

# Cardy-Verlinde formula in FRW Universe with inhomogeneous generalized fluid and dynamical entropy bounds near the future singularity

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We derive a formula for the entropy for a multicomponent coupled fluid, which under special conditions reduces to the Cardy-Verlinde form relating the entropy of a closed FRW universe to its energy together with its Casimir energy. The generalized fluid obeys an inhomogeneous equation of state. A viscous dark fluid is included, and also modified gravity is included in terms of its fluid representation. It is demonstrated how such an expression reduces to the standard Cardy-Verlinde formula corresponding to the 2d CFT entropy in some special cases. The dynamical entropy bound for a closed FRW universe with dark components is obtained. The universality of the dynamical entropy bound near a future singularity (of all known four types), as well as near the Big Bang singularity, is investigated. It is demonstrated that except from some special cases of Type II and Type IV singularities the dynamical entropy bound is violated near the singularity even if quantum effects are taken into account. The dynamical entropy bound seems to be universal for the case of a regular universe, including the asymptotic de Sitter universe.

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## I. INTRODUCTION

Recent astronomical data indicate that the current universe is expanding with cosmic acceleration caused by the so-called Dark Energy (DE). The well-known  $\Lambda$ CDM model where Dark Energy comes from the effective cosmological constant fits quite well the observational data. Nevertheless, it is not excluded that the current DE may have a phantom origin (or will become phantom-like DE in the near future). Moreover, the possibility of a quintessence-like DE with effective equation of state (EoS) parameter being very close to  $-1$  is widely discussed in the recent literature. It is quite likely that the current FRW universe is spatially-flat. However, the occurrence of a slightly spatially-curved FRW universe is not excluded by observational bounds.

The study of phantom-like or quintessence-like effective dark fluids opens for a number of new phenomena which are typical for such a DE universe. For instance, it is known that phantom DE may drive the future universe to a so-called finite-time Big Rip singularity (for earlier works on this, see Refs.[1, 2]). From another side, quintessence-like DE may bring the future universe to a milder future singularity (like the sudden singularity [3, 4] where the effective energy-density is finite). Actually, the study of Ref.[5] shows that there are four different types of future finite-time singularities where the Type I singularity corresponds to the Big Rip, the sudden singularity is of Type II, etc. The universe looks quite strange near to the singularity where curvature may grow up so that quantum gravity effects may be dominant [2]. In any case, the study of the universe under critical conditions (for instance, near a future singularity) may clarify the number of fundamental issues relating seemingly different physical theories.

Some time ago [6] it was shown that the first FRW equation for a closed FRW universe may have a more fundamental origin than what is expected from standard General Relativity. It was demonstrated that this equation may be rewritten so as to describe the universe entropy in terms of total energy and Casimir energy (the so-called Cardy-Verlinde (CV) formula). Moreover, it turns out that the corresponding formula has a striking correspondence with the Cardy formula for the entropy of a two-dimensional conformal field theory (2d CFT). Finally, the formula may be rewritten as a dynamical entropy bound from which a number of entropy bounds, proposed earlier, follow. The connection between the standard gravitational equation and the 2d CFT dynamical entropy bound indicates a very deep relation between gravity and thermodynamics. It raises the question about to which extent the CV formula is

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universal. Problems of this sort are natural to study when the universe is under critical conditions, such as near a singularity.

The present work is devoted to a study of the universality of the CV formula and the corresponding dynamical entropy bound in a DE universe filled with a generalized fluid, especially near the singularity regime. Generalization of the CV formula for a multicomponent fluid with interactions, assuming the EoS to be inhomogeneous, is presented. The viscous case is incorporated. It is shown that the standard CV formula with correct power (square root) is restored only for some very special cases. The dynamical entropy bound for such fluids near all the four types of the future singularity is considered. It is demonstrated that this dynamical entropy bound is most likely violated near the singularity (except from some cases of Type II and Type IV singularity). This situation is not qualitatively changed even if account is taken of quantum effects in conformally invariant theory. Using the formalism of modified gravity to describe an effective dark fluid, the corresponding CV formula is constructed also for  $F(R)$ -gravity. The corresponding dynamical entropy bound is derived. It is shown that the bound is satisfied for a de Sitter universe solution. Further discussion and outlook is given in the discussion section.

## II. GENERALIZATION OF CARDY-VERLINDE FORMULA IN FRW UNIVERSE FOR VARIOUS TYPES OF FLUIDS

This section is devoted to consideration of the Cardy-Verlinde formula for more general scenarios than those considered in previous works (see [6]-[7]). We consider a  $(n+1)$ -dimensional spacetime described by the FRW metric, written in comoving coordinates as

$$ds^2 = -dt^2 + \frac{a(t)^2 dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2, \quad (1)$$

where  $k = -1, 0, +1$  for an open, flat, or closed spatial Universe,  $a(t)$  is taken to have unit of length, and  $d\Omega_{n-1}^2$  is the metric of an  $n-1$  sphere. By inserting the metric (1) in the Einstein field equations the FRW equations are derived,

$$H^2 = \frac{16\pi G}{n(n-1)} \sum_{i=1}^m \rho_i - \frac{k}{a^2}, \quad \dot{H} = -\frac{8\pi G}{n-1} \sum_{i=1}^m (\rho_i + p_i) + \frac{k}{a^2}. \quad (2)$$

Here  $\rho_i = E_i/V$  and  $p_i$  are the energy-density and pressure of the matter component  $i$  that fills the Universe. In this paper we consider only the  $k = 1$  closed Universe. Moreover, we assume an equation of state (EoS) of the form  $p_i = w_i \rho_i$  with  $w_i$  constant for each fluid, and assume at first no interaction between the different components. Then, the conservation law for energy has the form

$$\dot{\rho}_i + nH(\rho_i + p_i) = 0, \quad (3)$$

and by solving (3), we find that the  $i$  fluid depends on the scale factor as

$$\rho_i \propto a^{-n(1+w_i)}. \quad (4)$$

Let us now review the case of Ref.[7], where just one fluid with EoS  $p = w\rho$  and  $w = \text{constant}$  is considered. The total energy inside the comoving volume  $V$ ,  $E = \rho V$ , can be written as the sum of an extensive part  $E_E$  and a subextensive part  $E_C$ , called the Casimir energy, and takes the form:

$$E(S, V) = E_E(S, V) + \frac{1}{2} E_C(S, V). \quad (5)$$

Under a rescaling of the entropy ( $S \rightarrow \lambda S$ ) and the volume ( $V \rightarrow \lambda V$ ), the extensive and subextensive parts of the total energy transform as

$$E_E(\lambda S, \lambda V) = \lambda E_E(S, V), \quad E_C(\lambda S, \lambda V) = \lambda^{1-2/n} E_C(S, V). \quad (6)$$

