IFT-UAM/CSIC-05-50
CERN-PH-TH/2005-248
hep-th/0603099
March 13^{th}, 2006

The supersymmetric configurations of $N = 2, d = 4$ supergravity coupled to vector supermultiplets

Patrick Meessen[♢ 1], *and Tomás Ortín*[♠ 2]

♢ *Physics Department, Theory, CERN, Geneva, Switzerland*

♠ *Instituto de Física Teórica UAM/CSIC*
Facultad de Ciencias C-XVI, C.U. Cantoblanco, E-28049-Madrid, Spain

Abstract

We classify all the supersymmetric configurations of ungauged $N = 2, d = 4$ supergravity coupled to n vector multiplets and determine under which conditions they are also classical solutions of the equations of motion. The supersymmetric configurations fall into two classes, depending on the timelike or null nature of the Killing vector constructed from Killing spinor bilinears. The timelike class configurations are essentially the ones found by Behrndt, Lüst and Sabra, which exhaust this class and are the ones that include supersymmetric black holes. The null class configurations include pp-waves and cosmic strings.

[1]E-mail: `Patrick.Meessen@cern.ch`
[2]E-mail: `Tomas.Ortin@cern.ch`

Contents

1 Introduction and main results

Classical supersymmetric solutions of supergravity theories play a key rôle in many of the recent developments in string theory, provide vacua, on which the theory can be quantized and may be interesting for phenomenology, and objects that live in those vacua such as p-branes, black holes etc. Therefore, the amount of effort that is being devoted to the classification of supersymmetric solutions of supergravity theories, both in higher [1]-[17] and lower [18]-[26] dimensions, can hardly be called a surprise.

$N = 2, d = 4$ supergravities, the cases under consideration, are particularly interesting theories: they are simple enough to be manageable and yet rich enough in structure, duality symmetries, interesting solutions and phenomena. Many, but not all, of them are also related to low-energy limit of Calabi-Yau compactifications of 10-dimensional type II superstring theories. There is a very extensive literature on these theories[3] but, except for the simplest cases of pure gauged and ungauged supergravity [28, 23], and for black-hole type solutions,[4] there have been no systematic attempts to classify all their supersymmetric solutions. In this work we start filling this gap by classifying all the supersymmetric configurations and solutions in the next-to-simplest case, namely pure supergravity coupled to n vector supermultiplets whose supersymmetric black-hole solutions have been studied intensively in the not so remote past. This work should also lay the groundwork for the more complicated cases we intend to study next.

In this work we use the method of Ref. [19], consisting in finding differential and algebraic equations satisfied by the tensors that can be built as bilinears of the Killing spinor, whose existence we assume from the onset. We then derive consistency conditions for these equations to admit solutions and determine necessary conditions for the backgrounds to be supersymmetric. Subsequently we show that the conditions are also sufficient, meaning that we have identified all the supersymmetric *configurations* of the theory. Finally we impose the equations of motion in order to find the supersymmetric *solutions*. Throughout this work we stress the difference between generic supersymmetric field *configurations* and classical *solutions* of the equations of motion. We will also make use of the *Killing spinor identities*, derived in Refs. [33, 34], to minimize the number of independent equations of motion that need to be checked explicitly in order to prove that a given supersymmetric configuration is a solution.

Let us briefly describe our results: the supersymmetric solutions of $N = 2, d = 4$ supergravity coupled to n vector supermultiplets belong to two main classes:

1. Those with a timelike Killing vector. They are essentially the field configurations found in Ref. [35], and include families of solutions of $N = 4, 8, d = 4$ supergravity such as those found and studied, for instance, in Refs. [49, 38, 37]. These solutions were shown in Ref. [36] to be the only supersymmetric ones with Killing spinors satisfying the constraint Eq. (4.17). Here we show that all the supersymmetric solutions

[3] A good review on these theories is Ref. [27].

[4] See, *e.g.* the reviews [29, 30, 31, 32] and references therein.

in the timelike class admit Killing spinors that satisfy that constraint and, therefore there are no more supersymmetric configurations nor solutions in this class.

These supersymmetric configurations are completely determined by a choice of symplectic section $\mathcal{V}/X$. The metric is then given by

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}dx^i dx^i\,, \tag{1.1}$$

where

$$|M|^{-2} = 2e^{\mathcal{K}[\mathcal{V}/X,(\mathcal{V}/X)^*]}\,, \tag{1.2}$$

and where $\mathcal{K}[\mathcal{V}/X,(\mathcal{V}/X)^*]$ means that the Kähler potential has to be computed using in the expression Eq. (C.24) the components of the symplectic section $\mathcal{V}/X$. $\omega = \omega_{\underline{i}}dx^i$ is a time-independent 1-form that has to satisfy the constraint

$$(d\omega)_{mn} = \epsilon_{mnp}e^{-\mathcal{K}[\mathcal{V}/X,(\mathcal{V}/X)^*]}\mathcal{Q}_p[\mathcal{V}/X,(\mathcal{V}/X)^*]\,, \tag{1.3}$$

where $\mathcal{Q}_p[\mathcal{V}/X,(\mathcal{V}/X)^*]$ is the pullback of the Kähler 1-form connection, computed in the same fashion.

The vector field strengths are given by

$$F = \tfrac{1}{\sqrt{2}}\{d[|M|^2\mathcal{R}(dt+\omega)] - {}^\star[|M|^2 d\mathcal{I}\wedge(dt+\omega)]\}\,. \tag{1.4}$$

where $\mathcal{R}$ and $\mathcal{I}$ stand, respectively, for the real and imaginary parts of the symplectic section $\mathcal{V}/X$.

The scalar fields Z^i can be computed by taking the quotients

$$Z^i = (\mathcal{V}/X)^i/(\mathcal{V}/X)^0\,. \tag{1.5}$$

The supersymmetric configurations are classical solutions iff the real section $\mathcal{I}$ is harmonic on $\mathbb{R}^3$. Rewriting the equation that determines ω as

$$(d\omega)_{mn} = 2\epsilon_{mnp}\langle\,\mathcal{I}\mid\partial_p\mathcal{I}\,\rangle\,, \tag{1.6}$$

we see that its integrability condition

$$\langle\,\mathcal{I}\mid\partial_p\partial_p\mathcal{I}\,\rangle = 0\,, \tag{1.7}$$

is satisfied for $\mathcal{I}$ harmonic on $\mathbb{R}^3$. In practice, though, the only functions globally harmonic in $\mathbb{R}^3$ are constant and the rest have singularities and for them the above condition becomes non-trivial to satisfy [39, 40]. We will discuss these conditions and their implications in a forthcoming paper [41].

The $2\bar{n}$ real harmonic functions then determine the solution, although one has to solve $\mathcal{R}$ in terms of the $\mathcal{I}$ in order to be able to write the whole solution explicitly in terms of the harmonic functions. This problem is equivalent to that of solving the *stabilization equations* and has no known generic solution except in a few cases, some of which we review in Appendix D.

2. Those with a null Killing vector [42]: Generically they have Brinkmann-type metrics

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{-\mathcal{K}}dzdz^* \,. \tag{1.8}$$

where $\mathcal{K}$ is the Kähler potential and $\hat{\omega}$ is determined by the equation

$$(d\hat{\omega})_{\underline{zz}^*} = 2ie^{-\mathcal{K}}\mathcal{Q}_{\underline{u}} \,, \tag{1.9}$$

where $\mathcal{Q}_\mu$ is the pullback of the Kahler 1-form connection (See Eq. (B.11)).

The scalar fields can be defined through a symplectic section with arbitrary dependence on u and z and the vector fields are determined by complex arbitrary functions $\phi(u), \psi^i(z, z^*, u)$

$$F = e^{-\mathcal{K}/2}\left(\mathcal{U}_i\psi^i + \tfrac{i}{2}\mathcal{V}^*\phi\right) du \wedge dz^* + \text{c.c.} \,. \tag{1.10}$$

The solutions of this case are harder to determine completely. There are, however, two interesting families of solutions:

(a) Cosmic strings. They have vanishing vector field strengths and scalars that are arbitrary holomorphic functions $Z^i(z)$.

$$\left\{\begin{array}{rcl} ds^2 & = & 2du(dv + Hdu) - 2e^{-[\mathcal{K}(Z,Z^*)-h-h^*]}dzdz^* \,, \\ Z^i & = & Z^i(z) \,, \\ h & = & h(z) \,, \\ \partial_{\underline{z}}\partial_{\underline{z}^*}H & = & 0 \,. \end{array}\right. \tag{1.11}$$

The functions h must have the right behavior under Kähler transformations to make the metric formally duality-invariant and the Killing spinors well defined. These solutions generalize the ones found in Ref. [42] in flat spacetime for arbitrary Kähler potentials. Observe that the harmonic function H describes a plane wave moving along the string.

(b) Plane waves. In the simplest case they have the form

$$
\left\{
\begin{array}{rcl}
ds^2 & = & 2du(dv + Hdu) - 2dzdz^* \,, \\
F^{\Lambda\,+} & = & \tfrac{i}{2}\mathcal{L}^{*\,\Lambda}\phi(u) du\wedge dz^* \,, \\
Z^i & = & Z^i(u)\,, \\
H & = & (\mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} + 2|\phi|^2)|z|^2 + f(z,u) + f^*(z^*,u)\,,
\end{array}
\right.
\tag{1.12}
$$

where Z^i, ϕ are arbitrary functions of u and f an arbitrary function of u and z.

This work is organized as follows: in section 2 we review the aspects of these theories relevant for this work: action, equations of motion, supersymmetry transformations and symplectic transformations. In section 3 we set up the problem we want to solve: Killing spinor equations, integrability conditions and conditions imposed on Killing spinor bilinears. In section 4 we solve the case in which the Killing vector bilinear is timelike and in section 5 the case in which it is null. The appendices contain the conventions (A) some formulae of Kähler (B) and special Kähler (C) geometry plus some explicit examples of supersymmetric solutions for chosen theories (D).

2 $N = 2, d = 4$ supergravity coupled to vector supermultiplets

In this section we are going to describe briefly the theory we are going to work with. Our main source for the formalism used in this section is Ref. [43], whose notation we use here quite closely although its origin goes back to Refs. [44, 45]. Our conventions for the metric, connection, curvature, gamma matrices and spinors are described in detail in the appendices of Ref. [26] which also contain many identities and results that will be used repeatedly throughout the text. These conventions are very similar, but not identical, to those employed in Ref. [43]. The differences and a dictionary of all the indices we use can be found in Appendix A.

The gravity multiplet of the $N = 2, d = 4$ theory consists of the graviton $e^a{}_\mu$, a pair of gravitinos $\psi_{I\,\mu}\,,\ (I = 1, 2)$ which we describe as Weyl spinors, and a vector field A_μ. Each of the n vector supermultiplets of $N = 2, d = 4$ supergravity that we are going to couple to the pure supergravity theory contains complex scalar $Z^i\,,\ \ (i = 1, \cdots, n)$, a pair of gauginos $\lambda^{I\,i}$, which we also describe as Weyl spinors and a vector field $A^i{}_\mu$. In the coupled theory, the $\bar{n} = n + 1$ vectors can be treated on the same footing and they are described collectively by an array $A^\Lambda{}_\mu\ \ (\Lambda = 1, \cdots, \bar{n})$. The coupling of scalars to scalars is described by a non-linear σ-model with Kähler metric $\mathcal{G}_{ij^*}(Z, Z^*)$ (see Appendix B), and the coupling to the vector fields by a complex scalar-field-valued matrix $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$. These two couplings are related by a structure called special Kähler geometry, described in Appendix C. The symmetries of these two sectors will be related and this relation will be discussed shortly.

The action for the bosonic fields of the theory is

$$S = \int d^4x \sqrt{|g|} \left[R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*\, j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu\star} F^{\Sigma}{}_{\mu\nu} \right] . \tag{2.1}$$

Observe that the canonical normalization of the vector fields kinetic terms implies that $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}$ is negative definite, as is guaranteed by special geometry [46].

For vanishing fermions, the supersymmetry transformation rules of the fermions are

$$\delta_\epsilon \psi_{I\,\mu} = \mathfrak{D}_\mu \epsilon_I + \epsilon_{IJ} T^{+}{}_{\mu\nu} \gamma^\nu \epsilon^J , \tag{2.2}$$

$$\delta_\epsilon \lambda^{Ii} = i\, \not{\partial} Z^i \epsilon^I + \epsilon^{IJ}\, \not{G}^{i\,+} \epsilon_J , \tag{2.3}$$

where $\mathfrak{D}_\mu$ is defined in Eq. (B.10), which acts on the spinors ϵ_I, since they are of Kähler weight 1/2, as

$$\mathfrak{D}_\mu \epsilon_I \equiv (\nabla_\mu + \tfrac{i}{2} \mathcal{Q}_\mu) \epsilon_I , \tag{2.4}$$

and $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form defined in Eq. (B.3). The 2-forms T and G^i are the combinations

$$T_{\mu\nu} \equiv \mathcal{T}_\Lambda F^{\Lambda}{}_{\mu\nu} , \tag{2.5}$$

$$G^i{}_{\mu\nu} \equiv \mathcal{T}^i{}_\Lambda F^{\Lambda}{}_{\mu\nu} , \tag{2.6}$$

where, in turn, $\mathcal{T}_\Lambda$ and $\mathcal{T}^i{}_\Lambda$ are, respectively, the graviphoton and the matter vector fields projectors, defined in Eqs. (C.20) and (C.21).

The supersymmetry transformations of the bosons are

$$\delta_\epsilon e^a{}_\mu = -\tfrac{i}{4} (\bar{\psi}_{I\,\mu} \gamma^a \epsilon^I + \bar{\psi}^I{}_\mu \gamma^a \epsilon_I) , \tag{2.7}$$

$$\begin{aligned} \delta_\epsilon A^\Lambda{}_\mu = {} & \tfrac{1}{4} (\mathcal{L}^{\Lambda\,*} \epsilon^{IJ} \bar{\psi}_{I\,\mu} \epsilon_J + \mathcal{L}^\Lambda \epsilon_{IJ} \bar{\psi}^I{}_\mu \epsilon^J) \\ & + \tfrac{i}{8} (f^\Lambda{}_i \epsilon_{IJ} \bar{\lambda}^{Ii} \gamma_\mu \epsilon^J + f^{\Lambda *}{}_{i^*} \epsilon^{IJ} \bar{\lambda}_I{}^{i^*} \gamma_\mu \epsilon_J) , \end{aligned} \tag{2.8}$$

$$\delta_\epsilon Z^i = \tfrac{1}{4} \bar{\lambda}^{Ii} \epsilon_I . \tag{2.9}$$

For convenience, we denote the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \nabla_{\nu}{}^{\star}F^{\Lambda\,\nu\mu}\,. \tag{2.10}$$

and the bosonic equations of motion by

$$\mathcal{E}_{a}{}^{\mu} \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^{a}{}_{\mu}}\,, \quad \mathcal{E}_{i} \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^{i}}\,, \quad \mathcal{E}_{\Lambda}{}^{\mu} \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^{\Lambda}{}_{\mu}}\,, \tag{2.11}$$

whose explicit forms can be found to be

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^{*}}[\partial_{\mu}Z^{i}\partial_{\nu}Z^{*\,j^{*}} - \tfrac{1}{2}g_{\mu\nu}\partial_{\rho}Z^{i}\partial^{\rho}Z^{*\,j^{*}}] \\
&\quad +8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Sigma\,-}{}_{\nu\rho}\,, \qquad (2.12)\\
\mathcal{E}_{i} &= \nabla_{\mu}(\mathcal{G}_{ij^{*}}\partial^{\mu}Z^{*\,i^{*}}) - \partial_{i}\mathcal{G}_{jk^{*}}\partial_{\rho}Z^{j}\partial^{\rho}Z^{*\,k^{*}} + \partial_{i}[\tilde{F}_{\Lambda}{}^{\mu\nu\star}F^{\Lambda}{}_{\mu\nu}]\,, \qquad (2.13)\\
\mathcal{E}_{\Lambda}{}^{\mu} &= \nabla_{\nu}{}^{\star}\tilde{F}_{\Lambda}{}^{\nu\mu}\,, \qquad (2.14)
\end{aligned}$$

where we have defined the dual vector field strength $\tilde{F}_{\Lambda}$ by

$$\tilde{F}_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta^{\star}F^{\Lambda}{}_{\mu\nu}} = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Sigma}{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}{}^{*}F^{\Sigma}{}_{\mu\nu}\,. \tag{2.15}$$

