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Supersymmetric $N = 2$ Einstein-Yang-Mills monopoles and covariant attractors

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We present two generic classes of supersymmetric solutions of $N = 2$, $d = 4$ supergravity coupled to non-Abelian vector supermultiplets with a gauge group that includes an $SU(2)$ factor. The first class consists of embeddings of the 't Hooft-Polyakov monopole and in the considered model is a globally regular, asymptotically flat spacetime. The other class of solutions consists of regular non-Abelian extreme black holes. There is a *covariant attractor* at the horizon of these non-Abelian black holes.

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The search for and study of supersymmetric supergravity solutions having the interpretation of long-range fields of string states has been one of the most fruitful fields of theoretical research for the last 15 years. All this work is having a big impact in the study of general relativity solutions since, after all, the supersymmetric supergravity solutions are nothing but particular examples of standard solutions of gravity coupled to standard bosonic matter fields. Minkowski and anti-de Sitter spacetimes, the extreme Reissner-Nordström black-hole solution, and gravitational pp-waves are some examples of supersymmetric supergravity solutions.

In 4-dimensional theories, most of the effort has been directed to finding and studying asymptotically flat black-hole solutions with Abelian charges (the Reissner-Nordström black hole [BH] being among them). The most general ones in ungauged $N = 2$, $d = 4$ supergravity coupled to vector supermultiplets were found in Ref. [1,2]. This, and the existence of the attractor mechanism [3], that fixes the values of the scalars at the horizon in terms of the conserved electric and magnetic charges only, and its relations to stringy black hole entropy calculations or to topological strings are two of the main results obtained so far.

These results have not been extended to black holes with non-Abelian charges and the little work that has been done concerns magnetic monopoles. This is due, in part, to the fact that little is known about this kind of solution in the nonsupersymmetric Einstein-Yang-Mills (EYM) theories: the only analytically known black-hole EYM solutions correspond just to the embedding of Abelian solutions whereas the purely non-Abelian solutions are only known numerically [4]. On the supergravity side, two main results have been the construction of supersymmetric, globally regular, gravitating monopole solutions in $N = 4$, $d = 4$ theories by Harvey and Liu [5] and Chamseddine and Volkov [6]. These have not been extended to black holes and, to the best of our knowledge, there is no microscopic interpretation of these massive charged objects that are not black holes but may be elementary constituents of them. Thus, we do not know whether and how the attractor mechanism works in non-Abelian black holes.

Our aim is to start filling this gap in our knowledge of supersymmetric supergravity solutions with non-Abelian Yang-Mills (YM) fields (whence also in EYM theories), by studying, in particular, black-hole and monopole-type solutions. We are going to present an extension of the results of [7,8], characterizing the most general static supersymmetric solutions in $N = 2$, $d = 4$ supergravity coupled to non-Abelian vector supermultiplets [9], to which we shall refer as $N = 2$ super-Einstein-Yang-Mills theory (SEYM). The bosonic sector of this theory differs from the standard pure EYM theory by the presence of charged scalar fields with couplings and a scalar potential dictated by local supersymmetry. The presence of scalars will allow us to study the existence of an attractor mechanism for their values at the horizon. This characterization simplifies the search for supersymmetric black-hole solutions and we are going to use it to construct explicit solutions in a specific model admitting an $SO(3)$ gauge group.

Monopoles in $N = 2$ gauge theories were first studied in [10], and the model we are going to study is its closest supergravity analogue: $SO(3)$ gauged model on $\overline{\mathbb{CP}}^3$. In fact, one can see that the YM limit of the model (see e.g. [11]) explicitly leads to the theory studied in [10]. $SO(3)$ monopoles in EYM were also studied in Ref. [12], but their model can only be related to a supergravity theory for a specific value of the dilaton coupling [13]; at this value their solution is the one found in Ref. [5].

We are going to see that the model considered (and presumably a lot more models, including the *stringy* ones) admits solutions in which the YM fields describe the 't Hooft-Polyakov monopole with a globally regular metric. We will also show, by finding an explicit analytic expression for them, that this model (and, again, probably many more models) admits solutions with non-Abelian YM fields having the same asymptotic behavior as the 't Hooft-Polyakov monopole and whose metrics are regular outside an event horizon. We will also describe how the attractor mechanism works in this example. The monopole solutions found long ago in Refs. [5,6] should be particular examples of this general class of monopole solutions. Furthermore, the $SU(2) \times U(1)$ black-hole solution of Ref. [14] should also belong to the class of black hedgehogs, although finding the exact correspondence is a difficult task.

