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The General, Duality-Invariant Family of Non-BPS Black-Hole Solutions of $N = 4$, $d = 4$ Supergravity

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Abstract

We present the most general family of stationary point-like solutions of pure $N = 4$, $d = 4$ Supergravity characterized by *completely independent* electric and magnetic charges, mass, angular momentum and NUT charge plus the asymptotic values of the scalars. It includes, for specific values of the charges all previously known BPS and non-BPS, extreme and non-extreme black holes and Taub-NUT solutions.

As a family of solutions, it is manifestly invariant under T and S duality transformations and exhibits a structure related to the underlying special geometry structure of the theory.

Finally, we study briefly the black-hole-type subfamily of metrics and give explicit expressions for their entropy and temperature.

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Introduction

The low-energy effective action of the heterotic string compactified on T^6 is that of pure $N = 4, d = 4$ Supergravity coupled to $N = 4$ super Yang-Mills. It is possible to truncate consistently this theory to the simpler pure supergravity theory. From the string theory point of view the truncation consists in introducing always equal numbers of Kaluza-Klein and winding modes for each cycle. The truncated theory still exhibits S and T dualities and, thus, pure $N = 4, d = 4$ Supergravity provides a simple framework in which to study classical solutions which still can be considered as solutions of the full effective String Theory. The bosonic sector of this theory is also known in the literature as “Dilaton-Axion Gravity” or as “Einstein-Maxwell Dilaton-Axion Theory” when only a single vector field is considered.

Perhaps the most interesting solutions of the 4-dimensional string effective action are the black-hole type ones[3] since they constitute the best testing ground for the Quantum Gravity theory contained in String Theory. It is believed that a good Quantum Gravity theory should be able to explain in terms of microscopic degrees of freedom the values of the macroscopic thermodynamical quantities found classically and semiclassically. There has been some success in this respect for supersymmetric (“BPS-saturated”) and near-supersymmetric black holes although the results are to be interpreted carefully since the supersymmetric limit is singular in many respects.

A great deal of effort has been put in finding the most general families of black-hole solutions whose thermodynamical properties should exhibit also invariance (or, rather, covariance) under the duality symmetries of the theory and covering the supersymmetric and non-supersymmetric cases and, further, covering stationary (not static) cases.

The first two examples of this kind of families of solutions were found in Ref. [3][4]. The first family of solutions corresponds to non-supersymmetric, static black-hole solutions and the second to supersymmetric, static, multi-black-hole solutions of $N = 4, d = 4$ supergravity. Under the dualities of the theory, solutions of each family transform into other solutions of the same family, with the same functional form. Thus, only the values of the charges and moduli transform. The supersymmetric solutions are given in terms of two constrained complex harmonic functions.

Different extensions and properties of these solutions in the context of Dilaton-Axion Gravity were later obtained in Refs. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

A main step forward was given in Ref. [19] where it was realized that the form of the above supersymmetric solutions was dictated the special geometry of the associated $N = 2, d = 4$ Supergravity theory. The two complex harmonic functions are associated to coordinates and certain components of the metric are associated to the Kähler potential and the holomorphic vector. It was found that similar Ansatzs could be used in other $N = 2, d = 4$ Supergravity theories with different matter multiplets and Kähler potentials.

Finally, in Refs. [20, 21] the most general supersymmetric black-hole-type solutions of

[3]For a review of black holes in toroidally compactified string theory see e.g. [1] and [2].

[4]In the much simpler context of pure $N = 2, d = 4$ Supergravity the IWP solutions of Refs. [4, 5, 6] also have this property.

pure $N=4, d=4$ Supergravity (SWIP solutions) where found. The only difference with those of Ref. [3] is that the complex harmonic functions are now completely arbitrary and unconstrained. This automatically allows for the introduction of angular momentum and NUT charge in the solutions. In fact the constraint simply meant that these charges were not allowed. The generating solution for regular, supersymemtric, $N=8$ supergravity black hole solutions has been found in Ref. [22].

Similar supersymmetric solutions were later found for other $N=2, d=4$ Supergravity theories [23] with vector multiplets[5].

For the non-supersymmetric solutions of Ref. [3] the story has been different since no clear relation with the underlying special geometry was established. In Ref. [25] a general recipe for obtaining non-supersymmetric solutions from supersymmetric solutions in $N=2, d=4$ Supergravity theories, previously used in other contexts, was shown to work for *static* black holes: one simply has to deform the metric with the introduction of a non-supersymmetry (non-extremality) function.

What has to be done in more general cases (stationary, for instance) is far from clear and general duality-invariant families of stationary non-supersymmetric solutions are not available in the literature and no recipe to build them is known.

In this paper we present such a general duality-invariant family of stationary non-supersymmetric solutions of pure $N=4, d=4$ Supergravity characterized by completely independent electric and magnetic charges, mass, angular momentum and NUT charge plus the asymptotic values of the scalar fields[6].

The rest of the paper is organized as follows: in Section 1 we describe the bosonic sector of $N=4, d=4$ Supergravity theory. In Section 2 we give and study the general family of solutions we relate it to others already known. In Section 3, we focus our attention in the black hole type subfamily of metrics and calculate the explicit values for their entropy and the temperature, showing that also these quantities can be put in a manifestly duality-invariant form. Section 4 contains our conclusions. The Appendices contain the definitions of the different charges we use and their duality-invariant combinations.

[5] For a review on supersymmetric black hole solutions of supergravity theories see e.g. Ref. [24].