The Maxwell and Bianchi identities can be rotated into each other by $GL(2\bar{n},\mathbb{R})$ transformations under which they are a $2\bar{n}$-dimensional vector:

$$\mathcal{E}^{\mu} \equiv \begin{pmatrix} \mathcal{B}^{\Lambda\,\mu} \\ \\ \mathcal{E}_{\Lambda}{}^{\mu} \end{pmatrix} \longrightarrow \begin{pmatrix} D & C \\ \\ B & A \end{pmatrix}\begin{pmatrix} \mathcal{B}^{\Lambda\,\mu} \\ \\ \mathcal{E}_{\Lambda}{}^{\mu} \end{pmatrix}\,, \tag{2.16}$$

where A, B, C and D are $\bar{n}\times\bar{n}$ matrices. These transformations act in the same form on the vector of $2\bar{n}$ 2-forms

$$F \equiv \begin{pmatrix} F^{\Lambda} \\ \\ \tilde{F}_{\Lambda} \end{pmatrix} \longrightarrow \begin{pmatrix} D & C \\ \\ B & A \end{pmatrix}\begin{pmatrix} F^{\Lambda} \\ \\ \tilde{F}_{\Lambda} \end{pmatrix}\,, \tag{2.17}$$

and, since, by definition,

$$\tilde{F}'_{\Lambda} = \Re\mathrm{e}\mathcal{N}'_{\Lambda\Sigma}F'^{\Sigma} + \Im\mathrm{m}\mathcal{N}'_{\Lambda\Sigma}{}^{\star}F'^{\Sigma}\,, \tag{2.18}$$

for the transformations to be consistently defined, the must act on the period matrix $\mathcal{N}$ according to

$$\mathcal{N}' = (A\mathcal{N} + B)(C\mathcal{N} + D)^{-1}\,. \tag{2.19}$$

Furthermore, the transformations must preserve the symmetry of the period matrix, which requires

$$A^T C = C^T A\,, \qquad D^T B = B^T D\,, \qquad A^T D - C^T B = 1\,, \tag{2.20}$$

i.e. the transformations must belong to $Sp(2\bar{n},\mathbb{R})$.

The above transformation rules for the vector field strength and period matrix imply

$$\Im\mathrm{m}\mathcal{N}' = (C\mathcal{N}^* + D)^{-1\,T}\Im\mathrm{m}\mathcal{N}(C\mathcal{N}+D)^{-1}\,, \qquad F'^{\Lambda\,+} = (C\mathcal{N}^*+D)_{\Lambda\Sigma}F^{\Sigma\,+}\,, \tag{2.21}$$

so the combination $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_{\mu}{}^{\rho}F^{\Lambda\,-}{}_{\nu\rho}$ that appears in the energy-momentum tensor is automatically invariant.

The above symplectic transformations of the period matrix $\mathcal{N}$ correspond to certain transformations of the complex scalar fields Z^i:

$$\mathcal{N}'(Z,Z^*) = [A\mathcal{N}(Z,Z^*)+B][C\mathcal{N}(Z,Z^*)+D]^{-1} \equiv \mathcal{N}(Z',Z'^*)\,. \tag{2.22}$$

These transformations have to be symmetries of the theory as well, which implies that they have to be isometries of the special Kähler manifold [47]. Thus only the isometries of the special Kähler manifold which are embedded in $Sp(2\bar{n},\mathbb{R})$ are symmetries of all the equations of motion of the theory (dualities of the theory). Observe that the Kähler potential is, in general, not invariant under the isometries of the Kähler metric, but undergoes a Kähler transformation. This means that all objects with non-zero Kähler weight transform non-trivially under duality.

The scalar equation $\mathcal{E}_i$ Eq. (2.13) can be written in a manifestly covariant form by rising the index with $\mathcal{G}^{j^*i}$, $\mathcal{E}^{i^*}$. The complex conjugate equation then takes on the form

$$\mathcal{E}^i = \mathfrak{D}_\mu\partial^\mu Z^i + \mathcal{G}^{ij^*}\partial_{j^*}[\tilde{F}_\Lambda{}^{\mu\nu\,*}F^\Lambda{}_{\mu\nu}]\,. \tag{2.23}$$

3 Supersymmetric configurations: general setup

Our first goal is to find all the bosonic field configurations $\{g_{\mu\nu}, F^\Lambda{}_{\mu\nu}, Z^i\}$ for which the Killing spinor equations (KSEs):

$$\delta_\epsilon\psi_{I\,\mu} \;=\; \mathfrak{D}_\mu\epsilon_I + \epsilon_{IJ}T^+{}_{\mu\nu}\gamma^\nu\epsilon^J = 0\,, \tag{3.1}$$

$$\delta_\epsilon\lambda^{Ii} \;=\; i\,\not{\partial}Z^i\epsilon^I + \epsilon^{IJ}\,\not{G}^{i\,+}\epsilon_J = 0\,, \tag{3.2}$$

admit at least one solution.[5] It must be stressed that the configurations considered need

[5]The generalized holonomy of the gravitino supersymmetry transformation indicates that the minimal number of solutions that these equations will admit is actually 4 [48], but we will not use this fact in our derivation. We will, in the end recover the result that supersymmetric solutions generically preserve 1/2 or all supersymmetries.

not be classical solutions of the equations of motion. Furthermore, we will not assume that the Bianchi identities are satisfied by the field strengths of a configuration.

Our second goal will be to identify among all the supersymmetric field configurations those that satisfy all the equations of motion (including the Bianchi identities).

Let us initiate the analysis of the KSEs by studying their integrability conditions.

3.1 Killing Spinor Identities (KSIs)

Using the supersymmetry transformation rules of the bosonic fields Eqs. (2.7–2.9) and using the results of Refs. [33, 34] we can derive the following relations (*Killing spinor identities*, KSIs) between the (off-shell) equations of motion of the bosonic fields Eqs. (2.12–2.14) that are satisfied by any field configuration $\{e^a{}_\mu, A^\Lambda{}_\mu, Z^i\}$ admitting Killing spinors:

$$\mathcal{E}_a{}^\mu\gamma^a\epsilon^I - 4i\epsilon^{IJ}\mathcal{L}^\Lambda\mathcal{E}_\Lambda{}^\mu\epsilon_J = 0\,, \tag{3.3}$$

$$\mathcal{E}^i\epsilon^I - 2i\epsilon^{IJ}f^{*\,i\Lambda}\,\not{\mathcal{E}}_\Lambda\epsilon_J = 0\,. \tag{3.4}$$

The vector field Bianchi identities Eq. (2.10) do not appear in these relations because the procedure used to derive them, assumes the existence of the vector potentials, and hence the vanishing of the Bianchi identities.

It is convenient to treat the Maxwell equations and Bianchi identities on an equal footing as to preserve the electric-magnetic dualities of the theory, for which it is convenient to have a duality-covariant version of the above KSIs. This can be found by performing duality rotations on the above identities or from the integrability conditions of the KSEs Eqs. (3.1,3.2), which is the method we are going to use.

Using the Kähler special geometry machinery, we obtain

$$\begin{aligned}\mathfrak{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} \;=\;& -\tfrac{1}{8}\{[R_{\mu\nu}{}^{ab} - 8\mathcal{T}_\Lambda\mathcal{T}^*_\Sigma F^{\Lambda\,+}{}_{[\mu|}{}^a F^{\Sigma\,-}{}_{|\nu]}{}^b]\gamma_{ab} + 4\mathcal{G}_{ij^*}\partial_{[\mu}Z^i\partial_{\nu]}Z^{*\,j^*}\}\epsilon_I \\ & +\epsilon_{IJ}\mathfrak{D}_{[\mu}T^+{}_{\nu]\rho}\gamma^\rho\epsilon^J = 0\,,\end{aligned} \tag{3.5}$$

which gives rise to

$$4\gamma^\nu\mathfrak{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} = (\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_\sigma{}^\sigma)\gamma^\nu\epsilon_I - 2i\epsilon_{IJ}\mathcal{L}^\Lambda(\not{\mathcal{E}}_\Lambda - \mathcal{N}_{\Lambda\Sigma}\,\not{\mathcal{B}}^\Sigma)\gamma_\mu\epsilon^J = 0\,. \tag{3.6}$$

Contracting the above identity with γ^μ, we obtain another one involving only the trace $\mathcal{E}_\sigma{}^\sigma$, which can be used to eliminate it completely from the KSIs. The result is the duality-covariant version of (the complex conjugate of) Eq. (3.3) we were after:

$$\mathcal{E}_a{}^\mu\gamma^a\epsilon_I - 4i\epsilon_{IJ}\mathcal{L}^\Lambda(\mathcal{E}_\Lambda{}^\mu - \mathcal{N}_{\Lambda\Sigma}\mathcal{B}^{\Sigma\mu})\epsilon^J = 0\,. \tag{3.7}$$

It turns out to be convenient to define the combination

$$\mathcal{H}^{\Lambda\mu} \equiv (\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}(\mathcal{E}_{\Sigma}{}^{\mu} - \mathcal{N}_{\Sigma\Omega}\mathcal{B}^{\Sigma\mu})\,. \tag{3.8}$$

Using it, the above KSIs Eqs. (3.6,3.7) take the form

$$(\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_{\sigma}{}^{\sigma})\gamma^{\nu}\epsilon_I - \mathcal{T}_{\Lambda}\,\not{\mathcal{H}}^{\Lambda}\gamma_{\mu}\epsilon_{IJ}\epsilon^{J} \;=\; 0\,, \tag{3.9}$$

$$\mathcal{E}_a{}^{\mu}\gamma^a\epsilon_I - 2\mathcal{T}_{\Lambda}\mathcal{H}^{\Lambda\mu}\epsilon_{IJ}\epsilon^{J} \;=\; 0\,. \tag{3.10}$$

Observe that the graviphoton-projected combination $\mathcal{T}_{\Lambda}\mathcal{H}^{\Lambda\,\mu}$ can be written as

$$\mathcal{T}_{\Lambda}\mathcal{H}^{\Lambda\,\mu} = 2i[\mathcal{L}^{\Lambda}\mathcal{E}_{\Lambda}{}^{\mu} - \mathcal{M}_{\Lambda}\mathcal{B}^{\Lambda\,\mu}] = 2i\,\langle\,\mathcal{E}^{\mu}\mid\mathcal{V}\,\rangle\,, \tag{3.11}$$

where $\mathcal{E}$ is the symplectic vector defined in Eq. (2.16).

The duality-covariant version of Eq. (3.4) can be obtained in a similar fashion, and reads

$$-\,i\,\not{\mathfrak{D}}\delta_{\epsilon}\lambda^{Ii} = \mathcal{E}^{i}\epsilon^{I} - 2i\mathcal{T}^{i}{}_{\Lambda}\,\not{\mathcal{H}}^{\Lambda}\epsilon^{IJ}\epsilon_J = 0\,. \tag{3.12}$$

Observe that the identities Eqs. (3.6,3.7) and (3.12) are necessary but not sufficient conditions to have supersymmetry.

From these identities we can derive further identities involving only tensors by multiplication with gamma matrices and conjugate spinor from the left, as to have only bilinears. As is usual, it is convenient to consider the case in which the vector bilinear $V^{\mu} \equiv i\bar{\epsilon}^{I}\gamma^{\mu}\epsilon_I$ is timelike and the case in which it is null, separately.

3.1.1 The timelike case: independent e.o.m.'s

When V^{μ} is timelike one can derive the following identities:

$$\mathcal{E}^{\mu\nu} \;=\; \mathcal{E}^{\rho\sigma}v_{\rho}v_{\sigma}v^{\mu}v^{\nu}\,, \tag{3.13}$$

$$\mathcal{T}_{\Lambda}\mathcal{H}^{\Lambda\,\mu} \;=\; -\tfrac{i}{2}e^{i\alpha}\mathcal{E}^{\rho\sigma}v_{\rho}v_{\sigma}v^{\mu}\,, \tag{3.14}$$

$$\mathcal{T}^{i}{}_{\Lambda}\mathcal{H}^{\Lambda\,\mu} \;=\; \tfrac{1}{2}e^{-i\alpha}\mathcal{E}^{i}v^{\mu}\,, \tag{3.15}$$

where we have defined the unit vector and the (local) phase

$$v^{\mu} \equiv V^{\mu}/2|X|\,, \qquad e^{i\alpha} \equiv X/|X|\,. \tag{3.16}$$

These identities contain a large amount of information about the supersymmetric configurations. In particular, they contain the necessary information about which equations of motion need to be checked explicitly in order to determine whether a given configuration solves the equations of motion: the first of these identities tells us that the only components of the Einstein equations that do not vanish automatically for supersymmetric configurations are those in the direction of $v^\mu v^\nu$; the rest vanish automatically. *I.e.* once supersymmetry is established, one does not need to check that those components of the Einstein equations are satisfied. Further, the second and third identities state that the only components of the combination of Maxwell equations and Bianchi identities $\mathcal{H}^{\Lambda\,\mu}$ that do not vanish automatically are the ones in the direction v^μ. For the graviphoton (second equation), they are related to the only non-trivial components of the Einstein equations and for the matter vector fields (third equation), they are related to the equations of motion of the scalars. Therefore, we see that iff the Maxwell equation and Bianchi identities are satisfied, then the equations of motion of the scalars and the Einstein equations are satisfied identically. The conclusion then must be that, in the timelike case, one only needs to solve the Maxwell equation and the Bianchi identities in order to be sure that a supersymmetric configuration is an actual (supersymmetric) solution of the equations of motion.

3.1.2 The null case

When V^μ is a null-vector (we will denote it by l^μ), using the auxiliary spinor η defined in the appendix of Ref. [26] to construct a standard complex null tetrad $\{l^\mu, n^\mu, m^\mu, m^{*\,\mu}\}$ we can derive the following identities:

$$(\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_\sigma{}^\sigma)l^\nu = (\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_\sigma{}^\sigma)m^\nu \;=\; 0\,, \tag{3.17}$$

$$\mathcal{E}_{\mu\nu}l^\nu = \mathcal{E}_{\mu\nu}m^\nu \;=\; 0\,, \tag{3.18}$$

$$\mathcal{T}_\Lambda \mathcal{H}^{\Lambda\,\mu} \;=\; 0\,, \tag{3.19}$$

$$\mathcal{T}^i{}_\Lambda \mathcal{H}^{\Lambda\,\mu} l_\mu = \mathcal{T}^i{}_\Lambda \mathcal{H}^{\Lambda\,\mu} m_\mu \;=\; 0\,, \tag{3.20}$$

$$\mathcal{E}^i \;=\; 0\,. \tag{3.21}$$

Thus, in this case, the equations of motion of the scalars are always automatically satisfied for a supersymmetric configuration. Only a few components of the Einstein and Maxwell equations and Bianchi identities may also be non-zero and these are the only ones that need to be checked if we want to have solutions. Observe that the vanishing of the graviphoton-projected combination $\mathcal{T}_\Lambda\mathcal{H}^{\Lambda\,\mu}$ does not imply the vanishing of the Maxwell equations or the Bianchi identities.

3.2 Solving the Killing spinor equations

To solve the KSEs we are going to follow these steps:

1. In section 3.3, we are going to derive equations for the tensor bilinears that can be built from the Killing spinors.[6] Solving these equations is not, in principle, sufficient for solving the KSEs, but it is certainly necessary, which is why they are analyzed first.