The occurrence of a *covariant attractor mechanism* is intriguing and work on a general proof is under way.

We start by describing the bosonic sector of $N = 2$, $d = 4$ supergravity coupled to n_V non-Abelian vector supermultiplets, i.e. $N = 2$ SEYM. It is a generalization of the EYM theory with $n_V + 1$ vector fields $A^\Lambda{}_\mu$, $\Lambda = 0, 1, \ldots n_V$ and n_V complex scalars Z^i, $i = 1, \ldots n$ that parametrize a special-Kähler manifold with metric $\mathcal{G}_{ij^*}(Z, Z^*)$ [15]. The theory has a non-Abelian gauge symmetry that acts on the vector and scalar fields, which are charged. In contrast to pure EYM theory, however, the SEYM theory has a scalar potential $V(Z, Z^*)$ and a scalar matrix $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$ that couples to the vector field strengths whose forms are dictated by supersymmetry.

More explicitly, the bosonic Lagrangian for these theories can be written in the form

$$e^{-1}\mathcal{L} = R + 2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*j^*} + 2F^{\Lambda\mu\nu} \star F_{\Sigma\mu\nu} - V. \tag{1}$$

Here, the gauge covariant derivative on the scalars is

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i + gA^\Lambda{}_\mu k_\Lambda{}^i, \tag{2}$$

where $k_\Lambda{}^i(Z)$ are the holomorphic Killing vectors of the scalar metric $\mathcal{G}_{ij^*}$. The electric field strengths $F^{\Lambda\mu\nu}$, and their magnetic duals

$$F_{\Lambda\mu\nu} \equiv \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} \star F^\Sigma{}_{\mu\nu}, \tag{3}$$

define a $2(n_V + 1)$-dimensional symplectic vector of 2-forms $F = (F^\Lambda, F_\Lambda)$. The real and imaginary parts of the matrix $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$ are field-dependent generalizations of the θ-angle and the coupling constant. Finally, the potential is given by

$$V(Z, Z^*) = -\frac{1}{4}g^2(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}\mathcal{P}_\Lambda\mathcal{P}_\Sigma, \tag{4}$$

where $\mathcal{P}_\Lambda$ is the momentum map satisfying $k_{\Lambda i^*} = i\partial_{i^*}\mathcal{P}_\Lambda$. Since $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}$ must be negative-definite and the $\mathcal{P}_\Lambda$ are real, we have $V \geq 0$.

A useful alternative description of the n_V scalars Z^i is through the $2(n_V + 1)$-dimensional complex symplectic section $\mathcal{V} \equiv (\mathcal{L}^\Lambda, \mathcal{M}_\Lambda)$. To eliminate the redundancy in this description of the scalars, $\mathcal{V}$ is subject to several constraints such as

$$\langle \mathcal{V}^*, \mathcal{V}\rangle \equiv \mathcal{L}^\Lambda\mathcal{M}^*_\Lambda - \mathcal{L}^{*\Lambda}\mathcal{M}_\Lambda = i, \tag{5}$$

and a gauge symmetry described in the references cited above. An important constraint is that

$$\mathcal{M}_\Lambda = \partial_{\mathcal{L}^\Lambda}\mathcal{F}(\mathcal{L}^\Sigma), \tag{6}$$

where $\mathcal{F}$, called the prepotential, is a homogeneous function of degree 2, that depends on the model under consideration and determines it uniquely, i.e. $\mathcal{G}_{ij^*}$, $\mathcal{V}$, and $\mathcal{N}_{\Lambda\Sigma}$ can be derived from it. The physical scalars are recovered from $\mathcal{V}$ via

$$Z^i = \mathcal{L}^i/\mathcal{L}^0. \tag{7}$$

We are interested in solutions of the above system and particularly in the supersymmetric (Bogomol'nyi-Prasad-Sommerfield monopole [BPS]) ones, which are easier to find. Actually, the general form of all of them can be found following the procedure of Refs. [7,8]. It turns out that a wide class of supersymmetric static solutions can be constructed starting from a solution of the standard Bogomol'nyi equation [16]