[6] When we talk about general solutions we are implicitly excluding the possibility of having primary scalar hair. Solutions with primary scalar hair are in all known cases (see, e.g. [26]), singular (providing evidence for the never proven "no-hair theorem") and, being interested in true black holes with event horizons covering all the physical singularities, these cases are not important for us and in the solutions which we are going to present the scalar charges are always completely determined by the $U(1)$ charges. Nevertheless, it should be pointed out that more general solutions (some of them supersymmetric) which include primary scalar hair must exist and should be related to the ones given here by formal T duality in the time direction [27].

1 $N = 4$, $d = 4$ Supergravity

1.1 Description of the System and Equations of Motion

The bosonic sector of pure $N = 4, d = 4$ Supergravity contains two real scalar fields (axion a and dilaton ϕ), six Abelian vector fields $A^{(n)}_\mu$ (which we generalize to an arbitrary number N) and the metric $g_{\mu\nu}$. The action reads[7]

$$S = \tfrac{1}{16\pi} \int d^4x \sqrt{|g|} \left\{ R + 2(\partial\phi)^2 + \tfrac{1}{2} e^{4\phi}(\partial a)^2 - e^{-2\phi} \sum_{n=1}^{N} F^{(n)} F^{(n)} + a \sum_{n=1}^{N} F^{(n)}\ {}^\star F^{(n)} \right\} . \tag{1.1}$$

The axion and dilaton are combined into a single complex scalar field, the *axidilaton* λ:

$$\lambda = a + ie^{-2\phi} . \tag{1.2}$$

For each vector field we can also define its $SL(2, \mathbb{R})$-dual, which with our conventions will be given by:

$$\tilde{F}^{(n)}{}_{\mu\nu} \equiv e^{-2\phi}\ {}^\star F^{(n)}{}_{\mu\nu} + aF^{(n)}{}_{\mu\nu} . \tag{1.3}$$

The equations of motion derived from the action (1.1) plus the Bianchi identities for the vector fields can be written as follows:

$$\nabla_\mu {}^\star \tilde{F}^{(n)\,\mu\nu} = 0 , \tag{1.4}$$

$$\nabla_\mu {}^\star F^{(n)\,\mu\nu} = 0 , \tag{1.5}$$

$$\nabla^2\phi - \tfrac{1}{2} e^{4\phi}(\partial a)^2 - \tfrac{1}{2} e^{-2\phi} \sum_{n=1}^{N} F^{(n)} F^{(n)} = 0 , \tag{1.6}$$

$$\nabla^2 a + 4\partial_\mu\phi\, \partial^\mu a - e^{-4\phi} \sum_{n=1}^{N} F^{(n)}\ {}^\star F^{(n)} = 0 , \tag{1.7}$$

$$R_{\mu\nu} + 2\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2} e^{4\phi}\partial_\mu a\partial_\nu a - 2e^{-2\phi} \sum_{n=1}^{N} \left(F^{(n)}{}_{\mu\rho} F^{(n)}{}_\nu{}^\rho - \tfrac{1}{4} g_{\mu\nu} F^{(n)} F^{(n)} \right) = 0 . \tag{1.8}$$

Observe that we have written the Maxwell equations as the Bianchi equations for the $SL(2, \mathbb{R})$ duals. Therefore N dual vector potentials $\tilde{A}^{(n)}_\mu$ defined by

[7] Our conventions coincide with those of Ref. [28]. In particular, we use mostly minus signature and Hodge duals are defined such that ${}^\star F^{(n)\,\mu\nu} = \frac{1}{2\sqrt{|g|}} \epsilon^{\mu\nu\rho\sigma} F^{(n)}_{\rho\sigma}$ with $\epsilon^{0123} = +1$.

$$\tilde{F}^{(n)}_{\mu\nu} = \partial_\mu \tilde{A}^{(n)}_\nu - \partial_\nu \tilde{A}^{(n)}_\mu \,, \tag{1.9}$$

exist locally.

The axidilaton parametrizes an $SL(2,\mathbb{R})/SO(2)$ coset [29], the equations of motion being invariant under global $SL(2,\mathbb{R})$ ("S duality") transformations. If Λ is an $SL(2,\mathbb{R})$ matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad ad - bc = 1 \,, \tag{1.10}$$

then the vector fields and their duals transform as doublets

$$\begin{pmatrix} \tilde{F}^{(n)}{}_{\mu\nu} \\ \\ F^{(n)}{}_{\mu\nu} \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} \tilde{F}^{(n)}{}_{\mu\nu} \\ \\ F^{(n)}{}_{\mu\nu} \end{pmatrix} , \tag{1.11}$$

and the axidilaton transforms according to

$$\lambda \longrightarrow \frac{a\lambda + b}{c\lambda + d} \,. \tag{1.12}$$

This is an electromagnetic duality rotation that acts on the dilaton. From the point of view of String Theory, this is the 4-dimensional string coupling constant. Hence the name S duality.

Furthermore the N (6 in the SUGRA theory) vector fields can be $SO(N)$-rotated. These are "T duality" transformations (perturbative from the String Theory point of view). The full duality group is, then $SL(2,\mathbb{R}) \otimes SO(N)$.