2. We are going to see in in the same section that these equations for the bilinears state that the vector bilinear we denote by V^μ, is always a Killing vector, whereas the other three are closed (locally exact) 1-forms, which need not be independent.

3. In the same section we will derive an expression for the contractions $V^\nu F^\Lambda{}_{\nu\mu}$ in terms of the scalar bilinear X and the scalars Z^i. These contractions determine to a large extent the form of the full vector field strengths, depending on the causal nature of the Killing vector V^μ, which can be timelike or null. These two cases have to be studied separately.

4. In the timelike case (section 4)

   (a) The contractions $V^\nu F^\Lambda{}_{\nu\mu}$ fully determine the vector field strengths (section 4.1).

   (b) The form of the metric is also fixed by the existence of a timelike Killing vector and the three exact 1-forms, which, in this case, are independent (section 4.2).

   (c) At this point these two fields are entirely expressed in terms of bilinear X and the scalars Z^i, which remain arbitrary, and we are going to check explicitly (section 4.3) that, in all cases, these field configurations are supersymmetric, provided that they satisfy the integrability condition Eq. (4.12).

   (d) This solves the timelike case, but, obviously, we are particularly interested in supersymmetric configurations which are *solutions*. We have seen in the previous section that the KSIs insure that this is equivalent to satisfying the Maxwell equations and the Bianchi identities, which, as we are going to see (section 4.4), is the case if the scalars satisfy the 'simple' Eqs. (4.30).

5. In the null case (section 5)

   (a) we will use the formalism of Ref. [49], exploiting the fact that the two ϵ_I must be proportional and can be written in the form $\epsilon_I = \phi_I \epsilon$. The KSEs can be split into equations involving ϵ and equations involving ϕ_Is.

   (b) A second spinor η needs to be introduced as to construct a null tetrad via spinor bilinears; the relative normalization of ϵ and η requires η to satisfy a differential equation whose integrability conditions need to be added to the KSEs integrability conditions. All these conditions and the conditions implied for the null tetrad are studied in section 5.2.

[6]The definitions and properties of these bilinears can be found in the appendices of Ref. [26].

(c) Since the solution admits a covariantly constant null vector, we can introduce a coordinate system and solve the consistency conditions (see section 5.3). In section 5.4 we use this coordinate system to analyze the KSEs, and show that a supersymmetric configuration preserves either half or all the supersymmetries.

(d) Section 5.5, then, analyzes the equations of motion, reducing them to two, seemingly involved, differential equations, namely Eqs. (5.88) and (5.91), and discusses some interesting subclasses of solutions.

3.3 Killing equations for the bilinears

From the gravitino supersymmetry transformation rule Eq. (2.2) we get the independent equations

$$\mathfrak{D}_\mu X \quad = \quad -iT^+{}_{\mu\nu}V^\nu \,, \tag{3.22}$$

$$\nabla_\mu V^I{}_{J\,\nu} \quad = \quad i\delta^I{}_J(XT^{*-}{}_{\mu\nu} - X^*T^+{}_{\mu\nu}) - i(\epsilon^{IK}T^{*\,-}{}_{\mu\rho}\Phi_{KJ}{}^\rho{}_\nu - \epsilon_{JK}T^+{}_{\mu\rho}\Phi^{IK}{}_\nu{}^\rho)\,. \tag{3.23}$$

The first equation relates the scalar bilinear X with the self-dual part of the graviphoton field strength and indicates that it contains all the information of the central charge of the theory [50].

The first term in the r.h.s. of the second equation is completely antisymmetric in $\mu\nu$ indices and has a non-vanishing trace in IJ indices, while the second term is completely symmetric in $\mu\nu$ indices and traceless in IJ indices. This implies that V^μ is a Killing vector and the 1-form $\hat{V} = V_\mu dx^\mu$ satisfies the equation

$$d\hat{V} = 4i(XT^{*\,-} - X^*T^+)\,, \tag{3.24}$$

while the remaining 3 independent 1-forms $\hat{V}^i \equiv \frac{1}{\sqrt{2}}V^I{}_{J\,\mu}\sigma^{i\,J}{}_I dx^\mu$ ($\sigma^{i\,I}{}_J\,,\ \ i = 1,2,3$ are the Pauli matrices) are exact

$$d\hat{V}^i = 0\,. \tag{3.25}$$

From the gauginos supersymmetry transformation rules, Eqs. (2.3), we get

$$V^I{}_K{}^\mu\partial_\mu Z^i + \epsilon^{IJ}\Phi_{KJ}{}^{\mu\nu}G^{i\,+}{}_{\mu\nu} \quad = \quad 0\,, \tag{3.26}$$

$$iM^{KI}\partial_\mu Z^i + i\Phi^{KI}{}_\mu{}^\nu\partial_\nu Z^i - 4i\epsilon^{IJ}V^K{}_J{}^j G^{i\,+}{}_{\mu\nu} \quad = \quad 0\,. \tag{3.27}$$

The trace of the first equation gives

$$V^\mu\partial_\mu Z^i = 0\,, \tag{3.28}$$

while the antisymmetric part of the second equation gives

$$2iX^{*}\partial_{\mu}Z^{i} + 4iG^{i\,+}{}_{\mu\nu}V^{\nu} = 0\,. \tag{3.29}$$

Using Eq. (C.19), we can derive

$$i\mathcal{L}^{*\,\Lambda}T^{+} + 2f^{\Lambda}{}_{i}G^{i\,+} = F^{\Lambda\,+}\,, \tag{3.30}$$

which in its turn allows us to combine Eqs. (3.22) and (3.29), as to obtain

$$V^{\nu}F^{\Lambda\,+}{}_{\nu\mu} = \mathcal{L}^{*\,\Lambda}\mathfrak{D}_{\mu}X + X^{*}f^{\Lambda}{}_{i}\partial_{\mu}Z^{i} = \mathcal{L}^{*\,\Lambda}\mathfrak{D}_{\mu}X + X^{*}\mathfrak{D}_{\mu}\mathcal{L}^{\Lambda}\,, \tag{3.31}$$

which, in the timelike case, is enough to completely determine F^{Λ} as a function of the scalars Z^{i}, X and V.

4 The timelike case

4.1 The vector field strengths

As is well-known, the contraction of a self-dual 2-form with a non-null vector completely determines the 2-form. In the timelike case we can use V^{μ} and we have

$$C^{\Lambda\,+}{}_{\mu} \equiv V^{\nu}F^{\Lambda\,+}{}_{\nu\mu} \quad\Rightarrow\quad F^{\Lambda\,+} = V^{-2}[\hat{V}\wedge\hat{C}^{\Lambda\,+} + i\,{}^{\star}(\hat{V}\wedge\hat{C}^{\Lambda\,+})]\,, \tag{4.1}$$

where $C^{\Lambda\,+}$ is given by Eq. (3.31). Therefore, we have the vector field strengths written in terms of the scalars Z^{i}, X and the vector V. Let us then consider the spacetime metric.

4.2 The metric

It is convenient to choose coordinates adapted to the timelike Killing vector V and also to use the exact 1-forms $\hat{V}^{i}$ (which, as was said before, are independent in the timelike case) to define the spacelike coordinates. Thus, we define a time coordinate t by

$$V^{\mu}\partial_{\mu} \equiv \sqrt{2}\partial_{t}\,, \tag{4.2}$$

and the spacelike coordinates x^{i} by

$$\hat{V}^{i} \equiv dx^{i}\,. \tag{4.3}$$

Since in this case $V^{\mu}V_{\mu} = 2|M|^{2} = 4|X|^{2} \neq 0$, the metric can always be constructed as

$$ds^{2} = |M|^{-2}[\hat{V}\otimes\hat{V} - \hat{V}^{I}{}_{J}\otimes\hat{V}^{J}{}_{I}] = |M|^{-2}[\tfrac{1}{2}\hat{V}\otimes\hat{V} - \hat{V}^{i}\otimes\hat{V}^{i}]\,, \tag{4.4}$$

which is manifestly invariant under all the transformations leaving invariant the equations of motion and the supersymmetry transformation rules. With the above choice of coordinates, the metric takes on the form

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}dx^i dx^i\,, \qquad i,j=1,2,3\,, \tag{4.5}$$

where $\omega = \omega_{\underline{i}}dx^i$ is a time-independent 1-form that satisfies an equation that can be found as follows: the choice of coordinates implies

$$\hat{V} = 2\sqrt{2}|X|^2(dt+\omega)\,, \tag{4.6}$$

which trivially implies that

$$d\omega = \tfrac{1}{2\sqrt{2}}d(|X|^{-2}\hat{V})\,. \tag{4.7}$$

Using Eqs. (3.24) and (3.22) we find the equation for ω

$$d\omega = -\tfrac{i}{2\sqrt{2}}{}^{\star}\left[(X\mathfrak{D}X^* - X^*\mathfrak{D}X)\wedge \frac{\hat{V}}{|X|^4}\right]\,. \tag{4.8}$$

With this equation we have succeeded in expressing completely the metric in terms of the scalars X, Z^i and the vector V, as we did with the vector field strengths.

It is convenient to define the $U(1)$ connection 1-form

$$\xi \equiv \tfrac{i}{2}\frac{X^*dX - XdX^*}{|X|^2}\,. \tag{4.9}$$

This ξ is similar to the ξ defined in the $N=4, d=4$ case in Ref. [26], but in this case it is exact. In terms of ξ and the pullback of the Kähler 1-form $\mathcal{Q}$, the equation for ω is

$$d\omega = \tfrac{1}{\sqrt{2}}{}^{\star}\left[(\xi - \mathcal{Q})\wedge \frac{\hat{V}}{|M|^2}\right]\,. \tag{4.10}$$

In terms of the 3-dimensional Euclidean metric, this equation takes the form

$$(d\omega)_{mn} = \frac{1}{|X|^2}\epsilon_{mnp}(\mathcal{Q}_p - \xi_p)\,, \tag{4.11}$$

and we will later rewrite it to a more standard form.

In what follows the integrability condition for this equation will be needed: It reads

$$\partial_m\left[\frac{(\mathcal{Q}_m - \xi_m)}{|X|^2}\right] = 0\,. \tag{4.12}$$

4.3 Solving the Killing spinor equations

We are now going to show that the field configurations of $N=2, d=4$ given by the metric Eqs. (4.5) and (4.11) and field strengths Eqs. (4.1) and (3.31) are supersymmetric for arbitrary values of the complex scalars X, Z^i ($X \neq 0$).

Let us start by the gauginos supersymmetry transformations Eqs. (3.2). Using Eqs. (4.1), we find

$$\not{F}^{\Lambda\,+} = -\frac{1}{|X|^2} C^{\Lambda\,+}{}_{\rho} V_{\sigma} \gamma^{\rho\sigma} \tfrac{1}{2}(1-\gamma_5)\,. \tag{4.13}$$

On the other hand, using the properties Eqs. (C.12) and (C.13), we find that

$$\mathcal{T}^{i}{}_{\Lambda} C^{\Lambda\,+}{}_{\mu} = \tfrac{1}{2} X^* \partial_\mu Z^i\,, \tag{4.14}$$

and, combining this with the previous result we get

$$\mathcal{T}^{i}{}_{\Lambda}\ \not{F}^{\Lambda\,+}\epsilon^{IJ}\epsilon_J = -\frac{M^{IJ}}{2|X|^2}\partial_\rho Z^i V_\sigma \gamma^{\rho\sigma}\epsilon_J = i\ \not{\partial} Z^i (i\gamma_0 e^{-i\alpha}\epsilon^{IJ}\epsilon_J)\,, \tag{4.15}$$

where α is the phase of the complex scalar bilinear X and we have used that in our Vierbein basis $\hat{V} = 2|X|e_0$, Eq. (3.28).

Eq. (3.2) takes, then the form

$$i\ \not{\partial} Z^i (\epsilon^I + i\gamma_0 e^{-i\alpha}\epsilon^{IJ}\epsilon_J) = 0\,, \tag{4.16}$$

and can always be solved by imposing the constraint

$$\epsilon_I + i\gamma_0 e^{i\alpha}\epsilon_{IJ}\epsilon^J = 0\,, \tag{4.17}$$

which breaks half of the available supersymmetries.

Let us now consider the 0^{th} component of the gravitino supersymmetry transformations Eq. (3.1): using Eq. (4.11), we find

$$\mathfrak{D}_0\epsilon_I = \frac{1}{\sqrt{2}|X|}\{\partial_t - X^*\mathfrak{D}_m X\gamma_{0m}\}\epsilon_I\,. \tag{4.18}$$

On the other hand, using Eqs. (C.11) and (C.12), we find

$$\mathcal{T}_\Lambda F^{\Lambda\,+}{}_{0m} = \tfrac{i}{\sqrt{2}}\mathfrak{D}_m X\,, \tag{4.19}$$

and combining this with the previous result we find that the 0^{th} component of Eq. (3.1) takes, up to a global factor, the form

$$\partial_t\epsilon_I - \frac{X^*\mathfrak{D}_m X}{|X|^2}\gamma_{0m}\left[\epsilon_I + i\gamma_0 e^{i\alpha}\epsilon_{IJ}\epsilon^J\right] = 0\,, \tag{4.20}$$

which is always solved by time-independent spinors satisfying the constraint (4.17).

Finally, let us consider the m^{th} component of Eq. (3.1): using essentially the same properties, we find on the one hand

$$\mathfrak{D}_m\epsilon_I = \sqrt{2}|X|\left\{\mathfrak{D}_m - \tfrac{i}{2}\epsilon_{mnp} - \frac{X^*\mathfrak{D}_p X}{|X|^2}\gamma_{0n}\right\}\epsilon_I\,, \tag{4.21}$$

and on the other,

$$\mathcal{T}_\Lambda F^{\Lambda\,+}{}_{ma}\gamma^a = \tfrac{1}{\sqrt{2}}(\delta_{mp} - i\epsilon_{mnp}\gamma_{0m})\mathfrak{D}_p X i\gamma_0\,. \tag{4.22}$$

Combining these two results and using the constraint Eq. (4.17) as to have an equation involving spinors of the same chirality, we find that the m^{th} component of Eq. (3.1), up to a multiplicative factor, reads

$$\partial_m(X^{-1/2}\epsilon_I) = 0\,, \tag{4.23}$$

which is solved by

$$\epsilon_I = X^{1/2}\epsilon_{I\,0}\,, \qquad \partial_\mu \epsilon_{I\,0} = 0\,, \qquad \epsilon_{I\,0} + i\gamma_0\epsilon_{IJ}\epsilon^J{}_0 = 0\,. \tag{4.24}$$

This is the form of the Killing spinor associated to the field configurations that we have found. All of them are, therefore, supersymmetric and preserve, at least, 1/2 of the possible supersymmetries.

Observe, however, that in this proof we have assumed that Eq. (4.11) can be solved. Thus, we have assumed implicitly that the integrability condition of this equation, Eq. (4.12), has been solved. This equation is the only condition that the field configurations considered need to satisfy in order to be supersymmetric and, in fact, to be well defined. We will reconsider this condition when we consider supersymmetric *solutions*, but we can already see that our result is different from the one in Ref. [35], where the pull-back of the Kähler form had to vanish in order for the solution to be supersymmetric.

4.4 Equations of motion

Following Refs. [49, 26] we are going to introduce an $Sp(2\bar{n},\mathbb{R})$ vector of electric and magnetic scalar potentials E defined by

$$\nabla_\mu E \equiv V^\nu F_{\nu\mu}\,, \tag{4.25}$$

where $F_{\mu\nu}$ the $Sp(2\bar{n},\mathbb{R})$ vector of field strengths defined in Eq. (2.17). Let us also define the real symplectic sections $\mathcal{I}$ and $\mathcal{R}$

$$\mathcal{R} \equiv \Re\mathrm{e}(\mathcal{V}/X)\,, \qquad \mathcal{I} \equiv \Im\mathrm{m}(\mathcal{V}/X)\,, \tag{4.26}$$

where $\mathcal{V}$ is the symplectic section defined in Appendix C.

