	*ACT*	*Aust*
Prim 2004–05	9,995	**8,583**	9,751	9,529	10,592	9,118	12,343	10,941	9,673
Prim 2005–06	9,769	**8,767**	9,809	9,734	10,684	9,510	12,866	10,903	9,699
**%Diff Nat Average*	*+0.7*	***-9.6***	*+1.1*	*+0.4*	*+10.2*	*-1.9*	*+32.7*	*+12.4*	
Sec 2004–05	12,591	**11,383**	11,721	12,146	14,167	11,519	16,645	13,453	12,264
Sec 2005–06	12,397	**11,329**	11,600	12,017	13,732	11,877	17,904	14,295	12,148
**%Diff Nat Average*	*+2.0*	***-6.7***	*-4.5*	*-1.1*	*+13.0*	*-2.2*	*+47.4*	*+17.7*	

Source: *Report on Government Services 2008* #Real dollars are previous year's expenditure in current year's dollars after basing expenditure on the ABS GDP price deflator 2005–06=100. Schools' data are total government expenditure on government schools divided by two year average FTE student population in 2004-2005 and 2005-2006. *Percentage difference between state/territory result and national average expenditure for 2005-06.

Student achievement – literacy and numeracy benchmarks

25. Percent achieving benchmark 2006[#]

	NSW	*Vic*	*Qld*	*SA*	*WA*	*Tas*	*NT*	*ACT*	*Aus*
Year 3									
Reading	93.1 ±1.7%	**91.6** ±2.0%	94.5 ±1.3%	93.1 ±1.8%	94.0 ±1.6%	94.1 ±1.3%	70.8 ±2.6%	96.4 ±0.8%	93.0 ±1.7%
Writing	93.8 ±1.9%	**96.6** ±0.5%	95.3 ±0.6%	92.2 ±2.4%	90.2 ±1.4%	90.8 ±1.6%	66.6 ±3.6%	93.8 ±1.8%	93.9 ±1.3%
Numeracy	95.8 ±0.8%	**96.6** ±0.7%	88.9 ±2.3%	91.5 ±1.3%	88.4 ±2.5%	88.5 ±1.8%	85.4 ±2.1%	94.3 ±1.6%	93.0 ±1.4%
Year 5									
Reading	90.3 ±1.1%	**88.8** ±1.4%	81.2 ±3.1%	88.0 ±1.4%	92.8 ±1.4%	94.1 ±1.0%	74.5 ±2.0%	95.6 ±0.6%	88.4 ±1.6%
Writing	93.9 ±2.0%	**97.6** ±0.1%	96.0 ±0.4%	92.7 ±2.8%	84.7 ±2.3%	87.5 ±1.8%	66.1 ±3.1%	96.5 ±1.2%	93.8 ±1.3%
Numeracy	92.6 ±1.2%	**94.6** ±0.9%	85.4 ±1.6%	88.3 ±1.7%	86.0 ±1.5%	88.7 ±1.5%	70.0 ±2.2%	93.0 ±1.4%	90.3 ±1.3%
Year 7									
Reading	88.4 ±0.9%	**94.6** ±0.5%	85.6 ±1.0%	93.3 ±0.4%	84.4 ±0.8%	86.5 ±1.1%	72.3 ±2.0%	94.2 ±0.9%	89.2 ±0.8%
Writing	93.0 ±2.1%	**96.4** ±0.5%	96.0 ±0.3%	87.7 ±5.0%	85.5 ±1.6%	81.7 ±2.0%	61.6 ±2.7%	91.4 ±2.9%	92.4 ±1.5%
Numeracy	72.7 ±1.6%	**84.6** ±0.7%	79.8 ±1.2%	87.3 ±0.8%	84.5 ±0.7%	80.4 ±1.2%	67.3 ±1.9%	89.5 ±1.2%	79.7 ±1.1%

Source: MCEETYA *National Report on Schooling in Australia*–Preliminary paper 2006 # Most recent data are for 2006. Shaded cells = 'at or above the national average'. NB: National benchmark data are subject to significant measurement error. Care should be taken when making comparisons between states.

26. Victorian data – student cohort groups (2006)

	Reading			**Writing**			**Numeracy**		
	Year 3	**Year 5**	**Year 7**	**Year 3**	**Year 5**	**Year 7**	**Year 3**	**Year 5**	**Year 7**
Male	89.3 ±2.4%	87.7 ±1.7%	93.6 ±0.6%	95.2 ±0.8%	96.7 ±0.2%	93.2 ±0.8%	95.2 ±0.7%	94.5 ±0.9%	84.9 ±0.7%
Female	93.8 ±1.6%	92.1 ±1.2%	96.2 ±0.5%	98.0 ±0.3%	98.6 ±0.1%	97.7 ±0.3%	96.6 ±0.7%	95.3 ±0.9%	84.3 ±0.9%
Indigenous	81.5 ±5.1%	69.7 ±4.9%	80.7 ±3.8%	91.8 ±2.9%	93.5 ±1.7%	83.3 ±3.9%	90.7 ±2.8%	84.4 ±4.3%	60.0 ±4.0%
LBOTE	90.1 ±2.2%	87.6 ±1.7%	93.2 ±0.8%	96.1 ±0.5%	97.0 ±0.2%	95.3 ±0.6%	94.7 ±0.8%	93.7 ±0.9%	83.5 ±0.9%

Shaded cells indicate 'at or above national average' for sub-group. NB: Measurement error as above.

27. Useful Websites and on-line resources

Report on Government Services: www.pc.gov.au/gsp/rogs
National Report on Schooling in Australia www.mceetya.edu.au
Schools Australia http://www.abs.gov.au/AUSSTATS/abs@.nsf/DetailsPage/4221.02007?OpenDocument
Statistical Tables–EMIS (Access via 'Start' menu)
EdStats https://portal.eduweb.vic.gov.au/reportsstats/edstats/Pages/default.aspx