2 The General Solution

We now present the family of solutions. All the fields in our solutions may be expressed in terms of two *fixed* complex harmonic functions of the three dimensional Euclidean space, $\mathcal{H}_1$ and $\mathcal{H}_2$, a set of N complex constants $k^{(n)}$, a "non-extremality" function W and a background 3-dimensional metric ${}^{(3)}\gamma_{ij}$. In all of them appear the physical constants defined in Appendix B. Only Υ, the axidilaton charge, is not independent. The harmonic functions are

$$\left\{ \begin{array}{rcl} \mathcal{H}_1 & = & \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta} \left(\lambda_0 + \dfrac{\lambda_0 \mathfrak{M} + \bar{\lambda}_0 \Upsilon}{\tilde{\rho}} \right) , \\ \\ \mathcal{H}_2 & = & \frac{1}{\sqrt{2}} e^{\phi_0} e^{i\beta} \left(1 + \dfrac{\mathfrak{M} + \Upsilon}{\tilde{\rho}} \right) , \end{array} \right. \tag{2.1}$$

where $\tilde{\rho}^2 \equiv x^2 + y^2 + (z + i\alpha)^2$ is the usual complex radial coordinate, and β is an arbitrary, unphysical real number related to the duality transformation of these functions under $SL(2,\mathbb{R})$ (see the explanation in Section 2.1). The complex constants are

$$k^{(n)} = -\tfrac{1}{\sqrt{2}} e^{-i\beta} \frac{\mathfrak{M}\Gamma^{(n)} + \overline{\Upsilon\Gamma^{(n)}}}{|\mathfrak{M}|^2 - |\Upsilon|^2} \,. \tag{2.2}$$

In supersymmetric cases (e.g. Ref. [21]) it is useful to introduce oblate spheroidal coordinates which are related to the ordinary Cartesian ones by:

$$\left\{ \begin{array}{rcl} x & = & \sqrt{r^2 + \alpha^2}\ \sin\theta \cos\varphi \,, \\ \\ y & = & \sqrt{r^2 + \alpha^2}\ \sin\theta \sin\varphi \,, \\ \\ z & = & r \cos\theta \,. \end{array} \right. \tag{2.3}$$

The three dimensional Euclidean metric is written in these coordinates in the following way:

$$d\vec{x}^{\,2} = \frac{r^2 + \alpha^2 \cos^2\theta}{r^2 + \alpha^2}\, dr^2 + \left(r^2 + \alpha^2 \cos^2\theta\right)\ d\theta^2 + \left(r^2 + \alpha^2\right) \sin^2\theta\ d\varphi^2 \,. \tag{2.4}$$

In terms of (2.3) the radial coordinate $\tilde{\rho}$ that appears in (2.1) may be expressed as $\tilde{\rho} = r + i\alpha\cos\theta$. Furthermore, in these new coordinates, the "non-extremality" function has a simple form:

$$W = 1 - \frac{{r_0}^2}{r^2 + \alpha^2 \cos^2\theta} \,, \tag{2.5}$$

where r_0, given by

$$r_0^2 = |\mathfrak{M}|^2 + |\Upsilon|^2 - \sum_{n=1}^{N} |\Gamma^{(n)}|^2 \,. \tag{2.6}$$

is usually called "extremality parameter" in the static cases. In stationary cases, though, $r_0 = 0$ means that the solution is supersymmetric but in general it is not an extreme black hole (nor a black hole). Thus, a more appropriate name is *supersymmetry parameter*. The *extremality parameter* will be $R_0^2 = r_0^2 - \alpha^2$.

Finally, the last ingredient is the background metric ${}^{(3)}\gamma_{ij}$

$$\begin{aligned} d\vec{x}^{\,2} &= {}^{(3)}\gamma_{ij} dx^i dx^j \\ &= \frac{r^2 + \alpha^2 \cos^2\theta - r_0^2}{r^2 + \alpha^2 - r_0^2}\, dr^2 + (r^2 + \alpha^2 \cos^2\theta - {r_0}^2)\, d\theta^2 + + (r^2 + \alpha^2 - {r_0}^2) \sin^2\theta d\varphi^2 \,, \end{aligned} \tag{2.7}$$

which differs from (2.4) in non-supersymmetric ($r_0 \neq 0$) cases and is not flat[8]. This is an important qualitative difference between the usual supersymmetric IWP-type [4, 5] metrics (e.g. those of Refs. [6, 20, 21]) and our solution.

[8]In (2.8) as well as in (2.7) the x^i label the coordinates r, θ and φ for $i = 1, 2, 3$ respectively.

We can now describe the solutions. They take the form

$$\left\{\begin{array}{rcl} ds^2 & = & e^{2U}W\left(dt+\omega_\varphi d\varphi\right)^2 - e^{-2U}W^{-1}\,{}^{(3)}\gamma_{ij}dx^i dx^j\,, \\ A^{(n)}{}_t & = & 2e^{2U}\,\Re\mathrm{e}\left(k^{(n)}\mathcal{H}_2\right)\,, \\ \tilde{A}^{(n)}{}_t & = & 2e^{2U}\,\Re\mathrm{e}\left(k^{(n)}\mathcal{H}_1\right)\,, \\ \lambda & = & \dfrac{\mathcal{H}_1}{\mathcal{H}_2}\,, \end{array}\right. \tag{2.8}$$

where

$$e^{-2U} = 2\,\Im\mathrm{m}\left(\mathcal{H}_1\bar{\mathcal{H}}_2\right) = 1 + 2\Re\mathrm{e}\left(\frac{\mathfrak{M}}{r+i\alpha\cos\theta}\right) + \frac{|\mathfrak{M}|^2-|\Upsilon|^2}{r^2+\alpha^2\cos^2\theta}\,, \tag{2.9}$$

and where

$$\begin{aligned} \omega_\varphi &= 2n\cos\theta + \alpha\sin^2\theta\left(e^{-2U}W^{-1}-1\right) \\ &= \frac{2}{r^2+\alpha^2\cos^2\theta - r_0^2}\times \\ &\quad\times\left\{n\cos\theta\left(r^2+\alpha^2-r_0^2\right)+\alpha\sin^2\theta\left[mr+\tfrac{1}{2}\left({r_0}^2+|\mathfrak{M}|^2-|\Upsilon|^2\right)\right]\right\}\,. \end{aligned} \tag{2.10}$$