Hence, by assuming that the Universe satisfies the first law of thermodynamics, the term corresponding to the Casimir energy  $E_C$  can be seen as a violation of the Euler identity according to the definition in Ref.[6]:

$$E_C = n(E + pV - TS). \quad (7)$$

Since the total energy behaves as  $E \sim a^{-nw}$  and by the definition (5), the Casimir energy also goes as  $E_C \sim a^{-nw}$ . The FRW Universe expands adiabatically,  $dS = 0$ , so the products  $E_C a^{nw}$  and  $E_E a^{nw}$  should be independent of the

volume  $V$ , and be just a function of the entropy. Then, by the rescaling properties (6), the extensive and subextensive part of the total energy can be written as functions of the entropy only [7],

$$E_E = \frac{\alpha}{4\pi a^{nw}} S^{w+1}, \quad E_C = \frac{\beta}{2\pi a^{nw}} S^{w+1-2/n}. \quad (8)$$

Here  $\alpha$  and  $\beta$  are undetermined constants. By combining these expressions with (5), the entropy of the Universe is written as a function of the total energy  $E$  and the Casimir energy  $E_C$ , [7],

$$S = \left( \frac{2\pi a^{nw}}{\sqrt{\alpha\beta}} \sqrt{E_C(2E - E_C)} \right)^{\frac{n}{n(w+1)-1}}, \quad (9)$$

which for  $w = 1/n$  (radiation-like fluid) reduces to [6],

$$S = \frac{2\pi a}{\sqrt{\alpha\beta}} \sqrt{E_C(2E - E_C)}, \quad (10)$$

which has the same form as the Cardy formula given in Ref.[8]. The first FRW equation (2) can be rewritten as a relation between thermodynamics variables, and yields

$$S_H = \frac{2\pi}{n} a \sqrt{E_{\text{BH}}(2E - E_{\text{BH}})}, \quad \text{where} \quad S_H = (n-1) \frac{HV}{4G}, \quad E_{\text{BH}} = n(n-1) \frac{V}{8\pi G a^2}. \quad (11)$$

It is easy to check that for the bound proposed in Ref.[6],  $E_C \leq E_{\text{BH}}$ , the equation for the entropy (10) coincides with the first FRW equation (11) when the bound is reached. We will see below that when there are several fluid components, the same kind of expression as in Ref.[6] cannot be found. Nor is there the same correspondence with the FRW equation when the bound is saturated.

### A. Multicomponent Universe

If  $m$  fluids are considered with arbitrary EoS,  $p_i = w_i \rho_i$ , the expression for the total entropy is simple to derive just by following the same method as above. The total entropy is given by the sum of the entropies for each fluid,

$$S = \sum_{i=1}^m S_i = \sum_{i=1}^m \left( \frac{2\pi a^{nw_i}}{\sqrt{\alpha\beta}} \sqrt{E_{iC}(2E_i - E_{iC})} \right)^{\frac{n}{n(w_i+1)-1}}. \quad (12)$$

This expression cannot be reduced to one depending only on the total energy unless very special conditions on the nature of the fluids are assumed. Let us for simplicity assume that there are only two fluids with EoS given by  $p_1 = w_1 \rho_1$  and  $p_2 = w_2 \rho_2$ ,  $w_1$  and  $w_2$  being constants. We can substitute the fluids by an effective fluid described by the EoS

$$p_{\text{eff}} = w_{\text{eff}} \rho_{\text{eff}}, \quad \text{where} \quad w_{\text{eff}} = \frac{p_1 + p_2}{\rho_1 + \rho_2} = w_1 + \frac{w_2 - w_1}{1 + \rho_1/\rho_2}, \quad (13)$$

and  $p_{\text{eff}} = \frac{1}{2}(p_1 + p_2)$ ,  $\rho_{\text{eff}} = \frac{1}{2}(\rho_1 + \rho_2)$ . Then, by using the energy conservation equation (3), we find  $\rho_1 \sim (a/a_0)^{-n(1+w_1)}$  and  $\rho_2 \sim (a/a_0)^{-n(1+w_2)}$ , where  $a_0$  is assumed to be the value of the scale factor at the time  $t_0$ . The effective EoS parameter  $w_{\text{eff}}$  can be expressed as a function of the scale factor  $a(t)$

$$w_{\text{eff}} = w_1 + \frac{w_2 - w_1}{1 + (a/a_0)^{n(w_2 - w_1)}}. \quad (14)$$

The total energy inside a volume  $V$  becomes

$$E_T = E_1 + E_2 \propto (a/a_0)^{-nw_1} + (a/a_0)^{-nw_2}. \quad (15)$$

As the energy is proportional to two different powers of the scale factor  $a$ , it is not possible to write it as a function of the total entropy only. As a special case, if the EoS parameters are  $w_1 = w_2 = w_{\text{eff}}$ , the formula for the entropy reduces to (9), and coincides with the CV formula when  $w_{\text{eff}} = 1/n$ .

As another case, one might consider that for some epoch of the cosmic history,  $w_1 \gg w_2$ . Taking also  $a \gg a_0$ , we could then approximate the total energy by the function  $E_T \propto a^{-nw_2}$ . From (5) the Casimir energy would also depend on the same power of  $a$ ,  $E_C \propto a^{-nw_2}$ . The expression (9) is again recovered with  $w = w_2$ .

Thus in general, when a multicomponent FRW Universe is assumed, the formula for the total entropy does not resemble the Cardy formula, nor does it correspond to the FRW equation when the Casimir bound is reached. It becomes possible to reconstruct the formula (10), and establish the correspondence with the Cardy formula, only if we make specific choices for the EoS of the fluids.