Then, using Eq. (3.31) we find

$$E = 2|X|^2\mathcal{R}\,, \tag{4.27}$$

and using the explicit form of $\hat{V}$ and the property Eq. (4.8) we can write

$$F = -\tfrac{1}{2}\{d[\mathcal{R}\hat{V}] - {}^\star[d\mathcal{I}\wedge\hat{V}]\}\,. \tag{4.28}$$

which immediately leads to the following form of the Bianchi identities and Maxwell equations:

$$dF = \tfrac{1}{2} d^{\star}[d\mathcal{I} \wedge \hat{V}]\,. \tag{4.29}$$

Rewriting these equations in standard Cartesian $\mathbb{R}^3$ language, we find that the Maxwell equations and Bianchi identities (whence, according to the KSIs, all the equations of motion of $N=2, d=4$ supergravity) are satisfied if

$$\partial_m \partial_m \mathcal{I} = 0\,, \tag{4.30}$$

i.e. if the imaginary parts of $\mathcal{L}^{\Lambda}/X$ and $\mathcal{M}_{\Lambda}/X$ are given by $2\bar{n}$ real harmonic functions on $\mathbb{R}^3$.

Let us now reconsider the integrability condition Eq. (4.12) of the differential equation that defines the 1-form ω. The equation for the 1-form ω (4.11) can be rewritten as to give

$$(d\omega)_{mn} = 2\epsilon_{mnp} \langle\, \mathcal{I} \mid \partial_p \mathcal{I}\,\rangle\,, \tag{4.31}$$

and its integrability condition takes on the simple form

$$\langle\, \mathcal{I} \mid \partial_p \partial_p \mathcal{I}\,\rangle = 0\,, \tag{4.32}$$

which is, as discussed in the introduction, a non-trivial condition due to the presence of singularities in harmonic function [39, 40].

Summarizing, we have just shown that the configurations of $N=2, d=4$ supergravity given by the metric Eqs. (4.5) and (4.11) and field strengths Eqs. (4.1) and (3.31) are solutions of the equations of motion iff the scalars X, Z^i satisfy the condition Eq. (4.30). The integrability condition of the equation for the 1-form ω, which was the only condition necessary to have supersymmetry, is automatically satisfied for supersymmetric solutions.

We still have to show how, given the real harmonic section $\mathcal{I}$, we can express the scalars, the vector field strengths and the metric in terms of this harmonic section: the metric Eq. (4.5) depends of the 1-form ω which can be calculated from $\mathcal{I}$ integrating Eq. (4.31) and on the absolute value of the bilinear scalar X, which can be computed from $\mathcal{I}$ and $\mathcal{R}$ observing that

$$\langle\, (\mathcal{V}/X)^* \mid \mathcal{V}/X\,\rangle = 2i \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = i \frac{1}{|X|^2}\,, \tag{4.33}$$

and that

$$\langle\, (\mathcal{V}/X)^* \mid \mathcal{V}/X\,\rangle = i e^{-\mathcal{K}[\mathcal{V}/X, (\mathcal{V}/X)^*]}\,, \tag{4.34}$$

where $\mathcal{K}[\mathcal{V}/X, (\mathcal{V}/X)^*]$ is the expression for the Kähler potential obtained in Eq. (C.24) where the coordinates $\mathcal{X}$ have been substituted by $\mathcal{L}/X$. This leads to the expression

$$|M|^2 = 2|X|^2 = 2e^{\mathcal{K}[\mathcal{V}/X, (\mathcal{V}/X)^*]}\,. \tag{4.35}$$

Observe that the form of the Kähler potential that has to be used here, namely Eq. (C.24), is fixed after a section has been chosen and no Kähler transformations are allowed. Otherwise, the whole construction would be inconsistent since, as we have discussed, the spacetime metric is invariant under all the symmetries of the equations of motion and, in particular, under Kähler transformations.

It is also possible to rewrite Eq. (4.31) for ω as

$$(d\omega)_{mn} = \epsilon_{mnp} e^{-\mathcal{K}[\mathcal{V}/X, (\mathcal{V}/X)^*]} \mathcal{Q}_p[\mathcal{V}/X, (\mathcal{V}/X)^*] \,. \tag{4.36}$$

It is clear that, in order to find $|M|^2$, and also in order to find the field strengths using Eqs. (4.28) and the scalars using, for instance, the special coordinates

$$Z^i = \frac{\mathcal{L}^i}{\mathcal{L}^0} = \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} \,, \tag{4.37}$$

we need the real section $\mathcal{R}$ expressed as a function of $\mathcal{I}$. Finding $\mathcal{R}(\mathcal{I})$ is equivalent to solving the so-called "*stabilization equations*" [51, 52] which are nothing but Eqs. (4.29) for static, spherical, asymptotically flat black holes evaluated at the black-hole horizon: the l.h.s. gives the $Sp(2\bar{n}, \mathbb{R})$ vector of charges $q^t \equiv \left(p^\Lambda \ , \ q_\Lambda\right)$ and this essentially determines the coefficient of the harmonic functions $\mathcal{I}$. Solving the stability equations amounts to finding the real part of the section $\mathcal{V}/X$ as a function of the imaginary part, i.e. of the charges. There seems to be no systematic procedure to find $\mathcal{R}(\mathcal{I})$ and the solution of this problem is only known in a few simple cases, some of which we review in Appendix D.

At this point we should compare our results with those of Refs. [35] and [36]. In th efirst of these references, the supersymmetric configurations we have found were proposed as an *Ansatz* and they were shown to be supersymmetric. In the second, the same solutions were found from the KSEs of superconformal gravity starting with an *Ansatz* for the contraint satisfied by the Killing spinors of the form Eq. (4.17)[7]. We have just shown that all the solutions[8] in the timelike class satisfy this constraint and there are no more solutions than those found by Behrndt, Lüst and Sabra in the timelike class, although there are more supersymmetric solutions in the null class, as we are going to see.

5 The null case

In the null case[9] the two spinors ϵ_I are proportional $\epsilon_I = \phi_I \epsilon$. The complex scalar functions ϕ_I carry a -1 $U(1)$ charge w.r.t. the purely imaginary connection

$$\zeta \equiv \phi^I d\phi_I \,, \tag{5.1}$$

opposite to that of the spinor ϵ, so the ϵ_I are neutral. On the other hand, the ϕ_Is are neutral with respect to the Kähler connection, and the Kähler weight of the spinor ϵ is the same as that of the spinor ϵ_I, i.e. 1/2.

[7] Their results include also R^2 corrections, but we are not concerned with them here.

[8] The exceptions are the maximally supersymmetric Minkowski and Bertotti-Robinson-type solutions.

[9] The technical details concerning the normalization of the spinors and the construction of the bilinears in this case are explained in the Appendix of Ref. [26].

We are now going to substitute $\epsilon_I = \phi_I \epsilon$ into the KSEs and we are going to use the normalization condition of the scalars $\phi_I \phi^I = 1$ to split the KSEs into three algebraic and one differential equation for ϵ; one of the algebraic equations for ϵ will be a differential equation for ϕ_I.

This substitution immediately yields

$$\partial_\mu \phi_I \epsilon + \phi_I \mathfrak{D}_\mu \epsilon - \epsilon_{IJ} \phi^J T^+{}_{\mu\nu} \gamma^\nu \epsilon^* = 0\,, \tag{5.2}$$

$$\phi^I \, \not\partial Z^i \epsilon^* + \epsilon^{IJ} \phi_J \, \not G^{i\,+} \epsilon = 0\,. \tag{5.3}$$

Acting on Eq. (5.2) with ϕ^I leads to

$$\mathfrak{D}_\mu \epsilon = -\phi^I \partial_\mu \phi_I \epsilon\,, \tag{5.4}$$

which takes the form

$$\tilde{\mathfrak{D}}_\mu \epsilon \equiv (\mathfrak{D}_\mu + \zeta_\mu) \epsilon = 0\,, \tag{5.5}$$

and becomes the only differential equation for ϵ. Observe that the covariant derivative $\tilde{\mathcal{D}}_\mu$ contains, apart from the connection ζ, the spin and Kähler connections. Plugging Eq. (5.5) into Eq. (5.2) as to eliminate $\mathcal{D}_\mu \epsilon$ we obtain

$$\tilde{\mathfrak{D}} \phi_I \epsilon + \epsilon_{IJ} \phi^J T^+{}_{\mu\nu} \gamma^\nu \epsilon^* = 0\,, \qquad (\tilde{\mathfrak{D}} \phi_I \equiv (\partial_\mu - \zeta_\mu) \phi_I)\,, \tag{5.6}$$

which is one of the algebraic constraints for ϵ and is a differential equation for ϕ_I.

Multiplying Eq. (5.3) with ϕ_I, we see that it splits into two algebraic constraints for ϵ:

$$\not\partial Z^i \epsilon^* = 0\,, \tag{5.7}$$

$$\not G^{i\,+} \epsilon = 0\,. \tag{5.8}$$

Finally, we add to the system an auxiliary spinor η, with the same chirality as ϵ but with all $U(1)$ charges opposite to those of ϵ and normalized by the condition

$$\bar{\epsilon} \eta = \tfrac{1}{2}\,. \tag{5.9}$$

This normalization condition will be preserved if and only if η satisfies

$$\tilde{\mathfrak{D}}_\mu \eta + a_\mu \epsilon = 0\,, \tag{5.10}$$

for some a_μ with $U(1)$ charges -2 times those of ϵ, *i.e.*

$$\tilde{\mathfrak{D}}_\mu a_\nu = (\nabla_\mu - 2\zeta_\mu - i\mathcal{Q}_\mu) a_\nu\,, \tag{5.11}$$

to be determined by the requirement that the integrability conditions of this differential equation have to be compatible with those of the differential equation for ϵ.

Observe that the null tetrad of vector bilinears one constructs from ϵ and η will in general have non-trivial charges and, in particular, non-trivial Kähler weight: taking into account the definition of the bilinear vectors in Ref. [26], which we reproduce here for convenience

$$l_\mu = i\sqrt{2}\bar{\epsilon^*}\gamma_\mu\epsilon\,, \quad n_\mu = i\sqrt{2}\bar{\eta^*}\gamma_\mu\eta\,, \quad m_\mu = i\sqrt{2}\bar{\epsilon^*}\gamma_\mu\eta = i\bar{\eta}\gamma_\mu\epsilon^*\,, \quad m^*_\mu = i\sqrt{2}\bar{\epsilon}\gamma_\mu\eta^* = i\bar{\eta^*}\gamma_\mu\epsilon\,. \tag{5.12}$$

we see that l and n have 0 $U(1)$ charges but m has -2 times the charges of ϵ and m^* has $+2$ times the charges of ϵ. The metric

$$ds^2 = 2\hat{l}\otimes\hat{n} - 2\hat{m}\otimes\hat{m}^*\,, \tag{5.13}$$

is invariant, though.

The orientation of the null tetrad is important: we choose the relation between a standard Cartesian tetrad $\{e^0, e^1, e^2, e^3\}$ and the complex null tetrad $\{e^u, e^v, e^z, e^{z^*}\} = \{\hat{l}, \hat{n}, \hat{m}, \hat{m}^*\}$ to be

$$\begin{pmatrix} e^u \\ e^v \\ e^z \\ e^{z^*} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & 1 & i \\ & & 1 & -i \end{array}\right) \begin{pmatrix} e^0 \\ e^1 \\ e^2 \\ e^3 \end{pmatrix}. \tag{5.14}$$

This translates into identical relations between gamma matrices:

$$\begin{pmatrix} \gamma^u \\ \gamma^v \\ \gamma^z \\ \gamma^{z^*} \end{pmatrix} = \begin{pmatrix} \not{l} \\ \not{n} \\ \not{m} \\ \not{m}^* \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} 1 & 1 & & \\ 1 & -1 & & \\ \hline & & 1 & i \\ & & 1 & -i \end{array}\right) \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}. \tag{5.15}$$

This choice implies for the chirality matrix

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^{uv}\gamma^{zz^*}\,. \tag{5.16}$$

5.1 Killing equations for the vector bilinears and first consequences

We are now ready to derive equations involving the bilinears, in particular the vector bilinears constructed from ϵ and the auxiliary spinor η introduced above. First we deal with the equations that do not involve derivative of the spinors. Acting with $\bar{\epsilon}$ on Eq. (5.6) and with $\bar{\epsilon}\gamma^\mu$ on Eq. (5.8) we get, respectively

$$T^{+}{}_{\mu\nu}l^{\nu} = 0\,, \tag{5.17}$$

$$G^{i\,+}{}_{\mu\nu}l^{\nu} = 0\,, \tag{5.18}$$

which together imply

$$F^{\Lambda\,+}{}_{\mu\nu}l^{\nu} = 0\,, \tag{5.19}$$

which in its turn implies

$$F^{\Lambda\,+} = \tfrac{1}{2}\phi^{\Lambda}\hat{l}\wedge\hat{m}^{*}\,, \tag{5.20}$$

where ϕ^{Λ} is some complex function. This form of $F^{\Lambda\,+}$ completely solves Eq. (5.8), as becomes paramount through the Fierz identity

$$l_{\mu}\gamma^{\mu\nu}\epsilon^{*} = 3l^{\nu}\epsilon^{*}\,. \tag{5.21}$$

Acting with $\bar{\eta}$ on Eq. (5.6) we get

$$\tilde{\mathfrak{D}}_{\mu}\phi_{I} + i\sqrt{2}\epsilon_{IJ}\phi^{J}T^{+}{}_{\mu\nu}m^{\nu} = 0\,, \tag{5.22}$$

and substituting Eq. (5.20) into it, we obtain

$$\tilde{\mathfrak{D}}_{\mu}\phi_{I} - \tfrac{i}{\sqrt{2}}\epsilon_{IJ}\phi^{J}\mathcal{T}_{\Lambda}\phi^{\Lambda}l_{\mu} = 0\,. \tag{5.23}$$

Finally, acting with $\bar{\epsilon}$ and $\bar{\eta}$ on Eq. (5.7) we get

$$l^{\mu}\partial_{\mu}Z^{i} = 0\,, \tag{5.24}$$

$$m^{\mu}\partial_{\mu}Z^{i} = 0\,, \tag{5.25}$$

which imply

$$dZ^{i} = A^{i}\hat{l} + B^{i}\hat{m}\,, \tag{5.26}$$

for some functions A^i and B^i that are v independent. Observe that, since dZ^i and $\hat{l}$ have no Kähler weight and $\hat{m}$ has Kähler weight $+2$, B^i must have Kähler weight -2. As shown in Refs. [49, 26], for a single scalar ($dZ = A\hat{l} + B\hat{m}$) we can always assume that either B is zero (case A) or A is zero (case B). However, for more than one scalar, it is not possible to remove all the A^is and we are going to have, in general, non-vanishing A^is and B^is, although we can consider particular cases in which either all the A^is or all the B^is vanish.

Observe that, due to the Fierz identity

$$
\not{l}\epsilon^* = \not{n}\epsilon^* = 0\,, \tag{5.27}
$$

the above expression solves Eq. (5.7) identically. These are all the algebraic equations for the bilinears. Now, from Eqs. (5.5) and (5.10) we find the differential equations

$$
\begin{aligned}
\nabla_\mu l_\nu &= 0\,, && (5.28)\\
\tilde{\mathfrak{D}}_\mu n_\nu &= \nabla_\mu n_\nu = -a^*_\mu m_\nu - a_\mu m^*_\nu\,, && (5.29)\\
\tilde{\mathfrak{D}}_\mu m_\nu &= (\nabla_\mu - 2\zeta_\mu - i\mathcal{Q}_\mu) m_\nu = -a_\mu l_\nu\,. && (5.30)
\end{aligned}
$$

5.2 Equations of motion and integrability constraints

Our immediate objective is to find information about the connection ζ_μ using the KSIs and the integrability equations of Eqs. (5.5) and (5.10).