ω_φ can be interpreted as the unique non-vanishing covariant component of a 1-form $\omega = \omega_i dx^i$ which for $\alpha \neq 0$ is a solution of the following equation in 3-dimensional space:

$$^{\star}d\omega - e^{-2U}\,{}^{\star}d\mu - W^{-1}\Re\mathrm{e}\left(\mathcal{H}_1 d\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 d\mathcal{H}_1\right) = 0\,, \tag{2.11}$$

where μ is also a 1-form whose only non-vanishing component μ_φ is given by

$$\mu_\varphi = \frac{r_0^2}{\alpha}\,\frac{r^2+\alpha^2-r_0^2}{r^2+\alpha^2\cos^2\theta-r_0^2}\,, \tag{2.12}$$

and where the 3-dimensional background metric ${}^{(3)}\gamma_{ij}$ has to be used in the Hodge duals.

For $\alpha = 0$ the μ term in Eq. (2.11) has to be eliminated and the equation tales the form

$$^{\star}d\omega - W^{-1}\Re\mathrm{e}\left(\mathcal{H}_1 d\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 d\mathcal{H}_1\right) = 0\,. \tag{2.13}$$

We can also write our solution in the standard form used to describe general rotating black holes, which will be useful to describe the structure of the singularities:

$$\begin{aligned} ds^2 &= \frac{\Delta-\alpha^2\sin^2\theta}{\Sigma}\,dt^2 + 2\alpha\sin^2\theta\frac{\Sigma+\alpha^2\sin^2\theta-\Delta}{\Sigma}\,dtd\varphi - \\ &\quad -\frac{\Sigma}{\Delta}\,dr^2 - \Sigma\,d\theta^2 - \frac{\left(\Sigma+\alpha^2\sin^2\theta\right)^2-\Delta\alpha^2\sin^2\theta}{\Sigma}\sin^2\theta\,d\varphi^2\,, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} \Delta &= r^2 - R_0{}^2 = r^2 + \alpha^2 - r_0{}^2\,, \\ \Sigma &= (r+m)^2 + (n + \alpha\cos\theta)^2 - |\Upsilon|^2\,. \end{aligned} \tag{2.15}$$

This completes the description of the general solution. Now we are going to describe its properties.

2.1 Duality Properties

We can study the effect of duality transformations in two ways which are fully equivalent in this family of solutions: we can study the effect of the transformations of the fields or simply the effect of the transformations on the physical constants. One of the main features of our family of solutions is precisely this equivalence: we can simply transform the physical constants (adding "primes") because the functional form of the solutions always will remain invariant.

Let us, then, study the effect of $SL(2,\mathbb{R})$ transformations of the charges $\mathfrak{M}, \Gamma^{(n)}, \Upsilon$ and moduli λ_0 in Appendix B.

Both the complex harmonic functions $\mathcal{H}_{1,2}$ and the complex constants $k^{(n)}$ are defined up to a phase: if we multiply $\mathcal{H}_{1,2}$ by a constant phase and the $k^{(n)}$'s by the opposite one, the solution remains unchanged. We have made this fact explicit by including the arbitrary angle β in their definition.

As it can take any value, in particular we can require it to change in the following way when performing an $SL(2,\mathbb{R})$ rotation:

$$e^{i\beta} \longrightarrow e^{i\arg(c\lambda_0 + d)} e^{i\beta}\,. \tag{2.16}$$

With this choice the $k^{(n)}$'s are left invariant, while the pair $\mathcal{H}_{1,2}$ transforms as a doublet:

$$\begin{pmatrix} \mathcal{H}_1' \\ \mathcal{H}_2' \end{pmatrix} = \Lambda \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}\,. \tag{2.17}$$

$\mathcal{H}_{1,2}$ appear only through two invariant combinations: e^{-2U} and $\Re\mathrm{e}\left(\mathcal{H}_1 d\bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 d\mathcal{H}_1\right)$. The remaining building blocks of the solution are μ and W which are invariant if the supersymmetry parameter r_0 is invariant. This (first proven in Ref. [30]) is shown in Appendix C.

Under $SO(N)$ the $k^{(n)}$'s transform as vectors, as they should, and everything else is invariant.

The relation of the form of these solutions to $N = 2$ special geometry is the same as in the supersymmetric case [19, 21] and we will not repeat here that discussion. The only difference is the introduction of the background metric ${}^{(3)}\gamma$, and the functions μ, W which "deform" the supersymmetric solution but have no special meaning from the special geometry point of view.