### B. Interacting fluids

As a second case we now consider a Universe, described by the metric (1), filled with two interacting fluids. One can write the energy conservation equation for each fluid as

$$\dot{\rho}_1 + nH(\rho_1 + p_1) = Q, \quad \dot{\rho}_2 + nH(\rho_2 + p_2) = -Q, \quad (16)$$

where  $Q$  is a function that accounts for the energy exchange between the fluids. This kind of interaction has been discussed previously in studies of dark energy and dark matter. The effective EoS parameter is given by the same expression (13) as before. With a specific choice for the coupling function  $Q$ , the equations (16) may be solved. One can in principle find the dependence of the energy densities  $\rho_{1,2}$  on the scale factor  $a$ ,

$$\rho_1 = a(t)^{-n(1+w_1)} \left( C_1 + \int a^{n(1+w_1)} Q(t) dt \right), \quad \rho_2 = a(t)^{-n(1+w_2)} \left( C_2 - \int a^{n(1+w_2)} Q(t) dt \right), \quad (17)$$

where  $C_1$  and  $C_2$  are integration constants. In general it is not possible to reproduce the CV formula, and the result will be a sum of different contributions, similar to the entropy expression given in (12). However, for the case where the effective EoS parameter (14) is a constant, the expression for the entropy will be given by equation (9) as before. This condition only holds when  $w_{\text{eff}} = w_1 = w_2$ , where the situation is thus equivalent to the one-fluid case, and the entropy reduces to the CV formula when  $w_{\text{eff}} = 1/n$ .

Let us consider a simple choice for the function  $Q$  that leads to the CV formula for a certain limit. Let  $Q = Q_0 a^m H$ , where  $m$  is a positive number,  $Q_0$  is a constant, and  $H(t)$  the Hubble parameter. Then the integral in (17) is easily calculated, and the energy densities depend on the scale factor according to

$$\rho_1 = C_1 a^{-n(1+w_1)} + k_1 a^m, \quad \rho_2 = C_2 a^{-n(1+w_2)} + k_2 a^m, \quad (18)$$

where  $k_{1,2} = Q_0/(n(1+w_{1,2}) + m)$ . If we restrict ourselves to the regime where  $a \gg C_{1,2}$  such that the first terms in the expressions for  $\rho_{1,2}$  are negligible, the effective EoS parameter becomes

$$w_{\text{eff}} = w_1 + \frac{w_2 - w_1}{1 + \frac{k_1}{k_2}}. \quad (19)$$

Then, the entropy of the universe is given by (9) with  $w = w_{\text{eff}}$ . The CV formula can be reproduced only with very specific choice of the free parameters, just as above.

We have thus shown that in general a formula for the entropy of the type (9) cannot be reconstructed for interacting fluids. Coincidence with the Cardy formula is obtained if the effective EoS parameter is radiation-like,  $w_{\text{eff}} = 1/n$ . Then the expression for the entropy turns out to be in agreement with the formula (10), corresponding to the first FRW equation (11) when the Casimir energy reaches the bound  $E_C = E_{\text{BH}}$ .

### C. Inhomogeneous EoS fluid and bulk viscosity

Let us now explore the case of an  $n + 1$ -dimensional Universe filled with a fluid satisfying an inhomogeneous EoS. This kind of EoS, generalizing the perfect fluid model, has been considered in several papers as a way to describe effectively the dark energy (see [9] and [10]). We assume an EoS expressed as a function of the scale factor,

$$p = w(a)\rho + g(a). \quad (20)$$

This EoS fluid could be taken to correspond to modified gravity, or to bulk viscosity (Ref.[9]). By introducing (20) in the energy conservation equation (3) we obtain

$$\rho'(a) + \frac{n(1+w(a))}{a}\rho(a) = -n\frac{g(a)}{a}. \quad (21)$$

Here we have performed a variable change  $t = t(a)$  such that the prime over  $\rho$  denotes derivative with respect to the scale factor  $a$ . The general solution of this equation is

$$\rho(a) = e^{-F(a)} \left( K - n \int e^{F(a)} \frac{g(a)}{a} da \right) \quad \text{where} \quad F(a) = \int^a \frac{1+w(a')}{a'} da', \quad (22)$$

and  $K$  is an integration constant. As shown above, only for some special choices of the functions  $w(a)$  and  $g(a)$ , the formula (10) can be recovered. Let us assume, as an example, that  $w(a) = -1$  and  $g(a) = -a^m$ , with  $m = \text{constant}$ . Then, the energy density behaves as  $\rho \propto a^m$ . Hence, by following the same steps as described above, the extensive and subextensive energy go as  $a^{m+n}$ , and by imposing conformal invariance and the rescaling properties (6), we calculate the dependence on the entropy to be

$$E_E = \frac{\alpha}{4\pi n a^{-(m+n)}} S^{-m/n}, \quad E_C = \frac{\beta}{4\pi n a^{-(m+n)}} S^{-(2n+m/n)}. \quad (23)$$

The expression for the entropy is easily constructed by combining these two expressions and substituting the extensive part by the total energy. This gives us the same expression as in (9) with  $w = -(n+m)/n$ . Note that for  $m = -(1+n)$ , the formula (10) is recovered and also its correspondence with the CFT formula. However for a generic power  $m$ , the CV formula cannot be reconstructed, like the cases studied above. Only for some special choices does the correspondence work, leading to the identification between the FRW equation and the Cardy formula.

Let us now consider an inhomogeneous EoS fluid due to bulk viscosity. From a hydrodynamical perspective it is natural to extend the formalism so as to incorporate viscosity effects. Working to the first order in the deviations from thermal equilibrium we are faced with two viscosity coefficients, namely the shear viscosity  $\eta$  and the bulk viscosity  $\zeta$ . In conformity with spatial isotropy we shall assume, as usual, that only the bulk viscosity contributes. (For a review of viscous cosmology and entropy, one may consult Ref.[11] and also Refs.[12].) The viscous fluid may be considered as a special kind of inhomogeneous EoS fluid, although of a different kind from that of Eq. (20) above. We set the number of spatial dimensions  $n$  equal to 3. The energy-momentum tensor can be written as

$$T_{\mu\nu} = \rho U_\mu U_\nu + \tilde{p} h_{\mu\nu}, \quad (24)$$

where  $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$  is the projection tensor and

$$\tilde{p} = p - 3H\zeta, \quad (25)$$

the effective pressure. For simplicity let us assume in this subsection the following simple fluid model:

$$w = \text{constant}, \quad g = 0, \quad \zeta = \text{constant}, \quad (26)$$

In comoving coordinates,  $U^0 = 1, U^i = 0$ . The first of the FRW equations (2) maintains its form (it is viscous insensitive), whereas the second equation becomes

$$\dot{H} = -4\pi G(\rho + \tilde{p}) + \frac{1}{a^2}, \quad (27)$$

showing that  $\tilde{p}$  is now the thermodynamically important pressure. We see that the relationship  $p = w\rho$ , or

$$\tilde{p} = w\rho - 3H\zeta, \quad (28)$$

can be considered as an EoS in the present case.