$$\epsilon_{pmn}F^\Lambda{}_{mn} = -\sqrt{2}\mathfrak{D}_p\mathcal{I}^\Lambda, \qquad m, n, p = 1, 2, 3, \tag{8}$$

for real "Higgs" scalars $\mathcal{I}^\Lambda$ such as those describing well-known YM monopoles. Given a solution $A^\Lambda{}_m$, $\mathcal{I}^\Lambda$ of this equation, it is enough to solve next

$$\mathfrak{D}_m\mathfrak{D}_m\mathcal{I}_\Lambda = \frac{1}{2}g^2[f_{\Lambda(\Sigma}{}^\Gamma f_{\Delta)\Gamma}{}^\Omega\mathcal{I}^\Sigma\mathcal{I}^\Delta]\mathcal{I}_\Omega, \tag{9}$$

for the real scalars $\mathcal{I}_\Lambda$ satisfying the condition [17]

$$\langle\mathcal{I}|\mathfrak{D}_m\mathcal{I}\rangle = 0, \qquad \mathcal{I} \equiv (\mathcal{I}^\Lambda, \mathcal{I}_\Lambda), \tag{10}$$

to determine a complete supersymmetric static solution of all the equations of motion of the theory. We now show how the physical fields of the theory are derived from this information.

The real symplectic vector $\mathcal{I}$ is, in these solutions, the imaginary part of $\mathcal{V}/X \equiv \mathcal{R} + i\mathcal{I}$. The real part $\mathcal{R}$ can be found from Eq. (6), which in this context are known as *stabilization equations*. Then, knowing $\mathcal{R}$ and $\mathcal{I}$, and therefore $\mathcal{V}/X$, we can find the physical scalars using Eq. (7)

$$Z^i = \mathcal{L}^i/\mathcal{L}^0 = (\mathcal{L}^i/X)/(\mathcal{L}^0/X) = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}. \tag{11}$$

The metric reads

$$ds^2 = 2|X|^2dt^2 - (2|X|^2)^{-1}dx^m dx^m, \tag{12}$$

and $F = (F^\Lambda, F_\Lambda)$ take the form

$$F = -\sqrt{2}\mathfrak{D}(|X|^2\mathcal{R}dt) - \sqrt{2}|X|^2 \star (dt \wedge \mathfrak{D}\mathcal{I}), \tag{13}$$

and both of them are uniquely determined by $\mathcal{R}$, $\mathcal{I}$, as Eq. (5) implies that

$$(2|X|^2)^{-1} = \langle\mathcal{R} \mid \mathcal{I}\rangle. \tag{14}$$

This provides a systematic procedure to generate supersymmetric solutions to the $N = 2$ SEYM theories that we have described. The well-known solutions of the Abelian case are obviously included. We will work out an example of special interest following the above steps.

Let us consider $N = 2$ SEYM systems. We split the index Λ into an a-index $a = 1, 2, 3$ on which an $SO(3)$ gauge group acts, and a u-index labeling the ungauged directions. In these directions, the $\mathcal{I}^u$s are harmonic functions on $\mathbb{R}^3$ which we will choose judiciously. In the gauged directions, making the standard *hedgehog* ansatz

$$\mathcal{I}^a = \mathcal{I}(r)n^a, \qquad A^a{}_m = \Phi(r)\varepsilon_{mn}{}^a n^n \\ n^a \equiv x^a r^{-1}, \qquad r \equiv \sqrt{x^b x^b}, \tag{15}$$

the Bogomol'nyi equation (8) admits a 2-parameter family of solutions given by [18]

$$\begin{aligned} \mathcal{I}(r) &= \sqrt{2}\mu g^{-1}\mathsf{H}_\rho(\mu r), \\ \mathsf{H}_\rho(r) &= \coth(r+\rho) - r^{-1}, \\ \Phi(r) &= \mu g^{-1}\mathsf{G}_\rho(\mu r), \\ \mathsf{G}_\rho(r) &= r^{-1} - \sinh^{-1}(r+\rho). \end{aligned} \tag{16}$$

In this family there are two particularly interesting solutions, namely $\rho = 0$ and $\rho \to \infty$.

In the $\rho = 0$ solution the functions G_0 and H_0 are regular and bound between 0 and 1. Thus, we see that $\mathcal{I}$ and Φ are regular at $r = 0$. The YM fields of this solution are those of the 't Hooft-Polyakov monopole [19].