2.2 Reduction to Other Known Solutions

We can now relate our solution to those less general found in the literature. We can consider two types of solutions: supersymmetric and non-supersymmetric. The most general family of supersymmetric black-hole type solutions of $N = 4, d = 4$ Supergravity (SWIP solutions) was found in Refs. [20, 21]. Due to the existence of supersymmetry, the family contains two arbitrary harmonic functions. To describe point-like solutions one chooses harmonic functions with a single pole. Our solutions reduce precisely to these when the supersymmetry parameter vanishes: $r_0 = 0$. As shown in Appendix C in the $r_0 = 0$ limit at least one of the two possible Bogomol'nyi bounds of $N = 4$ Supergravity are saturated. In this case $W = 1, \mu_\varphi = 0$, ${}^{(3)}\gamma$ becomes flat (in spheroidal coordinates) and we recover the structure of the SWIP solutions. The SWIP solutions always saturate one bound due to the constraints that the constants $k^{(n)}$ satisfy

$$\begin{aligned} \sum_{n=1}^{N} (k^{(n)})^2 &= 0\,, \\ \sum_{n=1}^{N} |k^{(n)}|^2 &= \tfrac{1}{2}\,. \end{aligned} \tag{2.18}$$

while, in our case there is no constraint on the charges (apart from the one on the scalar charge, associated to the non-hair theorem)

$$\begin{aligned} \sum_{n=1}^{N} (k^{(n)})^2 &= \frac{-\mathfrak{M}\overline{\Upsilon}}{(|\mathfrak{M}|^2 - |\Upsilon|^2)^2}\, {r_0}^2\,, \\ \sum_{n=1}^{N} |k^{(n)}|^2 &= \tfrac{1}{2}\left(1 - \frac{|\mathfrak{M}|^2 + |\Upsilon|^2}{(|\mathfrak{M}|^2 - |\Upsilon|^2)^2}\, {r_0}^2\right). \end{aligned} \tag{2.19}$$

In Ref. [21] it is shown how this solution reduces to supersymmetric solutions with angular momentum, NUT charge etc. Only some of the static ones (those with 1/4 of the supersymmetries unbroken) are black holes with a regular horizon. These include extreme Reissner-Nordström black holes and their axion-dilaton generalizations [31, 32, 33, 34, 30]. The rest have naked singularities.

As for the non-supersymmetric solutions, the non-extreme Taub-NUT axion-dilaton solutions of Ref. [7] are clearly covered by our general solution. Further, in Ref. [9] were found general point-like solutions for a theory with only one vector field ("axion-dilaton gravity"). We can see that our solutions reduce to these ones by setting $N = 1$. The principal difference is that, in this particular case, we can fit the analogous of expression (B.10) in the definition (2.6) for the extremality parameter, giving

$$r_{0,N=1}^2 = (|\mathfrak{M}| - |\Upsilon|)^2\,, \tag{2.20}$$

and inserting this into the metric (2.8) we get exactly Eqs. (31-35) of Ref. [9] up to a shift in the radial coordinate. Although one can immediately see that the functional form of the metric (2.8) does not change very much from that found in [9], somewhat different results appear when considering multiple vector fields, due to the constraints that the physical parameters obey when only one vector field is present. This analysis was already done in detail in [21], and we refer to this paper for further discussion.

A generalization of the non-extreme solutions of [9] for the same theory, but with an arbitrary number of vector fields (*i.e.*, the same theory we are treating), was found in [16]. However, the solutions reported there concern only the *static* case, and therefore the total number of independent physical parameters is $2N+4$. As it was shown in that paper, the metric for the static case is of the "Reissner-Nordström-type", but with a variable mass factor. It can be seen that, taking the static ($\alpha = 0$) limit of our metric (2.8), and shifting the radial coordinate by a quantity $m + \sqrt{|\Upsilon|^2 - n^2}$, we recover the same solution of [16] (Eq. (7.6) of that reference) up to redefinitions in the different constants parametrizing the solution.

Finally, we observe that setting the axidilaton charge equal to zero (which can be done with appropriate combinations of electric and magnetic charges) in Eqs. (2.15-2.14), we recover the Kerr-Newman solution in Boyer-Lindquist coordinates (but with a constant shift equal to the mass in the radial coordinate. See, *e.g.*, Refs.[35, 36]).

3 Black-Hole-Type Solutions

3.1 Singularities

We now carry out the analysis of the structure of our solutions. First, we proceed to study the different types of singularities of the metric. Due to the standard form of $g_{\mu\nu}$ in terms of Δ and Σ, the singularities in terms of these functions are those of all Kerr-type metrics, *i.e.*, we have coordinate singularities at

$$\Delta = 0\,, \qquad \theta = 0\,, \tag{3.1}$$

and a curvature singularity at

$$\Sigma = 0\,. \tag{3.2}$$

The first of Eqs. (3.1) gives the possible horizons. To study the different cases, let us shift the radial coordinate to recover the Boyer-Lindquist coordinates in which this kind of solutions are usually given. If we perform the following rescaling:

$$r \longrightarrow r - m \tag{3.3}$$

then Δ and Σ of (2.15) become

$$
\begin{aligned}
\Delta &= (r-m)^2 - {R_0}^2 \, , \\
\Sigma &= r^2 + \alpha^2 \cos^2\theta - |\Upsilon|^2 \, ,
\end{aligned}
\tag{3.4}
$$

where we also have made the NUT charge n equal to zero in order to obtain black-hole-type solutions. In studying the singularities given by $\Delta = 0$ we have three cases to consider:

a) $R_0^2 < 0 \, , \quad ({r_0}^2 < \alpha^2) \, .$

Here $\Delta = 0$ has no real solutions, we have a naked singularity at $\Sigma = 0$ and no true black hole interpretation is possible. This is the case of supersymmetric ($r_0 = 0$) rotating ($\alpha \neq 0$) "black holes".