Introduction of a viscosity means effectively the introduction of a length parameter, and so the conformal invariance of the formalism is lost. The question arises: Can the entropy arguments leading to the Cardy-Verlinde formula be carried over to the viscous case? The answer actually turns out to be affirmative, at least when  $\zeta$  is small. The most delicate point in the line of arguments is the assumed pure entropy dependence of the product  $Ea^{3w}$ . Now, if one uses the FRW equations to derive the ‘‘energy equation’’ ( $k = +1$  assumed)

$$\frac{d}{dt} \left( \rho a^{3(1+w)} \right) = 9\zeta H^2 a^{3(1+w)}, \quad (29)$$

one can combine this with the equation for entropy production

$$N\dot{\sigma} = \frac{9\zeta}{T} H^2, \quad (30)$$

where  $N$  is the particle (baryon) density and  $\sigma$  the entropy per particle. As discussed in some detail in [11] it follows that, since the total energy  $E \sim \rho a^3$  and the total entropy  $S \sim N\sigma a^3$ , the quantity  $Ea^{3w}$  becomes independent of the volume  $V$  and is a function of  $S$  only. This generalizes the pure entropy dependence of the product  $Ea$  found by Verlinde in the case of a non-viscous radiation dominated universe. We obtain the expression (9) with  $n = 3$  as the generalized Cardy-Verlinde formula and reducing to the standard formula (with square root) when the universe is radiation dominated.

The following point ought finally to be noted. The energy conservation equation  $T^{0\nu}{}_{;\nu} = 0$  implies

$$\dot{\rho} + 3H(\rho + \tilde{p}) = 0, \quad (31)$$

so that equations (21) can be taken over to the viscous case, only with the substitutions  $p_i \rightarrow \tilde{p}_i$ .

### III. ON THE COSMOLOGICAL BOUNDS NEAR FUTURE SINGULARITIES

In Ref.[6], Verlinde proposed a new universal bound on cosmology based on a restriction of the Casimir energy  $E_C$ ; cf. his entropy formula (10). This new bound postulated was

$$E_C \leq E_{\text{BH}}, \quad (32)$$

where  $E_{\text{BH}} = n(n-1)\frac{V}{8\pi G a^2}$ . It was deduced by the fact that in the limit when the Universe passes between strongly and weakly self-gravitating regimes, the Bekenstein entropy  $S_{\text{B}} = \frac{2\pi a}{n} E$  and the Bekenstein-Hawking entropy  $S_{\text{BH}} = (n-1)\frac{V}{4Ga}$ , which define each regime, are equal. This bound could be interpreted to mean that the Casimir energy never becomes able to reach sufficient energy,  $E_{\text{BH}}$ , to form a black hole of the size of the Universe. It is easy to verify that the strong ( $Ha \geq 1$ ) and weak ( $Ha \leq 1$ ) self-gravity regimes have the following restrictions on the total energy,

$$\begin{aligned} E &\leq E_{\text{BH}} & \text{for } Ha \leq 1, \\ E &\geq E_{\text{BH}} & \text{for } Ha \geq 1. \end{aligned} \quad (33)$$

From here it is easy to calculate the bounds on the entropy of the Universe in the case when the Verlinde formula (10) is valid; this is (as shown in the above sections) for an effective radiation dominated Universe  $w_{\text{eff}} \sim 1/n$ . The bounds for the entropy deduced in Ref.[6] for  $k = 1$  are

$$\begin{aligned} S &\leq S_{\text{B}} & \text{for } Ha \leq 1, \\ S &\geq S_{\text{H}} & \text{for } Ha \geq 1, \end{aligned} \quad (34)$$

where  $S_{\text{B}}$  is the Bekenstein entropy defined above, and  $S_{\text{H}}$  is the Hubble entropy given by (11). Note that for the strong self-gravity regime,  $Ha \geq 1$ , the energy range is  $E_C \leq E_{\text{BH}} \leq E$ . According to the formula (10) the maximum entropy is reached when the bound is saturated,  $E_C = E_{\text{BH}}$ . Then  $S = S_{\text{H}}$ , such that the FRW equation coincides with the CV formula, thus indicating a connection with CFT. For the weak regime,  $Ha \leq 1$ , the range of energies goes as  $E_C \leq E \leq E_{\text{BH}}$  and the maximum entropy is reached earlier, when  $E_C = E$ , yielding the result  $S = S_{\text{B}}$ . The entropy bounds can be extended to more general cases, corresponding to an arbitrary EoS parameter  $w$ . By taking the bound (32) to be universally valid one can easily deduce the new entropy bounds for each regime, from the expression of the entropy (9). These new bounds, discussed in Ref.[7], differ from the ones given in (34), but still establish a bound on the entropy as long as the bound on  $E_C$  expressed in (32) is taken to be valid. The entropy bounds can be related through the first FRW equation, yielding the following quadratic expression (for  $k = 1$ ),

$$S_{\text{H}}^2 + (S_{\text{B}} - S_{\text{BH}})^2 = S_{\text{B}}^2. \quad (35)$$

We would like to study what happens to the bounds, particularly to the fundamental bound (32), when the cosmic evolution is close to a future singularity; then the effective fluid dominating the cosmic evolution could have an unusual EoS. As shown below, for some class of future singularities such a bound could soften the singularities in order to avoid violation of the *universal* bound (32). It could be interpreted to mean that quantum effects become important when the bound is reached. However, as the violation of the bound could happen long before the singularity even in the presence of quantum effects, it could be a signal of breaking of the universality of the bound (32). Let us first of all give a list of the possible future cosmic singularities, which can be classified according to Ref.[5] as

- Type I (“Big Rip”): For  $t \rightarrow t_s$ ,  $a \rightarrow \infty$  and  $\rho \rightarrow \infty$ ,  $|p| \rightarrow \infty$ .
- Type II (“Sudden”): For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$  and  $\rho \rightarrow \rho_s$ ,  $|p| \rightarrow \infty$ . (see Refs.[3, 4])
- Type III: For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$  and  $\rho \rightarrow \infty$ ,  $|p| \rightarrow \infty$ .
- Type IV: For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$  and  $\rho \rightarrow \rho_s$ ,  $p \rightarrow p_s$  but higher derivatives of Hubble parameter diverge.