In the limit $\rho \to \infty$ the solution becomes

$$\begin{aligned} A^a{}_m &= \varepsilon_{mb}{}^a n^b (gr)^{-1}, \\ \mathcal{I}^a &= -\sqrt{2}(\mathcal{I}_\infty + (gr)^{-1})n^a, \\ \mathcal{I}_\infty &\equiv -\mu g^{-1}. \end{aligned} \tag{17}$$

These fields are singular at $r = 0$: this singularity makes the solution uninteresting in flat spacetime and is, probably, the reason why it has not been considered before in the literature. However, the coupling to gravity may cover it by an event horizon in which case we would obtain a non-Abelian black-hole solution which we call a "black hedgehog."

The next step is to obtain the $\mathcal{I}_a$ from Eq. (9) [20]. A solution is found by observing that, if $\mathcal{I}_a \sim n^a$, the right-hand side of said equation vanishes identically. Equation (9) then reduces to the integrability condition of Eq. (8), so that

$$\mathcal{I}_a = \frac{1}{2} g\mathcal{J}\mathcal{I}^a, \tag{18}$$

where $\mathcal{J}$ is an arbitrary constant.

The fact that $\mathcal{I}_a$ has the same functional form as $\mathcal{I}^a$ has consequences for the staticity condition Eq. (10): the condition (10) acts nontrivially only on the ungauged part, i.e.

$$\begin{aligned} 0 &= \mathcal{I}_u d\mathcal{I}^u - \mathcal{I}^u d\mathcal{I}_u + \mathcal{I}_a \mathfrak{D}\mathcal{I}^a - \mathcal{I}^a\mathfrak{D}\mathcal{I}_a \\ &= \mathcal{I}_u d\mathcal{I}^u - \mathcal{I}^u d\mathcal{I}_u. \end{aligned} \tag{19}$$

This equation is the same that appears in the Abelian case and can be solved in exactly the same way [21].

At this point the solutions are completely determined. In order to find the explicit form of the physical fields, we must find $\mathcal{R}$ by solving the stabilization equations which depend on the specific supergravity model considered. Let us then consider a simple $SO(3)$ gauged model.

As mentioned before, we can describe a particular model by specifying the prepotential $\mathcal{F}$, Eq. (6), and the gauge group. The prepotential for the model with special-Kähler manifold $\overline{\mathbb{CP}}^n$ reads

$$\mathcal{F} = \frac{i}{4}\eta_{\Lambda\Sigma}\mathcal{L}^\Lambda\mathcal{L}^\Sigma, \qquad \eta = \mathrm{diag}(-, [+]^n). \tag{20}$$

The Kähler potential is

$$e^{-\mathcal{K}} = |\mathcal{X}^0|^2 - |\mathcal{X}^i|^2 = 1 - |Z^i|^2 \equiv 1 - |Z|^2, \tag{21}$$

which results in the Fubini-Study metric on $\overline{\mathbb{CP}}^n$. Observe that due to Eq. (21) the coordinates Z^i are constrained to $0 \leq |Z|^2 < 1$.

The stabilization equations (6) can be readily solved for this model:

$$\mathcal{R}_\Lambda = -\frac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma, \qquad \mathcal{R}^\Lambda = 2\eta^{\Lambda\Sigma}\mathcal{I}_\Sigma, \tag{22}$$

which allows us to write the metrical factor in Eq. (14) in terms of the $\mathcal{I}^\Lambda$ and $\mathcal{I}_\Lambda$ as

$$g_{tt}^{-1} = -g_{rr} = \frac{1}{2}[\mathcal{I}^{02} - \mathcal{I}^{i2} + 4\mathcal{I}_0^2 - 4\mathcal{I}_i^2]. \tag{23}$$

Consider the case $n = 3$ with gauge group $SO(3)$ acting on the indices $a = 1, 2, 3$ and with the ungauged direction $u = 0$; the solution for A^a_m, $\mathcal{I}^a$, $\mathcal{I}_a$ is given by Eqs. (15), (17), and (18). $\mathcal{I}^0$ and $\mathcal{I}_0$ are arbitrary harmonic functions in $\mathbb{R}^3$ that we will choose in such a way as to solve Eq. (10) and get regular solutions. We find

$$-g_{rr} = \frac{1}{2}\{\mathcal{I}^{02} + 4\mathcal{I}_0^2 - 2\mu^2[g^{-2} + \mathcal{J}^2]\mathsf{H}_\rho^2(\mu r)\}. \tag{24}$$