b) $R_0^2 > 0 \, , \quad ({r_0}^2 > \alpha^2) \, .$

In this case we have two horizons placed at

$$
r_\pm = m \pm R_0 \, . \tag{3.5}
$$

To see if in this case we have a true black hole we must verify that the singularity is always hidden by the event horizon. The region where the singularity is placed is given by the following equation:

$$
\Sigma = 0 \;\Leftrightarrow\; r^2_{\rm sing} = |\Upsilon|^2 - \alpha^2 \cos^2\theta \, . \tag{3.6}
$$

This is not the usual "ring singularity", but a more complicated 2-dimensional *surface* in general. Depending on the values of the charges, this can have the topology of the surface of a torus (maybe degenerate in certain cases to the surfaces of two concentric ellipsoids). Whatever its shape is, it is always confined in the region

$$
r^2_{\rm sing} \leq |\Upsilon|^2 \, , \tag{3.7}
$$

while, on the other hand, the would-be event horizon

$$
r_+ = m + R_0 > m \, , \tag{3.8}
$$

and it will cover the singularity if $m > |\Upsilon|$. Using the value of $|\Upsilon|$ in terms of the other charges it is easy to prove

$$
\left(|\mathfrak{M}| - |\Upsilon|\right)^2 > \alpha^2 \, . \tag{3.9}
$$

We can now distinguish two cases:

i) $|\mathfrak{M}| - |\Upsilon| > |\alpha| \, .$

In this case (setting $n = 0$) the horizon covers the singularity and the object is a true black hole. Using the expressions in Appendix C it is possible to prove that this happens when both

$$|\mathfrak{M}| > |\mathcal{Z}_{1,2}|\,, \tag{3.10}$$

which is the case allowed by supersymmetry (but not supersymmetric).

ii) $|\mathfrak{M}| - |\Upsilon| < |\alpha|$.

In this case there are naked singularities. This is the case forbidden by supersymmetry since one can show that in it both Bogomol'nyi bounds are simultaneously violated

$$|\mathfrak{M}| < |\mathcal{Z}_{1,2}|\,. \tag{3.11}$$

c) $R_0^2 = 0\,, \quad ({r_0}^2 = \alpha^2)\,.$

This is the extremal case, and here we have a single would-be horizon placed at

$$r_\pm = m\,. \tag{3.12}$$

Again, we can distinguish two cases

i) $|\mathfrak{M}| - |\Upsilon| > |\alpha|\,, \quad |\mathfrak{M}| > |\mathcal{Z}_{1,2}|\,.$

In this case the singularity is inside the horizon and we have a true extreme rotating black hole. This is the case allowed by supersymmetry (not supersymmetric unless $\alpha = 0$).

i) $|\mathfrak{M}| - |\Upsilon| < |\alpha|\,, \quad |\mathfrak{M}| < |\mathcal{Z}_{1,2}|\,.$

The singularity is outside the "horizon" and this is not a black hole.

3.2 Entropy and temperature

We can now calculate the physical quantities associated to the true black-hole-type solutions. The entropy of the BH can be worked out by a straightforward computation of the area of the event horizon. This gives the following result:

$$A_{horizon} = 4\pi\left({r_+}^2 + \alpha^2 - |\Upsilon|^2\right)\,, \tag{3.13}$$

so that for the Bekenstein-Hawking entropy of the black hole we get, in units such that $G = \hbar = c = 1$

$$S = \pi\left(2m^2 + 2mR_0 - I_2\right)\,, \tag{3.14}$$

where I_2 is the quadratic duality invariant defined in Eq. (A.7). It is useful to have the expression of the entropy in terms of the mass and supersymmetry central charges

$$S = \pi \left\{ (m^2 - |\mathcal{Z}_1|^2) + (m^2 - |\mathcal{Z}_2|^2) + 2\sqrt{(m^2 - |\mathcal{Z}_1|^2)(m^2 - |\mathcal{Z}_2|^2) - J^2}\,. \right\} . \qquad (3.15)$$

For vanishing angular momentum $J = 0$, this expression can be further simplified to

$$S = \pi \left[\left(m^2 - |\mathcal{Z}_1|^2\right)^{1/2} + \left(m^2 - |\mathcal{Z}_2|^2\right)^{1/2} \right]^2 , \qquad (3.16)$$

which means that, if we believe the extrapolation of this formula to all extreme cases, the entropy vanishes if and only if both Bogomol'nyi bounds are saturated and 1/2 of the supersymmetries are unbroken [34].

When any one of the two possible Bogomol'nyi bounds is saturated (for $J = 0$) the entropy is proportional to the difference between the modulus of the two central charges, which is proportional to the quartic duality invariant I_4 defined in Eq. (A.8), which is moduli-independent.