Note that the above list was suggested in the case of a flat FRW Universe. As we consider in this paper a closed Universe ( $k = 1$ ), we should make an analysis to see if the list of singularities given above is also valid in this case. It is straightforward to see that all the singularities listed above can be reproduced for a particular choice of the effective EoS. To show how the cosmic bounds behave for each type of singularity, we could write an explicit solution of the FRW equations, expressed as a function of time depending on free parameters that will be fixed for each kind of singularity. Then, the Hubble parameter may be written as follows

$$H(t) = \sqrt{\frac{16\pi G}{n(n-1)}\rho - \frac{1}{a^2}} = H_1(t_s - t)^m + H_0, \quad (36)$$

where  $m$  is a constant properly chosen for each type of singularity. Note that this is just a solution that ends in the singularities mentioned above, but there are other solutions which also reproduce such singularities. We will study how the cosmic bounds behave near each singularity listed above.

As pointed out in Ref.[2, 5], around a singularity quantum effects could become important as the curvature of the Universe grows and diverges in some of the cases. In other words, approaching the finite-time future singularity the curvature grows and universe reminds the early universe where quantum gravity effects are dominant ones because of extreme conditions. Then one has to take into account the role of such quantum gravity effects which should define the behaviour of the universe just before the singularity. Moreover, they may act so that to prevent the singularity occurrence. In a sense, one sees the return of quantum gravity era. However, the consistent quantum gravity theory does not exist so far. Then, in order to estimate the influence of quantum effects to universe near to singularity one can use the effective action formulation. We will apply the effective action produced by conformal anomaly (equivalently, the effective fluid with pressure/energy-density corresponding to conformal anomaly ones) because of several reasons. It is known that at high energy region (large curvature) the conformal invariance is restored so one can neglect the masses. Moreover, one can use large  $N$  approximation to justify why large number of quantum fields may be considered as effective quantum gravity. Finally, in the account of quantum effects via conformal anomaly we keep explicitly the graviton (spin 2) contribution. The conformal anomaly  $T_A$  has the following well-known form

$$T_A = b \left( F + \frac{2}{3} \square R \right) + b' G + b'' \square R. \quad (37)$$

Here we assume for simplicity a 3+1 dimensional spacetime. Then,  $F$  is the square of a 4D Weyl tensor and  $G$  is the Gauss-Bonnet invariant,

$$F = \frac{1}{3} R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl}, \quad G = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}. \quad (38)$$

The coefficients  $b$  and  $b'$  in (37) are described by the number of  $N$  scalars,  $N_{1/2}$  spinors,  $N_1$  vector fields,  $N_2$  gravitons and  $N_{\text{HD}}$  higher derivative conformal scalars. They can be written as

$$b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\text{HD}}}{120(4\pi)^2}, \quad b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\text{HD}}}{360(4\pi)^2}. \quad (39)$$

As  $b''$  is arbitrary it can be shifted by a finite renormalization of the local counterterm. The conformal anomaly  $T_A$  can be written as  $T_A = -\rho_A + 3p_A$ , where  $\rho_A$  and  $p_A$  are the energy and pressure densities respectively. By using (37) and the energy conservation equation  $\rho_A + 3H(\rho_A + p_A) = 0$ , one obtains the following expression for  $\rho_A$  [5, 13],

$$\begin{aligned} \rho_A &= -\frac{1}{a^4} \int dt a^4 H T_A \\ &= -\frac{1}{a^4} \int dt a^4 H \left[ -12b\dot{H}^2 + 24b'(-\dot{H}^2 + H^2\dot{H} + H^4) - (4b + 6b'')(\ddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H}) \right]. \end{aligned} \quad (40)$$

The quantum corrected FRW equation is given by

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_A) - \frac{1}{a^2}. \quad (41)$$

We study now how the bounds behave around the singularity in the classical case when no quantum effects are added, and then include the conformal anomaly (37) quantum effects in the FRW equations. We will see that for some cases the violation of the cosmic bound can be avoided.

### A. Big Rip Singularity

This type of singularity has been very well studied and has become very popular as it is a direct consequence in the majority of the cases when the effective EoS parameter is less than  $-1$ , the so-called phantom case [1, 2]. Observations currently indicate that the phantom barrier could have already been crossed or it will be crossed in the near future, so a lot of attention has been paid to this case. It can be characterized by the solution (36) with  $m \leq -1$ , and this yields the following dependence of the total energy density on the scale factor near the singularity, when  $a \gg 1$ , for a closed Universe ( $k = 1$ ),

$$\rho = \frac{n(n-1)}{16\pi G} H^2 + \frac{1}{a^2} \sim a^{-n(1+w)} \quad \text{for } t \rightarrow t_s, \quad (42)$$

where we have chosen  $H_1 = 2/n|1+w|$  with  $w < -1$ ,  $m = -1$  and  $H_0 = 0$  for clarity. This solution drives the Universe to a Big Rip singularity for  $t \rightarrow t_s$ , where the scale factor diverges. If the singularity takes place, the bound (32) has to be violated before this happens. This can be seen from equation (42), as the Casimir energy behaves as  $E_C \propto a^{n|w|}$  while the Bekenstein-Hawking energy goes as  $E_{\text{BH}} \propto a^{n-2}$ . Then, as  $w < -1$ , the Casimir energy grows faster than the BH energy, so close to the singularity where the scale factor becomes very big, the value of  $E_C$  will be much higher than  $E_B$ , thus violating the bound (32). Following the postulate from Ref.[6] one could interpret the bound (32) as the limit where General Relativity and Quantum Field Theory converge, such that when the bound is saturated quantum gravity effects should become important. QG corrections could help to avoid the violation of the bound and may be the Big Rip singularity occurrence. As this is just a postulate based on the CV formula, which is only valid for special cases as shown in the sections above, the bound on  $E_C$  could not be valid for any kind of fluid.