Using the results of the previous section, we can write the Einstein equations the form

$$
\begin{aligned}
\mathcal{E}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathcal{E}^\rho{}_\rho &= R_{\mu\nu} + \left[2\mathcal{G}_{ij^*} A^i A^{*\,j^*} - 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\phi^\Lambda\phi^{*\,\Sigma}\right] l_\mu l_\nu \\
&\quad +2\mathcal{G}_{ij^*} B^i B^{*\,j^*} m_{(\mu} m^*_{\nu)} + 2\mathcal{G}_{ij^*} A^i B^{*\,j^*} l_{(\mu} m^*_{\nu)} \qquad (5.31)\\
&\quad +2\mathcal{G}_{ij^*} B^i A^{*\,j^*} l_{(\mu} m_{\nu)}\,.
\end{aligned}
$$

Comparing with the KSI Eq. (3.17), we end up with tho following two conditions

$$
\begin{aligned}
R_{\mu\nu} l^\nu &= 0\,, && (5.32)\\
R_{\mu\nu} m^\nu - \mathcal{G}_{ij^*}(A^i l_\mu + B^i m_\mu) B^{*\,j^*} &= 0\,. && (5.33)
\end{aligned}
$$

Commuting the derivative and projecting with gamma matrices and spinors in the usual way, and using

$$
(d\mathcal{Q})_{\mu\nu} m^{*\nu} = i\mathcal{G}_{ij^*} B^i B^{*\,j^*} m^*_\mu\,, \qquad (d\mathcal{Q})_{\mu\nu} l^\nu = (d\mathcal{Q})_{\mu\nu} n^\nu = 0\,, \tag{5.34}
$$

which follow from the definition of the Kähler connection and Kähler form Eq. (5.26), it is easy to find, from Eq. (5.5)

$$
\begin{aligned}
\{R_{\mu\nu} + 2(d\zeta)_{\mu\nu}\} l^\nu &= 0\,, && (5.35)\\
\{R_{\mu\nu} + 2(d\zeta)_{\mu\nu}\} m^{*\,\nu} - \mathcal{G}_{ij^*} B^i (A^{*\,j^*} l_\mu + B^{*\,j^*} m^*_\mu) &= 0\,, && (5.36)
\end{aligned}
$$

and from Eq. (5.10)

$$\{R_{\mu\nu} - 2(d\zeta)_{\mu\nu}\}m^{\nu} - \mathcal{G}_{ij^*}(A^i l_\mu + B^i m_\mu)B^{*\,j^*} + 2(\tilde{\mathfrak{D}}a)_{\mu\nu}l^{\nu} \;\;=\;\; 0\,, \tag{5.37}$$

$$\{R_{\mu\nu} - 2(d\zeta)_{\mu\nu}\}n^{\nu} + 2(\tilde{\mathfrak{D}}a)_{\mu\nu}m^{*\,\nu} \;\;=\;\; 0\,. \tag{5.38}$$

Comparing these three sets of equations, we find that they are compatible if

$$(d\zeta)_{\mu\nu}l^{\nu} = (d\zeta)_{\mu\nu}m^{\nu} = 0\,, \quad \Rightarrow d\zeta = 0\,, \quad \Rightarrow \zeta = d\alpha\,, \tag{5.39}$$

locally, and, eliminating ζ by a local phase redefinition, $\tilde{\mathfrak{D}}a$ becomes just $\mathfrak{D}a$ and we get

$$(\mathfrak{D}\hat{a})_{\mu\nu}l^{\nu} \;\;=\;\; 0\,, \tag{5.40}$$

$$(\mathfrak{D}\hat{a})_{\mu\nu}m^{*\,\nu} \;\;=\;\; -\tfrac{1}{2}R_{\mu\nu}n^{\nu}\,, \tag{5.41}$$

so that

$$\mathfrak{D}\hat{a} = -\tfrac{1}{2}R_{z^*u}\hat{m}\wedge\hat{m}^* + \tfrac{1}{2}R_{uu}\hat{l}\wedge\hat{m} + C\hat{l}\wedge\hat{m}^*\,, \tag{5.42}$$

where C is a function that needs to be chosen as to make this equation (and, hence, Eq. (5.10)) integrable. We also have to satisfy the integrability equations (5.32) and (5.33).

Another consequence of the elimination of ζ_μ is that $\tilde{\mathfrak{D}}\phi_I$ becomes just $d\phi_I$, whence Eq. (5.23) implies that $d\phi_I \sim \hat{l}$ and the graviphoton combination

$$\phi \equiv \mathcal{T}_\Lambda\phi^\Lambda\,, \qquad d\phi \sim \hat{l}\,. \tag{5.43}$$

Observe that a similar statement cannot be made about the matter combinations

$$\psi^i \equiv \mathcal{T}^i{}_\Lambda\phi^\Lambda\,. \tag{5.44}$$

The variables ϕ, ψ^i will be convenient for further calculations, and the relation between them and the ϕ^Λ can be obtained from Eq. (3.30):

$$\phi^\Lambda = i\mathcal{L}^{*\,\Lambda}\phi + 2f^\Lambda{}_i\psi^i\,. \tag{5.45}$$

Using these variables, the symplectic vector of field strengths defined in Eq. (2.17) takes the form

$$F = \left(\mathcal{U}_i\psi^i + \tfrac{i}{2}\mathcal{V}^*\phi\right)\hat{l}\wedge\hat{m}^* + \text{c.c.}\,, \tag{5.46}$$

and the symplectic vector containing the Bianchi identities and Maxwell equations, defined in Eq. (2.16) is, in differential-form language

$$
{}^{\star}\hat{\mathcal{E}} = dF = -\hat{l} \wedge \left[d\left(\mathcal{U}_i\psi^i + \tfrac{i}{2}\mathcal{V}^*\phi\right) \wedge \hat{m}^* + \left(\mathcal{U}_i\psi^i + \tfrac{i}{2}\mathcal{V}^*\phi\right) d\hat{m}^* + \text{c.c.}\right] . \tag{5.47}
$$

Since $d\phi \sim \hat{l}$, it drops out of the above equations. Next, we substitute

$$
d\mathcal{V}^* = \mathcal{U}^*{}_{i^*} dZ^{*\,i^*} + \tfrac{1}{2}\mathcal{V}^* d\mathcal{K}\, . \tag{5.48}
$$

Finally, using Eqs. (5.30) and (5.26) we find

$$
\hat{l} \wedge d\hat{m}^* = \hat{l} \wedge (-\tfrac{1}{2}d\mathcal{K}) \wedge \hat{m}^*\, , \tag{5.49}
$$

which, after substituting and assuming independence of v, leads to

$$
{}^{\star}\hat{\mathcal{E}} = e^{\mathcal{K}/2} d(e^{-\mathcal{K}/2}\psi^i\mathcal{U}_i) \wedge \hat{l} \wedge \hat{m}^* + \text{c.c.} \tag{5.50}
$$

We are now in a position to check the KSIs that involve the Maxwell equations and Bianchi identities: first of all, Eqs. (3.20) are satisfied automatically, and Eq. (3.19) can be put in the form

$$
\langle\, {}^{\star}\hat{\mathcal{E}} \mid \mathcal{V}\,\rangle = 0\, . \tag{5.51}
$$

Rewriting

$$
{}^{\star}\hat{\mathcal{E}} = [e^{\mathcal{K}} d(e^{-\mathcal{K}} d\psi^i)\mathcal{U}_i - e^{\mathcal{K}/2}\psi^i m^{*\,\mu}\partial_\mu Z^j \mathfrak{D}_j\mathcal{U}_i\, \hat{m}] \wedge \hat{l} \wedge \hat{m}^* + \text{c.c.} \tag{5.52}
$$

and using $\langle \mathcal{U}_i \mid \mathcal{V}\rangle = \langle \mathcal{U}^*{}_{i^*} \mid \mathcal{V}\rangle = \langle \mathfrak{D}_j\mathcal{U}_i \mid \mathcal{V}\rangle = \langle \mathfrak{D}_{j^*}\mathcal{U}^*{}_{i^*} \mid \mathcal{V}\rangle = 0$ we see that the above equation is always satisfied.

The only component of these equations is, then,

$$
m^{*\,\mu}\partial_\mu(e^{-\mathcal{K}/2}\psi^i\mathcal{U}_i) - \text{c.c.} = 0\, . \tag{5.53}
$$

Finally, let us consider the scalar equation of motion, which takes the form

$$
\mathcal{E}^{i^*} = m^{*\,\mu}\mathfrak{D}_\mu B^{*\,i^*} - B^{*\,i^*} l^\mu a^*_\mu\, . \tag{5.54}
$$

According to Eq. (3.21), this combination has to vanish in order to have supersymmetry, and in the next section we are going to see that this happens if the B^is are covariantly holomorphic in a complex coordinate, denoted by z, and $l^\mu a_\mu = 0$.

5.3 Metric

In order to advance and check the KSIs involving the Ricci tensor we need an explicit form of the metric. This form is dictated by the existence of a covariantly constant null Killing vector, Eq. (5.28), which tells us that the spacetime is a Brinkmann *pp*-wave, [53, 54]. Since l^μ is a Killing vector and $d\hat{l} = 0$ we can introduce the coordinates u and v such that

$$\hat{l} = l_\mu dx^\mu \equiv du\,, \tag{5.55}$$

$$l^\mu \partial_\mu \equiv \frac{\partial}{\partial v}\,. \tag{5.56}$$

We can also define a complex coordinate z by

$$\hat{m} = e^U dz\,, \tag{5.57}$$

where U may depend on z, z^* and u. Eq. (5.24) then states that the scalars Z^i are functions of z and u only:

$$Z^i = Z^i(z,u)\,, \tag{5.58}$$

and, therefore, the functions A^i and B^i defined in Eq. (5.26) are

$$A^i = \partial_{\underline{u}} Z^i\,, \qquad e^U B^i = \partial_{\underline{z}} Z^i\,, \;\Rightarrow \partial_{\underline{z}^*}(e^U B^i) = 0\,. \tag{5.59}$$

Finally, the most general form that $\hat{n}$ can take in this case is

$$\hat{n} = dv + H du + \hat{\omega}\,, \qquad \hat{\omega} = \omega_{\underline{z}} dz + \omega_{\underline{z}^*} dz^*\,, \tag{5.60}$$

where all the functions in the metric are independent of v and where either H or the 1-form $\hat{\omega}$ could, in principle, be removed by a coordinate transformation but we have to check that the tetrad integrability equations (5.28)-(5.30) are satisfied by our choices of e^U, H and $\hat{\omega}$. Eq. (5.13) and the above choice of coordinates, lead to the metric[10]

$$ds^2 = 2du(dv + H du + \hat{\omega}) - 2e^{2U} dz dz^*\,. \tag{5.61}$$

Let us then consider the tetrad integrability equations (5.28)-(5.30): the first equation is solved because the metric does not depend on v. The third equation, with the choice of coordinate z, Eq. (5.57), implies[11]

$$\hat{a} = \left[\dot{U} - i\mathcal{Q}_{\underline{u}}\right]\hat{m} + D\hat{l}\,, \tag{5.62}$$

$$U = -\mathcal{K}/2\,, \tag{5.63}$$

where $D(z, z^*, u)$ is a functions to be determined and over-dots denote partial derivation w.r.t. u. Combining both equations we get

[10]The components of the connection and the Ricci tensor of this metric can be found in the Appendix of Ref. [26].

[11]Actually, the most general solution is $U = -\mathcal{K}/2 + h(u)$, but we can always eliminate $h(u)$ by a redefinition of z that does not change the structure of the metric.

$$\hat{a} = -A^i\partial_i\mathcal{K}\hat{m} + D\hat{l}\,. \tag{5.64}$$

Finally, the second tetrad integrability equation (5.29) implies

$$D = e^{\mathcal{K}/2}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})\,, \tag{5.65}$$

$$(d\hat{\omega})_{\underline{z}\underline{z}^*} = 2ie^{-\mathcal{K}}\mathcal{Q}_{\underline{u}}\,, \tag{5.66}$$

whence $\hat{a}$ is given by

$$\hat{a} = -A^i\partial_i\mathcal{K}\hat{m} + e^{\mathcal{K}/2}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})\hat{l}\,. \tag{5.67}$$

Observe that this implies $a_\mu l^\mu = 0$. On the other hand, the last of Eqs. (5.59) together with Eq. (5.63)

$$\partial_{\underline{z}^*}(e^{-\mathcal{K}/2}B^i) = \mathfrak{D}_{\underline{z}^*}B^i = 0\,. \tag{5.68}$$

Thus, the scalar equation of motion (5.54) is identically satisfied and so is the KSI (3.19).

Having a coordinate system, we can check the integrability conditions Eqs. (5.32,5.33). The first of these is automatically satisfied for Brinkmann metrics. The second splits into

$$\begin{aligned} R_{uz^*} + \mathcal{G}_{ij^*}A^iB^{*\,j^*} &= 0\,, \\ R_{zz^*} + \mathcal{G}_{ij^*}B^iB^{*\,j^*} &= 0\,. \end{aligned} \tag{5.69}$$

The coefficients of the Ricci tensor for Brinkmann metrics were given in the Appendix of Ref. [26]: substituting Eqs. (5.63) and (5.66) into those expressions and using the holomorphicity of the Z^is the above equations are seen to be satisfied identically.[12]

Having an expression for $\hat{a}$, Eq. (5.67), we can impose the integrability condition Eq. (5.42), resulting in

$$C = -e^{\mathcal{K}/2}\partial_{\underline{z}^*}[e^{\mathcal{K}/2}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})]\,, \tag{5.70}$$

$$R_{uu} = -2\partial_{\underline{u}}(A^i\partial_i\mathcal{K}) - 2e^{\mathcal{K}/2}\mathfrak{D}_{\underline{z}}[e^{\mathcal{K}/2}(\partial_{\underline{z}^*}H - \dot{\omega}_{\underline{z}^*})] + 2(A^i\partial_i\mathcal{K})^2\,, \tag{5.71}$$

$$R_{uz^*} = -2e^{\mathcal{K}/2}\partial_{\underline{z}^*}(A^i\partial_i\mathcal{K}) = -2\mathcal{G}_{ij^*}A^iB^{*\,j^*}\,. \tag{5.72}$$

[12] We would like to point out a typo in Tod's article [49]: The metric has the part $2m_{(\mu}m^*_{\nu)}$, which together with [49, (A.9–10)], indicates that the metric factor should be $e^{-2\phi}\omega\bar{\omega}$ and not Eq. [49, (A.17)], which is just the inverse. After taking this into account, Tod's results fully agree with the ones presented here.

The second equation is satisfied automatically. The last equation is, however, incompatible with the integrability equation Eq. (5.69) and with the actual value of R_{uz^*} for the Brinkmann metric unless

$$\partial_{\underline{u}} Z^i \partial_{\underline{z}^*} Z^{*j^*} \mathcal{G}_{ij^*} = 0\,. \tag{5.73}$$

This conditions is a consequence of the choice of η, *i.e.* of our frame and coordinate choice, and should be of no importance whatsoever to the problem of solving the KSEs. Actually, it is easy to see that it can always be satisfied by a shift in η that preserves the normalization condition $\bar{\epsilon}\eta = 1/2$:

$$\eta' = \eta + \delta\epsilon\,, \;\;\Rightarrow\;\; \left\{ \begin{array}{rcl} \hat{l}' & = & \hat{l}\,, \\ \hat{n}' & = & \hat{n} + \delta^* \hat{m} + \delta \hat{m}^* + |\delta|^2 \hat{l}\,, \\ \hat{m}' & = & \hat{m} + \delta \hat{l}\,. \end{array} \right. \tag{5.74}$$

If $B^i = \partial_z Z^i = 0$ then the condition is automatically satisfied. If $B^i = \partial_z Z^i \neq 0$ then $\mathcal{G}_{ij^*} B^i B^{*j^*} \neq 0$ and we just have to perform the above shift with

$$\delta = -\frac{\mathcal{G}_{ij^*} A^i B^{*j^*}}{\mathcal{G}_{ij^*} B^i B^{*j^*}}\,, \tag{5.75}$$

in order to trivialize the condition (5.73).