The temperature can be calculated imposing the regularity of the metric near the event horizon in imaginary time. Following the standard prescription [37], we must shift the time t and the rotation parameter α to the values $t \to i\tau$ and $\alpha \to i\tilde{\alpha}$ respectively. This yields the Euclidean section of the metric, and the absence of conical singularities at the event horizon in imaginary time requires the identification $(\tau, \varphi) \sim (\tau + \beta_H, \varphi - \tilde{\Omega}_H \beta_H)$, where $\tilde{\Omega}_H$ is the Euclidean angular velocity of the event horizon and β_H is the inverse Hawking temperature. For the (real) angular velocity of the horizon we have the following result:

$$\Omega_H = \frac{\alpha}{r_+^2 + \alpha^2 - |\Upsilon|^2} , \qquad (3.17)$$

and so we obtain, in a perfectly straightforward way, the value for the Hawking temperature of the black hole:

$$T_H = \frac{R_0}{2S} . \qquad (3.18)$$

For $J = 0$ the temperature always vanishes in the supersymmetric limit, except in the case in which 1/2 of the supersymmetries are going to be left unbroken. In that case the limit is simply not well defined.

4 Conclusions

We have given a new set of solutions of pure $N = 4$, $d = 4$ supergravity which are beyond the BPS limit (in both directions) and which constitute the most general stationary point-like solution of this theory, since all the conserved charges are present in our solution, and all of them can take completely arbitrary values[9]. These solutions include black holes as well as Taub-NUT spacetimes, BHs being non-extremal in the general case. We have also

[9]The only possible addition would be primary scalar hair, but we are not interested in that kind of solutions.

shown that our family of solutions, and also the thermodynamic quantities associated to the BHs, are duality-invariant.

From a more technical point of view, we hope that the Ansatz providing the solution (basically characterized by the introduction of the 'non-extremality' function W and the non-flat three-dimensional metric (2.7) as "background" space) will prove helpful for the task of finding more non-extreme black holes s in other models, in particular, in those arising from more realistic compactifications of string theory, like compactifications on Calabi-Yau spaces, orbifolds, etc.

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A Conserved Charges and Duality Invariants

The non-geometrical conserved charges of this theory are associated to the $U(1)$ vector fields. There are electric charges $\tilde{q}^{(n)}$ whose conservation law is associated to the Maxwell equation and are defined, for point-like objects by the asymptotic behavior of the $t-r$ components of the $SL(2,\mathbb{R})$-dual field strengths

$$
{}^\star\tilde{F}^{(n)}{}_{tr} \sim \frac{\tilde{q}^{(n)}}{\rho^2}\,, \tag{A.1}
$$

and magnetic charges $\tilde{p}^{(n)}$ whose conservation law is associated to the Bianchi identity and are defined, for point-like objects by the asymptotic behavior of the $t-r$ components of the field strengths

$$
{}^\star F^{(n)}{}_{tr} \sim \frac{\tilde{p}^{(n)}}{\rho^2}\,. \tag{A.2}
$$

These charges can be arranged in $SO(N)$ vectors $\vec{\tilde{q}}, \vec{\tilde{p}}$ and these can be arranged in $SL(2,\mathbb{R})$ doublets

$$
\begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix}. \tag{A.3}
$$

Under S and T duality transformations Λ, R, this charge vector transforms according to

$$\begin{pmatrix} \vec{\tilde{q}}' \\ \\ \vec{\tilde{p}}' \end{pmatrix} = \Lambda \otimes R \begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix} . \tag{A.4}$$

It is also useful to introduce the $SL(2,\mathbb{R})$ matrix of scalar fields

$$\mathcal{M} = e^{2\phi} \begin{pmatrix} |\lambda|^2 & a \\ \\ a & 1 \end{pmatrix} , \qquad \mathcal{M}^{-1} = e^{2\phi} \begin{pmatrix} 1 & -a \\ \\ -a & |\lambda|^2 \end{pmatrix} , \tag{A.5}$$

which transforms under $SL(2,\mathbb{R})$ according to

$$\mathcal{M}' = \Lambda \mathcal{M} \Lambda^T , \tag{A.6}$$

and it is an $SO(N)$ singlet.

We can now construct two expressions which are manifestly $SL(2,\mathbb{R}) \otimes SO(N)$-invariant and that will be useful later to express physical results in a manifestly duality-invariant way. The first one is quadratic in the charges:

$$I_2 \equiv \begin{pmatrix} \vec{\tilde{q}} & \vec{\tilde{p}} \end{pmatrix} \mathcal{M}_0^{-1} \begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix} \tag{A.7}$$

where $\mathcal{M}_0$ is the constant asymptotic value of $\mathcal{M}$.

The second invariant we will use is :

$$I_4 \equiv \det \left[\begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix} \begin{pmatrix} \vec{\tilde{q}} & \vec{\tilde{p}} \end{pmatrix} \right] = \left(\vec{\tilde{q}} \cdot \vec{\tilde{q}}\right)^2 \left(\vec{\tilde{p}} \cdot \vec{\tilde{p}}\right)^2 - \left(\vec{\tilde{q}} \cdot \vec{\tilde{p}}\right)^2 , \tag{A.8}$$

which is quartic in the charges. Observe that I_2 is moduli-dependent and I_4 is moduli-independent.

B Physical Parameters

Here we explain our notation for charges and moduli used in the solutions. m stands for the ADM mass, and n for the NUT charge. They appear combined into the complex constant $\mathfrak{M}$ defined by

$$\mathfrak{M} = m + in . \tag{B.1}$$

The rotation parameter α is $\alpha = J/m$, where J is the angular momentum.

These charges are singlets under all duality transformations.