Let us now include the conformal anomaly (37) as a quantum effect that becomes important around the Big Rip. In such a case there is a phase transition and the Hubble evolution will be given by the solution of the FRW equation (41). Let us approximate to get some qualitative results, assuming 3+1 dimensions. Around  $t_s$  the curvature is large, and  $|\rho_A| \gg (3/\kappa^2)H^2 + k/a^2$ . Then  $\rho \sim -\rho_A$ , and from (40) we get

$$\dot{\rho} + 4H\rho = H \left[ -12b\dot{H}^2 + 24b'(-\dot{H}^2 + H^2\dot{H} + H^4) - (4b + 6b'')(\ddot{H} + 7H\dot{H} + 4\dot{H}^2 + 12H^2\dot{H}) \right]. \quad (43)$$

We assume that the energy density, which diverges in the classical case, behaves now as

$$\rho \sim (t_s - t)^\lambda, \quad (44)$$

where  $\lambda$  is some negative number. By using the energy conservation equation  $\dot{\rho} + 3H(1+w)\rho = 0$ , the Hubble parameter goes as  $H \sim 1/(t_s - t)$ . We can check if this assumption is correct in the presence of quantum effects by inserting both results in Eq. (43). We get

$$\rho \sim 3H^4(-13b + 24b'). \quad (45)$$

Hence as  $b > 0$  and  $b' < 0$ ,  $\rho$  becomes negative, which is an unphysical result. Thus  $\rho$  should not go to infinity in the presence of the quantum correction. This is the same result as obtained in Ref.[5] where numerical analysis showed that the singularity is moderated by the conformal anomaly, so that the violation of the bound that naturally occurs in the classical case can be avoided/postponed when quantum effects are included.

### B. Sudden singularity

This kind of singularity is also problematic with respect to the bounds, but as the energy density  $\rho$  does not diverge, the violation of the bound may be avoided for some special choices. The sudden singularity can be described by the solution (36) with  $0 < m < 1$ , and constants  $H_{0,1} > 0$ . Then the scale factor goes as

$$a(t) \propto \exp \left[ -\frac{H_1}{m+1}(t_s - t)^{m+1} + H_0 t \right], \quad (46)$$

which gives  $a(t) \sim e^{h_0 t}$  (de Sitter) close to  $t_s$ . From the first FRW equation the total energy density becomes

$$\rho = H^2(t) + \frac{1}{a^2} = [H_1(t_s - t)^m + H_0]^2 + \exp \left[ 2\frac{H_1}{m+1}(t_s - t)^{m+1} - 2H_0 t \right], \quad (47)$$



which tends to a constant  $\rho \sim H_0^2 + e^{-2H_0 t_s}$  for  $t \rightarrow t_s$ . Then the Casimir energy grows as  $E_C \propto H_0^2 a^n + a^{n-2}$ , while  $E_{\text{BH}} \propto a^{n-2}$  close to  $t_s$ . The BH energy grows slower than the Casimir energy, and the bound is violated for a finite  $t$ . However, by a specific choice of the coefficients, the violation of the bound (32) could be avoided. For  $H_0 = 0$ , and by some specific coefficients, the bound could be obeyed. In general, it is very possible that  $E_C$  exceeds its bound. In the presence of quantum corrections, the singularity can be avoided but the bound can still be violated, depending on the free parameters for each model. We may assume that in the presence of the conformal anomaly for  $n = 3$ , the energy density grows as [5]

$$\rho = \rho_0 + \rho_1 (t_s - t)^\lambda, \quad (48)$$

where  $\rho_0$  and  $\rho_1$  are constants, and  $\lambda$  is now a positive number. Then the divergences on the higher derivatives of the Hubble parameter can be avoided, as is shown in Ref.[5]. Nevertheless,  $E_C$  still grows faster than  $E_{\text{BH}}$ , such that the Universe has to be smaller than a critical size in order to hold the bound (32) as is pointed in Ref.[7] for the case of a vacuum dominated universe.

### C. Type III singularity

This type of singularity is very similar to the Big Rip, in spite of the scale factor  $a(t)$  being finite at the singularity. The solution (36) reproduces this singularity by taking  $-1 < m < 0$ . The scale factor goes as

$$a(t) = a_s \exp \left[ -\frac{H_1}{m+1} (t_s - t)^{m+1} \right], \quad (49)$$

where for simplicity we take  $H_0 = 0$ . Then, for  $t \rightarrow t_s$ , the scale factor  $a(t) \rightarrow a_s$ . To see how  $E_C$  behaves near the singularity, let us write it in terms of the time instead of the scale factor,

$$E_C \propto a_s^n H_1^2 (t_s - t)^{2m} + a_s^{n-2}, \quad (50)$$

where  $m < 0$ . Hence, the Casimir energy diverges at the singularity, while  $E_{\text{BH}} \propto a_s^{n-2}$  takes a finite value for the singularity time  $t_s$ , so the bound is clearly violated long before the singularity. Then, in order to maintain the validity of the bound (32), one might assume, as in the Big Rip case, that GR is not valid near or at the bound. Even if quantum effects are included, as was pointed in Ref.[5], for this type of singularity the energy density diverges more rapidly than in the classical case, so that the bound is also violated in the presence of quantum effects.

### D. Type IV singularity

For this singularity, the Hubble rate behaves as

$$H = H_1(t) + (t_s - t)^\alpha H_2(t). \quad (51)$$

Here  $H_1(t)$  and  $H_2(t)$  are regular function and do not vanish at  $t = t_s$ . The constant  $\alpha$  is not integer and larger than 1. Then the scale factor behaves as

$$\ln a(t) \sim \int dt H_1(t) + \int dt (t_s - t)^\alpha H_2(t). \quad (52)$$

Near  $t = t_s$ , the first term dominates and every quantities like  $\rho$ ,  $p$ , and  $a$  etc. are finite and therefore the bound (32) would not be violated near the singularity.

### E. Big Bang singularity

When the matter with  $w \geq 0$  coupled with gravity and dominates, the scale factor behaves as

$$a \sim t^{\frac{2}{n(1+w)}}. \quad (53)$$

Then there appears a singularity at  $t = 0$ , which may be a Big Bang singularity. Although the Big Bang singularity is not a future singularity, we may consider the bound (32) when  $t \sim 0$ . Since  $n(1+w) > 2$ , the energy density behaves

as  $\rho \sim a^{-n(1+w)}$  and therefore the Casimir energy behaves as  $E_C \sim a^{-nw}$ . On the other hand, we find  $E_{\text{BH}} \sim a^{n-2}$ . Then when  $n > 2$  or when  $n \geq 2$  and  $w > 0$ ,  $E_C$  dominates when  $a \rightarrow 0$ , that is, when  $t \rightarrow 0$ , and the bound (32) is violated. This tells us, as expected, that quantum effects become important in the early universe.