5.4 Solving the Killing spinor equations

We are now going to see that field configurations given by a metric of the form (Eqs. (5.61) and (5.63))

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{-\mathcal{K}} dz dz^*\,, \tag{5.76}$$

where $\hat{\omega}$ satisfies (Eq. (5.66))

$$(d\omega)_{\underline{z}\underline{z}^*} = 2ie^{-\mathcal{K}} \mathcal{Q}_{\underline{u}}\,, \tag{5.77}$$

scalars of the form (Eq. (5.26))

$$dZ^i = A^i \hat{l} + B^i \hat{m}\,, \tag{5.78}$$

and vector field strengths of the form (Eq. (5.20))

$$F^{\Lambda\,+} = \tfrac{1}{2}\phi^\Lambda \hat{l} \wedge \hat{m}^*\,, \tag{5.79}$$

are always supersymmetric, even though we derived these equations as necessary conditions for supersymmetry.

With the above form of the scalars and vector field strengths the KSE $\delta_\epsilon \lambda^{iI} = 0$ takes the form

$$iA^i \not{l}\epsilon^I + iB^i \not{m}\epsilon^I - \tfrac{1}{2}\epsilon^{IJ}\mathcal{T}^i{}_\Lambda\phi^\Lambda \not{m}^*\not{l}\epsilon_J = 0\,, \tag{5.80}$$

and can be solved by imposing two conditions on the spinors:

$$\not{l}\epsilon^I = 0\,, \qquad \not{m}\epsilon^I = 0\,, \tag{5.81}$$

which formally coincide with the Fierz identities Eqs. (5.27), although now, since there is no *à priori* relation between l, m and ϵ^I, they are not identities but *constraints on* ϵ^I. This fact should be enough to show that they are compatible, but we are going to go further and show that they are equivalent. Multiplying the first condition by $\not{n}$ and the second by $\not{m}^*$ we obtain the more conventional-looking conditions

$$\begin{aligned} \not{n}\,\not{l}\epsilon^I &= (1-\gamma^{uv})\epsilon^I = 0\,, \\ \not{m}^*\,\not{m}\epsilon^I &= -(1+\gamma^{zz^*})\epsilon^I = 0\,. \end{aligned} \tag{5.82}$$

If ϵ^I satisfies the second condition, using $\gamma_5 = -\gamma^{uv}\gamma^{zz^*}$

$$\gamma^{zz^*}\epsilon^I = -\epsilon^I\,,\ \Rightarrow\ \gamma^{uv}\gamma^{zz^*}\epsilon^I = \gamma^{uv}\epsilon^I\,,\ \Rightarrow -\gamma^5\epsilon^I = \gamma^{uv}\epsilon^I\,, \tag{5.83}$$

which, due to the chirality of ϵ^I, leads to the first condition.

Let us now consider the KSE $\delta_\epsilon\psi_{I\,a} = 0$. Taking into account Eqs. (5.81), our tetrad choice and Eq. (5.66), we find that the Killing spinors ϵ_I must be independent of v, z, z^* and must satisfy

$$\dot{\epsilon}_I + \tfrac{1}{2}\epsilon_{IJ}\phi\gamma^{z^*}\epsilon^J = 0\,, \tag{5.84}$$

where $\phi = \mathcal{T}_\Lambda\phi^\Lambda = \phi(u)$. Observe that this equation can always be integrated, even though the explicit form of the ϵ_I may be hard to find.

If $\phi(u)$ is a real function, however, the general solution is readily found to be

$$\epsilon_I = e^{i\Phi}\epsilon_{I\,0} + \tfrac{1}{\sqrt{2}}\epsilon_{IJ}\gamma^{z^*}e^{-i\Phi}\epsilon^J{}_0,\,, \tag{5.85}$$

where

$$\gamma^{z^*}\epsilon_{I\,0} = \gamma^u\epsilon_{I\,0} = 0\,, \qquad (\epsilon_{I\,0})^* = \epsilon^I{}_0 \qquad \dot{\Phi} = -i\phi/\sqrt{2}\,. \tag{5.86}$$

Thus, all the configurations identified are supersymmetric and preserve, at least 1/2 of the available supersymmetries. One can see, moreover, that the only configurations that preserve more than 1/2 are in fact maximally supersymmetric: Minkowski space and the maximally supersymmetric wave of minimal $N = 2$ $D = 4$ supergravity found by Kowalski-Glikman [55], embedded such that only the graviphoton is non-trivial.

5.5 Equations of motion

Let us start with the Maxwell equations and Bianchi identities, given in Eq. (5.50). There is only one non-trivial component which is not automatically satisfied for supersymmetric configurations, namely Eq. (5.53), and we can rewrite it as

$$e^{\mathcal{K}/2}\mathfrak{D}_{\underline{z}}(e^{-\mathcal{K}/2}\psi^i)\mathcal{U}_i + \psi^i\partial_{\underline{z}}Z^j\mathfrak{D}_j\mathcal{U}_i - \text{c.c.} = 0\,, \tag{5.87}$$

where one should keep in mind that the combination $e^{-\mathcal{K}/2}\psi^i$ is a weight -1 vector field. Taking the symplectic product with $\mathcal{U}_k$ and using Eqs. (C.3,C.4) and (C.6), one finds

$$\mathfrak{D}_{\underline{z}^*}(e^{-\mathcal{K}/2}\psi^{*\,i^*}) - ie^{-\mathcal{K}/2}\psi^j\partial_{\underline{z}}Z^k\mathcal{C}_{jk}{}^{i^*} = 0\,. \tag{5.88}$$

A somewhat lighter equation can be derived by defining

$$\psi^i \;=\; e^{\mathcal{K}}\mathcal{G}^{ij^*}\;P_{j^*} \;\;\rightarrow\;\; \partial_{\underline{z}^*}P^*_i \;=\; i\mathcal{C}_{ij}{}^{k^*}\;\partial_z Z^j\;P_{k^*}\;, \tag{5.89}$$

where P_{i^*} is of Kähler weight $(0,2)$. This equation determines ψ^i, but it is extremely difficult to find a general solution, although we will give some solutions in Appendix D.

The only non-automatically satisfied component of the Einstein equations is the uu one

$$\mathcal{E}_{uu} = R_{uu} + 2\mathcal{G}_{ij^*}A^iA^{*\,j^*} - i\Im\text{m}\mathcal{N}_{\Lambda\Sigma}\phi^\Lambda\phi^\Sigma = 0\,. \tag{5.90}$$

Using Eq. (5.45), and the value of R_{uu} this equation takes the form

$$\begin{aligned}
-2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}H + \tfrac{1}{2}e^{-4U}(\partial_{\underline{z}^*}\omega_{\underline{z}} - \partial_{\underline{z}}\omega_{\underline{z}^*})^2 + e^{-2U}(\partial_{\underline{z}^*}\dot{\omega}_{\underline{z}} + \partial_{\underline{z}}\dot{\omega}_{\underline{z}^*}) & \\
+2(\ddot{U} + \dot{U}\dot{U}) + 2\mathcal{G}_{ij^*}(A^iA^{*\,j^*} + 8\psi^i\psi^{*\,j^*}) + 4|\phi|^2 \;\;=\;\; 0\,. &
\end{aligned} \tag{5.91}$$

A supersymmetric solution in this class is, then, fully determined by the real function $H(z,z^*,u)$ and the complex functions $\omega_{\underline{z}}(z,z^*,u), \phi(u), \psi^i(z,z^*,u), Z^i(z,u)$ satisfying Eqs. (5.77,5.87) and (5.91). There are two simple and interesting families of solutions

1. $Z^i = Z^i(z)$. ($A^i = 0$). This implies that $\mathcal{Q}_{\underline{u}} = 0$ and we can safely take $\hat{\omega} = 0$. The Einstein equation takes the form

   $$e^{\mathcal{K}}\partial_{\underline{z}}\partial_{\underline{z}^*}H = 8\mathcal{G}_{ij^*}\psi^i\psi^{*\,j^*} + 2|\phi|^2\,, \tag{5.92}$$

   and can be integrated once the solutions to the Maxwell and Bianchi equations, ψ^i, are given ($\phi(u)$ is an arbitrary complex function).

   If we set to zero the vector field strengths $\phi = \psi^i = 0$, the Einstein equation reduces to the statement that H is a real harmonic function on $\mathbb{C}$.

   The solutions in this subclass are combinations of *pp*-waves associated to the harmonic function H and cosmic strings of the kind considered in Ref. [42], *i.e.* determined by n holomorphic functions $Z^i = Z^i(z)$. Now the metric is determined by supersymmetry to be

$$
\begin{cases}
ds^2 &= 2du(dv + Hdu) - 2e^{-2\mathcal{K}(Z,Z^*)}dzdz^*\,, \\
Z^i &= Z^i(z)\,, \\
\partial_{\underline{z}}\partial_{\underline{z}^*}H &= 0\,.
\end{cases}
\tag{5.93}
$$

In order to study the behaviour of these solutions under the symmetries of the theory, it is convenient to express them in an arbitrary system of holomorphic coordinates

$$
\begin{cases}
ds^2 &= 2du(dv + Hdu) - 2e^{-[\mathcal{K}(Z,Z^*)-h-h^*]}dzdz^*\,, \\
Z^i &= Z^i(z)\,, \\
h &= h(z)\,, \\
\partial_{\underline{z}}\partial_{\underline{z}^*}H &= 0\,.
\end{cases}
\tag{5.94}
$$

The Killing spinors of these solutions are

$$
\epsilon_I = e^{-\frac{1}{4}(h-h^*)}\epsilon_{I\,0}\,, \qquad \gamma^{z^*}\epsilon_{I\,0} = 0\,. \tag{5.95}
$$

The isometries of the Kähler metric (which are the duality symmetries of our theory) leave invariant the Kähler potential up to Kähler transformations Eq. (B.7). Of course, these duality transformations leave invariant the spacetime metric, but the relation between the g_{zz^*} component and the Kähler potential will change unless the holomorphic function h transforms according to

$$
h' = h + f\,, \tag{5.96}
$$

which makes the Killing spinors transform precisely as objects of Kähler weight 1/2, as they should. Actually, for the metric to be form-invariant, it is enough that $\Re\mathrm{e}(h') = \Re\mathrm{e}(h') + \Re\mathrm{e}(f)$ while the spinors will behave as objects of Kähler weight 1/2 if $\Im\mathrm{m}(h') = \Im\mathrm{m}(h') + \Im\mathrm{m}(f)$. These two conditions are independent. Only the first was required in the construction of Ref. [42], but supersymmetry requires the second.[13]

The holomorphic functions $Z^i(z)$ will in general be multi-valued and will have non-trivial monodromies. Only those which are isometries of the Kähler metric can be allowed. The metric will be invariant and the Killing spinors will also have the correct monodromy if h transforms as above.

[13]We thank Jelle Hartong for useful conversations on this point.

2. $Z^i = Z^i(u) = 0$. This implies that $\mathcal{K}$ and, therefore, U are functions of u only, whence the latter can be eliminated from the metric by a change of coordinates. Since the pullback of Kähler 1-form depends on u only, we can solve Eq. (5.66) for $\hat{\omega}$:

$$\hat{\omega} = ie^{-\mathcal{K}}\mathcal{Q}_{\underline{u}}(zdz^* - z^*dz)\,, \tag{5.97}$$

which can, however, be eliminated by further change of coordinates. The remaining Einstein equation takes the form

$$2\partial_{\underline{z}}\partial_{\underline{z}^*}H = 2\mathcal{G}_{ij^*}(A^iA^{*\,j^*} + 8\psi^i\psi^{*\,j^*}) + 4|\phi|^2\,. \tag{5.98}$$

Eq. (5.88) can in this case be solved, leading to the statement that ψ^i only depends on u and z^*. Introducing then the functions Υ^i, defined through the relation $\partial_{z^*}\Upsilon^i = \psi^i$, the above equation can be integrated with great ease, giving

$$H = (\mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} + 2|\phi|^2)|z|^2 + 8\mathcal{G}_{ij^*}\Upsilon^i\Upsilon^{*j^*} + f(z,u) + f^*(z^*,u)\,. \tag{5.99}$$

The supersymmetric solutions of this class take, therefore, the form

$$\left\{\begin{array}{rcl} ds^2 &=& 2du(dv + Hdu) - 2dzdz^*\,,\\[2ex] F^{\Lambda\,+} &=& \left[\frac{i}{2}\mathcal{L}^{*\,\Lambda}\phi(u) + f_i^{\Lambda}\psi^i(u,z^*)\right]du\wedge dz^*\,,\\[2ex] Z^i &=& Z^i(u)\,, \end{array}\right. \tag{5.100}$$

where Z^i, ϕ are arbitrary functions of u and H is given above.

Acknowledgments

T.O. would like to thank interesting conversations with I. Bena and R. Kallosh and the PH-TH Division of CERN for their hospitality and financial support during the last stages of this work and M.M. Fernández for her lasting support. Likewise, PM thanks the Instituto de Física Teórica for its hospitality. This work has been supported in part by the Spanish grant BFM2003-01090.

A Conventions

In this paper we use basically the notation of Ref. [43] and the conventions of Ref. [26], to which we have adapted the formulae of Ref. [43]. The main differences between the conventions of those two references are the signs of spin connection, the completely antisymmetric tensor ϵ^{abcd} and γ_5. Thus, chiralities are reversed and self-dual tensors are replaced by anti-self-dual tensors and vice-versa. The curvatures are identical. Finally, the normalization of the 2-form components differs by a factor of 2: for us

$$F = dA = \tfrac{1}{2}F_{\mu\nu}dx^{\mu} \wedge dx^{\nu} \Rightarrow F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,, \tag{A.1}$$

which amounts to a difference of a factor of 2 in the vectors supersymmetry transformations Eq. (2.8). Further, all fermions and supersymmetry parameters from Ref. [43] have been rescaled by a factor of $\frac{1}{2}$, which introduces additional factors of $\frac{1}{4}$ in all the bosonic fields supersymmetry transformations Eqs. (2.7-2.9).

The meaning of the different indices used in this paper is explained in Table 1. We use the shorthand $\bar{n} \equiv n+1$.

Type	Associated structure
$\mu,\ \nu,\ \ldots$	Curved space
$a,\ b,\ \ldots$	Tangent space
$m, n, \ldots$	Cartesian $\mathbb{R}^3$-indices
$i, j, \ldots;\ i^*, j^*, \ldots$	Complex scalar fields and their conjugates. There are n of them.
$\Lambda, \Sigma, \ldots$	$\mathfrak{sp}(\bar{n})$ indices ($\bar{n} = n+1$)
$I, J, \ldots$	$N=2$ spinor indices

Table 1: Meaning of the indices used in this paper.