The asymptotic behavior of the axidilaton is characterized by the constant asymptotic value $\lambda_0 = a_0 + ie^{-2\phi_0}$, where a_0 is the constant asymptotic value of the axion and ϕ_0 that of the dilaton. The axidilaton "charge" is denoted by Υ and, thus

$$\lambda \sim \lambda_0 - ie^{-2\phi_0}\frac{2\Upsilon}{\rho}\,. \tag{B.2}$$

(where ρ is a radial coordinate).

λ_0 transforms as λ under duality transformations and Υ is as $SO(N)$ singlet and, under $SL(2,\mathbb{R})$

$$\Upsilon' = e^{-2i\arg(c\lambda_0+d)}\Upsilon\,. \tag{B.3}$$

We find it convenient to use, instead of the conserved charges $\tilde{q}^{(n)}$ and $\tilde{p}^{(n)}$ defined in Appendix A, the constants $Q^{(n)}$ and $P^{(n)}$ defined by

$$F_{tr}{}^{(n)} \sim \frac{e^{\phi_0}Q^{(n)}}{\rho^2}\,, \qquad {}^\star F_{tr}{}^{(n)} \sim -\frac{e^{\phi_0}P^{(n)}}{\rho^2}\,, \tag{B.4}$$

and combined into the complex constants $\Gamma^{(n)}$ which can be arranged into an $SO(N)$ vector

$$\Gamma^{(n)} = Q^{(n)} + iP^{(n)}\,, \qquad \vec{\Gamma} = \vec{Q} + i\vec{P}\,. \tag{B.5}$$

In our solutions these charges are simple combinations of the conserved charges and moduli:

$$\begin{pmatrix} \vec{Q} \\ \\ \vec{P} \end{pmatrix} = \mathcal{V}_0 \begin{pmatrix} \vec{\tilde{q}} \\ \\ \vec{\tilde{p}} \end{pmatrix}\,, \tag{B.6}$$

where

$$\mathcal{V}_0 = e^{\phi_0}\begin{pmatrix} -1 & a_0 \\ \\ 0 & -e^{-2\phi_0} \end{pmatrix}\,, \qquad \mathcal{V}_0^T\mathcal{V}_0 = \mathcal{M}_0^{-1}\,. \tag{B.7}$$

$\vec{\Gamma}$ is an $SO(N)$ vector and transforms under $SL(2,\mathbb{R})$ according to

$$\vec{\Gamma}\,' = e^{i\arg(c\lambda_0+d)}\vec{\Gamma}\,, \tag{B.8}$$

so the duality invariants can be written

$$\begin{aligned} I_2 &= |\vec{\Gamma}\cdot\vec{\Gamma}|^2 = \textstyle\sum_{n=1}^{N}|\Gamma^{(n)}|^2\,, \\ I_4 &= -\tfrac{1}{4}\det\left[\begin{pmatrix} \vec{\Gamma} \\ \\ \vec{\bar{\Gamma}} \end{pmatrix}\begin{pmatrix} \vec{\Gamma} & \vec{\bar{\Gamma}} \end{pmatrix}\right]\,. \end{aligned} \tag{B.9}$$

Our solutions do not have any primary scalar hair and the axidilaton charge is always completely determined by the electric and magnetic charges, and mass and NUT charge through

$$\Upsilon = -\tfrac{1}{2}\sum_{n=1}^{N}\frac{(\bar{\Gamma}^{(n)})^2}{\mathfrak{M}}\,. \tag{B.10}$$

The absolute value of this expression is duality invariant and can be rewritten in terms of the basic invariants (A.7,A.8) as follows:

$$|\Upsilon|^2 = \frac{1}{4|\mathfrak{M}|^2}(I_2^2 - 4I_4)\,. \tag{B.11}$$

C Central Charge Matrix Eigenvalues

The supersymmetry parameter r_0 can be expressed in terms of the two different skew eigenvalues of the central charge matrix of $N=4, d=4$ Supergravity [38, 39] $\mathcal{Z}_{1,2}$. Their absolute values can be expressed in terms of the electric and magnetic charges in the following way:

$$|\mathcal{Z}_{1,2}|^2 = \tfrac{1}{2}\sum_{n=1}^{N}|\Gamma^{(n)}|^2 \pm \tfrac{1}{2}\left[\left(\sum_{n=1}^{N}|\Gamma^{(n)}|^2\right)^2 - \left|\sum_{n=1}^{N}\left(\Gamma^{(n)}\right)^2\right|^2\right]^{1/2}, \tag{C.1}$$

and in terms of the invariants I_2, I_4 defined in Eqs. (A.7,A.8) as follows

$$|\mathcal{Z}_{1,2}|^2 = \tfrac{1}{2}I_2 \pm I_4^{1/2}\,. \tag{C.2}$$

With the help of these expressions and those of the previous Appendices we can write the supersymmetry parameter r_0 as follows:

$$r_0^2 = \frac{1}{|\mathfrak{M}|^2}\left(|\mathfrak{M}|^2 - |\mathcal{Z}_1|^2\right)\left(|\mathfrak{M}|^2 - |\mathcal{Z}_2|^2\right)\,. \tag{C.3}$$

This last equation makes explicit the fact that, if and only if r_0^2=0, one of the two possible supersymmetry Bogomol'nyi bounds

$$|\mathfrak{M}|^2 \geq |\mathcal{Z}_{1,2}|^2\,, \tag{C.4}$$

is saturated.

The following expression is also useful

$$|\mathcal{Z}_1\mathcal{Z}_2|^2 = |\mathfrak{M}|^2|\Upsilon|^2\,. \tag{C.5}$$

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