Above, we have thus explored what happens near the future cosmic singularities. We have seen that in general, and with some very special exceptions on the case of Type II and Type IV, the bound will be violated if one assumes the validity of GR close to the singularity. Even if quantum corrections are assumed, it seems that the bound will be violated, although in the Big Rip case the singularity may be avoided when quantum effects are incorporated. It is natural to suggest, in accordance with Verlinde, that the bound on the Casimir energy means a finite range for the validity of the classical theory. When this kind of theory becomes saturated, some other new quantum gravity effects have to be taken into account. We conclude that the universality of the bound (32) is not clear and may hold just for some specific cases, like the radiation dominated Universe.

#### IV. $F(R)$ -GRAVITY AND THE CARDY-VERLINDE FORMULA

It is known that modified gravity (for general introduction, see [14]) may be presented in the form of generalized fluid with inhomogeneous EoS [15]. This type of theories which became popular recently may pretend to unify the early-time inflation theory with the theory describing the late-time acceleration [16]. We specify here a modified  $F(R)$ -gravity modeled as an effective fluid and construct the corresponding CV formula for it. The action that describes  $F(R)$ -gravity is given by

$$S = \frac{1}{2\kappa^2} \int d^{n+1}x \sqrt{-g} (F(R) + L_m) , \quad (54)$$

where  $L_m$  represents the matter Lagrangian and  $\kappa^2 = 8\pi G$ . The field equations are obtained by varying the action (54) with respect to the metric  $g_{\mu\nu}$ ,

$$R_{\mu\nu} F'(R) - \frac{1}{2} g_{\mu\nu} F(R) + g_{\mu\nu} \square F'(R) - \nabla_\mu \nabla_\nu F'(R) = \kappa^2 T_{\mu\nu}^{(m)} . \quad (55)$$

Here  $T_{\mu\nu}^{(m)}$  is the energy-momentum tensor for the matter filling the Universe, and we have assumed a 1+3 spacetime for simplicity. For closed 3+1 FRW Universe, the modified FRW equations are expressed as

$$\begin{aligned} \frac{1}{2} F(R) - 3(H^2 + \dot{H}) F'(R) + 3H F''(R) \dot{R} &= \kappa^2 \rho_m , \\ -\frac{1}{2} F(R) + \left[ 3H^2 + \dot{H} + \frac{2}{a^2} \right] F'(R) - [(\partial_{tt} F'(R)) + 2H(\partial_t F'(R))] &= \kappa^2 p_m , \end{aligned} \quad (56)$$

where primes denote derivatives respect to  $R$  and dots with respect to  $t$ . These equations can be rewritten in order to be comparable with those of standard GR. For such a propose the geometric terms can be presented as an effective energy-density  $\rho_{F(R)}$  and a pressure  $p_{F(R)}$ ,

$$\begin{aligned} H^2 + \frac{1}{a^2} &= \frac{\kappa^2}{3F'(R)} \rho_m + \frac{1}{3F'(R)} \left[ \frac{RF'(R) - F(R)}{2} - 3H\dot{R}F''(R) \right] , \\ 2\dot{H} + 3H^2 + \frac{1}{a^2} &= -\frac{\kappa^2}{F'(R)} p_m - \frac{1}{F'(R)} \left[ \dot{R}^2 F'''(R) + 2H\dot{R}F''(R) + \ddot{R}F''(R) + \frac{1}{2}(F(R) - RF'(R)) \right] . \end{aligned} \quad (57)$$

Then, an EoS for the geometric terms can be defined as  $p_{F(R)} = w_{F(R)} \rho_{F(R)}$ . We can define an effective energy-density  $\rho = \rho_m/F'(R) + \rho_{F(R)}$  and pressure  $p = p_m/F'(R) + p_{F(R)}$ . Hence, for some special cases the formula for the entropy developed in the second section can be obtained in  $F(R)$ -gravity (for an early attempt deriving a CV formula in a specific version of  $F(R)$ -gravity, see [17]). For example, for an  $F(R)$  whose solution gives  $\rho \propto a^{-3(1+w_{\text{eff}})}$ , the formula for the entropy (9) is recovered although in general, as in the cases studied above, no such expression can be given. On the other hand, one could assume that the geometric terms do not contribute to the matter sector. Supposing a constant EoS matter fluid, the expression for the entropy is given by (9), although the cosmic Cardy formula (11) has not the same form and in virtue of the modified first FRW equation (56) the form of the Hubble entropy  $S_H$ , the total energy  $E$ , and the Bekenstein energy  $E_{\text{BH}}$ , will be very different. It is not easy to establish correspondence between two such approaches. Note that using the effective fluid representation the generalized CV formula may be constructed for any modified gravity.

Now we consider the case where  $F(R)$  behaves as

$$F(R) \sim R^\alpha, \quad (58)$$

when the curvature is small or large. Then if the matter has the EoS parameter  $w > -1$ , by solving (56) we find

$$a \sim \begin{cases} t^{\frac{2\alpha}{n(1+w)}} & \text{when } \frac{2\alpha}{n(1+w)} > 0 \\ (t_s - t)^{\frac{2\alpha}{n(1+w)}} & \text{when } \frac{2\alpha}{n(1+w)} < 0 \end{cases}. \quad (59)$$

Then there may appear a singularity at  $t = 0$ , which corresponds to the Big Bang singularity, or at  $t = t_s$ , which corresponds to the Big Rip singularity. Since the Casimir energy behaves as  $E_C \sim a^{-nw}$  but  $E_{\text{BH}} \sim a^{n-2}$ , only when  $t \rightarrow 0$ ,  $E_C$  dominates in case that  $n > 2$  and  $w \geq 0$  or in case that  $n \geq 2$  and  $w > 0$ . Even in the phantom phase where  $\frac{2\alpha}{n(1+w)} < 0$ , the bound (32) is not violated.