B Kähler geometry

A Kähler manifold $\mathcal{M}$ is a complex manifold on which there exist complex coordinates z^i and $z^{*\,i^*} = (z^i)^*$ and a function $\mathcal{K}$, called the *Kähler potential*, such that the line element is

$$ds^2 = 2\mathcal{G}_{ii^*}\; dz^i dz^{*\,i^*}\,, \tag{B.1}$$

with

$$\mathcal{G}_{ii^*} = \partial_i \partial_{i^*}\mathcal{K}\,. \tag{B.2}$$

The *Kähler (connection) 1-form* $\mathcal{Q}$ is defined by

$$\mathcal{Q} \equiv (2i)^{-1}(dz^i\partial_i\mathcal{K} - dz^{*\,i^*}\partial_{i^*}\mathcal{K})\,, \tag{B.3}$$

and the *Kähler 2-form* $\mathcal{J}$ is its exterior derivative

$$\mathcal{J} \equiv d\mathcal{Q} = i\mathcal{G}_{ii^*}dz^i \wedge dz^{*\,i^*}\,. \tag{B.4}$$

The Levi-Cività connection on a Kähler manifold is given by

$$\Gamma_{jk}{}^{i} = \mathcal{G}^{ii^*}\partial_j\mathcal{G}_{i^*k}\,, \qquad \Gamma_{j^*k^*}{}^{i^*} = \mathcal{G}^{i^*i}\partial_{j^*}\mathcal{G}_{k^*i}\,. \tag{B.5}$$

The Riemann curvature tensor has as only non-vanishing components $R_{ij^*kl^*}$, but we will not need their explicit expression. The Ricci tensor is given by

$$R_{ii^*} = \partial_i\partial_{i^*}\left(\tfrac{1}{2}\log\det\mathcal{G}\right)\,. \tag{B.6}$$

The Kähler potential is not unique: it is defined only up to *Kähler transformations* of the form

$$\mathcal{K}'(z,z^*) = \mathcal{K}(z,z^*) + f + f^*\,, \tag{B.7}$$

where f is any holomorphic function of the complex coordinates z^i. Under these transformations, the Kähler metric and Kähler 2-form are invariant, while the components of the Kähler connection 1-form transform according to

$$\mathcal{Q}'_i = \mathcal{Q}_i - \tfrac{i}{2}\partial_i f\,. \tag{B.8}$$

By definition, objects with Kähler weight $(q,\bar{q})$ transform under the above Kähler transformations with a factor $e^{-(qf+\bar{q}f^*)/2}$ and the Kähler-covariant derivative $\mathfrak{D}$ acting on them is given by

$$\mathfrak{D}_i \equiv \nabla_i + iq\mathcal{Q}_i\,, \qquad \mathfrak{D}_{i^*} \equiv \nabla_{i^*} - i\bar{q}\mathcal{Q}_{i^*}\,, \tag{B.9}$$

where ∇ is the standard covariant derivative associated to the Levi-Cività connection on $\mathcal{M}$.

When $(q,\bar{q}) = (1,-1)$, this defines a complex line bundle $L^1 \to \mathcal{M}$ over the Kähler manifold $\mathcal{M}$ whose first, and only, Chern class equals the Kähler 2-form $\mathcal{J}$. A complex line bundle with this property is known as a *Kähler-Hodge (KH) manifold* and provides the formal starting point for the definition of a special Kähler manifold[14] that is explained in the next Appendix.

We will often use the spacetime pullback of the Kähler-covariant derivative on tensor fields with Kähler weight $(q,-q)$ (weight q, for short) for which it takes the simple form

$$\mathfrak{D}_\mu = \nabla_\mu + iq\mathcal{Q}_\mu\,, \tag{B.10}$$

where ∇_μ is the standard spacetime covariant derivative plus possibly the pullback of the Levi-Cività connection on $\mathcal{M}$; $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form, *i.e.*

$$\mathcal{Q}_\mu = (2i)^{-1}(\partial_\mu z^i\partial_i\mathcal{K} - \partial_\mu z^{*i^*}\partial_{i^*}\mathcal{K})\,. \tag{B.11}$$

[14] Some basic references for this material are [56, 50, 46] and the review [27]. The definition of special Kähler manifold was made in Ref. [57], formalizing the original results of Ref. [45].

C Special Kähler geometry

Let us now consider a flat $2\bar{n}$-dimensional vector bundle $E \to \mathcal{M}$ with structure group $Sp(\bar{n};\mathbb{R})$, and take a section $\mathcal{V}$ of the product bundle $E \otimes L^1 \to \mathcal{M}$ and its complex conjugate $\overline{\mathcal{V}}$, which formally is a section of the bundle $E \otimes L^{-1} \to \mathcal{M}$. Then, a *special Kähler manifold*, is a bundle $E \otimes L^1 \to \mathcal{M}$, for which there exists a section $\mathcal{V}$ such that

$$\mathcal{V} = \left(\begin{array}{c} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{array} \right) \quad \to \quad \left\{ \begin{array}{lcl} \langle \mathcal{V} \mid \mathcal{V}^* \rangle & \equiv & \mathcal{L}^{*\,\Lambda}\mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}^*_\Lambda = -i\,, \\ \mathfrak{D}_{i^*}\mathcal{V} & = & (\partial_{i^*} + \frac{1}{2}\partial_{i^*}\mathcal{K})\mathcal{V} = 0\,, \\ \langle \mathfrak{D}_i \mathcal{V} \mid \mathcal{V} \rangle & = & 0\,. \end{array} \right. \tag{C.1}$$

If we then define

$$\mathcal{U}_i \equiv \mathfrak{D}_i \mathcal{V} = \left(\begin{array}{c} f^\Lambda{}_i \\ h_{\Sigma\, i} \end{array} \right)\,, \quad \mathcal{U}^*{}_{i^*} = (\mathcal{U}_i)^*\,, \tag{C.2}$$

then it follows from the basic definitions that

$$\begin{array}{rclcrcl} \mathfrak{D}_{i^*}\,\mathcal{U}_i & = & \mathcal{G}_{ii^*}\,\mathcal{V} & & \langle \mathcal{U}_i \mid \mathcal{U}^*{}_{i^*} \rangle & = & i\mathcal{G}_{ii^*}\,, \\ \langle \mathcal{U}_i \mid \mathcal{V}^* \rangle & = & 0\,, & & \langle \mathcal{U}_i \mid \mathcal{V} \rangle & = & 0\,. \end{array} \tag{C.3}$$

Taking the covariant derivative of the last identity $\langle \mathcal{U}_i \mid \mathcal{V} \rangle = 0$ we find immediately that $\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = -\langle\, \mathcal{U}_j \mid \mathcal{U}_i \rangle$. It can be shown that the r.h.s. of this equation is antisymmetric while the l.h.s. is symmetric, so that

$$\langle \mathfrak{D}_i \mathcal{U}_j \mid \mathcal{V} \rangle = \langle \mathcal{U}_j \mid \mathcal{U}_i \rangle = 0\,. \tag{C.4}$$

The importance of this last equation is that if we group together $\mathcal{E}_\Lambda = (\mathcal{V},\mathcal{U}_i)$, we can see that $\langle \mathcal{E}_\Sigma \mid \mathcal{E}^*{}_\Lambda \rangle$ is a non-degenerate matrix. This then allows us to construct an identity operator for the symplectic indices, such that for a given section of $\mathcal{A} \ni \Gamma\,(E,\mathcal{M})$ we have

$$\mathcal{A} = i\langle \mathcal{A} \mid \mathcal{V}^* \rangle \mathcal{V} - i\langle \mathcal{A} \mid \mathcal{V} \rangle\; \mathcal{V}^* + i\langle \mathcal{A} \mid \mathcal{U}_i \rangle \mathcal{G}^{ii^*}\; \mathcal{U}^*{}_{i^*} - i\langle \mathcal{A} \mid \mathcal{U}^*{}_{i^*} \rangle \mathcal{G}^{ii^*} \mathcal{U}_i\,. \tag{C.5}$$

As we have seen $\mathfrak{D}_i \mathcal{U}_j$ is symmetric in i and j, but what more can be said about it: as one can easily see, the inner product with $\mathcal{V}^*$ and $\mathcal{U}^*{}_{i^*}$ vanishes due to the basic properties. Let us then define the Kähler-weight 2 object

$$\mathcal{C}_{ijk} \equiv \langle \mathfrak{D}_i\; \mathcal{U}_j \mid \mathcal{U}_k \rangle \quad \to \quad \mathfrak{D}_i\; \mathcal{U}_j = i\mathcal{C}_{ijk}\mathcal{G}^{kl^*}\mathcal{U}^*{}_{l^*}\,, \tag{C.6}$$

where the last equation is a consequence of Eq. (C.5). Since the $\mathcal{U}$'s are orthogonal, however, one can see that $\mathcal{C}$ is completely symmetric in its 3 indices. Furthermore one can show that

$$\mathfrak{D}_{i^*}\ \mathcal{C}_{jkl} = 0\,, \qquad \mathfrak{D}_{[i}\ \mathcal{C}_{j]kl} = 0\,. \tag{C.7}$$

Observe that these equations imply the existence of a function $\mathcal{S}$, such that

$$\mathcal{C}_{ijk} = \mathfrak{D}_i\mathfrak{D}_j\mathfrak{D}_k\ \mathcal{S}\,. \tag{C.8}$$

The function $\mathcal{S}$ is given by [58]

$$\mathcal{S} \sim \mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,, \tag{C.9}$$

where $\mathcal{N}$ is the period or monodromy matrix. This matrix is defined by the relations

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,, \qquad h_{\Lambda\, i} = \mathcal{N}^*{}_{\Lambda\Sigma}f^\Sigma{}_i\,. \tag{C.10}$$

The relation $\langle \mathcal{U}_i \mid \overline{\mathcal{V}}\rangle = 0$ then implies that $\mathcal{N}$ is symmetric, which then also trivializes $\langle \mathcal{U}_i \mid \mathcal{U}_j\rangle = 0$.

From the other basic properties in (C.3) we find

$$\mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^{*\,\Sigma} \;=\; -\tfrac{1}{2}\,, \tag{C.11}$$

$$\mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}f^\Sigma{}_i \;=\; \mathcal{L}^\Lambda \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}f^{*\,\Sigma}{}_{i^*} = 0\,, \tag{C.12}$$

$$f^\Lambda{}_i\ \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}f^{*\,\Sigma}{}_{i^*} \;=\; -\tfrac{1}{2}\mathcal{G}_{ii^*}\,. \tag{C.13}$$

Further identities that can be derived are

$$(\partial_i\mathcal{N}_{\Lambda\Sigma})\mathcal{L}^\Sigma \;=\; -2i\Im\mathrm{m}(\mathcal{N})_{\Lambda\Sigma}\ f^\Sigma{}_i\,, \tag{C.14}$$

$$\partial_i\mathcal{N}^*{}_{\Lambda\Sigma}\ f^\Sigma{}_j \;=\; -2\mathcal{C}_{ijk}\mathcal{G}^{kk^*}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}f^{*\,\Sigma}{}_{k^*}\,, \tag{C.15}$$

$$\mathcal{C}_{ijk} \;=\; f^\Lambda{}_i f^\Sigma{}_j\partial_k\mathcal{N}^*_{\Lambda\Sigma}\,, \tag{C.16}$$

$$\mathcal{L}^\Sigma\partial_{i^*}\mathcal{N}_{\Lambda\Sigma} \;=\; 0\,, \tag{C.17}$$

$$\partial_{i^*}\mathcal{N}^*{}_{\Lambda\Sigma}\ f^\Sigma{}_i \;=\; 2i\mathcal{G}_{ii^*}\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma\,. \tag{C.18}$$

An important identity one can derive, and that will be used various times in the main text, is given by

$$U^{\Lambda\Sigma} \equiv f^\Lambda{}_i\mathcal{G}^{ii^*}f^{*\,\Sigma}{}_{i^*} = -\tfrac{1}{2}\Im\mathrm{m}(\mathcal{N})^{-1|\Lambda\Sigma} - \mathcal{L}^{*\,\Lambda}\mathcal{L}^\Sigma\,, \tag{C.19}$$

whence $(U^{\Lambda\Sigma})^* = U^{\Sigma\Lambda}$.

We can define the graviphoton and matter vector projectors

$$\mathcal{T}_\Lambda \equiv 2i\mathcal{L}_\Lambda = 2i\mathcal{L}^\Sigma \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,, \tag{C.20}$$

$$\mathcal{T}^i{}_\Lambda \equiv -f^*{}_\Lambda{}^i = -\mathcal{G}^{ij^*} f^*{}^\Sigma{}_{j^*} \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,. \tag{C.21}$$

Using these definitions and the above properties one can show the following identities for the derivatives of the period matrix:

$$\begin{aligned} \partial_i \mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{T}_{i(\Lambda}\mathcal{T}_{\Sigma)}\,, \\ \partial_{i^*}\mathcal{N}_{\Lambda\Sigma} &= 4\mathcal{C}^*{}_{i^*j^*k^*}\mathcal{T}^{i^*}{}_{(\Lambda}\mathcal{T}^{j^*}{}_{\Sigma)}\,. \end{aligned} \tag{C.22}$$

C.1 Prepotential: Existence and more formulae

Let us start by introducing the explicitly holomorphic section $\Omega = e^{-\mathcal{K}/2}\mathcal{V}$, which allows us to rewrite the system Eqs. (C.1) as

$$\Omega = \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Sigma \end{pmatrix} \rightarrow \begin{cases} \langle \Omega \mid \Omega^* \rangle \equiv \mathcal{X}^{*\Lambda}\mathcal{F}_\Lambda - \mathcal{X}^\Lambda \mathcal{F}^*_\Lambda = -i\, e^{-\mathcal{K}}\,, \\ \partial_{i^*}\Omega = 0\,, \\ \langle \partial_i \Omega \mid \Omega \rangle = 0\,. \end{cases} \tag{C.23}$$

Observe that the first of Eqs. (C.23) together with the definition of the period matrix $\mathcal{N}$ imply the following expression for the Kähler potential:

$$e^{-\mathcal{K}} = -2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{X}^\Lambda \mathcal{X}^{*\Sigma}\,. \tag{C.24}$$

If we now assume that $\mathcal{F}_\Lambda$ depends on Z^i through the $\mathcal{X}$'s, then from the last equation we can derive that

$$\partial_i \mathcal{X}^\Lambda \left[2\mathcal{F}_\Lambda - \partial_\Lambda \left(\mathcal{X}^\Sigma \mathcal{F}_\Sigma\right)\right] = 0\,. \tag{C.25}$$

If $\partial_i \mathcal{X}^\Lambda$ is invertible as an $n \times \bar{n}$ matrix, then we must conclude that

$$\mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F}(\mathcal{X})\,, \tag{C.26}$$

where $\mathcal{F}$ is a homogeneous function of degree 2, called the *prepotential.*

Making use of the prepotential and the definitions (C.10), we can calculate

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{F}^*_{\Lambda\Sigma} + 2i\frac{\Im\mathrm{m}\mathcal{F}_{\Lambda\Lambda'}\mathcal{X}^{\Lambda'}\Im\mathrm{m}\mathcal{F}_{\Sigma\Sigma'}\mathcal{X}^{\Sigma'}}{\mathcal{X}^\Omega \Im\mathrm{m}\mathcal{F}_{\Omega\Omega'}\mathcal{X}^{\Omega'}}\,. \tag{C.27}$$

Having the explicit form of $\mathcal{N}$, we can also derive an explicit representation for $\mathcal{C}$ by applying Eq. (C.17). One finds

$$\mathcal{C}_{ijk} = e^{\mathcal{K}}\partial_i\mathcal{X}^\Lambda\partial_j\mathcal{X}^\Sigma\partial_k\mathcal{X}^\Omega\mathcal{F}_{\Lambda\Sigma\Omega}\ , \tag{C.28}$$

so that the prepotential really determines all structures in special geometry.

A last remark has to be made about the existence of a prepotential: clearly, given a holomorphic section Ω a prepotential need not exist. It was shown in Ref. [46], however, that one can always apply an $Sp(\bar{n},\mathbb{R})$ transformation such that a prepotential exists. Clearly the $N=2$ SUGRA action is not invariant under the full $Sp(\bar{n},\mathbb{R})$, but the equations of motion and the supersymmetry equations are. This means that for the purpose of this article we can always, even if this is not done, impose the existence of a prepotential.