Let us try to find a globally regular embedding of the 't Hooft-Polyakov monopole in the $\overline{\mathbb{CP}}^3$ model: as the function $\mathsf{H}_0(\mu r)$ is bound, it is enough for $\mathcal{I}^0$ and $\mathcal{I}_0$ to be constant in order to ensure that the scalars satisfy their constraint. Actually, taking them to be nonconstant would produce scalars violating said constraint and would introduce singularities. Fixing the values of $\mathcal{I}^0$ and $\mathcal{I}_0$ by imposing asymptotic flatness we find

$$-g_{rr} = 1 + \mu^2[g^{-2} + \mathcal{J}^2](1 - \mathsf{H}^2(\mu r)), \tag{25}$$

which implies that the metric is globally regular and describes an object of mass

$$\mathsf{M} = \mu[g^{-2} + \mathcal{J}^2]. \tag{26}$$

Let us now consider the black hedgehog case: since the function $\mathsf{H}_\infty(\mu r)$ is singular, either $\mathcal{I}^0$ or $\mathcal{I}_0$ has to be a nonconstant harmonic function as to produce scalar fields that satisfy the constraint $0 \leq |Z|^2 < 1$.

Choosing for simplicity

$$\mathcal{I}^0 = \mathcal{I}^0_\infty + p^0 r^{-1}, \tag{27}$$

we get

$$
\begin{aligned}
-g_{rr} == & \frac{1}{2}\{\mathcal{I}^{0}_{\infty}{}^{2} - 2\mu^{2}[g^{-2} + \mathcal{J}^{2}]\} \\
& + \{\mathcal{I}^{0}_{\infty} p^{0} - 2|\mu|[g^{-2} + \mathcal{J}^{2}]\} r^{-1} \\
& + \frac{1}{2}\{p^{02} - 2[g^{-2} + \mathcal{J}^{2}]\} r^{-2}.
\end{aligned} \tag{28}
$$

The first term can be normalized to 1 as to recover Minkowski asymptotically. With this normalization the coefficient of the second term is the mass and should be positive; the coefficient of the last term, if positive, is the area of an event horizon divided by 4π. Under the provisos of positivity of mass and area, the above metric can be seen to describe the geometry outside the outer horizon of a regular BH and the coefficient of the last term is identified with its entropy (see e.g. [21]).

It is always possible to choose the parameters such as to obtain a regular black hole. A simple choice is

$$
\mathcal{I}^{0}_{\infty} = \sqrt{2}\sqrt{1 + \mu^{2}[g^{-2} + \mathcal{J}^{2}]}, \qquad p^{0} = |\mu|^{-1} \mathcal{I}^{0}_{\infty}, \tag{29}
$$

and gives a mass and event horizon area

$$
\mathsf{M} = 2|\mu|^{-1} \qquad \mathsf{A} = 4\pi|\mu|^{-2}. \tag{30}
$$

This black-hole solution has a truly non-Abelian magnetic charge and it is the first of this kind whose analytic form is known. It is clear that the presence of scalars with nontrivial couplings dictated by supersymmetry plays a crucial role in the simplicity of their final form.

The asymptotic values of the scalars seem to violate the *no-hair theorem*, as the BH is naively specified by more than its mass and conserved charges. As in the Abelian case, however, they should be considered *secondary hair* in the sense of Ref. [22]. This interpretation is in accord with the fact that on the horizon the scalars are

$$
Z^{a} = \frac{\sqrt{2}}{p^{0}}(g^{-1} - i\mathcal{J}) n^{a}, \tag{31}
$$

which are independent of their asymptotic values, but are *not* constant over the horizon. Actually, since these scalars are charged, the most we can ask for is that they be constant up to $SO(3)$ gauge transformations, which is the case. The scalar fields have a *covariant attractor* on the horizon and their gauge-invariant combination $|Z|^2$ has a standard attractor, as in the Abelian case. The entropy, proportional to the area, can be expressed in terms of the conserved charges only, allowing, as in the Abelian theories, for a microscopic interpretation.

Monopole and black-hole solutions similar to those found here should also occur in other models with $SO(3)$ gauge group, but a completely general and explicit construction is not possible and the details need to be worked out case by case.

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