Let us now consider de Sitter space solution in  $F(R)$ -gravity (for review of CV formula in dS or AdS spaces, see [18]). As was pointed in Ref.[19], almost every function  $F(R)$  admits a de Sitter solution. This can be easily seen from the first FRW equation in (56). A de Sitter solution is given by a constant Hubble parameter  $H(t) = H_0$ ; then by inserting in (56) we obtain the following algebraic equation,

$$3H_0^2 = \frac{F(R_0)}{2F'(R_0)}. \quad (60)$$

Here  $R_0 = 12H_0^2$  and the contribution of matter is neglected. Then, for positive roots  $H_0$  of this equation, the corresponding  $F(R)$  leads to the de Sitter solution which may describe inflation or dark energy. In this case the formula for the entropy (9) can be reproduced for  $w = -1$ , and even the universal bound (32) can hold by taking a critical size of the Universe. The formula that relates the cosmic bounds in (35) is easily obtained also in  $F(R)$ -gravity for a de Sitter solution. In such a case one can identify

$$S_{\text{H}} = \frac{H_0 V}{2G}, \quad S_{\text{B}} = \frac{aV}{24G} \frac{F(R_0)}{F'(R_0)}, \quad S_{\text{BH}} = \frac{V}{2Ga}, \quad (61)$$

which corresponds to the first FRW equation written as  $S_{\text{H}}^2 + (S_{\text{B}} - S_{\text{BH}})^2 = S_{\text{B}}^2$ . Thus, one can conclude that dynamical entropy bounds are not violated for modified gravity with de Sitter solutions. Note that quantum gravity effects may be presented also as an effective fluid contribution. In case when de Sitter space turns out to be the solution, even with the account of quantum gravity the above results indicate that dynamical cosmological/entropy bounds are valid. In other words, the argument indicates the universality of dynamical bounds. It seems that their violation is caused only by future singularities if they are not cured by quantum gravity effects. Note that a large number of modified gravity theories do not contain future singularities; they are cured by higher derivatives terms.

## V. DISCUSSIONS

In summary, we have derived a generalized CV formula for multicomponent, interacting fluids, generalized in the sense that an inhomogeneous EoS (including viscous fluid) was assumed. We also considered modified  $F(R)$ -gravity, using its fluid representation. We showed that for some special cases the formula is reduced to the standard CV formula expressing the correspondence with 2d CFT theory. The dynamical entropy bound for all above cases was found. The universality of dynamical entropy bound near all four types of the future singularity, as well as the initial Big Bang singularity, was investigated. It was proved that except from some special cases of Type II and Type IV singularity the dynamical entropy bound is violated near the singularity. Taking into account quantum effects of conformally invariant matter does not improve the situation.

One might think that the dynamical entropy bound is universal and that its violation simply indicates that the situation will be changed with the introduction of quantum gravity effects. However, arguments given below indicate that it is not the case and that the future singularity is the domain where all known physical laws and equations are not valid. Indeed, taking account of quantum effects such as done in section VI does not improve the situation with respect to non-universality of the dynamical entropy bound. From another side, it was shown that the dynamical entropy bound is valid for the de Sitter solution. Having in mind that quantum gravity corrections may always be presented as a generalized effective fluid, one sees that the dynamical cosmological bound is not valid near the singularity (even when account is taken of Quantum Gravity). It is only when modified gravity (with or without quantum corrections) is regular in the future, like the models of Refs.[4, 20, 21] where the future universe is asymptotically de Sitter, that the dynamical bound remains valid. Hence, the problem of non-universality of dynamical entropy bound is related to the more fundamental question about the real occurrence of a future singularity. It remains a challenge to find any observational indications for the structure of the future universe.

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### Appendix A: Account of Casimir effect in the CV formula

A different way to account for the Casimir energy is to relate the effect to the single length parameter in the theory, that is the scale factor, and to assume that we can exploit the expression for the Casimir effect in a perfectly conducting spherical shell. The method consists essentially in identifying the scale factor with the radius of the shell. This proposal has been studied in Ref.[22], and as shown, could have some important effects on the cosmological history. Here, we want to deduce the expression for the entropy following this alternative approach. The expression for the Casimir energy according to [22] can be expressed as follows

$$E_C = \frac{C}{2La}, \quad (\text{A1})$$

where  $L$  is an auxiliary length that has been introduced due to the non-dimensional nature of  $a$ . This is the same form as the one encountered for a perfectly conducting shell (see Ref.[23]), where it was found that

$$C = 0.09235. \quad (\text{A2})$$

Hence, we assume that value of  $C$  is much less than the unity, which is physically reasonable in view of the conventional febleness of the Casimir force. Also, for simplicity we shall assume a 1 + 3 FRW Universe. The expression (A2) corresponds to a Casimir pressure

$$p_c = \frac{-1}{8\pi(La)^2} \frac{\partial E_c}{\partial(La)} = \frac{C}{8\pi L^4 a^4}, \quad (\text{A3})$$

and leads consequently to a Casimir energy density  $\rho_c \propto 1/a^4$ , which means that the Casimir fluid has an EoS parameter given by  $w_C = 1/3$  in order to obey the energy conservation equation (3). Then, the Casimir energy density and pressure are given by

$$p_c = \frac{C}{8\pi L^4 a^4}, \quad \rho_c = \frac{3C}{8\pi L^4 a^4}. \quad (\text{A4})$$

Now we assume that the Universe is filled with a perfect fluid  $p_m = w_m \rho_m$ , where  $\rho = E_m/V$ . By following the rescaling properties (6), and assuming now  $w_m = 1/3$ , the extensive and the Casimir energy are written as a function of the entropy,

$$E_m = \frac{\alpha}{4\pi a^w} S^{2/3}, \quad E_C = \frac{\beta C}{2\pi La} S^{2/3}. \quad (\text{A5})$$

The total energy is  $E = E_m + 1/2E_C$ . Then, the expression for the entropy is very similar to the one obtained in Ref.[6],

$$S = \frac{2\pi La}{\sqrt{\alpha\beta C}} \sqrt{E_C(2E - E_C)}, \quad (\text{A6})$$

where  $\alpha$  and  $\beta$  are arbitrary constants. We see that this expression, except from the constants, is equal to (10); the constants can be absorbed by  $\alpha\beta$ . Then, this approach also supports the formula (10), but like that, it is not possible to extend the formalism to matter fluids with arbitrary  $w_m$ . In that case the expression for the entropy would not be constant and the first law of thermodynamics would be violated.

Hence, we have obtained a twofold description of the Casimir effect. The Casimir energy appears explicitly in the entropy formula. Moreover, all quantities are constructed from an effective Casimir fluid.

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