D Some explicit cases

D.1 Quadratic prepotential

This is a simple, but important, case in which there is a prepotential and it takes the form

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma\,, \tag{D.1}$$

where $\mathcal{F}_{\Lambda\Sigma}$ is a complex, symmetric, constant matrix that coincides with the matrix of second derivatives of $\mathcal{F}$. Its imaginary part must be negative definite. The period matrix is given by Eq. (C.27). Observe that

$$\mathcal{F}_\Lambda = \mathcal{F}_{\Lambda\Sigma}\mathcal{X}^\Sigma\,. \tag{D.2}$$

The Kähler potential is

$$e^{-\mathcal{K}} = -2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{X}^{*\,\Lambda}\mathcal{X}^\Sigma\,. \tag{D.3}$$

To construct the general solution of the timelike case, we need to relate the real section $\mathcal{R}$ and $\mathcal{I}$ defined in Eqs. (C.10). This can be done by using the property Eq. (D.2), rescaling it by $e^{\mathcal{K}/2}/X$:

$$\mathcal{M}_\Lambda/X = \mathcal{F}_{\Lambda\Sigma}\mathcal{L}^\Sigma/X\,, \tag{D.4}$$

and then, taking the imaginary part of this equation and using the invertibility of $\Im\mathrm{m}\mathcal{F}_{\Lambda\Sigma}$, we find the solution

$$\Re\mathrm{e}(\mathcal{L}^\Lambda/X) = \Im\mathrm{m}(\mathcal{F})^{-1|\Lambda\Sigma}[\Im\mathrm{m}(\mathcal{M}_\Sigma/X) - \Re\mathrm{e}(\mathcal{F}_{\Sigma\Omega}\Im\mathrm{m}(\mathcal{L}^\Omega/X)]\,, \tag{D.5}$$

which implies

$$\mathcal{L}^\Lambda/X = \Im\mathrm{m}(\mathcal{F})^{-1|\Lambda\Sigma}[\Im\mathrm{m}(\mathcal{M}_\Sigma/X) - \mathcal{F}^*{}_{\Sigma\Omega})\Im\mathrm{m}(\mathcal{L}^\Omega/X)]\,. \tag{D.6}$$

In other words, this implies that we can take the components of the section $\mathcal{L}^\Lambda/X$ to be arbitrary complex harmonic functions $\mathcal{H}^\Lambda$.

Now, $|M|^2$ which appears in the metric Eq. (4.5) is given, according to Eq. (4.35), by the Kähler potential Eq. (D.3) where the scalars $\mathcal{X}^\Lambda$ are substituted by the complex harmonic functions $\mathcal{H}^\Lambda$, i.e.

$$|M|^2 = 2|X|^2 = -[\Im\mathrm{m}\mathcal{F}_{\Lambda\Sigma}\mathcal{H}^{*\Lambda}\mathcal{H}^\Sigma]^{-1} . \tag{D.7}$$

The other term that appears in the metric ω, is given in terms of the real section $\mathcal{I}$ by Eq. (4.31). Substituting the imaginary parts of the harmonic functions $\mathcal{H}^\Lambda$ in that formula, we get

$$(d\omega)_{mn} = -i\epsilon_{mnp}\Im\mathrm{m}\mathcal{F}_{\Lambda\Sigma}[\partial_p\mathcal{H}^\Lambda\mathcal{H}^{*\Sigma} - \partial_p\mathcal{H}^{*\Lambda}\mathcal{H}^\Sigma] = -\epsilon_{mnp}e^{-\mathcal{K}}\mathcal{Q}_p[\mathcal{V}/X,(\mathcal{V}/X)^*] , \tag{D.8}$$

where $\mathcal{Q}_p[\mathcal{V}/X,(\mathcal{V}/X)^*]$ stands for the pull-back of the Kähler 1-form substituting the X^Λs by the harmonic functions $\mathcal{H}^\Lambda$. For the $\bar{n}=2$ case, which can be embedded in pure $N=4, d=4$ supergravity, these expressions were first found in Ref. [37]. We stress that these functions are completely arbitrary and that there is no further constraint on them. Different choices lead to solutions describing different physical systems. In Ref. [37] the most general choice that leads to a stationary, axisymmetric, asymptotically flat spacetime in the $\bar{n}=2$ case was studied. These spacetimes correspond, in general, to charged, rotating "black holes" (sometimes with singular horizon), with NUT charge.

As one can see from Eq. (C.28), the fact that we are dealing with a quadratic prepotential implies that $\mathcal{C}_{ijk}=0$. This then means that Eq. (5.88) is generically solved by

$$\psi^i \;=\; e^{\mathcal{K}}\,\mathcal{G}^{ij^*}\;P_{j^*}(u,z^*)\;, \tag{D.9}$$

so that in the case of a quadratic prepotential there are only two differential equations that need to be solved: eqs. (5.77) and (5.91).

D.2 STU-like models

By an STU-like model, we mean a theory with a prepotential of the type

$$\mathcal{F} \;=\; -d_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} \qquad (i,j,k=1,\ldots,n)\;, \tag{D.10}$$

where d_{abc} is a totally symmetric tensor. The proper STU-model is defined by $n=3$ and $d_{123}=1$ as the only non-vanishing coefficients of d. The coefficients d_{ijk} are related to the $\mathcal{C}_{ijk}$ by

$$\mathcal{C}_{ijk} \;=\; e^{\mathcal{K}}\,d_{ijk} \quad ; \quad e^{-\mathcal{K}} \;=\; 8\,d_{ijk}\,\Im m(Z^i)\Im m(Z^j)\Im m(Z^k)\;. \tag{D.11}$$

In [59] Shmakova found a generic, conditional solution to the stabilization equation for STU-like models. Writing $\mathcal{I}^T=(p^\Lambda,q_\Lambda)$, the stabilization equations read $p^\Lambda=\mathrm{Im}(\mathcal{X}^\Lambda)$

and

$$q_0 \ = \ |\mathcal{X}^0|^{-4}\ d_{ijk}\ \mathrm{Im}\left((\overline{\mathcal{X}}^0)^2\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k\right) \ , \tag{D.12}$$

$$q_i \ = \ |\mathcal{X}^0|^{-2}\ d_{ijk}\ \mathrm{Im}\left(\overline{\mathcal{X}}^0\mathcal{X}^j\mathcal{X}^k\right) \ . \tag{D.13}$$

Clearly the stabilization equations for p^Λ are solved by the Ansatz

$$\mathcal{X}^0 \ = \ p^0\,(i \ - \ Y_0) \quad , \quad \mathcal{X}^i \ = \ Y^i \ - \ p^iY_0 \ + \ ip^i \ , \tag{D.14}$$

and plugging this Ansatz into the stabilization equation (D.13), we can see that it can be solved by redefining $Y^i = \sqrt{1+Y_0^2}\ \tilde{Y}^i$ iff a solution to

$$d_{ijk}\tilde{Y}^j\tilde{Y}^k \ = \ \tfrac{1}{3}\ p^0q_i \ + \ d_{ijk}\ p^jp^k \ , \tag{D.15}$$

can be found. Assuming that such a solution exists, we can then analyze Eq. (D.12) by direct substitution, as to find

$$1 \ + \ Y_0^2 \ = \ \frac{4\tilde{\Delta}^2}{4\tilde{\Delta}^2 - [p^0p^\Lambda q_\Lambda + 2\Delta]^2} \ , \tag{D.16}$$

where we have defined $\Delta = d_{ijk}p^ip^jp^k$ and $\tilde{\Delta} = d_{ijk}\tilde{Y}^i\tilde{Y}^j\tilde{Y}^k$. Armed with this knowledge one can then, using Eqs. (4.33), calculate

$$\frac{1}{|X|^2} \ = \ \frac{4\sqrt{4\tilde{\Delta}^2 - [p^0p^\Lambda q_\Lambda + 2\Delta]^2}}{p^0} \ . \tag{D.17}$$

It should be clear that Eq. (D.15) acts as the keystone for this construction.

There are 2 cases for which a solution to Eq. (D.15) is near trivial to find. The first one is the STU-model, so that $n=3$ and $d_{123}=1$ is the only non-vanishing coefficient of d where one can see that the general solution reads

$$\tilde{Y}^i \ = \ \sqrt{\frac{d_{ijk}\Upsilon_j\Upsilon_k}{4\Upsilon_i}} \quad ; \quad \Upsilon_i \ \equiv \ \tfrac{1}{3}p^0q_i - d_{ijk}p^ip^j \ . \tag{D.18}$$

Of course, the other case is when Eq. (D.15) reduces to a purely quadratic equation, which happens when $d_{ijk} = \delta_{ij}\delta_{jk}$, where for simplicity we have chosen possible constants to be unity. The solution then reads

$$\tilde{Y}^i \ = \ \sqrt{\Upsilon_i} \ . \tag{D.19}$$

With this knowledge and a choice for the harmonic functions, needed to calculate ω through Eq. (4.31), the solution is fully specified in the timelike case.

The null case is in general a far harder nut to crack, and as one might have suspected, we have been unable to find a generic solution to Eq. (5.88) for the STU-like models. A particular, but non-trivial, solution we were able to find is for the case $n=1$, $d_{111}=1$ and reads

$$\psi^1 \ = \ \left(a\ \Im m(t)^{-3} \ + \ ib\ \Im m(t)\right)\ \partial_{z^*}t^* \ , \tag{D.20}$$

where we have used the notation $t = Z^1$.

References

[1] J.P. Gauntlett and S. Pakis, JHEP **0304** (2003) 039 [arXiv:hep-th/0212008].

[2] J.P. Gauntlett, J.B. Gutowski and S. Pakis, JHEP **0312** (2003) 049 [arXiv:hep-th/0311112].

[3] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Class. Quant. Grav. **21** (2004) 4335 [arXiv:hep-th/0402153].

[4] O.A.P. Mac Conamhna, Phys. Rev. D **70** (2004) 105024 [arXiv:hep-th/0408203].

[5] J. Figueroa-O'Farrill, P. Meessen and S. Philip, Class. Quant. Grav. **22** (2005) 207 [arXiv:hep-th/0409170].

[6] J. Gillard, U. Gran and G. Papadopoulos, Class. Quant. Grav. **22** (2005) 1033 [arXiv:hep-th/0410155].

[7] M. Cariglia and O.A.P. Mac Conamhna, Phys. Rev. D **73** (2006) 045011 [arXiv:hep-th/0411079].

[8] M. Cariglia and O.A.P. Mac Conamhna, Phys. Rev. Lett. **94** (2005) 161601 [arXiv:hep-th/0412116].

[9] M. Cariglia and O.A.P. Mac Conamhna, Phys. Rev. Lett. **94** (2005) 161601.

[10] U. Gran, J. Gutowski and G. Papadopoulos, arXiv:hep-th/0501177.

[11] U. Gran, G. Papadopoulos and D. Roest, Class. Quant. Grav. **22** (2005) 2701 [arXiv:hep-th/0503046].

[12] O.A.P. Mac Conamhna, arXiv:hep-th/0504028.

[13] U. Gran, J. Gutowski and G. Papadopoulos, arXiv:hep-th/0505074.

[14] O.A.P. Mac Conamhna, arXiv:hep-th/0505230.

[15] U. Gran, J. Gutowski, G. Papadopoulos and D. Roest, arXiv:hep-th/0507087.

[16] U. Gran, P. Lohrmann and G. Papadopoulos, arXiv:hep-th/0510176.

[17] K. Behrndt, M. Cvetic and T. Liu, arXiv:hep-th/0512032.

[18] G.W. Gibbons and C.M. Hull, Phys. Lett. B **109**, 190 (1982).

[19] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, Class. Quant. Grav. **20** (2003) 4587 [arXiv:hep-th/0209114].

[20] J.P. Gauntlett and J.B. Gutowski, Phys. Rev. D **68** (2003) 105009; Erratum, *ibid.* D **70** (2004) 089901 [arXiv:hep-th/0304064].

[21] J.B. Gutowski, D. Martelli and H.S. Reall, Class. Quant. Grav. **20** (2003) 5049 [arXiv:hep-th/0306235].

[22] A. Chamseddine, J. Figueroa-O'Farrill and W. Sabra, arXiv:hep-th/0306278.

[23] M.M. Caldarelli and D. Klemm, JHEP **0309** (2003) 019 [arXiv:hep-th/0307022].

[24] M. Cariglia and O.A.P. Mac Conamhna, Class. Quant. Grav. **21** (2004) 3171 [arXiv:hep-th/0402055].

[25] M. Cariglia and O.A.P. Mac Conamhna, Phys. Rev. D **70** (2004) 125009 [arXiv:hep-th/0407127].

[26] J. Bellorín and T. Ortín, Nucl. Phys. B **726** (2005) 171 [arXiv:hep-th/0506056].

[27] A. van Proeyen, lectures given at the Institute Henri Poincaré, Paris, November 2000. http://itf.fys.kuleuven.ac.be/~toine/LectParis.pdf

[28] K.P. Tod, Phys. Lett. B **121** (1983) 241.

[29] T. Mohaupt, Fortsch. Phys. **49**, 3 (2001) [arXiv:hep-th/0007195].

[30] L. Andrianopoli and R. D'Auria, arXiv:hep-th/0203076.

[31] B. de Wit, arXiv:hep-th/0511261.

[32] T. Mohaupt, arXiv:hep-th/0512048.

[33] R. Kallosh and T. Ortín, arXiv:hep-th/9306085.

[34] J. Bellorín and T. Ortín, Phys. Lett. B **616** (2005) 118 [arXiv:hep-th/0501246].

[35] K. Behrndt, D. Lüst and W.A. Sabra, Nucl. Phys. B **510** (1998) 264 [arXiv:hep-th/9705169].

[36] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, JHEP **0012** (2000) 019 [arXiv:hep-th/0009234].

[37] E. Bergshoeff, R. Kallosh and T. Ortín, Nucl. Phys. B **478** (1996) 156 [arXiv:hep-th/9605059].

[38] M. Cvetic and A. A. Tseytlin, Phys. Rev. D **53** (1996) 5619 [Erratum-ibid. D **55** (1997) 3907] [arXiv:hep-th/9512031].

[39] F. Denef, JHEP **0008** (2000) 050 [arXiv:hep-th/0005049].

[40] B. Bates and F. Denef, arXiv:hep-th/0304094.

[41] J. Bellorín, P. Meessen and T. Ortín, in preparation.

[42] B.R. Greene, A.D. Shapere, C. Vafa and S.T. Yau, Nucl. Phys. B **337** (1990) 1.

[43] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, J. Geom. Phys. **23** (1997) 111 [arXiv:hep-th/9605032].

[44] B. de Wit and A. Van Proeyen, Nucl. Phys. B **245** (1984) 89.

[45] B. de Wit, P.G. Lauwers and A. van Proeyen, Nucl. Phys. B **255** (1985) 569.

[46] B. Craps, F. Roose, W. Troost and A. van Proeyen, Nucl. Phys. B **503** (1997) 565 [arXiv:hep-th/9703082].

[47] M.K. Gaillard and B. Zumino, Nucl. Phys. B **193** (1981) 221.

[48] A. Batrachenko and W.Y. Wen, Nucl. Phys. B **690** (2004) 331 [arXiv:hep-th/0402141].

[49] K.P. Tod, Class. Quant. Grav. **12** (1995) 1801.

[50] A. Ceresole, R. D'Auria, S. Ferrara and A. van Proeyen, Nucl. Phys. B **444** (1995) 92 [arXiv:hep-th/9502072].

[51] S. Ferrara and R. Kallosh, Phys. Rev. D **54** (1996) 1514 [arXiv:hep-th/9602136].

[52] S. Ferrara and R. Kallosh, Phys. Rev. D **54** (1996) 1525 [arXiv:hep-th/9603090].

[53] H.W. Brinkmann, *Proc. Natl. Acad. Sci. U.S.* **9** (1923) 1.

[54] H.W. Brinkmann, *Math. Annal.* **94** (1925) 119.

[55] J. Kowalski-Glikman, Phys. Lett. B **150** (1985) 125.

[56] A. Ceresole, R. D'Auria and S. Ferrara, Nucl. Phys. Proc. Suppl. **46** (1996) 67 [arXiv:hep-th/9509160].

[57] A. Strominger, Commun. Math. Phys. **133** (1990) 163.

[58] L. Castellani, R. D'Auria and S. Ferrara, Class. Quant. Grav. **7** (1990) 1767.

[59] M. Shmakova, Phys. Rev. D **56** (1997) 540 [arXiv:hep-th/9612076].